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## SEMISIMPLE DEGREE OF SYMMETRY OF MANIFOLD WITH THE HOMOTOPY TYPE OF PRODUCT

$$(S^1)^r \times (S^2)^s \times (S^3)^t$$

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### Introduction

In this note, we shall consider the topological semisimple degree of symmetry of manifold with the homotopy type of a product  $(S^1)^r \times (S^2)^s \times (S^3)^t$  of spheres. Here the *topological semisimple degree of symmetry* is, by definition, the maximum of dimension of compact connected semisimple Lie group which acts on the manifold topologically and almost effectively.

This note is motivated by works on the degree of symmetry of manifold with large low homotopy groups as  $K(\pi, 1)$ -manifolds or product of 2-spheres (see [1], [5], [6], [9], [14] or [15]). We shall prove the following

**Theorem A.** *Let  $M$  be a  $3m$ -dimensional closed topological manifold with the same integral cohomology ring as a product of 3-dimensional spheres. If a compact connected simple Lie group  $G$  acts on  $M$  topologically and almost effectively, then  $G$  is  $SU(2)$  or  $SO(3)$ .*

By a slight modification of the method of the proof of Theorem A, we can prove the following

**Theorem B.** *Let  $M$  be a closed topological manifold with the homotopy type of  $N = (S^1)^r \times (S^2)^s \times (S^3)^t$ . If a compact connected simple Lie group  $G$  acts on  $M$  topologically and almost effectively, then  $G$  is  $SU(2)$  or  $SO(3)$ .*

Moreover we shall prove the following

**Theorem C.** *Let  $M$  be as in Theorem B and  $G$  a compact connected Lie group which acts on  $M$  topologically and almost effectively. Then  $G$  is locally isomorphic to  $T^u \times (SU(2))^v$  with  $u+v \leq r+2(s+t)$ .*

In this note, we shall consider only topological almost effective action and "manifold" means "connected paracompact Hausdorff manifold".

The authors would like to thank the referee for his valuable suggestions. In this note, we shall use the following notations;

1.  $T^n$  and  $T$  denote  $n$ -dimensional and 1-dimensional toral group, respectively and we call a 1-dimensional toral group a torus.
2. If a compact Lie group  $G$  acts on a topological space  $X$  and  $H$  is a subgroup of  $G$ , then  $X^H$  denotes the fixed point set of  $H$ .
3.  $X \underset{A}{\sim} Y$  means topological spaces  $X$  and  $Y$  have isomorphic cohomology rings with coefficient group  $A$ .
4.  $Z$  and  $Q$  denote the ring of integers and the field of rational numbers, respectively.
5.  $H^i(X)$  denotes  $i$ -th cohomology group of  $X$  with coefficient in  $Q$ .
6.  $\dim H^*(X; A) = \sum_{i \geq 0} \dim_A H^i(X; A)$  for a field  $A$ .
7. If a compact Lie group  $G$  acts on  $X$ , then  $X_G$  denotes the orbit space of  $E_G \times X$  under the action of  $G$  defined by  $g(e, x) = (eg^{-1}, gx)$ , where  $E_G \rightarrow B_G$  is the universal  $G$ -bundle.

**1. Preliminaries**

In this section, we shall recall some basic facts about the Leray spectral sequence of the orbit map. Let  $G$  be a compact connected Lie group and act on a connected completely regular space  $X$ . Let  $\pi: X \rightarrow X/G = X^*$  be the orbit map and  $\{E_r^{p,q}, d_r\}$  the Leray spectral sequence of  $\pi$ . Then we have  $E_2^{p,q} = H^p(X^*; H^q(\pi))$ , where  $H^q(\pi)$  is the sheaf generated by the presheaf  $U^* \rightarrow H^q(\pi^{-1}(U^*))$  for open set  $U^*$  in  $X^*$  (see [4]). Recall the following facts about the Leray spectral sequence.

- i)  $E_2^{p,q} \Rightarrow H^*(X)$ .
- ii)  $E_2^{0,q} = \Gamma(H^q(\pi)) =$  the vector space of all sections of  $H^q(\pi)$ .
- iii) The stalk of  $H^q(\pi)$  at  $x^* = \pi(x)$  is  $H^q(G(x))$ .
- iv) The edge homomorphism  $e: H^q(X) \rightarrow E_2^{0,q}$  is given by  $e(a)(x^*) = i_x^*(a)$ , where  $i_x: G(x) \rightarrow X$  is the inclusion and  $x^* = \pi(x)$ .
- v) The following diagram is commutative;

$$\begin{array}{ccc}
 H^p(X^*) & \xrightarrow{\pi^*} & H^p(X) \\
 \downarrow & & \uparrow \\
 E_2^{p,0} & \longrightarrow & E_\infty^{p,0} .
 \end{array}$$

See [4] for the details.

We have the following Propositions.

**Proposition 1.** *Let  $k$  be the dimension of a principal orbit of the action of  $G$  on  $X$ . If the action has a singular orbit, then the edge homomorphism  $e: H^k(X) \rightarrow E_2^{0,k}$  is trivial. In particular, we have  $E_\infty^{0,k} = 0$ .*

See [14] for the proof.

**Proposition 2.** *Let  $k$  be as above. If there is a point  $x$  in  $X$  such that  $i_x^*: H^k(X) \rightarrow H^k(G(x))$  is trivial, then the edge homomorphism  $e: H^k(X) \rightarrow E_2^{0,k}$  is trivial.*

This Proposition is proved by the same method as the proof of Proposition 1.

