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PROPER DUPIN HYPERSURFACES GENERATED
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Introduction

A connected oriented hypersurface $M$ of the space form $\overline{M} = E^n$, $S^n$ or $H^n$ is called a Dupin hypersurface, if for any curvature submanifold $S$ of $M$ the corresponding principal curvature $\lambda$ is constant along $S$. Here by a curvature submanifold we mean a connected submanifold $S$ with a smooth function $\lambda$ on $S$ such that for each point $x \in S$, $\lambda(x)$ is a principal curvature of $M$ at $x$ and $T_xS$ is equal to the principal subspace in $T_xM$ corresponding to $\lambda(x)$. A Dupin hypersurface is said to be proper, if all principal curvatures have locally constant multiplicities. A connected oriented hypersurface of $\overline{M}$ is called an isoparametric hypersurface, if all principal curvatures are locally constant. Obviously an isoparametric hypersurface is a proper Dupin hypersurface. Another example of a Dupin hypersurface (Pinkall [6]) is an $\varepsilon$-tube $M^\varepsilon$ around a symmetric submanifold $M$ of $\overline{M}$ of codimension greater than 1, which is said to be generated by $M$. Recall that a connected submanifold $M$ of $\overline{M}$ is a symmetric submanifold, if for each point $x \in M$ there is an involutive isometry $\sigma$ of $\overline{M}$ leaving $M$ and $x$ invariant such that $(-1)$-eigenspace of $(\sigma_x)_*$ is equal to $T_xM$. The most simple example is the tube $M^\varepsilon$ around a complete totally geodesic submanifold $M$. This is a complete isoparametric hypersurface with two principal curvatures, which is further homogeneous in the sense that the group $\text{Aut}(M^\varepsilon) = \{\phi \in I(\overline{M}); \phi(M^\varepsilon) = M^\varepsilon\}$ acts transitively on $M^\varepsilon$. Here $I(\overline{M})$ denotes the group of isometries of $\overline{M}$. In this note we will determine all the symmetric submanifolds whose tube is a proper Dupin hypersurface, in the following theorem.

Theorem. Let $M$ be a non-totally geodesic symmetric submanifold of a space form $\overline{M}$ of codimension greater than 1. Then the tube $M^\varepsilon$ around $M$ is a proper Dupin hypersurface if and only if either

(i) $M$ is a complete extrinsic sphere of $\overline{M}$ (see Section 2 for definition) of codimen-
sion greater than 1; or
(ii) $M$ is one of the following symmetric submanifolds of $S^n$:
(a) the projective plane $P_d(F) \subset S^{d+1}$, $d = \dim_R F$, over $F = \mathbb{R}, \mathbb{C}$, quaternions $\mathbb{H}$ or octonions $\mathbb{O}$;
(b) the complex quadric $Q_3(\mathbb{C}) \subset S^9$;
(c) the Lie quadric $Q^{m+1} \subset S^{2m+1}$, $m \geq 2$;
(d) the unitary symplectic group $Sp(2) \subset S^{15}$.
(Explicit embeddings of these spaces will be given in Section 2.) In case (i), $M^*$ is a Dupin cyclide, i.e., a proper Dupin hypersurface with two principal curvatures, but it is not an isoparametric hypersurface. In case (ii), $M^*$ is a homogeneous isoparametric hypersurface with three or four principal curvatures, and it is an irreducible Dupin hypersurface in the sense of Pinkall [6].

