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On the Existence of Unknotted Polygons on 2-Manifolds in E^3

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Introduction

All sets considered in this paper lie in the 3-dimensional Euclidean space E^3. Let P be a simple closed polygon and N an arbitrary set. Then P will be called an *N-unknotted polygon,* if P is the boundary-polygon of a polyhedral disk $D(P)$ whose interior is contained in N, and $D(P)$ will be called an *associated disk.* An E^3-unknotted polygon is usually called an *unknotted polygon.* The purpose of this paper is to prove the following theorems:

Theorem 1. *Let M be a closed polyhedral 2-manifold whose genus is different from* 0. *Then there exists an unknotted polygon on M not homologous to* 0 *in M.*

Theorem 2. *Let M be a closed polyhedral 2-manifold whose genus is differentfrom* 0. *Then there exists an $(E^3\text{-}M)$-unknotted polygon on M not homotopic to* 0 *in M.*

As an extension to Theorem 1 we have further

Theorem 3. *Let M be a closed polyhedral 2-manifold of genus p. Then there exists p mutually disjoint unknotted polygons such that they are linearly independent in the homology group of M.*

§ 1.

First we shall introduce several definitions.

Let M be a closed polyhedral 2-manifold. A family $\mathfrak{L}_\theta$ of the planes normal to a given unit vector θ will be said to be *admissible* with respect to M, if the vertices of M lie in different planes of the family. In general the intersection of M with representative plane L of an admissible family $\mathfrak{L}_\theta$ is the union of a finite number of mutually disjoint simple closed polygons. The intersection of M with an exceptional plane L of the family is either

(1) the union of an isolated point q and a finite collection of mutually disjoint simple closed polygons in the complement of q, or

(2) p simple closed polygons, of which p' pieces $(1 < p' \leqq p)$ have a *singular point* q in common, the polygons being otherwise mutually disjoint.

The isolated and singular points are to be found among the vertices of M, so that there are but a finite number of exceptional planes in the family $\mathfrak{L}_\theta$.

A simple closed polygon P contained in the intersection of M with one of the planes of the admissible family $\mathfrak{L}_\theta$ will be called a *singular polygon* on M with respect to $\mathfrak{L}_\theta$, if P contains a singular point.

A connected subset M' of M will be said to be *elementary*, if the intersection of M' with any planes of $\mathfrak{L}_\theta$ is either a simple closed polygon or a single point or an empty set.

A continuum M' on M which is homeomorphic to a continuum on the 2-sphere will be called a *semi-planary continuum*. A simple closed polygon P of a semi-planary continuum T will be called a boundary-polygon of T, if P does not divide T.

If E is elementary, then $\bar{E}$ is a semi-planary continuum. It is evident that if T and T' are semi-planary continua and $T \frown T'$ is a boundary-polygon of both T and T', then the union $T \smile T'$ is also a semi-planary continuum.

Now we shall prove the following

Lemma. *Let M be a closed polyhedral 2-manifold and $\mathfrak{L}_\theta$ an abmissible family with respect to M. If any singular polygon on M with respect to $\mathfrak{L}_\theta$ is homologous to* 0 *in M, then M is a polyhedral 2-sphere.*

PROOF. Let $\mathfrak{P}$ be the collection of all singular polygons on M with respect to $\mathfrak{L}_\theta$. Let q be a given point of M not lying on any singular polygon. Put $P_i < P_j$ for every pair of P_i, $P_j \in \mathfrak{P}$, if P_j separates P_i from the point q in M. Then $\mathfrak{P}$ is a partially ordered set.

Since any $P \in \mathfrak{P}$ is homologous to 0 in M, the complement of P in M consists of two component. Let $T(P)$ be one of the two components such that $T(P) \frown q = 0$.