**Proposition 3.** *Let  $M$  be a closed  $3m$ -dimensional manifold such that there are  $m$  elements  $a_1, \dots, a_m$  in  $H^3(M)$  with non-zero cup product  $a_1 \cdots a_m \neq 0$ . Assume the group  $SU(2)$  acts on  $M$  with a finite principal isotropy subgroup and a singular orbit. Then there is a point  $x$  in  $M$  whose isotropy subgroup  $SU(2)_x$  is a torus.*

See [14] for the proof.

**Proposition 4.** *Let  $M$  be as in Proposition 3. Assume the group  $SU(3)$  or  $Sp(2)$  acts on  $M$  with a finite principal isotropy subgroup. Then there is a singular orbit.*

This Proposition follows the following Lemma and the same argument as in the proof of Proposition 5 in [14].

**Lemma 5.** *Let  $M$  be a closed manifold and a compact simple Lie group  $G$  act on  $M$  almost freely, i.e. all isotropy subgroups are finite. Then we have  $\dim_Q E_2^{0,3} \leq 1$ , where  $\{E_r^{p,q}, d_r\}$  is the Leray spectral sequence of the orbit map  $\pi: M \rightarrow M/G$ .*

Proof. It follows from a result in [7] (The argument in the proof of Theorem 1 in [7]) that  $H^3(\pi)$  is locally constant, i.e. for every point  $x^*$  in  $M/G$ , there is a neighborhood  $U^*$  of  $x^* = \pi(x)$  such that  $H^3(\pi)|_{U^*} = U^* \times H^3(G(x))$ . Thus we have  $E_2^{0,3} = \Gamma(H^3(\pi)) \cong H^3(G(x)) = Q$ . Q.E.D.

Moreover we have the following

**Proposition 6** (cf. Chap. VII in [3]). *Let  $M$  be a closed manifold with  $M \underset{Z}{\sim} S^{n_1} \times \dots \times S^{n_m}$  ( $n_i = 1, 3$  for  $i = 1, \dots, m$ ). Assume a torus  $T$  acts on  $M$  with a fixed point and  $M$  is totally non-homologous to zero in the fiber bundle  $M \rightarrow M_T \rightarrow B_T$  over  $Q$ . Then we have*

$$M^T \underset{Z}{\sim} S^{k_1} \times \dots \times S^{k_m} \text{ with } k_i \leq n_i, k_i = 1, 3 \text{ for all } i.$$

Moreover if  $b_1, \dots, b_s \in H^3(M^T: Z)$  are 3-dimensional generators of  $H^*(M^T: Z)$ , then  $b_1, \dots, b_s$  are in the image of the homomorphism  $H^3(M: Z) \rightarrow H^3(M^T: Z)$  induced by the inclusion.

REMARK 1. Proposition 6 is valid for a finite dimensional space if coefficient group  $Z$  is replaced by  $Q$  (see [3] Chap. VII).

2. The assumption that  $M$  is totally non-homologous to zero in the fiber bundle  $M \rightarrow M_T \rightarrow B_T$  holds when  $T$  has a fixed point and  $n_1 = \dots = n_m = 3$ .

**2. Proof of Theorem A**

In this section, we shall prove Theorem A. To prove it, it is sufficient to show that the group  $SU(3)$  or  $Sp(2)$  can not act on  $M$ , because the exceptional group  $G_2$  and simple Lie group of rank  $\geq 3$  contain  $SU(3)$  or  $Sp(2)$ .

Since the case of  $Sp(2)$  is completely parallel to the case of  $SU(3)$ , we shall prove that  $SU(3)$  cannot act on  $M$ .

From now on, we assume that  $M$  is a closed  $3m$ -dimensional manifold with  $M \simeq S^3 \times \dots \times S^3$  ( $m$ -times) and admits an action of  $SU(3)$ . Put  $G = SU(3)$  and let  $\phi: G \times M \rightarrow M$  be the given action of  $SU(3)$ .

Let  $K = SU(2)$  be the standard subgroup of  $SU(3)$  and  $\psi: K \times M \rightarrow M$  the action obtained from the restriction.

First we shall prove the following

**Proposition 7.** *The action  $\psi$  has a finite principal isotropy subgroup and there is a point  $x$  in  $M$  whose isotropy subgroup  $K_x$  is a torus.*

Proof. We shall prove the first part. Assume the contrary. Then the identity component of a principal isotropy subgroup is a torus. Hence the center  $C$  of  $K$  is contained in every isotropy subgroup, which means  $M^C = M$ . It follows that  $C$  is contained in the ineffective kernel of the action  $\phi$ . This implies that  $C$  is contained in the center of  $G$ . It is easy to see that this is impossible.

Next we shall prove the second part. Assume that there is no point whose isotropy subgroup is a torus. It follows from Proposition 3 that  $\psi$  is almost free. Consider the Leray spectral sequence  $\{E_r^{p,q}, d_r\}$  of the orbit map  $\pi: M \rightarrow M/K = M^*$ . It is proved that the edge homomorphism  $e: H^3(M) \rightarrow E_2^{0,3}$  is non-trivial. In fact, assume the contrary, we have  $E_\infty^{0,3} = 0$ . It follows that  $H^3(M) = E_\infty^{0,3} = H^3(M^*)$ , which is easily seen to be a contradiction. Then Proposition 2 shows that the homomorphism  $i_x^*: H^3(M) \rightarrow H^3(K(x))$  is not trivial for every point  $x$  in  $M$ . It follows from the following commutative diagram that  $H^3(G(x))$  is non-zero.

$$\begin{array}{ccc}
 H^3(M) & \xrightarrow{j_x^*} & H^3(G(x)) \\
 i_x^* \searrow & & \swarrow k^* \\
 & & H^3(K(x))
 \end{array}$$

Here  $i_x$  and  $j_x$  are inclusions and  $k$  is the natural map.