1. Principal curvatures of tubes

Let $M$ be a connected submanifold of a space form $\bar{M}$ of codimension $q > 1$, $NM$ and $U(NM)$ the normal bundle and the unit normal bundle of $M$, respectively. Denote by $A_\xi$ the shape operator of $M$. Suppose that the map $f^*: U(NM) \to \bar{M}$, $\varepsilon > 0$, defined by

$$f^*(u) = \text{Exp}(\varepsilon u) \quad \text{for} \quad u \in U(NM)$$

is an embedding, and set $M^* = f^*(U(NM)) \subset \bar{M}$. Then (cf. Cecil-Ryan [1]) we have the following

**Lemma 1.1.** Let $\lambda_1, \ldots, \lambda_p$ be the eigenvalues of $A_\xi$, $u \in U(NM)$, with multiplicities $m_1, \ldots, m_p$, respectively. Then the principal curvatures of $M^*$ at $f^*(u)$ with respect to the outward unit normal are given as follows.

$$-\frac{\lambda_i}{1 - \lambda_i \varepsilon}, \quad 1 \leq i \leq p,$$

and

$$-\frac{1}{\varepsilon} \quad \text{for} \quad \bar{M} = E^n,$$

$$\frac{\sin \varepsilon + \lambda_i \cos \varepsilon}{\cos \varepsilon - \lambda_i \sin \varepsilon}, \quad 1 \leq i \leq p,$$

$$\frac{-\sinh \varepsilon + \lambda_i \cosh \varepsilon}{\cosh \varepsilon - \lambda_i \sinh \varepsilon}, \quad 1 \leq i \leq p,$$  

$$\frac{-\cot \varepsilon}{\cosh \varepsilon - \lambda_i \sinh \varepsilon}, \quad \text{for} \quad \bar{M} = S^n,$$

$$-\frac{1}{\varepsilon} \quad \text{for} \quad \bar{M} = H^n,$$

with multiplicities $m_1, \ldots, m_p$, $q - 1$, respectively.

**Corollary 1.2.** Suppose that $M^*$ is a proper Dupin hypersurface. Then, for each point $x \in M$, the number of eigenvalues of $A_\xi$, $\xi \in N_x M - \{0\}$, is a constant independent of $\xi$.

In what follows in this section, let $T$ and $N$ be finite dimensional real vector spaces with inner product $\langle \cdot, \cdot \rangle$, and $A: N \ni \xi \mapsto A_\xi \in \text{Sym}(T)$ a linear map
from $N$ to the space $\text{Sym}(T)$ of symmetric endomorphisms of $T$ satisfying
\begin{equation}
(1.1) \quad \text{the number } \nu(\xi) \text{ of eigenvalues of } A_{\xi}, \xi \in N - \{0\}, \text{ is a constant } p \text{ independent of } \xi.
\end{equation}

**Lemma 1.3.** Assume that $N$ is an orthogonal sum:
\[ N = N_1 \oplus N_2 \quad \text{with } N_1 \neq \{0\}, \dim N_2 = 1, \]
and there are a linear map $A^{(1)}: N_1 \to \text{Sym}(T)$ and a vector $\eta_2 \in N_2$ such that
\[ A_{\xi_1 + \xi_2} = A^{(1)}_{\xi_1} + \langle \xi_2, \eta_2 \rangle I \quad \text{for any } \xi_1 \in N_1, \xi_2 \in N_2. \]
Then there exists a vector $\eta_1 \in N_1$ such that
\[ A^{(1)}_{\xi_1} = \langle \xi_1, \eta_1 \rangle I \quad \text{for any } \xi_1 \in N_1. \]

Proof. For any $\xi_2 \in N_2, \xi_2 \neq 0$, we have $A_{\xi_2} = \langle \xi_2, \eta_2 \rangle I$. Thus one has $p=1$. Hence, for any $\xi_1 \in N_1, \xi_1 \neq 0, A^{(1)}_{\xi_1} = A_{\xi_1}$ is a scalar operator on $T$. Now the linearity of $A^{(1)}$ implies the existence of $\eta_1$ above. \(\text{q.e.d.}\)

**Lemma 1.4.** Assume that $N$ is an orthogonal sum as in Lemma 1.3, and also $T$ is an orthogonal sum:
\[ T = T_1 \oplus T_2 \quad \text{with } T_1 \neq \{0\}, T_2 \neq \{0\}. \]
Furthermore assume that there are a linear map $A^{(1)}: N_1 \to \text{Sym}(T_1)$ and different vectors $\eta_2, \eta'_2 \in N_2$ such that
\[ A_{\xi_1 + \xi_2} = (A^{(1)}_{\xi_1} + \langle \xi_2, \eta_2 \rangle I_{T_1}) \oplus \langle \xi_2, \eta'_2 \rangle I_{T_2} \quad \text{for any } \xi_1 \in N_1, \xi_2 \in N_2. \]
Then $A^{(1)}=0$.