Now we shall prove that for each $P \in \mathfrak{P}$ $\overline{T(P)}$ is a semi-planary continuum. If P is one of the minimal elements of $\mathfrak{P}$, then $T(P)$ is elementary. Therefore $\overline{T(P)}$ is a semi-planary continuum. Suppose that P is not minimal and that for each $P' < P$ $\overline{T(P')}$ is a semi-planary continuum. Put

$$\mathfrak{P}(P) = \{P' \mid P' < P\}.$$

Let $\mathfrak{P}'(P)$ be the set of all maximal elements of $\mathfrak{P}(P)$. Since $T(P)$

$\overline{-\bigcup_{P'\in\mathfrak{P}'(P)} T(P')}$ is elementary, $\overline{T(P)-\bigcup_{P'\in\mathfrak{P}'(P)} T(P')}$ is a semi-planary continuum. For any $P_0'\in\mathfrak{P}'(P)$

$$P_0' = \overline{T(P)-\bigcup_{P'\in\mathfrak{P}'(P)} T(P')}\frown\overline{T(P_0')}$$

is a boundary-polygon of both semi-planary continua $\overline{T(P)-\bigcup_{P'\in\mathfrak{P}'(P)} T(P')}$ and $\overline{T(P')}$. Then it follows that $\overline{T(P)}$ is a semi-planary continuum. Thus for each $P\in\mathfrak{P}$ $\overline{T(P)}$ is a semi-planary continuum.

Let $\mathfrak{P}'$ be the set of all maximal elements of $\mathfrak{P}$. Since $M-\bigcup_{P'\in\mathfrak{P}'} \overline{T(P')}$ is elementary, $\overline{M-\bigcup_{P'\in\mathfrak{P}'} T(P')}$ is a semi-planary continuum. For any $P_0'\in\mathfrak{P}'$

$$P_0' = \overline{M-\bigcup_{P'\in\mathfrak{P}'} T(P')}\frown\overline{T(P_0')}$$

is a boundary-polygon of both semi-planary continua $\overline{M-\bigcap_{P'\in\mathfrak{P}'} T(P')}$ and $\overline{T(P_0')}$. Then it follows that the closed polyhedral 2-manifold M is a semi-planary continuum. From this it follows that M is a polyhedral 2-sphere and the proof of Lemma is complete.

Proof of Theorem 1. Let M be a 2-manifold whose genus is different from 0. Let $\mathfrak{L}_\theta$ be an admissible family with respect to M. By Lemma there exists at least one singular polygon P on an exceptional plane such that P is not homologous to 0 in M. Since any simple closed polygon on a plane is unknotted, P is the required polygon, and the proof of Theorem 1 is complete.

§ 2.

Proof of Theorem 2. By Theorem 1 there exists a simple closed polygon P on M not homologous to 0 such that P is the boundary-polygon of an associated disk $D(P)$. If the interior of $D(P)$ does not meet M, then P is the required polygon. Now suppose that $M\frown(D(P)-P)\neq 0$. We may assume without loss of generality that the intersection $M\frown(D(P)-P)$ is the union of a finit number of mutually disjoint simple closed polygons Q_i. Let $D(Q_i)$ be the polyhedral disk bounded by Q_i in $D(P)$. For each Q_i homotopic to 0 in M there exists one and only one polyhedral disk $D[Q_i]$ on M, whose boundary-polygon is Q_i. Put $Q_i < Q_j$, if $D[Q_i]\subset D[Q_j]$. Let Q_0 be one of the minimal elements (homotopic to 0 in M) with respect to the above ordering. Let Q_0 be a simple closed polygon in $D(P)$ sufficiently near to Q_0 and not intersecting $D(Q_0)$. Then there exists a polyhedral disk $D'[Q_0']$ whose boundary-polygon is Q_0' such that $D'[Q_0']$ is sufficiently near to $D[Q_0]$ and