It follows from a result in [14] (Proposition 8 in [14]) that  $G_x$  is finite for every point  $x$  in  $M$ , which contradicts Proposition 4. Q.E.D.

Next we shall prove the following

**Proposition 8.** *Let  $C$  be the center of  $K$ . Then  $M^C$  is a proper subset of  $M$  and*

$$M^C \simeq S^1 \times \dots \times S^1 \times S^3 \times \dots \times S^3 \text{ (} m\text{-times)}.$$

*Proof.* Assume  $M^C=M$ . Then the same argument as in Proposition 7 leads a contradiction. To prove the second part, recall the following

**Lemma** (cf. Chap. VII in [3]). *Let  $G=Z_2=Z/2Z$  or  $T$  act on a compact space  $X$ . Then we have  $\dim H^*(X: A) \geq \dim H^*(X^G: A)$ , where  $A=Z_2$  (if  $G=Z_2$ ) or  $A=Q$  (if  $G=T$ ). The equality holds if and only if  $X$  is totally non-homologous to zero in  $X_G$ .*

It follows from the Lemma that we have

$$(1) \quad \dim H^*(M) = \dim H^*(M: Z_2) \geq \dim H^*(M^C: Z_2) \geq \dim H^*(M^C).$$

Let  $M^C=L_1 \cup \dots \cup L_t$  be the decomposition into connected components and  $T$  a maximal torus of  $K$ . Since  $M^T$  is connected (see [3] Chap. VII), we may assume  $M^T \subseteq L_1$ . We prove that  $M^T$  is a proper subset of  $L_1$ . Assume  $M^T=L_1$ . Since  $L_1$  is  $K$ -invariant,  $M^T$  is also  $K$ -invariant, which means  $M^T=M^K$ . In fact, for every point  $x$  in  $M^T$  and every element  $g$  of  $K$ ,  $gx \in M^T$  and hence we have  $Tgx=gx$ . It follows that  $g^{-1}Tg \subseteq K_x$  for every  $g \in K$  and hence  $K = \bigcup_g g^{-1}Tg \subseteq K_x$ , which implies  $x \in M^K$ . This contradicts Proposition 7, which proves  $M^T \subsetneq L_1$ . It follows from the above Lemma that we have

$$(2) \quad \dim H^*(L_1) \geq \dim H^*((L_1)^T) \geq \dim H^*(M^T).$$

Since  $M$  is totally non-homologous to zero in  $M_T$  (see Remark 2 in section 1), we have the following equality;

$$(3) \quad \dim H^*(M) = \dim H^*(M^T).$$

It follows from (1), (2) and (3) that we have

$$(4) \quad \begin{aligned} \dim H^*(M) &= \dim H^*(M: Z_2) = \dim H^*(M^C: Z_2) = \dim H^*(M^C) \\ &= \dim H^*(L_1) = \dim H^*(M^T), \end{aligned}$$

which implies that  $M^C$  is connected,  $M$  and  $M^C$  are totally non-homologous to zero in  $M_C$  and  $(M^C)_T$  over  $Z_2$  and  $Q$ , respectively, and  $H^*(M^C: Z)$  has no 2-torsion. Hence the homomorphism  $i_c^*: H^3(M_C: Z_2) \rightarrow H^3(M: Z_2)$  is surjective, where  $i_c: M \rightarrow M_C$  is the inclusion. Put  $H^3(M: Z_2) = \langle a_1, \dots, a_m \rangle$  with  $a_1 \cdots a_m \neq 0$ . Choose elements  $a'_i \in H^3(M_C: Z_2)$  such that  $i_c^*(a'_i) = a_i$  for  $i=1, \dots, m$ . Consider the homomorphism  $j_c^*: H^3(M_C: Z_2) \rightarrow H^3((M^C)_C: Z_2) = \sum_{i+j=3} H^i(B_C: Z_2) \otimes H^j(M^C: Z_2)$ . Then we have

$$j^*(a'_i) = 1 \otimes b_{i3} + t \otimes b_{i2} + t^2 \otimes b_{i1} + t^3 \otimes b_{i0},$$

where  $b_{ij} \in H^i(M^c: Z_2)$  and  $H^*(B_C: Z_2) = Z_2[t]$  ( $\deg t = 1$ ). By the same argument as in [3] (see the proof of Theorem 10.2 in Chap. VII in [3]), it can be shown that  $a'_i$  can be chosen so that  $b_{i2} = b_{i0} = 0$  and  $b_{i3}^2 = b_{i1}^2 = 0$ . In fact, let  $E_r^{*,*}(Z_2)$  and  $E_r^{*,*}(Z)$  be the spectral sequences of  $M_C \rightarrow B_C$  with respect to  $Z_2$  and  $Z$  coefficients, respectively. Note that  $E_2^{p,q}(Z) = H^p(B_C: Z) \otimes H^q(M: Z)$ , because  $C$  acts on  $H^*(M: Z)$  trivially. As noted above,  $E_r^{*,*}(Z_2)$  degenerates. Since the reduction  $E_2^{p,q}(Z) \rightarrow E_2^{p,q}(Z_2)$  is a monomorphism for  $p > 0$ ,  $E_r^{*,*}(Z)$  also degenerates. It follows that the edge homomorphism  $H^*(M_C: Z) \rightarrow E_2^{0,*}(Z) = H^*(M: Z)$  is surjective and hence the homomorphism  $H^*(M_C: Z) \rightarrow H^*(M: Z_2)$  is surjective. This shows that  $a'_i$  can be chosen from the image of  $H^*(B_C: Z) \otimes H^*(M_C: Z) \rightarrow H^*(B_C: Z_2) \otimes H^*(M^c: Z_2)$  for all  $i$ . This implies  $b_{i2} = b_{i0} = 0$ . Since  $H^*(M^c: Z)$  has no 2-torsion, the reduction  $H^*(M^c: Z) \rightarrow H^*(M^c: Z_2)$  is surjective and hence  $x^2 = 0$  for every  $x$  in  $H^{\text{odd}}(M^c: Z_2)$ . Since  $a'_1 \cdots a'_m \neq 0$  and  $j^*$  is injective (Theorem (1.5) in Chap. VII in [3]), we have

