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On Groups and the Lattices of Subgroups.

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INTRODUCTION

Let G be a group and $L(G)$ be the lattice formed of all subgroups of G. We say $L(G)$ belongs to G. G is called modular if $L(G)$ is modular, while if $L(G)$ is upper or lower semi-modular we call G upper or lower semi-modular respectively. The structure of modular groups was studied by K. IWASAWA. He gave necessary and sufficient conditions for a group to be modular when it is a finite group [1]) or it is such infinite group as has at least one element of infinite order [2]), while he showed the structure of such modular groups as have no element of infinite order [3]) under the following condition: any factor-group of any subgroup of them is a finite group if the lattice belonging to the factor-group is of finite dimensions. Let us say a group satisfies the condition (A) if it satisfies the above condition. In this paper we call a lattice L of infinite dimensions *upper semi-modular* (=u. s. modular) as well as in the case of finite dimensions if it satisfies the following condition: If $a, b \in L$ and a covers $a \frown b$, that is, $dim\ (a/a \frown b)=1$, then $a \smile b$ covers b. The lower semi-modularity (=l. s. modularity) is defined as a dualism to it.

The purpose of this paper is to study the structure of infinite u. s. modular groups under similar principles as IWASAWA's in the case of modular groups. It is shown that an u. s. modular group whose elements are all of finite order has similar structure as that of finite one if it satisfies (A) (Chapter I), and that such u. s. modular group as has at least one element of infinite order is nothing but a modular group if it has no perfect subgroup and satisfies (A) (Chapter II).

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1) K. IWASAWA(1), *Über die endlichen Gruppen und die Verbände ihrer Untergruppen,* Journal of the Faculty of Science, Imperial University of Tokyo, Vol. 4 (1941).

2) K. IWASAWA(2), *On groups and the lattices of their subgroups,* (in Japanese) III, Zenkoku Sizyô Sûgaku-Danwa-Kai No. 225 (1941).

3) K. IWASAWA(3), *On groups and the lattices of their subgroups,* (in Japanese) II, Isô-Sûgaku Vol. 4-1 (1942)

In the first place we notice that the following Theorem holds.

THEOREM 1. *A group G of finite order is u. s. modular if and only if it has such structure as follows:*

$$G = A_1 \times A_2 \times \dots \times A_r \times R_1 \times R_2 \times \dots \times R_s,$$

where the orders of any two direct factors are relatively prime, and R_i are Sylow-subgroups of G and are u. s. modular, (hence they are modular since a nilpotent group is always l. s. modular, therefore their structure is well known), and A_i are non-nilpotent directly indecomposable u. s. modular groups. A non-nilpotent directly indecomposable u. s. modular group A of finite order has such structure as follows:

$$A = (P_1 \times P_2 \times \dots \times P_t) \cup Q,$$

where P_i and Q are the Sylow-subgroups of A of order p_i and q respectively, and $p_i > q$ $i = 1, \dots, t$, and

1) *every P_i is an elementary abelian group,*

2) *Q is cyclic: $Q = \{ b \}$,*

3) *every $P_i \cup Q$ is non-nilpotent and P_i is normal in it, and furthermore, to every P_i correspond two numbers r_i and β_i such that $r_i \not\equiv 1$ mod p_i, $b\, a_i\, b^{-1} = a_i^{r_i}$ for any $a_i \in P_i$, $r_i^{q^{\beta_i}} \equiv 1$ mod p_i and $1 \leq \beta_i \leq \beta$, if q^β is the order of Q, and*

4) *any two β_i in 3) are distinct.*

This is obtained by a slight modification of the theorem which is given in IWASAWA (1) in the case of modular groups.

DEFINITION. We say that the groups, the non-nilpotent directly indecomposable u. s. modular groups that were given in Theorem 1, have J-type.

DEFINITION. If an element in a group is of infinite order or is of finite order, we call it U-element or E-element respectively.

In this paper $\{a_1, a_2, \dots\}$ with elements a_i in a group denotes the subgroup generated by them, while $\{\{a_1, a_2, \dots\}\}$ denotes the least normal subgroup that contains them. When there occurs no confusion we denote at times by A/B with two subgroups A, B of a group G such that $A \supset B$ a quotient in $L(G)$, and by $dim\ (A/B)$ its dimension.

CHAPTER I. U. S. MODULAR GROUPS WITH NO U-ELEMENT

1. In this chapter we determine the type of such u. s. modular infinite groups as have no U-element and satisfy the condition (A).

Let G be such, then we have

LEMMA 1. If for a prime p, every element of p-power order in G is commutative with any element of order prime to p in G, then all elements of p-power order in G form a characteristic subgroup of G.

PROOF. Let a, b be any two elements of p-power order in G. According to (A), $\{a, b\}$ is a finite group and either is a p-group or has J-type. In our case the p-Sylow-subgroup of it is necessarily normal in it, hence contains a and b. Then $\{a, b\}$ is a p-group. Therefore, our assertion is obvious, q. e. d.

LEMMA 2. If there exist in G some two elements a, b' of prime-power order p_1^{α}, $q^{\beta'}$ respectively that are not commutative, and if $p_1 > q$, then there exist such (finite or infinite) elementary abelian normal p_i-subgroups P_i, $i = 1, \dots n$ that $(P_1 \cup \dots \cup P_n) \cup \{b\}$ contains any element whose order has no other prime factor than $p_1, p_2, \dots p_n$, q, where $\{b\}$ is a maximal cyclic q-group containing $\{b'\}$. Furthermore, any finite subgroup of $(P_1 \cup \dots \cup P_n) \cup \{b\}$ containing $\{b\}$ has J-type. Hence of course $(P_1 \cup \dots \cup P_n) = P_1 \times \dots \times P_n$, and for every P_i a positive integer r_i is uniquely determined such that $b a_i b^{-1} = a_i^{r_i}$ for any $a_i \in P_i$, and $n \leq \beta$ if the order of b is q^{β}.