Proof. For any $\xi_2 \in N_2, \xi_2 \neq 0$, we have
\[ A_{\xi_2} = \langle \xi_2, \eta_2 \rangle I_{T_1} \oplus \langle \xi_2, \eta'_2 \rangle I_{T_2}, \]
with $\langle \xi_2, \eta_2 \rangle \neq \langle \xi_2, \eta'_2 \rangle$, and hence $p=2$. We fix an arbitrary $\xi_1 \in N_1, \xi_1 \neq 0$.

First we assume that the eigenvalues $\lambda_1, \ldots, \lambda_k, k \geq 1$, of $A^{(1)}_{\xi_1}$ are all nonzero. Then, for $\xi = \alpha \xi_1 + \xi_2$ with $\xi_2 \in N_2, \xi_2 \neq 0$, and sufficiently small nonzero $\alpha \in \mathbb{R}$, the numbers $\alpha \lambda_1 + \langle \xi_2, \eta_2 \rangle, \ldots, \alpha \lambda_k + \langle \xi_2, \eta_2 \rangle, \langle \xi_2, \eta_2 \rangle$ are different each other, and hence $\nu(\xi) = k+1$. Thus, by (1.1) we get $k=1$, i.e., $A^{(1)}_{\xi_1} = \lambda_1 I_{T_1}, \lambda_1 \neq 0$. Take $\xi_2 \in N_2, \xi_2 \neq 0$, and $\beta \in \mathbb{R}$ with
\[ \beta \lambda_1 + \langle \xi_2, \eta_2 \rangle = \langle \xi_2, \eta'_2 \rangle. \]
Then, for $\xi = \beta \xi_1 + \xi_2 \neq 0$, we have $A_{\xi} = \langle \xi_2, \eta'_2 \rangle I$, and hence $\nu(\xi) = 1$. This is a contradiction to $p=2$.

We next assume that $A^{(1)}_{\xi_1}$ has eigenvalue 0, together with possible nonzero
eigenvalues $\lambda_1, \ldots, \lambda_k, k \geq 0$. Then, for $\xi = \alpha \xi_1 + \xi_2$ with $\xi_1, \xi_2 \in N_2, \xi_2 \neq 0$, and sufficiently small $\alpha \neq 0$, one has $\nu(\xi) = k + 2$. Thus, by (1.1) we get $k = 0$, i.e., $A^{(1)} = 0$.

Since $\xi_1 \in N_1, \xi_1 \neq 0$, is arbitrary, we obtain $A^{(0)} = 0$. q.e.d.

**Lemma 1.5.** Assume that both $N$ and $T$ have orthogonal decompositions:

$$N = N_1 \oplus N_2 \quad \text{with} \quad N_1 \neq \{0\}, N_2 \neq \{0\},$$

$$T = T_1 \oplus T_2 \quad \text{with} \quad T_1 \neq \{0\}, T_2 \neq \{0\},$$

and there are linear maps $A^{(0)}: N_1 \rightarrow \text{Sym}(T_1)$ and $A^{(2)}: N_2 \rightarrow \text{Sym}(T_2)$ such that

$$A_{\xi_1 + \xi_2}^{(0)} = A_{\xi_1}^{(1)} \oplus A_{\xi_2}^{(2)} \quad \text{for any} \quad \xi_1 \in N_1, \xi_2 \in N_2.$$

Then $A = 0$.