$$0 \neq j^*(a'_1 \cdots a'_m) = (\sum_q t^q \otimes b_{1q}) \cdots (\sum_q t^q \otimes b_{mq})$$

and hence some product  $b_{1q_1} \cdots b_{mq_m}$  is non-zero. Thus  $b_{1q_1}, \dots, b_{mq_m}$  generate an exterior algebra, which must equal  $H^*(M^c: Z_2)$ , since both have dimension equal to  $2^m = \dim H^*(M: Z_2)$ . It follows from the equality  $\dim H^*(M^c: Z_2) = \dim H^*(M^c)$  that  $H^*(M^c)$  is also an exterior algebra  $\Lambda_Q(b_{1q_1}, \dots, b_{mq_m})$ . Since  $M^T$  has the same rational cohomology ring as  $S^1 \times \cdots \times S^1 \times S^3 \cdots \times S^3$  ( $m$ -times) and  $M^c$  is totally non-homologous to zero in  $(M^c)_T$ , degree of  $b_{jq_j}$  is odd for every  $j$ . Q.E.D.

Now we shall prove Theorem A. Consider the following commutative diagram:

$$\begin{array}{ccccc}
 (\#) & H^{3m-3}(M, M-M^c) & \xrightarrow{j^*} & H^{3m-3}(M) & \xrightarrow{i^*} & H^{3m-3}(M-M^c) & \rightarrow & H^{3m-2}(M, M-M^c) \\
 & \Big| \cong & & \Big| \cong & & & & \\
 & H_3(M^c) & \xrightarrow{k^*} & H_3(M) & & & & 
 \end{array}$$

where  $i, j$  and  $k$  are inclusions. Since it follows from Proposition 8 that  $k^*: H^3(M) \rightarrow H^3(M^c)$  is not injective,  $k_*$  is not surjective and hence  $j^*$  is not surjective. Thus there are elements  $a_1, \dots, a_{m-1}$  in  $H^3(M)$  such that  $i^*(a_1 \cdots a_{m-1}) \neq 0$ . Since isotropy subgroups of the action of  $K$  on  $M-M^c$  are all cyclic of odd order,  $(M-M^c)/K$  is a rational manifold of dimension  $3m-3$ , which is clearly not compact, and hence we have  $H^{3m-3}((M-M^c)/K) = 0$ . Consider the Leray spectral sequence of the orbit map  $\pi: M-M^c \rightarrow (M-M^c)/K$ . Assume the edge homomorphism  $e: H^3(M-M^c) \rightarrow E_2^{0,3}$  is trivial. Then  $\pi^*: H^3((M-M^c)/K) \rightarrow H^3(M-M^c)$  is an isomorphism and hence there are elements

$b_j \in H^3((M - M^c)/K)$  such that  $\pi^*(b_j) = i^*(a_j)$  ( $j = 1, \dots, m - 1$ ). Then we have

$$0 \neq i^*(a_1 \cdots a_{m-1}) = \pi^*(b_1 \cdots b_{m-1}) = 0,$$

since  $b_1 \cdots b_{m-1} \in H^{3m-3}((M - M^c)/K) = 0$ . This is a contradiction, which means the edge homomorphism  $e$  is not trivial, in other words,  $i_x^*: H^3(M - M^c) \rightarrow H^3(K(x))$  is not trivial for some point  $x$  in  $M - M^c$ .

Consider the following commutative diagram:

$$\begin{array}{ccccc} H^3(M, M - M^c) & \rightarrow & H^3(M) & \xrightarrow{i^*} & H^3(M - M^c) & \rightarrow & H^4(M, M - M^c) \\ & & j_x^* \searrow & & \swarrow i_x^* & & \\ & & & & H^3(K(x)) & & \end{array},$$

where  $i_x, j_x$  and  $i$  are inclusions. It follows from Propositions 1 and 7 that  $j_x^*$  is trivial. If  $\dim M^c \neq 3m - 4$ , then it can be shown that  $H^4(M, M - M^c) = 0$ . In fact, if  $\dim M^c = 3m - 2$ , it follows from Proposition 8 that  $M^c \sim_{\mathcal{Q}} S^1 \times S^3 \times \cdots \times S^3$  ( $m$ -times) and hence  $H^4(M, M - M^c) = H_{3m-4}(M^c) = H^2(M^c) = 0$ . It is easy to see that  $H^4(M, M - M^c) = 0$  if  $\dim M^c < 3m - 4$ . Thus if  $\dim M^c \neq 3m - 4$ , then  $i_x^*$  is trivial, which is a contradiction. Assume  $\dim M^c = 3m - 4$ . Let  $b_1, \dots, b_m \in H^*(M)$  be generators of  $H^*(M)$ . It is easy to see that  $i^*(b_i) = \pi^*(c'_i)$  ( $i = 1, \dots, m$ ) where  $c'_i \in H^3((M - M^c)/K)$ . It follows from the diagram (#) that we may assume  $i^*(b_1 \cdots b_{m-1}) \neq 0$ . But we have  $i^*(b_1 \cdots b_{m-1}) = \pi^*(c'_1 \cdots c'_{m-1}) = 0$ , because  $c'_1 \cdots c'_{m-1} \in H^{3m-3}((M - M^c)/K) = 0$ . This contradiction completes the proof of Theorem A.

