Takeuchi, M.
Osaka J. Math.
28 (1991), 153–161

PROPER DUPIN HYPERSURFACES GENERATED BY SYMMETRIC SUBMANIFOLDS

Dedicated to Professor Tadashi Nagano on his sixtieth birthday

MASARU TAKEUCHI

(Received March 6, 1990)

Introduction

A connected oriented hypersurface M of the space form $\bar{M}=E^n$, S^n or H^n is called a *Dupin hypersurface*, if for any curvature submanifold S of M the corresponding principal curvature λ is constant along S. Here by a *curvature submanifold* we mean a connected submanifold S with a smooth function λ on S such that for each point $x\in S$, $\lambda(x)$ is a principal curvature of M at x and T_xS is equal to the principal subspace in T_xM corresponding to $\lambda(x)$. A Dupin hypersurface is said to be *proper*, if all principal curvatures have locally constant multiplicities. A connected oriented hypersurface of $\bar{M}$ is called an *isoparametric hypersurface*, if all principal curvatures are locally constant. Obviously an isoparametric hypersurface is a proper Dupin hypersurface. Another example of a Dupin hypersurface (Pinkall [6]) is an ε-tube M^ε around a symmetric submanifold M of $\bar{M}$ of codimension greater than 1, which is said to *be generated by* M. Recall that a connected submanifold M of $\bar{M}$ is a *symmetric submanifold*, if for each point $x\in M$ there is an involutive isometry σ of $\bar{M}$ levaing M and x invariant such that (-1)-eigenspace of $(\sigma_*)_x$ is equal to T_xM. The most simple example is the tube M^ε around a complete totally geodesic submanifold M. This is a complete isoparametric hypersurface with two principal curvatures, which is further *homogeneous* in the sense that the group

$$\mathrm{Aut}(M^\varepsilon)=\{\phi\in I(\bar{M});\ \phi(M^\varepsilon)=M^\varepsilon\}$$

acts transitively on M^ε. Here $I(\bar{M})$ denotes the group of isometries of $\bar{M}$. In this note we will determine all the symmetric submanifolds whose tube is a proper Dupin hypersurface, in the following theorem.

Theorem. *Let M be a non-totally geodesic symmetric submanifold of a space form $\bar{M}$ of codimension greater than* 1. *Then the tube M^ε around M is a proper Dupin hypersurface if and only if either*
(i) *M is a complete extrinsic sphere of $\bar{M}$ (see Section 2 for definition) of codimen-*

sion greater than 1; *or*

(ii) *M is one of the following symmetric submanifolds of S^n*:

(*a*) *the projective plane $P_2(\boldsymbol{F})\subset S^{3d+1}$, $d=\dim_{\boldsymbol{R}}\boldsymbol{F}$, over $\boldsymbol{F}=\boldsymbol{R}, \boldsymbol{C}$, quaternions $\boldsymbol{H}$ or octonions $\boldsymbol{O}$*;

(*b*) *the complex quadric $Q_3(\boldsymbol{C})\subset S^9$*;

(*c*) *the Lie quadric $Q^{m+1}\subset S^{2m+1}$, $m\geq 2$*;

(*d*) *the unitary sympletic group $Sp(2)\subset S^{15}$.*

(*Explicit embeddings of these spaces will be given in Section* 2.) *In case* (i), *M^ε is a Dupin cyclide, i.e., a proper Dupin hypersurface with two principal curvatures, but it is not an isoparametric hypersurface. In case* (ii), *M^ε is a homogeneous isoparametric hypersurface with three or four principal curvatures, and it is an irreducible Dupin hypersurface in the sense of Pinkall* [6].

1. Principal curvatures of tubes

Let M be a connected submanifold of a space form $\bar{M}$ of codimension $q>1$, NM and $U(NM)$ the normal bundle and the unit normal bundle of M, respectively. Denote by A_ξ the shape operator of M. Suppose that the map f^ε: $U(NM)\to\bar{M}$, $\varepsilon>0$, defined by

$$f^\varepsilon(u)=\mathrm{Exp}(\varepsilon u) \qquad \text{for } u\in U(NM)$$

is an embedding, and set $M^\varepsilon=f^\varepsilon(U(NM))\subset\bar{M}$. Then (cf. Cecil-Ryan [1]) we have the following