PROOF. According to (A), $\{a_1, b'\}$ has J-type, hence the order of a_1 is p_1 and $b' a_1 b'^{-1} = a_1^{r_1'}$. For any element a_1' of p_1-power order $\{a_1', a_1, b'\}$ has also J-type. Since q-Sylow-subgroup is not normal in it, we can see p_1-Sylow-subgroup of it is normal in it, and so contains both a_1, a_1', whence $a_1 a_1' = a_1' a_1$, and the order of a_1' is p_1. Then all elements of p_1-power order in G form an elementary abelian p_1-group P_1', and any finite subgroup of $P_1' \cup \{b'\}$ that contains $\{b'\}$ has J-type.

Now, if another element a_2 of p_2-power order with $p_2 \neq p_1$ is not commutative with b', then $\{a_1, a_2, b'\}$ has J-type. Since q-Sylow-subgroup is not normal in it, $a_1 a_2 = a_2 a_1$ and $p_2 > q$. Then we can conclude as above that all elements of p_2-power order in G form an elementary abelian

p_2-group P_2', and that $P_1' \cup P_2' = P_1' \times P_2'$. Continuing this consideration we obtain such group $(P_1' \times \dots \times P_s')$ as is the minimal one of those subgroups of G that contain all elemets ot order prime to q that are not commutative with b', where $s \leq \beta'$ (cf. the condition about β_i in Theorem 1).

Now suppose that there exists an element b'' of q-power order which is not contained in $(P_1' \times \dots \times P_s') \cup \{b'\}$. Let a be any element ($\neq 1$) in P_1', then $\{a, b', b''\}$ has J-type and is greater than $\{a, b'\}$. Hence there exists such b_1 that $\{a, b_1\} \ni b''$ and $b_1^{q^\varepsilon} = b'$ for some $\varepsilon > 0$. Furthermore, there exists a number r_1 and $b_1 a b_1^{-1} = a^{r_1}$, $r \not\equiv 1$, $r_1^{q^{\beta_1}} \equiv 1 \mod p_1$, where we take for β_1 the least value that satisfies the above condition. Since r_1 attains at most $p_1 - 2$ values, the values of β_1 cannot exceed a sufficiently large constant for any b'' and b_1 in G. But $a^{r_1'} = bab'^{-1} = b_1^{q^\varepsilon} a b_1^{-q^\varepsilon} = a^{r_1^{q^\varepsilon}}$ and $r_1' \not\equiv 1 \mod p_1$, hence $r_1^{q^\varepsilon} \not\equiv 1 \mod p_1$. This shows that $\varepsilon < \beta_1$, i. e., ε is bounded. Hence there exists a maximal finite cyclic q-group $\{b\}$ containing $\{b'\}$. Then considering about b instead of b' we have as above the desired group $(P_1 \times \dots \times P_n) \cup \{b\}$, q. e. d.

For brevity we say also that the infinite group given in Lemma 2 has J-type. Then from Lemma 1, Lemma 2 we have

THEOREM 2. *G is a direct product (containing finite or infinite number of factors) of u. s. modular p-groups satisfying (A) (cf. the next remark) and J-type groups, where any of these factors may be of finite or infinite order and any two elements from different factors have relatively prime orders. Conversely such direct product of groups is always u. s. modular and satisfies (A).*

PROOF. A direct product of u. s. modular factors is again u. s. modular if any pair of elements from different factors have always relatively prime orders. And that a J-type group is u. s. modular is quite obvious, q. e. d.

REMARK. Infinite u. s. modular p-groups satisfying (A) are always modular. This is obvious, because K. IWASAWA determined the structure of infinite modular p-groups satisfying (A) using only the property that every finite factor-group of any subgroup is modular, which property have our u. s. modular p-groups also. (cf. IWASAWA [3])

CHAPTER II. U. S. MODULAR GROUPS CONTAINING U-ELEMENTS

2. In this chapter we study the structure of such u. s. modular groups as contain at least one U-element and satisfy (A).

LEMMA 3. An u. s. (l. s.) modular lattice of infinite dimensions has no composition series of finite length that combines I and O.

PROOF. Trivial.

THEOREM 3. *Let G be any u. s. modular group, then all E-elements in G form a characteristic subgroup of G.*

PROOF. Let a, b be E-elements in G and $A = \{a\}, B = \{b\}$. Then the quotient $A/A \frown B$ is of finite dimensions. Let one of its lattice-theoretical composition series be $A \supset A_1 \supset A_2 \supset \dots \supset A_r \supset A \frown B$. Then $A \smile B \supseteq A_1 \smile B \supseteq \dots \supseteq A_r \smile B \supseteq B$ contains a composition series of $A \smile B/B$. Hence $A \smile B/B$ and therefore $A \smile B/1$ itself is of finite dimensions (Lemma 1). Hence every element in $A \smile B$ is E-element. Then our assertion is obvious, q. e. d.

REMARK. In this proof (A) is not used.

LEMMA 4. Let G be u. s. modular and satisfy (A), and let $a, b \in G$, a be U-element and b be E-element. Then all E-elements in $\{a\} \smile \{b\}$ are contained in $\{b\}$, i. e., $\{b\}$ is normal in it. If further G is l. s. modular as well as u. s. modular, then the same assertion holds without (A).

PROOF. We can assume that the order of $\{b\}$ is a prime number without loss of generality. Suppose that $E = \{b, aba^{-1}, a^{-1}ba, a^2ba^{-2}, a^{-2}ba^2, \dots\} \neq \{b\}$.