### 3. Proof of Theorem B

In this section, we shall prove Theorem B. Let  $M$  be as in Theorem B and  $f: M \rightarrow N = S^1 \times \cdots \times S^1 \times S^2 \times \cdots \times S^2 \times S^3 \times \cdots \times S^3$  a homotopy equivalence.

Put  $s = s' + s''$ . By the same argument as in [14], we can construct a principal  $T^{s'}$ -bundle  $\bar{M}$  over  $M$  and a homotopy equivalence  $\bar{f}: \bar{M} \rightarrow \bar{N} = S^1 \times \cdots \times S^1 \times S^3 \times \cdots \times S^3$ . It follows from results in [10] and [13] (Theorem 17 in [10] and Theorem 4 in [13]) that every action of a torus or compact connected semisimple Lie group on  $M$  can be lifted over  $\bar{M}$ . Thus, to prove Theorem B, it is sufficient to show that  $\bar{M}$  cannot admit any action of a simple Lie group of rank  $\geq 2$ . Hence we may assume  $N = S^1 \times \cdots \times S^1 \times S^3 \times \cdots \times S^3$ . Let  $N_1 = S^1 \times \cdots \times S^1$  ( $r$ -times) and  $f_1$  be the composition of  $f$  and the projection  $N \rightarrow N_1$ . Consider the universal covering  $\tilde{M} \rightarrow M$ . Note that  $\tilde{M} \rightarrow M$  is the pull-back of  $R^r \rightarrow N_1$  by  $f_1$ .

Let  $a_1, \dots, a_r \in H^1(M; Z)$  and  $b_1, \dots, b_s \in H^3(M; Z)$  be the generators of the ring  $H^*(M; Z)$  defined by generators of  $H^*(N; Z)$ . We use the same notations



for generators of  $H^*(M)$ .

Assume a compact simply connected semisimple Lie group  $G$  acts on  $M$ . It follows from a result in [8] (Theorem 4.3 in [8]) that the given action of  $G$  on  $M$  is lifted over  $\tilde{M}$ . It follows from the same argument as in [15] (see Lemma 2 in [15]) that the natural map  $\tilde{M}/G \rightarrow M/G$  is a covering projection and the following diagram is commutative;

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{\pi}} & \tilde{M}/G \\ p \downarrow & & q \downarrow \\ M & \xrightarrow{\pi} & M/G \end{array}$$

and moreover the classifying map  $g: M/G \rightarrow N_1$  of the covering  $\tilde{M}/G \rightarrow M/G$  satisfies that  $f_1$  is homotopic to the composition  $g \circ \pi$ . Hence  $g^*: H^*(N_1; Z) \rightarrow H^*(M/G; Z)$  is a monomorphism.

REMARK. Consider the spectral sequence  $\{E_r^{p,q}, d_r\}$  of the covering  $\tilde{M} \rightarrow M$  with  $E_2^{p,q} = H^p(Z'; H^q(\tilde{M}))$ . It is clear that the group  $Z'$  of the covering transformation acts on  $H^*(\tilde{M})$  trivially and we have  $E_2^{p,q} = H^q(\tilde{M})$ .

We shall prove the following Propositions which are slightly more general than Propositions used in section 2.

**Proposition 9.** *Let  $M$  be as above. Assume the group  $SU(2)$  acts on  $M$  with a finite principal isotropy subgroup and a singular orbit. Then there is a point  $x$  in  $M$  whose isotropy subgroup is a torus.*

Proof. Assume the contrary. Put  $K = SU(2)$ . Let  $\{E_r^{p,q}, d_r\}$  be the Leray spectral sequence of the orbit map  $\pi: M \rightarrow M/K$ . Since  $H^i(K(x)) = 0$  for  $i = 1, 2$  and for every point  $x$ , we have  $E_2^{3,0} = E_\infty^{3,0}$  and  $H^i(M) = H^i(M/K)$  via the homomorphism  $\pi^*$  for  $i = 1, 2$ . Since there is a singular orbit, it follows from Proposition 1 that  $E_\infty^{0,3} = 0$ , which implies  $H^3(M) = H^3(M/K)$  via  $\pi^*$ . It is easy to see that this leads a contradiction. Q.E.D.

**Proposition 10.** *Let  $M$  be as above. Assume the group  $G = SU(3)$  or  $Sp(2)$  acts on  $M$  with a finite principal isotropy subgroup. Then there is a singular orbit.*

Proof. Let  $\phi$  be the given action and  $\tilde{\phi}$  the action on  $\tilde{M}$  which is a lifting of  $\phi$ . Assume  $\phi$  has no singular orbit. Then  $\tilde{\phi}$  has also no singular orbit. Since  $\tilde{M}$  is simply-connected, it follows from a result in [7] (Theorem 1 in [7]) that the second term of the Leray spectral sequence of the orbit map  $\tilde{\pi}: \tilde{M} \rightarrow \tilde{M}/G$  is given by  $H^p(\tilde{M}/G) \otimes H^q(G)$ . We have the following exact sequence;

$$(*) \quad 0 \rightarrow E_{\infty}^{3,0} \xrightarrow{\tilde{\pi}^*} H^3(\tilde{M}) \rightarrow E_{\infty}^{0,3} \rightarrow 0.$$