**Proof.** We fix arbitrary $\xi_1 \in N_1, \xi_1 \neq 0$, and $\xi_2 \in N_2, \xi_2 \neq 0$.

Case (a): Both $A_{\xi_1}^{(1)}$ and $A_{\xi_2}^{(2)}$ have only nonzero eigenvalues $\lambda_1, \ldots, \lambda_k, k \geq 1$, and $\mu_1, \ldots, \mu_l, l \geq 1$, respectively. Then, for $\xi = \xi_1 + \alpha \xi_2$ with sufficiently small $\alpha \neq 0$, one has $\nu(\xi) = k + l$. On the other hand, one has $\nu(\xi_1) = k + 1$. Thus, by (1.1) we get $l = 1$. In the same way we get $k = 1$. It follows that $p = 2$ and $A_{\xi_1}^{(1)} = \lambda_1 I_{T_1}, A_{\xi_2}^{(2)} = \mu_1 I_{T_2}$ with $\lambda_1, \mu_1 \neq 0$. Now, for $\xi = \mu_1 \xi_1 + \lambda_1 \xi_2$, we get $A_{\xi} = (\lambda_1, \mu_1) I$. This is a contradiction to $p = 2$.

Case (b): One of the $A_{\xi_1}^{(0)}$, say $A_{\xi_1}^{(1)}$, has only nonzero eigenvalues $\lambda_1, \ldots, \lambda_k, k \geq 1$, and the other $A_{\xi_2}^{(2)}$ has eigenvalue 0 together with possible nonzero eigenvalues $\mu_1, \ldots, \mu_l, l \geq 0$. Then, for $\xi = \alpha \xi_1 + \xi_2$ with sufficiently small $\alpha \neq 0$, one has $\nu(\xi) = k + l + 1$. Together with $\nu(\xi_2) = l + 1$, we get $k = 0$. This is a contradiction to $k \geq 1$.

Case (c): Both $A_{\xi_1}^{(1)}$ and $A_{\xi_2}^{(2)}$ have eigenvalue 0, together with possible nonzero eigenvalues $\lambda_1, \ldots, \lambda_k, k \geq 0$, and $\mu_1, \ldots, \mu_l, l \geq 0$, respectively. Then, for $\xi = \xi_1 + \alpha \xi_2$ with sufficiently small $\alpha \neq 0$, one has $\nu(\xi) = k + l + 1$. Together with $\nu(\xi_1) = k + 1, \nu(\xi_2) = l + 1$, we get $k = l = 0$, i.e., $A_{\xi_1}^{(1)} = 0$ and $A_{\xi_2}^{(2)} = 0$.

Thus we conclude that $A = 0$. q.e.d.

**2. Proof of Theorem**

We first explain some terminologies. The Riemannian metric of $\bar{M}$ will be denoted by $\langle \mathbf{s}, \mathbf{s} \rangle$. A connected submanifold $M$ of $\bar{M}$ is called an **extrinsic sphere**, if the mean curvature normal $\eta$ of $M$ is nonzero and parallel (with respect to the normal connection in $NM$), and moreover each shape operator $A_{\xi}$ is the scalar operator $\langle \xi, \eta \rangle I$. A submanifold of a space form $\bar{M}$ is said to be **strongly full**, if it is full in $\bar{M}$, and further it is not contained in any extrinsic sphere of $\bar{M}$ of codimension 1.

Let now $M$ be a symmetric submanifold as in Theorem, and suppose that $M'$ is a proper Dupin hypersurface.
First we assume that $M$ is not full in $\bar{M}$. Then there exists a complete totally geodesic hypersurface $\bar{M}^{s-1}$ of $\bar{M}$ with $M \subset \bar{M}^{s-1}$. Applying Lemma 1.3 to the shape operator $A^{(1)}$ of $M \subset \bar{M}^{s-1}$ and $\eta_{2}=0$, we see that $A_{1}^{(1)}=\langle \xi_{1}, \eta_{1} \rangle I$ for any normal vector $\xi_{1}$ to $M \subset \bar{M}^{s-1}$. Here $\eta_{1}$ is the mean curvature normal of $M \subset \bar{M}^{s-1}$, which is parallel since the second fundamental form of $M \subset \bar{M}$ is parallel (cf. Naitoh-Takeuchi [4]). Thus $M$ is a complete totally geodesic submanifold or a complete extrinsic sphere of $\bar{M}$. Since the first case is excluded from the assumption, we obtain the case (i) in Theorem. In this case, the principal curvatures of $M'$ at $f'(u), u \in U(NM)$, are calculated by Lemma 1.1 as follows.