1) If E is a finite p-group, then E is an elementary abelian group, because in an u. s. modular p-group, any two elements of prime order are commutative. Let r be the minimal number (>0) such that $a^rba^{-r} \in \{b\}$. Then

$$\{a^r\} \smile E \supset \{a^r\} \smile \{b\} \supset \{a^r\}, \qquad \dots\dots\dots\dots (1)$$

for $\{b\}$ is normal in $\{a^r\} \smile E$. Since $E \frown \{a\} = 1$ and E is normal in $\{a\} \smile \{b\}$, we have

$$dim\,(\{a\} \smile E/\{a^r\} \smile E) = dim\,(\{a\}/\{a^r\}). \quad \dots\dots (2)$$

According to the u. s. modularity $dim\,(\{a\} \smile E/\{a\}) = 1$, because $\{a\} \smile E = \{a\} \smile \{b\}$. Then we have by (2)

$$dim(\{a\} \bigvee E/\{a^r\}) = dim(\{a\} \bigvee E/\{a\}) + di$$
$$= 1 + dim(\{a\} \bigvee E/\{a^r\}$$

But by (1) $dim(\{a^r\} \bigvee E/\{a^r\}) \geqq 2$, then we have

$$dim(\{a\} \bigvee E/\{a^r\}) = dim(\{a\} \bigvee E/\{a^r\} \bigvee E) + dim(\{a^r\} \bigvee E/\{a^r\})$$
$$\geqq dim(\{a\} \bigvee E/\{a^r\} \bigvee E) + 2$$

in contradiction with above. Hence E is not a finite group.

2) Let E be an infinite p-group. As above E is also an elementary abelian group. Put $E_1 = \{b, a^2ba^{-2}, a^{-2}ba^2, \ldots, a^{2n}ba^{-2n}, a^{-2n}ba^{2n}, \ldots\}$. Then $E_1 \neq E$, (and of course E_1 is normal in $\{a^2\} \bigvee \{b\}$). To see this, assume $E = E_1$. Then for some integer n $\{b, a^2ba^2, a^{-2}ba^2, \ldots\ldots,$ $a^{2n}ba^{-2n}\} \ni aba^{-1}$. If the above group is the minimal one of those that are of above quality, then $E_0 = \{b, aba^{-1}, a^{-1}ba, a^2ba^{-2}, a^{-2}ba^2, \ldots\ldots,$ $a^mba^{-m}\} \ni a^{2n}ba^{-2n}$, where $m \begin{cases} = -(2n-1) & if\ n > 0 \\ = 2\,|\,n\,| & if\ n < 0. \end{cases}$

Then $E_0 = E$, i.e., E is of finite order in contradiction with the hypothesis. Hence $E \neq E_1$. According to the u.s. modularity $dim(\{a\} \bigvee \{b\}/\{a^2\}) = dim(\{a\} \bigvee \{b\}/\{a\}) + dim(\{a\}/\{a^2\}) = 2$, but since $dim(\{a^2\} \bigvee \{b\}/\{a^2\}) = 1$, we conclude $dim(\{a\} \bigvee \{b\}/\{a^2\} \bigvee \{b\}) = 1$. Then since $\{a^2\} \bigvee \{b\} \not\ni aba^{-1} \notin E$, must hold $\{a\} \bigvee \{b\} = \{a^2\} \bigvee \{b\} \bigvee \{aba^{-1}\} \leqq \{a^2\} \bigvee E$. But the converse is obvious, therefore we have $\{a\} \bigvee \{b\} = \{a\} \bigvee E = \{a^2\} \bigvee E$. But this is impossible because E is normal in $\{a\} \bigvee \{b\}$ and $\{a\} \bigwedge E = \{a^2\} \bigwedge E = 1$. Hence E is not a p-group.

3) Suppose that E has a J-type group as one of its direct factors. Then we can find always a characteristic subgroup E_1 of E such that $E \neq E_1 \neq 1$. Then since E_1 is normal in $\{b\} \bigvee \{a\}$ and $E \bigwedge \{a\} = E_1 \bigwedge \{a\} = 1$, necessarily $\{a\} \bigvee E \supset \{a\} \bigvee E_1 \supset \{a\}$. But this is a contradiction, because $dim(\{a\} \bigvee E/\{a\}) = dim(\{a\} \bigvee \{b\}/\{a\}) = 1$. Hence the first part of the theorem is proved.

If G is l.s. modular then $dim(E/\{a\} \bigwedge E) = 1$, because $dim(\{a\} \bigvee E/\{a\}) = 1$. But $\{a\} \bigwedge E = 1$, then must $E = \{b\}$. This complete the proof of the second part of our theorem, q.e.d.

REMARK. The second part of this theorem shows the reason why we can determine the structure of modular groups containing U-elements without using (A).

THEOREM 4. *Let G be u.s. modular and satisfy (A). If $G \neq E$, then E is abelian, where E is the characeteristic subgroup of G formed of all E-elements.*

PROOF. Now that Lemma 4 is proved, the proof in IWASAWA (2) in case where G is modular remains valid term by term in our case, q. e. d.

Now we have seen the structure of E, then we shall study the structure of G/E and G itself in the rest of this paper. Hereafter, throughout coming paragraphs, E preserves the same significance as in Theorem 4, if no other mention is made.

3. We shall study the u.s. modular groups in the following two cases:

I) when for any two elements $\bar{a}$, $\bar{b}$ in G/E, $\{\bar{a}\} \frown \{\bar{b}\} \neq \bar{1}$, and

II) when there are a pair of elements $\bar{a}$, $\bar{b}$ in G/E such that $\{\bar{a}\} \frown \{\bar{b}\} = \bar{1}$.

The purpose of this paragraph is to show that in the case I) G/E is abelian, and that G is modular if it satisfies (A). According to IWASAWA (2) the following lemma holds.