Since  $E_{\infty}^{3,0} = H^3(\tilde{M}/G)$  and  $\dim E_{\infty}^{0,3} = 1$ , we have  $\dim \text{Im } \tilde{\pi}^* \geq s-1$ . Consider the spectral sequence  $\{\bar{E}_r^{p,q}, \bar{d}_r\}$  of the covering  $\tilde{M}/G \rightarrow M/G$  with  $\bar{E}_2^{p,q} = H^p(Z' : H^q(\tilde{M}/G))$ . Since  $\bar{E}_2^{p,0} \rightarrow H^p(M/G)$  is a monomorphism, we have that  $q^* : H^3(M/G) \rightarrow H^3(\tilde{M}/G)^{Z'}$  is an epimorphism since  $H^i(\tilde{M}/G) = 0$  for  $i=1, 2$ , where  $H^3(\tilde{M}/G)^{Z'}$  is the subgroup of  $H^3(\tilde{M}/G)$  which is invariant under the action of the group  $Z'$  of covering transformations. It follows from the Remark noted above Proposition 9 and sequence (\*) that  $H^3(\tilde{M}/G)^{Z'} = H^3(\tilde{M}/G)$ . Consider the following commutative diagram;

$$\begin{array}{ccc} H^3(M/G) & \xrightarrow{\pi^*} & H^3(M) \\ q^* \downarrow & & p^* \downarrow \\ H^3(\tilde{M}/G) & \xrightarrow{\tilde{\pi}^*} & H^3(\tilde{M}) \end{array}$$

We have shown that we may assume that  $p^*(b_1), \dots, p^*(b_{s-1})$  are in  $\text{Im } \tilde{\pi}^*$ . Put  $p^*(b_i) = \tilde{\pi}^*(b'_i)$ , where  $b'_i \in H^3(\tilde{M}/G)$  for  $i=1, \dots, s-1$ . Since  $q^* : H^3(M/G) \rightarrow H^3(\tilde{M}/G)$  is an epimorphism, we have  $b'_i = q^*(b_i)$ ,  $b_i \in H^3(M/G)$  for  $i=1, \dots, s-1$ . Put  $b'_i = \sum \beta_{imn} a'_m a'_n + A$ , where  $A$  is indecomposable, for  $i=1, \dots, s-1$ . It follows from the fact that  $H^i(\tilde{M}/G) = 0$  for  $i=1, 2$  that we have  $q^*(b'_i) = q^*(A)$ , which implies that  $b'_i$  may be chosen as  $b'_i = A$ . Since  $p^* : H^3(M) \rightarrow H^3(\tilde{M})$  is injective on  $\langle b_1, \dots, b_{s-1} \rangle$ ,  $\pi^*(A)$  is in  $\langle b_1, \dots, b_{s-1} \rangle$  and  $p^*(\pi^*(A)) = \tilde{\pi}^* q^*(A) = \tilde{\pi}^*(b'_i) = p^*(b_i)$  for  $i=1, \dots, s-1$ , we have  $b_i = \pi^*(A)$ . This implies that  $b_1, \dots, b_{s-1}$  are in  $\text{Im } \pi^*$ . As before, put  $b_i = \pi^*(b'_i)$  for  $i=1, \dots, s-1$ . Since  $H^i(G(x)) = 0$  for  $i=1, 2$  and for every point  $x$  in  $M$ , we have  $H^1(M/G) = H^1(M)$ . Put  $a_i = \pi^*(a'_i)$  for  $i=1, \dots, r$ . Then we have

$$0 \neq a_1 \cdots a_r b_1 \cdots b_{s-1} = \pi^*(a'_1 \cdots a'_r b'_1 \cdots b'_{s-1}) = 0,$$

because  $\dim M/G \leq \dim M - 8$ , which is a contradiction. This completes the proof. Q.E.D.

To prove Theorem B, it is sufficient to show that  $SU(3)$  or  $Sp(2)$  can not act on  $M$  (see the proof of Theorem A). Since the case of  $G = Sp(2)$  is completely parallel to the case of  $SU(3)$ , we shall consider only the case of  $G = SU(3)$  and prove that  $SU(3)$  cannot act on  $M$ .

From now on, we assume that  $G = SU(3)$  acts on  $M$ . Let  $K = SU(2)$  be the standard subgroup of  $G$ . Let  $\psi$  be the action of  $K$  obtained from the restriction. We shall prove the following

**Proposition 11.** *The action  $\psi$  has a finite principal isotropy subgroup and there is a point  $x$  in  $M$  whose isotropy subgroup is a torus.*

Proof. The proof of the first part is the same as Proposition 7. We shall prove the second part. Assume the contrary. It follows from Proposition 9 that  $\psi$  is almost free and hence the lifting  $\tilde{\psi}$  of  $\psi$  is also almost free. It is not difficult to see that the Leray spectral sequence of the orbit map  $\tilde{\pi}: \tilde{M} \rightarrow \tilde{M}/K$  collapses and hence we have  $H^*(\tilde{M}) = H^*(\tilde{M}/K) \otimes H^*(S^3)$ . This implies that the homomorphism  $i_{\tilde{x}}^*: H^3(\tilde{M}) \rightarrow H^3(K(\tilde{x}))$  is non-trivial for every point  $\tilde{x}$  in  $\tilde{M}$ . It is clear that the natural homomorphism  $H^3(K(x)) \rightarrow H^3(K(\tilde{x}))$  ( $p(x) = \tilde{x}$ ) is an isomorphism. We have the following commutative diagram;

$$\begin{array}{ccc} H^3(M) & \xrightarrow{p^*} & H^3(\tilde{M}) \\ i_x^* \downarrow & & i_{\tilde{x}}^* \downarrow \\ H^3(K(x)) & \longrightarrow & H^3(K(\tilde{x})) \end{array}$$

where  $p^*$  is an epimorphism. It follows that  $i_{\tilde{x}}^*$  is non-trivial. This implies that the homomorphism  $j_x^*: H^3(M) \rightarrow H^3(G(x))$  is non-trivial for every point  $x$  in  $M$ , which means that  $G_x$  is finite (see Proposition 8 in [14]) for every point  $x$  in  $M$ , which contradicts Proposition 10. This completes the proof. Q.E.D.