$$\begin{align*}
\frac{\langle u, \eta \rangle}{1-\langle u, \eta \rangle \varepsilon} \quad \text{and} \quad -\frac{1}{\varepsilon} & \quad \text{for } \bar{M}=E^{n}, \\
\sin \varepsilon+\frac{\langle u, \eta \rangle}{\cos \varepsilon} \cos \varepsilon & \quad \text{and} \quad -\cot \varepsilon \quad \text{for } \bar{M}=S^{n}, \\
\frac{-\sinh \varepsilon+\langle u, \eta \rangle \cosh \varepsilon}{\cosh \varepsilon-\langle u, \eta \rangle \sinh \varepsilon} & \quad \text{and} \quad -\coth \varepsilon \quad \text{for } \bar{M}=H^{n},
\end{align*}$$

where $\eta$ is the nonzero mean curvature normal of $M \subset \bar{M}$. Thus $M'$ is a non-isoparametric Dupin cyclide in $\bar{M}$.

Next we assume that $M$ is full, but not strongly full. Then there exists a complete extrinsic sphere $\bar{M}^{s-1}$ of $\bar{M}$ of codimension 1 such that $M$ is a strongly full submanifold in $\bar{M}^{s-1}$. Applying Lemma 1.3 to the shape operator $A^{(1)}$ of $M \subset \bar{M}^{s-1}$ and the mean curvature normal $\eta_{2}$ of $\bar{M}^{s-1} \subset \bar{M}$, we see that $M$ is a totally geodesic submanifold or an extrinsic sphere of $\bar{M}^{s-1}$. This is a contradiction to that $M$ is strongly full in $\bar{M}^{s-1}$.

Thus it remains to determine $M$ in the case where $M$ is a strongly full symmetric submanifold of $\bar{M}$. We will use the classification of such submanifolds in Takeuchi [10] (see also Naitoh-Takeuchi [4]).

(I) Case $\bar{M}=E^{n}$: One has $M=E^{n} \times M' \subset E^{n} \times S^{n}(r) \subset E^{n} \times E^{s+1}=E^{n}, n_{1}, n_{2} \geq 1, n_{1}+n_{2}=n-1$, where $M'$ is a symmetric submanifold of the hypersphere $S^{n}(r)$ with radius $r>0$ in $E^{s+1}$ such that $M' \subset E^{s+1}$ is substantial. Applying Lemma 1.4 to the shape operator $A^{(1)}$ of $M' \subset S^{n}(r)$, we see that $M'$ is totally geodesic in $S^{n}(r)$. This is a contradiction to that $M' \subset E^{s+1}$ is substantial.

(II) Case $\bar{M}=H^{n}$: We regard $H^{n}$ as

$$H^{n}=\{(x_{i}) \in R^{n+1}; -x_{1}^{2}+x_{2}^{2}+\cdots+x_{n+1}^{2}=-1, x_{i}>0\}.$$

Then $M=H^{n}(r_{1}) \times M' \subset H^{n}(r_{1}) \times S^{n}(r_{2}) \subset H^{n}, n_{1}, n_{2} \geq 1, n_{1}+n_{2}=n-1, r_{1}, r_{2}>0, r_{1}^{2}-r_{2}^{2}=1$, where

$$H^{n}(r_{1})=\{(x_{i}) \in R^{n+1}; -x_{1}^{2}+x_{2}^{2}+\cdots+x_{n+1}^{2}=-r_{1}^{2}, x_{i}>0\},$$

and $M'$ is a symmetric submanifold of $S^{n}(r_{2}) \subset R^{n+1}$ such that $M' \subset R^{n+1}$ is
substantial. In the same way as in (I), we see that \( M' \) is totally geodesic in \( S^{n}(r_2) \), which leads to a contradiction.