LEMMA 5. Let G be any group, $a, b \in G$, and a be U-element. If $\{a\} \frown \{b\} \neq 1$ and $ba^{\alpha}b^{-1} = a^{\beta}$ for some α, β, then $\alpha = \beta$, i.e., $ba^{\alpha} = a^{\alpha}b$.

The proof is very easy and omitted (see the first part of the proof of Lemma 2 in IWASAWA (2)).

THEOREM 5. *Let G be u.s. modular and have no other E-element than the unit. If $a, b \in G$ and $\{a\} \frown \{b\} \neq 1$, then $\{a\} \smile \{b\}$ is a cyclic group.*

PROOF. If we can prove $\{a\} \smile \{b\}$ is abelian, then it is quite obvious that it is cyclic, for according to our hypothesis $\{a\} \smile \{b\}$ is free abelian. Hence we shall prove $bab^{-1} = a$. To see this suppose that $bab^{-1} \neq a$. Put $\{a^{\mu}\} = \{a\} \frown \{b\}$. Then $\{a^{\mu}\}$ is contained in the center of $\{a\} \smile \{b\}$, therefore $(bab^{-1})^{\mu} = a^{\mu}$. This shows we can put $\{c\} = \{a\} \frown \{bab^{-1}\}$, and $\{c\} \neq 1$. If $a^{\alpha} = ba^{\beta}b^{-1} = c$, then according to Lemma 3 $c = a^{\alpha} = ba^{\alpha}b^{-1}$. Let p be a prime number and $p \mid \alpha$. Put $a_1 = a^{\alpha/p}$, and $a_2 = (bab^{-1})^{\alpha/p}$, then $a_1^{p} = a_2^{p} = c$, and $\{a_1\} \frown \{a_2\} = \{c\}$. According to the

u. s. modularity $\{a_1\} \bigcup \{a_2\}/\{c\}$ contains no U-element. Hence for some $\nu \neq 0$ $(a_1a_2^{-1})^\nu \in \{c\}$. Now put $(a_1 a_2^{-1})^\nu = c^\gamma$. But $a_1a_2^{-1} \neq 1$, hence it is a U-elemnent, whence $c^\gamma \neq 1$, i. e.,. $\gamma \neq 0$. (We can assume there $\gamma > 0$, taking $a_2a_1^{-1}$ instead of $a_1a_2^{-1}$ if necessary.) Put $A = \{a_1c^\gamma\}$, $B = \{a_2c^\gamma\}$. Then $A \bigwedge B = \{(a_1c^\gamma)^p\} = \{(a_2c^\gamma)^p\}$, To see this suppose $(a_1c^\gamma)^{p_1} = (a_2c^\gamma)^{p_2}$ ($\neq 1$ obviously). Then $a_1^{p_1}c^{\gamma p_1} = a_2^{p_2}c^{\gamma p_2}$, hence $a_1^{p_1} = a_2^{p_2}c^{\gamma(p_2-p_1)} = a_2^{p_2+p\gamma(p_2-p_1)}$. Thus we see $p_1 \geqq p$ according to the definition of p. But of course $(a_1c^\gamma)^p \in A \bigwedge B$. Hence our assertion is true. This shows $dim\,(A/A \bigwedge B) = dim\,(B/A \bigwedge B) = 1$. On the other hand, $A \bigvee B \ni (a_1c^\gamma)(a_1c^\gamma)^{-1} = a_1a_2^{-1}$, hence of course $A \bigvee B \ni (a_1a^{-1}_2)^\nu = c^\gamma$. Thus we have $A \bigvee B = \{a_1c^\gamma, a_2c^\gamma, c^\gamma\} = \{a_1, a_2\} = \{a_1\} \bigvee \{a_2\}$. But $\{a_1\} \neq A$, for $a_1c^\gamma = a_1^{1+p\gamma}, p \neq 0, \gamma > 0$. Hence

$$\begin{aligned} dim\,(A \bigvee B/A) = dim\,(\{a_1\} \bigvee \{a_2\}/A) &= dim\,(\{a_1\} \bigvee \{a_2\}/\{a_1\}) \\ &\quad + dim\,(\{a_1\}/A) \\ &= 1 + dim\,(\{a_1\}/A) > 1. \end{aligned}$$

But this is a contradiction, because G is u. s. modular and $dim\,(A/A \bigwedge B) = 1$, q. e. d.

By Theorem 3 we see G/E is an abelian group of rank 1 if G has the quality in case I). Now we determine the structure of such groups.

LEMMA 6. Let G be a non-abelian u. s. modular group satisfying (A). If $G/E = \{z\}$ is free cyclic, then G has such structure as follows:

$$z\,a\,z^{-1} = a^{\alpha(p)} \qquad \ldots\ldots \quad (*)$$

holds for all $a \in P$, where P is the maximal p-subgroup of E, therefore of course a direct factor of E (observe that E is abelian), and $\alpha(p)$ is a p-adic number satisfying $\alpha(p) \equiv 1 \mod p$ (but if $p = 2$ especially, then $\alpha(2) \equiv 1 \mod 4$), and is uniquely determined mod p^n except its inverse, where p^n is the highest one of all orders of elements in P. (p^n may be infinite, but the meaning of our assertion is obvious.) Conversely if we extend an abelian group containing only E-elements by a free cyclic group with (*), then we have a quasi-Hamiltonean group, i. e., any two subgroups of which are commutative, and therefore it is a modular group.

PROOF. The proof in IWASAWA (2) in case where G is modular remains valid in our case with natural modifications of terminologies, if we observe that an u. s. modular group of order $p^2 q$ with prime p, q such that $p > q$, is modular, q. e. d.

By Lemma 6 we are lead to the following theorem about the desired general case.