**Proposition 12.** *Let  $T$  and  $C$  be a maximal torus and the center of  $K$ , respectively. Then  $M^T$  and  $M^C$  are proper subsets and*

- (1)  $M^T \underset{r\text{-times}}{\sim} S^1 \times \dots \times S^1 \times S^{k_1} \times \dots \times S^{k_s}$  ( $k_i = 1$  or  $3$ )
- (2)  $M^C \underset{r\text{-times}}{\sim} S^1 \times \dots \times S^1 \times S^{n_1} \times \dots \times S^{n_s}$  ( $n_i = 1$  or  $3$ ).

Proof. We have shown that  $M^T$  and  $M^C$  are not empty. The same argument as in Proposition 7 shows that  $M^C$  is a proper subset.

We shall prove (1). To prove it, it is sufficient to show that  $M$  is totally non-homologous to zero in the fiber bundle  $M \rightarrow M_T \rightarrow B_T$ . Consider the covering  $M_i$  over  $M$  which is the pull-back of the covering  $N_i = R^i \times S^1 \times \dots \times S^1 \times S^3 \times \dots \times S^3 \rightarrow N$  by  $f$ , where  $R^i$  denotes the  $i$ -dimensional Euclidean space. Clearly  $M_j$  is a covering over  $M_{j-1}$ , which is the pull-back of  $N_j \rightarrow N_{j-1}$  by a homotopy equivalence of  $f_{j-1}: M_{j-1} \rightarrow N_{j-1}$ . Note that  $T$  acts on  $M_j$  and  $M_{j-1}$  in the way that projection  $p_j: M_j \rightarrow M_{j-1}$  is  $T$ -equivariant.

It follows from Remark 2 in section 1 that  $M_r$  is totally non-homologous to zero in the fiber bundle  $M_r \rightarrow (M_r)_T \rightarrow B_T$ .

To prove (1), it is sufficient to prove the following assertion.

**Assertion 1.** *If  $M_{j+1}$  is totally non-homologous to zero in the fiber bundle  $M_{j+1} \rightarrow (M_{j+1})_T \rightarrow B_T$ , then  $M_j$  is totally non-homologous to zero in the fiber bundle  $M_j \rightarrow (M_j)_T \rightarrow B_T$  for  $j = r - 1, \dots, 0$ .*

This assertion is equivalent to the following assertion.

**Assertion 2.** *If  $i_{j+1}^*: H^*((M_{j+1})_T) \rightarrow H^*(M_{j+1})$  is surjective, then  $i_j^*: H^*((M_j)_T) \rightarrow H^*(M_j)$  is surjective for  $j=r-1, \dots, 0$ . Here  $i_j: M_j \rightarrow (M_j)_T$  is an inclusion.*

Now we shall prove the assertion 2. We have the following observations.

(i) The mapping  $(p_{j+1})_T: (M_{j+1})_T \rightarrow (M_j)_T$  is a covering projection and the group  $Z$  of covering transformations acts on  $H^*((M_{j+1})_T)$  trivially.

The first part is clear and the second part is proved as follows. Consider the spectral sequence  $\{E_r^{p,q}, d_r\}$  of the fiber bundle  $M_{j+1} \rightarrow (M_{j+1})_T \rightarrow B_T$ . Let  $t': (M_{j+1})_T \rightarrow (M_{j+1})_T$  be a covering transformations. Then it is clear that  $t'$  can be represented as  $(t)_T$ , where  $t: M_{j+1} \rightarrow M_{j+1}$  is a covering transformation of the covering  $M_{j+1} \rightarrow M_j$ . Then  $(t)_T$  induces the homomorphism  $1 \otimes t^*: E_2^{p,q} = H^p(B_T) \otimes H^q(M_{j+1}) \rightarrow E_2^{p,q} = H^p(B_T) \otimes H^q(M_{j+1})$ . Since  $t^*: H^q(M_{j+1}) \rightarrow H^q(M_{j+1})$  is easily seen to be trivial,  $(t)_T^*$  is also trivial, which proves the observation.

(ii) To prove the assertion it is sufficient to show that  $i_j^*: H^k((M_j)_T) \rightarrow H^k(M_j)$  is surjective for  $k=1$  and  $3$ . Here this holds the case  $k=1$ .

The first part follows from the fact that  $H^*(M_j)$  is generated by 1-dimensional and 3-dimensional elements. The second part follows easily.

(iii) The homomorphism  $(p_{j+1})_T^*: H^3((M_j)_T) \rightarrow H^3((M_{j+1})_T)$  is surjective.

Consider the spectral sequence  $\{E_2^{p,q}, d_r\}$  of the covering  $(M_{j+1})_T \rightarrow (M_j)_T$ . It is clear that  $E_2^{p,q} = 0$  if  $p \geq 2$ , which means that  $E_2^{0,q} = E_\infty^{0,q}$ . It follows that  $(p_{j+1})_T^*: H^3((M_j)_T) \rightarrow E_2^{0,3} = H^3((M_{j+1})_T)^Z$  which equals  $H^3((M_{j+1})_T)$  as shown in (i) is surjective. Now consider the following commutative diagram;

$$\begin{array}{ccc} H^3((M_j)_T) & \xrightarrow{i_j^*} & H^3(M_j) \\ (p_{j+1})_T^* \downarrow & & \downarrow p_{j+1}^* \\ H^3((M_{j+1})_T) & \xrightarrow{i_{j+1}^*} & H^3(M_{j+1}) \end{array}$$