(III) Case \( \overline{M} = S^n \): In this case, \( M \) is a symmetric \( R \)-space and the inclusion \( M \subset S^n \) is induced from the substantial standard embedding \( M \subset R^{n+1} \) (Ferus [2]). If \( M \) is a reducible symmetric \( R \)-space, one has \( M = M_1 \times M_2 \subset S^n(r_1) \times S^n(r_2) \subset S^n \) with \( n_1, n_2 \geq 1, n_1 + n_2 = n - 1, r_1, r_2 > 0, r_1^2 + r_2^2 = 1 \). Let one of the \( M_i \), say \( M_i \), be equal to \( S^n(r_i) \). Then, applying Lemma 1.4 to the shape operator \( A^{(i)} \) of \( M_i \subset S^n(r_i) \), we see that \( M_2 \) is totally geodesic in \( S^n(r_2) \). This is a contradiction to that \( M \subset R^{n+1} \) is substantial. Otherwise, one has \( \dim M_1 < n_1 \) and \( \dim M_2 < n_2 \). Since the shape operator of \( M \subset S^n(r_1) \times S^n(r_2) \) also satisfies (1.1), we can apply Lemma 1.5 to the shape operators \( A^{(i)} \) of \( M_i \subset S^n(r_i) \) to see that both \( M_i \) are totally geodesic in \( S^n(r_i) \). This is also a contradiction to that \( M \subset R^{n+1} \) is substantial.

Thus it remains to consider an irreducible symmetric \( R \)-space \( M \). For this we recall the construction of the standard embedding of \( M \) (cf. Ferus [2], Takeuchi [10], [11]). Let

\[ g = g_1 + g_0 + g_0, \quad [g_\rho, g_\sigma] \subset g_{\rho + \sigma} \]

be a simple symmetric graded Lie algebra over \( R \), with a Cartan involution \( \tau \) satisfying \( \tau g_\rho = g_{-\rho} \), \( -1 \leq \rho \leq 1 \). The characteristic element \( e \in g_0 \) is the unique element with

\[ g_\rho = \{ x \in g ; [e, x] = \rho x \}, \quad -1 \leq \rho \leq 1. \]

Let

\[ g = \mathfrak{k} + \mathfrak{p}, \quad g_0 = \mathfrak{k}_0 + \mathfrak{p}_0 \quad \text{with} \quad e \in \mathfrak{p}_0 \]

be the Cartan decompositions associated to \( \tau \). We denote by \( K \) the compact connected subgroup of \( GL(\mathfrak{p}) \) generated by \( \text{ad}_e \mathfrak{k} \), and set

\[ K_0 = \{ k \in K ; k \cdot e = e \}. \]

Then we have identifications: \( \mathfrak{k} = \text{Lie} K \) and \( \mathfrak{k}_0 = \text{Lie} K_0 \). Making use of the Killing form \( B \) of \( g \), we define a \( K \)-invariant inner product \( \langle \cdot, \cdot \rangle \) on \( \mathfrak{p} \) by

\[ \langle x, y \rangle = \frac{1}{2 \dim g_{-1}} B(x, y) \quad \text{for} \quad x, y \in \mathfrak{p}, \]

to identify \( \mathfrak{p} \) with the euclidean space \( R^{n+1} \), \( n = \dim \mathfrak{p} - 1 \). Then \( e \) is in the unit sphere \( S^n \) of \( R^{n+1} \), and

\[ M = K/K_0 = K \cdot e \]

gives the required embedding. Let \( \mathfrak{a} \) be a maximal abelian subalgebra in \( \mathfrak{p} \) including \( e \), and set \( r = \dim \mathfrak{a} \). Then one has \( \mathfrak{a} \subset \mathfrak{p}_0 \). Let \( W = N_\mathfrak{p}(\mathfrak{a})/Z_\mathfrak{p}(\mathfrak{a}) \subset O(\mathfrak{a}) \).
be the Weyl group of $g$, where
\[ N_K(a) = \{ k \in K; k \cdot a = a \}, \]
\[ Z_K(a) = \{ k \in K; k \cdot h = h \text{ for any } h \in a \}. \]