$$D'[Q_0'] \frown D(P) = Q_0' \text{ and } D'[Q_0'] \frown M = 0.$$

Put

$$m'(D(P)) = (D(P) - D(Q_0')) \cup D'[Q_0'].$$

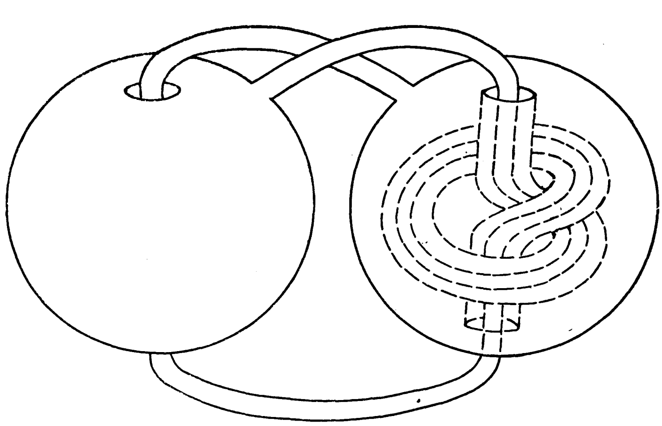
This is a modification of $D(P)$. If we repeat this modification step by step as long as possible, then we have an associated disk $m_1(D(P))$.

If $Q_i \subset D[Q_j]$ for some Q_j homotopic to 0 in M, then Q_i is homotopic to 0 in M. From this it follows that $m_1(D(P))$ consists of only a finite number s of mutually disjoint simple closed polygons not homotopic to 0 in M.

If $s = 0$, then $m_1(D(P)) \frown M = 0$. Therefore P is the required polygon. Now we assume that $s > 0$. Let Q be one of the innermost simple closed polygons in the associated disk $m_1(D(P))$. Then it is easy to see that Q is an $(E^3\text{-}M)$-unknotted polygon on M not homotopic to 0 in M. Therefore Q is the required polygon and the proof of Theorem 2 is complete.

From Theorem 2 it follows immediately the following

Corollary. If M is a closed polyhedral 2-manifold of genus 1 in S^3, then one of the components of the complementary domain of M is homeomorphic to the interior of a solid torus.



This Corollary has been proved by J. W. Alexander (See [1]).

REMARK. There exists a closed polyhedral 2-mainfold of genus 2 containing no $(E^3\text{-}M)$-unknotted polygon not homologous to 0 in M (See Fig.).

§ 3.

PROOF OF THEOREM 3. We shall prove Theorem 3 by induction on the genus p. If $p=0$, then Theorem 3 is evident. If $p=1$, then Theorem 3 is equivalent to Theorem 1. Suppose that Theorem 3 is true for any closed polyhedral 2-mainfold whose genus is less than $p(>1)$. Let M be a closed polyhedral 2-mainfold of genus p. By Theorem 2 there exists an $(E^3\text{-}M)$-unknotted polygon P on M not homotopic to 0 in M and an associated disk $D(P)$. Let P' be a simple closed polygon in M sufficiently near to P without intersecting P and let $D'(P')$ be a polyhedral disk whose boundary-polygon is P' such that

$$D'(P')\frown D(P)=0 \text{ and } D'(P')\frown M=P'.$$

Let R be the ring bounded by P and P' in M. Put

$$m_2(M)=(M-R)\smile D(P)\smile D'(P').$$

We consider the following two cases.

CASE A. If P is not homologous to 0 in M, then $m_2(M)$ is a closed polyhedral 2-mainfold of genus $p-1$. By the hypothesis of induction there exists $p-1$ mutually disjoint simple closed polygons such that they are linearly independent in the homology group of M. Furthermore we modify these $p-1$ polygons into $p-1$ polygons $P_1, P_2, \dots, P_{p-1}$ such that they meet neither P nor P'. Then the p polygons $P_1, P_2, \dots, P_{p-1}, P_p=P$ are the required ones. Hence Theorem 3 is true for the case A.

CASE B. If P is homologous to 0 in M, then $m_2(M)$ consists of two closed polyhedral 2-mainfolds M' and M'' of genus p' and p'' respectively, where $p'>0, p''>0$ and $p'+p''=p$. Then there exist p' mutually disjoint simple closed polygons of M' and p'' mutually disjoint simple closed polygons of M'' such that they are linearly independent in the homology groups of M' and M'' respectively. By the reasoning similar to the case A we see that Theorem 3 is also true for the case B, and the proof of Theorem 3 is complete.

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References

[1] J. W. Alexander, On the subdivision of 3-space by a polyhedron, Proc. Nat. Acad. U. S. A. **10** (1924).

[2] E. E. Moise, Affine structures in 3-mainfolds II, Ann. of Math. **55** (1952).