THEOREM 6. *Let G be a non-abelian u. s. modular group satisfying (A). If G/E is abelian and of rank 1, then it has such structure as follows:*

Let P be the p-component of E, i. e., the largest p-subgroup of E, (which is a direct factor of course), and let $G = \{E, z_1, z_2, \dots\}$, where z_i are all U-elements. Then for any $a\,(\neq 1) \in P$ holds

$$z_i\, a\, z_i^{-1} = a^{\alpha_i(p)}$$

where $\alpha_i(p)$ are p-adic numbers that have the same property as $\alpha(p)$ in lemma 6. Furthermore,

$$\alpha_{i+1}(p)^{p_i} \equiv \alpha_i(p) \bmod p^n \text{ (as to the significance of } p^n \text{ see Lemma 6)}$$

$$z_{i+1}^{p_i} = z_i e_i$$

$$z_{i+1} z_i z_{i+1}^{-1} = z_i\, e_i^{\,1-\alpha_{i+1}}$$

where p_i are prime numbers determined only depending on i, and $e_i \in E$, and α_{i+1} are such numbers that $z_{i+1}\, e_i\, z_{i+1}^{-1} = e_i^{\alpha_{i+1}}$. Conversely if we extend an abelian group E without U-elements with above relations for a sequence $\{p_i\}$ of prime numbers and a sequence $\{e_i\}$ of elements of E and a set of systems $\{\alpha_i(p);\ i = 1, 2 \dots\}$ of p-adic numbers corresponding to each prime component of E, we have always a quasi-Hamiltonean group, i. e., have a modular group.

PROOF. The proof in IWASAWA (2) in case where G is modular remains valid exactly term by term in this case, q. e. d.

4. In the following we study the structure of such u. s. modular groups as have the property in case II) (see the remark at the beginning of paragraph 3). In this case it seems to be difficult to determine the structure of them if there is no other assumption about it. However, for example, under the assumption that they have no perfect subgroups, we can prove they are necessarily abelian. In the following we shall show it. In case where the group is modular it was shown in IWASAWA (2) that the same is true without the assumption about perfect subgroups, (and of

course without (A)). Now we proceed step by step.

LEMMA 7. Let $G = \{a\} \bigvee \{b\}$ be u.s. modular and $E = 1$, i.e., let G have no other E-element than the unit. If $ba^{\beta} b^{-1} = a^{\beta\alpha}$, then must hold $bab^{-1} = a$, i.e., G is abelian.

PROOF. If $\{a\} \bigwedge \{b\} \neq 1$, then the assertion is obvious according to Theorem 5. Hence assume $\{a\} \bigwedge \{b\} = 1$.

1) Firstly we show $bab^{-1} = a^{\alpha}$. Since $\{bab^{-1}\} \bigwedge \{a\} \neq 1$ and $E = 1$, we see $\{bab^{-1}\} \bigvee \{a\}$ is cyclic (Theorem 5). Then it is obvious.

2) $\alpha \neq -1$. Otherwise, $bab^{-1} = a^{-1}$. Then $b^2ab^{-2} = a$, i.e., b^2 belongs to the center of G. But since $\{a^4\}$ is normal in G, we see $\bar{G} = \{a\} \bigvee \{b\}/\{a^4\} \bigvee \{b^2\}$ is of order 8. Obviously $\bar{G}$ is not u.s. modular, and it is a contradiction. Hence $\alpha \neq -1$.

3) By 1), 2) it suffices to show that never $|\alpha| > 1$. Suppose $|\alpha| > 1$. Put $A = \{ab\} \bigvee \{b^{mp}\}$, where p is a prime number and m is a positive integer. Put $H = \bigvee_{n=0}^{\infty} \{b^{-n}ab^n\}$, then it is the least normal subgroup of G that contains a. Furthermore, $H \bigwedge A$ is also normal in G, because $\bigvee_{n=0}^{\infty} b^{-nr} (A \bigwedge H)\, b^{nr} = \bigvee_{n=0}^{\infty} b^{-n} (A \bigwedge H)\, b^n$ for any number $r > 0$, and this is normal in G, where if we put $r = mp$, then it coincides with $A \bigwedge H$. Now put $B = (A \bigwedge H) \bigvee \{b^m\}$. Then $B \bigwedge H = A \bigwedge H$ and $dim\,(B/A \bigwedge B) = 1$. We shall show that if we take suitable p, m, then $dim\,(A \bigvee B/A) > 1$ in contradiction with the u.s. modularity. To see this it suffices to show there exists such normal subgroup N of G that $A \bigwedge H \subset N \subset (A \bigvee B) \bigwedge H$, for it implies obviously $A \subset AN \subset A \bigvee B$. But we can observe for any positive integer n holds $(\{ab\} \bigvee \{b^n\}) \bigwedge H = \{\{a^{1+\alpha+\cdots+\alpha^{n-1}}\}\} = \{\{a^{\alpha^n-1/\alpha-1}\}\}$. To see this, let $w = \prod_{\nu=1}^{r} (a\,b)^{s_\nu}\, b^{nt_\nu}$ $(\in \{a\,b\} \bigvee \{b^n\})$ be contained in H, where $s_\nu, t_\nu \gtreqless 0$. Then $w = a^x b^y$, where

$$x = \textstyle\sum_{\nu=1}^{r} (\alpha^{s_\nu} - 1/\alpha - 1)\ \alpha^{n(t_1 + \cdots + t_{\nu-1}) + s_1 + \cdots + s_{\nu-1}}$$
$$y = n\,(t_1 + \ldots + t_\nu) + s_1 + \ldots + s_\nu$$

But since $w \in H$, necessarily $y = 0$, hence we can see easily $x/\alpha^n - 1$ is an integer, i.e., $w \in \{\{a^{\alpha^n-1/\alpha-1}\}\}$. Thus we have $(\{ab\} \bigvee \{b^n\}) \bigwedge H \leq \{\{a^{\alpha n-1/\alpha-1}\}\}$. But the converse is obvious, hence we have $A \bigwedge H = \{\{a^{\alpha^{mp}-1/\alpha-1}\}\}$, and $(A \bigvee B) \bigwedge H = \{ab\} \bigvee \{b^m\} = \{\{a^{\alpha^m-1/\alpha-1}\}\}$. But since

$$a^{\alpha^{mp}-1/\alpha-1} = (a^{\alpha^{m}-1/\alpha-1})^{\alpha^{m(p-1)}+\alpha^{m(p-2)}+\dots+\alpha^{m}+1},$$

in order to see there exists such N that $A \frown H \subset N \subset (A \smile B) \frown H$, it is sufficient to show that for suitable p and m $\alpha^{m(p-1)} + \alpha^{m(p-2)} + \dots + \alpha^m + 1$ is not a prime number. (Observe that this is prime to α). But it is trivial that we can find such p and m, q. e. d.