We define $g$ to be a half of the cardinality $\#W$ of $W$. Denote by $\Sigma \subseteq a$ the root system of $g$ relative to $a$, and set
\[ \Sigma_i = \{ \gamma \in \Sigma; \langle \gamma, e \rangle = 1 \}. \]

Let $p_i$ be the orthogonal complement to $p_0$ in $p$. Then one has
\[ T_\varepsilon M = p_i = \sum_{\gamma \in \Sigma_i} \oplus p', \]
where $p'$ is the subspace of $p$ defined by
\[ p' = \{ x \in p; [h, [h, x]] = \langle h, \gamma \rangle x \text{ for any } h \in a \}. \]

Thus the normal space $N_\varepsilon M$ to $M \subset S^n$ at $e$ is given by
\[ N_\varepsilon M = a_0 \oplus a'_0, \]
where $a_0$ and $a'_0$ are the orthogonal complement to $a$ in $p_0$ and the one to $Re$ in $a$, respectively. The shape operator $A$ of $M \subset S^n$ at $e$ can be calculated by the same way as in Takagi-Takahashi [8] to get
\[ A_h x = -\langle h, \gamma \rangle x \quad \text{for } h \in a_0, x \in p', \gamma \in \Sigma_i. \]

Now we come back to our problem. If $r=1$, one has $M = S^n$. This case is excluded because of $\text{codim} M > 1$. If $r \geq 2$, one has $\#\Sigma_i > 1$, since $\#\Sigma_i = 1$ would imply $r = 1$. Therefore, if we denote the orthogonal projection $a \rightarrow a_\sigma$ by $\sigma$, we have $\#\sigma(\Sigma_i) > 1$, noting that $\sigma(\gamma) = \gamma - e$ for each $\gamma \in \Sigma_i$. It follows that if $r \geq 3$ there exist $h, h' \in a_0 - \{0\}$ such that
\[ \#\{-\langle h, \gamma \rangle; \gamma \in \Sigma_i\} = \#\{-\langle h', \gamma \rangle; \gamma \in \Sigma_i\}. \]

This is a contradiction to Corollary 1.2 by virtue of (2.1). Thus we must have $r = 2$. In this case, by the classification of irreducible symmetric $R$-spaces (Kobayashi-Nagano [3], Takeuchi [9]) we see that only the following four cases are possible.

(a) Case $g = 3$: $M = P_3(F)$, the projective plane over $F = R, C, H$ or $O$, and the standard embedding $P_3(F) \subset R^{3d+2}$ is the generalized Veronese embedding (Tai [7]).

Case $g = 4$:
(b) $M$ is the complex quadric of complex dimension 3:
\[ Q_3(C) = \{ [z] \in P_1(C); 'zz = 0 \}, \]
and \( \mathfrak{p} \) is identified with the space \( A_5(\mathbb{R}) \) of real alternating \( 5 \times 5 \) matrices with inner product:

\[
\langle X, Y \rangle = -\frac{1}{2} \text{tr}(XY) \quad \text{for } X, Y \in A_5(\mathbb{R}).
\]

Any \([z] \in Q_3(\mathbb{C})\) can be written as

\[
z = x + \sqrt{-1}y \quad \text{with } x, y \in S^4 \subset \mathbb{R}^5, \langle x, y \rangle = 0.
\]

The map \([z] \mapsto x'y - y'x\) is the standard embedding.