We can show that  $i_j^*$  is surjective as follows. Take an element  $a$  in  $H^3(M_j)$  which is not decomposable into a product of lower dimensional elements and choose an element  $a'$  in  $H^3((M_j)_T)$  such that  $p_{j+1}^*(a) = i_{j+1}^*(p_{j+1})_T^*(a')$ . Then we have  $p_{j+1}^* i_j^*(a') = p_{j+1}^*(a)$ . Put

$$a' = \sum a_{ijk} a'_i a'_j a'_k + \sum b_{jk} a'_j b'_k + b$$

where  $a'_i, a'_j, a'_k$  and  $a'_j'$  are of dimension 1,  $b'_k$  is of dimension 2 and  $b$  is of dimension 3 such that  $b'_k$  and  $b$  are not decomposable into a product of lower dimensional elements. Then  $i_j^*(a') = i_j^*(b)$ , since  $a$  is not decomposable. Thus we have  $p_{j+1}^* i_j^*(b) = p_{j+1}^*(a)$ , which means  $a = i_j^*(b)$  because  $p_{j+1}^*$  is injective on the subgroup of indecomposable elements. This proves the assertion 2. Then a result in [3] (Theorem 10.2 in Chap. VII in [3]) shows (1). By the

same argument as in Proposition 8, we can show (2). Q.E.D.

Now we shall prove Theorem B. The proof is almost parallel to the proof of Theorem A, but it is more complicated. By the same argument as in Theorem A, we can prove that  $M - M^c \underset{Q}{\sim} (M - M^c)/K \times S^3$ , and hence  $i_x^*: H^3(M - M^c) \rightarrow H^3(K(x))$  is non-trivial for every point  $x$  in  $M - M^c$ . Let  $a_1, \dots, a_r \in H^1(M)$  and  $b_1, \dots, b_s \in H^3(M)$  be generators of  $H^*(M)$ . Put  $\dim M = m$ . If  $\dim M^c < m - 4$ , then the same argument as in Theorem A leads a contradiction.

Assume  $\dim M^c = m - 4$ . Since  $H^i(M, M - M^c) = 0$  for  $i = 1, 3$ , we have the following exact sequence;

$$0 \rightarrow H^i(M) \xrightarrow{i^*} H^i(M - M^c) \rightarrow H^{i+1}(M, M - M^c) \quad i = 1, 3.$$

It follows that  $i^*(a_i) = \pi^*(a'_i)$  and  $i^*(b_j) = \pi^*(b'_j)$  for  $i = 1, \dots, r$  and  $j = 1, \dots, s$ , where  $\pi: M - M^c \rightarrow (M - M^c)/K$  is the orbit map and  $a'_i \in H^1((M - M^c)/K)$ ,  $b'_j \in H^3((M - M^c)/K)$ . From the diagram (#) and the same argument as in Theorem A, we may assume  $i^*(a_1 \cdots a_r b_1 \cdots b_{s-1}) \neq 0$  or  $i^*(a_1 \cdots a_{r-3} b_1 \cdots b_s) \neq 0$ . This is proved easily to be impossible.

Finally assume  $\dim M^c = m - 2$ . Then we have  $M^c \underset{Q}{\sim} S^1 \times \cdots \times S^1 \times S^3 \times \cdots \times S^3$ . As proved above, we can prove that  $i^*(a_i) = \pi^*(a'_i)$ , where  $a'_i \in H^1((M - M^c)/K)$ , for  $i = 1, \dots, r$ .

Let  $a^* \in H_i(M)$  denote the dual element of  $a \in H^i(M)$  and  $[M] = (a_1 \cdots a_r b_1 \cdots b_s)^*$  the fundamental class of a closed manifold  $M$ .

We have the following commutative diagram;

$$\begin{array}{ccccc}
 H^i(M, M - M^c) & \longrightarrow & H^i(M) & \xrightarrow{i^*} & H^i(M - M^c) \\
 \downarrow & & \cap [M] \downarrow & \searrow j_x^* & \downarrow i_x^* \\
 H_{m-i}(M^c) & \xrightarrow{k_*} & H_{m-i}(M) & & H^i(K(x))
 \end{array}$$

(#):

where  $k: M^c \rightarrow M$  is the inclusion.

Let  $k^*(b_s) = 0$ . Then we have  $k^*(a_1 \cdots a_r b_1 \cdots \hat{b}_j \cdots b_s) = 0$  for  $j = 1, \dots, s - 1$ , here  $\hat{b}_j$  denotes the removal of  $b_j$ . This implies that  $(a_1 \cdots a_r b_1 \cdots \hat{b}_j \cdots b_s)^*$  is not  $\text{Im } k_*$  for  $j = 1, \dots, s - 1$ . It follows from (#)<sub>3</sub> that  $i^*(b_j) \neq 0$  for  $j = 1, \dots, s - 1$ , because  $b_j \cap [M] = (-1)^{s-j} (a_1 \cdots a_r b_1 \cdots \hat{b}_j \cdots b_s)^*$ .

Since  $j_x^*$  is trivial, we have  $i^*(b_j) = \pi^*(b'_j)$  ( $b'_j \in H^3((M - M^c)/K)$ ) for  $j = 1, \dots, s - 1$ .

It follows from (#)<sub>m-3</sub> and the fact  $k^*(b_s) = 0$  that  $i^*(a_1 \cdots a_r b_1 \cdots b_{s-1}) \neq 0$  because  $(a_1 \cdots a_r b_1 \cdots b_{s-1}) \cap [M] = (-1)^{s-1} b_s^*$ . Then we have

$$0 \neq i^*(a_1 \cdots a_r b_1 \cdots b_{s-1}) = \pi^*(a'_1 \cdots a'_r b'_1 \cdots b'_{s-1}) = 0,$$



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