LEMMA 8. Let $G = \{a\} \smile \{b\}$ and have no other E-element than the unit. If $ba^{\beta}b^{-1} = a^{\alpha}$ with $(\alpha, \beta) \neq \alpha, \beta$, then G is not u. s. modular.

PROOF. Assume that G is u. s. modular. Then we can suppose $(\alpha, \beta) = 1$ without loss of generality (cf. lemma 7 proof 1)). Put $\{\{a\}\} = H$. Then H is an abelian group of rank 1, hence we can write $bab^{-1} = a^{\alpha/\beta}$. Then we can proceed quite analogously as in Lemma 7, q. e. d.

LEMMA 9. Let $G = \{a\} \smile \{b\}$ be u. s. modular, $E = 1$ and $\{a\} \frown \{b\} = 1$. If G has an abelian subgroup A whose index in G is finite, then G is itself abelian.

PROOF. Since G/A is a finite group, for some number $\alpha, \beta\ (\neq 0)$ $a^{\alpha}, b^{\beta} \in A$, i. e., $a^{\alpha}b^{\beta} = b^{\beta}a^{\alpha}$. Put $b^{\beta} = b_1$, then $\{b_1ab_1^{-1}\} \frown \{a\} \neq 1$. Hence according to Theorem 5 it is a cyclic group. But $\{b_1ab_1^{-1}\}/\{a^{\alpha}\}$ and $\{a\}/\{a^{\alpha}\}$ have the same order. Hence we have $b_1ab_1^{-1} = a^{-1}$. But according to Lemma 7 necessarily $b_1ab_1^{-1} = a$. Thus we have $\{aba^{-}\} \frown \{b\} \neq 1$. Hence as above we can conclude $ab = ba$, q. e. d.

LEMMA 10. Let G be u. s. modular and satisfy (A), and let $E \neq 1$. If G/E is an abelian group whose rank is not 1, then G is itself abelian.

PROOF 1) Let $\{h, a, b\} = G$, where let $h^{p^m} = 1$ and a, b be U-elements. Suppose that $G/\{h\} = \{a\} \times \{b\}$ (i. e. free abelian) and G is not abelian. Let G' be the commutator-subgroup of G, then $G' \subseteq \{h\}$. Therefore, we can assume $G' = \{h^{p^{m-1}}\}$ without loss of generality, (if necessary, take some factor-group of G). Then $aha^{-1} = h^{r_1}$, $bhb^{-1} = h^{r_2}$ for some numbers r_1, r_2 and $r_1 \equiv r_2 \equiv 1 \bmod p$ (cf. Lemma 6).

a) Suppose that $r_1 = r_2 = 1$. Then $\{h\}$ is contained in the center of G. But since $bab^{-1}a^{-1} \in \{h^{p^{m-1}}\}$, $ba^pb^{-1} = a^p$. Thus we see $\{b, a^p, h\}$ is abelian. According to $\{b\} \frown \{a^p\} = 1$ we see $\{b\} \smile \{a^p\} \not\supseteq \{h^{p^{m-1}}\}$, hence $dim(\{b, a^p, h^{p^{m-1}}\}/\{b, a^p\}) = 1$. But since $dim(\{b, a\}/\{b, a^p\}) \leq 1$

(u. s. modularity) and $\{b, a\} \geqq \{b, a^p, h^{p^{m-1}}\}$, we see $\{b, a\} = \{b, a^p, h^{p^{m-1}}\}$. This shows $\{b, a\}$ is abelian, and it implies G is itself abelian in contradiction with the first hypothesis.

b) Suppose that $r_2 \neq 1$. Since $G' = \{h^{p^{m-1}}\}$, we can assume $r_2 = 1 + p^{m-1}$ (if necessary we take a power of b instead of b). Firstly we show if we take some $ah^s = c$, then holds $cbc^{-1}b^{-1} = 1$. If $aba^{-1}b^{-1} = 1$, then it is trivial. Hence suppose $aba^{-1}b^{-1} = h^{p^{m-1}}$ (if necessary, take. a power of h). If $b(ah^s)b^{-1}(ah^s)^{-1} = 1$, then $sr_1(r_2 - 1) \equiv p^{m-1} \mod p^m$. But since $r_2 - 1 = p^{m-1}$, it suffices to solve $sr_1 \equiv 1 \mod p$. In any case we can find desired c. Obviously $c^{p^m} = a^{p^m}$, and this shows $\{c\} \frown \{b\} = 1$. Hence we have

$$m = dim(\{c\}/\{c^{p^m}, b\} \frown \{c\}) \geqq dim(\{a\}/\{c^{p^m}, b\} \frown \{a\}),$$

while $dim(\{a, b\}/\{c^{p^m}, b\}) \leqq m$ (u. s. modularity). And obviously $dim(\{c, b\}/\{c^{p^m}, b\}) = m$, for $\{c, b\}$ is free abelian. But $\{a, b\} \supset \{c, b\}$, for the left-hand side is non-abelian. This is a contradiction.