(c) \( M \) is the Lie quadric of dimension \( m+1 \), \( m \geq 2 \):

\[
Q^{m+1} = \{ [z] \in P_{m+2}(\mathbb{R}); -z_1^2 - z_2^2 + \cdots + z_{m+2}^2 = 0 \},
\]

and \( \mathfrak{p} \) is identified with the space \( M_{m+1,2}(\mathbb{R}) \) of real \( (m+1) \times 2 \) matrices with inner product:

\[
\langle X, Y \rangle = \text{tr}(t'XY) \quad \text{for } X, Y \in M_{m+1,2}(\mathbb{R}).
\]

Any \([z] \in Q^{m+1}\) can be written as

\[
z = \left( \begin{array}{c} x \\ y \end{array} \right) \quad \text{with } x \in S^1 \subset \mathbb{R}^2, y \in S^m \subset \mathbb{R}^{m+1}.
\]

The map \([z] \mapsto y'x\) is the standard embedding.

(d) \( M \) is the unitary symplectic group of degree 2:

\[
Sp(2) = \{ z \in M_2(\mathbb{H}); 'zz = 1_2 \},
\]

\( M_2(\mathbb{H}) \) being the space of quaternion \( 2 \times 2 \) matrices, and \( \mathfrak{p} \) is identified with \( M_2(\mathbb{H}) \) with inner product:

\[
\langle X, Y \rangle = \frac{1}{2} \text{Re tr}(t'XY) \quad \text{for } X, Y \in M_2(\mathbb{H}).
\]

The inclusion \( Sp(2) \subset M_2(\mathbb{H}) \) is the standard embedding.

In these cases, any tube around \( M \) is obtained as \( M' \) with \( 0 < \varepsilon < \pi / g \), and each \( M' \) is a homogeneous isoparametric hypersurface of \( S^n \) with \( g \) principal curvatures. In order to show this, first note that \( K \) acts on \( U(NM) \) transitively. In fact, since the semisimple part of \( \mathfrak{g}_0 \) has rank 1, \( K_0 \) acts on the unit sphere in \( \mathfrak{a}_0 \oplus \mathfrak{q}_0 = N_\varepsilon M \) transitively. We choose a unit vector \( f \in \mathfrak{a}_0 \), and thus \( f \in U_\varepsilon(NM) \). Then the stabilizer \( Z_\varepsilon \) of \( f \) in \( K \) is given by

\[
Z_\varepsilon = Z_{K_\varepsilon}(f) = Z_K(\varepsilon).
\]

Now for each \( \varepsilon \in \mathbb{R} \) the map \( f^*: U(NM) \rightarrow S^n \) is \( K \)-equivariant, and hence \( M' = f^*(U(NM)) \) is the \( K \)-orbit in \( S^n \) through

\[
h^\varepsilon = (\cos \varepsilon) e + (\sin \varepsilon) f.
\]
Note that $h'$ is $W$-regular if and only if $\varepsilon \in \pi/g \mathbb{Z}$. It follows that $M' = M''$ if and only if $h'$ and $h''$ are $W$-conjugate, and that $f'$ is an embedding if and only if $M'$ is a regular $K$-orbit in $S^n$, which is the same as that $h'$ is $W$-regular. Moreover, any regular $K$-orbit is a homogeneous isoparametric hypersurface in $S^n$ with $g$ principal curvatures (Takagi-Takahashi [8], Ozeki-Takeuchi [5]). These imply our claim.

It is known (Pinkall [6]) that an isoparametric hypersurface $\overline{M} \subseteq S^n$ is an irreducible Dupin hypersurface, if $\text{Aut}(\overline{M}) \subset O(n+1)$ acts irreducibly on $\mathbb{R}^{n+1}$. But, $\text{Aut}(M')$ for our tube $M'$ acts irreducibly on $\mathbb{R}^{n+1}$, because the subgroup $K$ of $\text{Aut}(M')$ acts on $p$ irreducibly by virtue of simplicity of $g$. Thus we get the last assertion in Theorem.

We finally note that a Dupin cyclide as in case (i) is always a reducible Dupin hypersurface.

References