2) Let G be any group that has properties given in the theorem. According to Lemma 4, Theorem 4, Lemma 6 we can reduce our case to 1), q. e. d.

LEMMA 11. Let $G = \{a\} \smile \{b\}$ be u. s. modular and $E = 1$. If there exists such normal subgroup B that $\{b\} \leqq B \subset G$ and $B \frown \{a\} \doteq 1$, and if G has no perfect subgroup, then G is abelian.

PROOF. If $B = \{b\}$, then G is abelian according to Lemma 7. Hence assume $B \neq \{b\}$.

I) Let B' be the commutator-subgroup of B and G/B' be not abelian.

1) Suppose that $B' \frown \{b\} = 1$.

i) If the rank of B/B' is not finite, then it is a free abelian group. To see this put $a^nba^{-n} = b_n$. Let us denote by $\bar{c}$ the element of B/B' corresponding to $c \in B$. Let r be the minimal number such that $\{\bar{b}_0, \bar{b}_1, \ldots, \bar{b}_r\} \frown \{\bar{b}_{r+1}\} \neq \bar{1}$. Then $\{\bar{b}_0\} \frown \{\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_{r+1}\} \neq \bar{1}$. This shows the rank of B/B' is not greater than $r+1$ in contradiction with our hypothesis. But this is impossible according to Lemma 4, as can easily be seen if we consider $(G/B')/(B^*/B')$ with such B^*/B' as formed of all m-power elements of B/B'.

ii) Suppose that the rank of B/B' is finite. Let us denote by B^*/B' the normal subgroup formed of all E-elements in G/B', where of course $B \supset B^*$ and $\{b\} \frown B^* = 1$. If the rank of B/B^* is 1, then according to Lemma 7, Lemma 8 G/B is abelian, whence the rank of G/B^* is 2 and G/B' is itself abelian according to Lemma 10. But this is a contradiction. Hence we assume the rank of B/B^* is not 1. Now we write $\bar{c}$ if c $(\in G)$ is considered mod B^*. Put $b_n = a^n b a^{-n}$ for $n \gtreqless 0$. Let $G_0 = \{\bar{b},$ $\bar{b}_1, \ldots, \bar{b}_{r-1}\}$ be free abelian and $\bar{b}_0^{m}\bar{b}_1^{x_1}\ldots\bar{b}_{r-1}^{x_{r-1}}\bar{b}_r^{l} = \bar{1}$ with $l \neq 0$. Put G_s' $= G_0 \smile \{\bar{b}_r\} \smile \{\bar{b}_{-1}\} \smile \{\bar{b}_{r+1}\} \smile \{\bar{b}_{-2}\} \smile \ldots \smile \{\bar{b}_{-s}\} \smile \{\bar{b}_{r+s}\}$ and $G_s =$ $G'_{s-1} \smile \{\bar{b}_{-s}\}$, then $G_0 \subset G_0' \subset G_1 \subset G_1' \subset G_2 \subset \ldots$, where every order of G_i/G'_{i-1} or G'_i/G_i is a divisor of l or m and $\smile_{i=1}^{\infty} \bar{G}_i = \bar{B}$. Let $\bar{B}^p$ be the subgroup of $\bar{B}$ formed of all such elements as can be expressed in the form $\bar{c}^p$ for some $\bar{c}$ in $\bar{B}$. Then if $(m, p) = 1$ and $(l, p) = 1$, we see $\bar{B} \neq$ $\bar{B}^p$. $\bar{B}/\bar{B}^p$ is an elementary abelian group and is not cyclic (observe $r \neq$ 1), but this is a contradiction, because $\{\bar{b}\} \smile \bar{B}^p/\bar{B}^p$ is normal in $\bar{G}/\bar{B}^p$ according to Lemma 4.

2) Suppose that $B' \frown \{b\} = \{b^m\} \neq 1$. In this case $B'' \frown \{b\} = 1$, where B'' is the commutator-subgroup of B'. To see this suppose $B'' \frown$ $\{b\} = \{b^n\} \neq 1$. Let us write $\bar{c}$, $\bar{A}$ if an element c in G and a subgroup A are considered mod B''. Put $G_1 = B' \smile \{a\}$. $\bar{B}_m = \{\bar{b}^m\}$ is normal in $\{\bar{b}^m\} \smile \{\bar{a}\}$ and $\bar{B}_m \subseteq \bar{B}'$. But since $B = \{b\} \smile B'$, $\bar{B}_m$ is normal both in $\bar{B}$ and in $\bar{G}_1$, therefore is normal in $\bar{G}$. Put $dim\ (\{\bar{b}\}/\{\bar{b}^m\}) = \alpha$ and $dim\ (\{\bar{b}^m\}/\bar{1}) = \beta$, then

$$dim\,(\bar{G}/\{\bar{a}\}) \leq dim\,(G/\{b^n\} \smile \{a\}) \leq \alpha + \beta \text{ (u.s. modularity).}$$

But since $dim\,(\bar{G}/\{\bar{a}\}) = dim\,(\bar{G}/\bar{G}_1) + dim\,(\bar{G}_1/\bar{B}_m \smile \{\bar{a}\}) + dim\,(\bar{B}_m \smile$

$$\{\bar{a}\}/\{\bar{a}\}) = \alpha + \beta + dim\,(\bar{G}_1/\bar{B}_m \smile \{\bar{a}\}),$$

we can conclude $dim\,(\bar{G}_1/\bar{B}_m \smile \{\bar{a}\}) = 0$, whence we have $\bar{B}_m = \bar{B}'$, observing they are both normal in $\bar{G}$. This shows $\{\bar{b}\} = \bar{B}$ in contradiction with $B' \neq B''$. Thus we see $B'' \frown \{b\} = 1$. Let now B^*/B'' be the group formed of all E-elements in G/B'' then of course $B^* \subset B$ and $B^* \neq B'$. Let us write $\bar{c}$, $\bar{A}$ if c $(\in G)$ and A $(\leq G)$ are considered mod B^*. According to 1) $\{\bar{a}\} \smile \{\bar{b}^m\}$ is abelian, then $\{aba^{-1}\} \frown \{b\} \neq 1$, therefore $\bar{G}$ is abelian by Lemma 7 and Lemma 8. But this is a contradiction, for

$B' \supset B^*$ and G/B' is not abelian.

II) Suppose G/B' is abelian.

1) If $B' \supset \{b^m\} \neq 1$, then as in I) 2) we can see $B'' \frown \{b\} = 1$ and $\bar{G} = G/B^*$ is abelian. But this is a contradiction, for $B^* \subset B' = G'$, where G' is the commutator-subgroup of G. (As to other notations see I) 2).

2) If $B' \frown \{b\} = 1$. Then G/B' is free abelian and is of rank 2. In this case cannot hold $\{aba^{-1}b^{-1}\} \frown B'' \neq 1$. To see this assume that the relation holds, then B'/B'' has no U-element, therefore G/B'' is abelian according to Lemma 10 in contradiction with $B' \neq B''$. Hence $\{aba^{-1}b^{-1}\} \frown B'' = 1$. Let B^*/B'' be the subgroup formed of all E-elements in G/B'', then $B^* \subset B'$. Let us write $\bar{c}$, $\bar{A}$ if $c\ (\in G)$ and $A\ (\leq G)$ are considered mod B^*. According to I) $\{\bar{a}\bar{b}\bar{a}^{-1}\bar{b}^{-1}\} \smile \{\bar{a}\}$ and $\{\bar{a}\bar{b}\bar{a}^{-1}\bar{b}^{-1}\} \smile \{\bar{b}\}$ are both abelian. This shows $\bar{B}'$ is contained in the center of $\bar{G}$. But since $\bar{B}/\bar{B}' \cong \{b\}$, $\bar{B}$ is itself abelian in contradiction with $B' \neq B^*$, q. e. d.

LEMMA 12. Let $G = \{a\} \smile \{b\}$ be u. s. modular, $E = 1$, and $\{a\} \frown \{b\} = 1$. Let $G \supseteq M \supseteq B \supseteq \{b\}$ be a series of subgroups and B be normal in G. If G has no perfect subgroup, and

i) if $B = \{b\}$, then G is abelian (cf. Lemma 5) and $M = \{b\} \smile \{a^m\}$ for some $m \neq 0$.

ii) If $B \neq \{b\}$, then $B = \{b\} \smile \{a^n\}$, $M = \{b\} \smile \{a^m\}$ for some n, $m \neq 0$.

PROOF. In case i) the assertion is trivial. In case ii) we have $B \frown \{a\} \neq 1$ according to Lemma 11. Put $B \frown \{a\} = \{a^n\}$. Let M_1 be a maximal subgroup of G containing B, then for some prime p $M_1 \frown \{a\} = \{a^p\}$. But according to the u. s. modularity $\{a^p\} \smile \{b\}$ is maximal in G, hence we have $M_1 = \{a^p\} \smile \{b\}$. Continuing this consideration we see our assertion is true, q. e. d.

THEOREM 7. *Let G be u. s. modular and $E = 1$. If G has no perfect subgroup, then G is abelian.*

PROOF. According to Theorem 5 we are only to prove $\{a\} \frown \{b\} = 1$ implies $ab = ba$. Hence we can assume $G = \{a\} \smile \{b\}$ without loss of generality. Suppose that G is not abelian. Let G' be the commutator-subgroup of G, then $G \neq G' \neq 1$. We can suppose $a \notin G'$, then $\{a\} \smile G' \neq \{a\}$ and $\{a\} \smile G'$ is normal in G, hence $\{a\} \smile G' = \{a\} \smile \{b^u\}$ for some $u \neq 0$ according to Lemma 12. Since $\{a\} \bigcup G'/G'$ is cyclic, we can take such v that $a^v \equiv b^u$ mod G'. Then $b_1 = b^u a^{-v} \in G'$ and $\{a\} \bigcup G' = \{a\} \bigcup \{b_1\}$, where $\{a\} \bigcap \{b_1\} = 1$, because if $\{a\} \bigcap \{b_1\} \neq 1$, then according to Theorem 5 $\{a\} \bigcup G'$ is an abelian group whose index in G is finite, but this implies, according to Lemma 9, G is itself abelian in contradiction with our first hypothesis. According to Lemma 12 either $G' = \{b_1\}$ or $\{b_1\} \bigcup \{a^w\}$ for some $w \neq 0$. If $G' = \{b_1\}$, then $\{a\} \bigcup \{b_1\}$ is abelian according to Lemma 7. But since the index of $\{a\} \bigcup \{b_1\}$ in G is finite, G is abelian by Lemma 9 in contradiction with our first hypothesis. Hence $G' = \{a^w\} \bigcup \{b_1\}$ with $w \neq 0$, and G/G' is a finite group. Continuing this consideration we have the commutator-series $G \supset G' \supset G'' \supset G''' \ldots$, where $G \neq G' \neq G'' \neq G'''$ and G/G', G'/G'', $G''/G''' \ldots$ are all finite groups. But this is impossible, because any u. s. modular group of finite order is meta-abelian. (See Theorem 1). Hence G is abelian, q. e. d.

Combining Lemma 10, Theorem 7 and Theorem 6 we have

THEOREM 8. *Let G be an u. s. modular group satisfying (A), and $G \neq E$. If G has no perfect subgroup, then either G has such structure as is shown in Theorem 6 or it is abelian. In short the family of all u. s. modular groups satisfying the above conditions coincides with the family of all modular groups that have at least one U-element.*

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