Lemma 1.1. *Let $\lambda_1, \cdots, \lambda_p$ be the eigenvalues of A_u, $u\in U(NM)$, with multiplicities $m_1, \cdots, m_p$, respectively. Then the principal curvatures of M^ε at $f^\varepsilon(u)$ with respect to the outward unit normal are given as follows.*

$$\frac{\lambda_i}{1-\lambda_i\varepsilon},\ 1\leq i\leq p, \ \text{and}\ -\frac{1}{\varepsilon} \qquad \text{for}\ \bar{M}=E^n,$$

$$\frac{\sin\varepsilon+\lambda_i\cos\varepsilon}{\cos\varepsilon-\lambda_i\sin\varepsilon},\ 1\leq i\leq p, \ \text{and}\ -\cot\varepsilon \qquad \text{for}\ \bar{M}=S^n,$$

$$\frac{-\sinh\varepsilon+\lambda_i\cosh\varepsilon}{\cosh\varepsilon-\lambda_i\sinh\varepsilon},\ 1\leq i\leq p, \ \text{and}\ -\coth\varepsilon \qquad \text{for}\ \bar{M}=H^n,$$

with multiplicities $m_1, \cdots, m_p$, $q-1$, respectively.

Corollary 1.2. *Suppose that M^ε is a proper Dupin hypersurface. Then, for each point $x\in M$, the number of eigenvalues of A_ξ, $\xi\in N_xM-\{0\}$, is a constant independent of ξ.*

In what follows in this section, let T and N be finite dimensional real vector spaces with inner product $\langle\cdot,\cdot\rangle$, and A: $N\ni\xi\mapsto A_\xi\in\mathrm{Sym}(T)$ a linear map from N to the space $\mathrm{Sym}(T)$ of symmetric endomorphisms of T satisfying

(1.1) the number $\nu(\xi)$ of eigenvalues of A_ξ, $\xi\in N-\{0\}$, is a constant p independent of ξ.

Lemma 1.3. *Assume that N is an orthogonal sum:*

$$N=N_1\oplus N_2 \qquad \text{with} \quad N_1\neq\{0\},\ \dim N_2=1,$$

and there are a linear map $A^{(1)}\colon N_1\to\mathrm{Sym}(T)$ and a vector $\eta_2\in N_2$ such that

$$A_{\xi_1+\xi_2}=A^{(1)}_{\xi_1}+\langle\xi_2,\eta_2\rangle I \qquad \text{for any} \quad \xi_1\in N_1,\ \xi_2\in N_2.$$

Then there exists a vector $\eta_1\in N_1$ such that

$$A^{(1)}_{\xi_1}=\langle\xi_1,\eta_1\rangle I \qquad \text{for any} \quad \xi_1\in N_1.$$

Proof. For any $\xi_2\in N_2$, $\xi_2\neq 0$, we have $A_{\xi_2}=\langle\xi_2,\eta_2\rangle I$. Thus one has $p=1$. Hence, for any $\xi_1\in N_1$, $\xi_1\neq 0$, $A^{(1)}_{\xi_1}=A_{\xi_1}$ is a scalar operator on T. Now the linearity of $A^{(1)}$ implies the existence of η_1 above. q.e.d.

Lemma 1.4. *Assume that N is an orthogonal sum as in Lemma* 1.3, *and also T is an orthogonal sum:*

$$T=T_1\oplus T_2 \qquad \text{with} \quad T_1\neq\{0\},\ T_2\neq\{0\}.$$

Furthermore assume that there are a linear map $A^{(1)}\colon N_1\to\mathrm{Sym}(T_1)$ and different vectors $\eta_2,\eta_2'\in N_2$ such that

$$A_{\xi_1+\xi_2}=(A^{(1)}_{\xi_1}+\langle\xi_2,\eta_2\rangle I_{T_1})\oplus\langle\xi_2,\eta_2'\rangle I_{T_2} \qquad \text{for any} \quad \xi_1\in N_1,\ \xi_2\in N_2.$$

Then $A^{(1)}=0$.

Proof. For any $\xi_2\in N_2$, $\xi_2\neq 0$, we have

$$A_{\xi_2}=\langle\xi_2,\eta_2\rangle I_{T_1}\oplus\langle\xi_2,\eta_2'\rangle I_{T_2},$$

with $\langle\xi_2,\eta_2\rangle\neq\langle\xi_2,\eta_2'\rangle$, and hence $p=2$. We fix an arbitrary $\xi_1\in N_1$, $\xi_1\neq 0$.

First we assume that the eigenvalues $\lambda_1,\cdots,\lambda_k$, $k\geq 1$, of $A^{(1)}_{\xi_1}$ are all nonzero. Then, for $\xi=\alpha\xi_1+\xi_2$ with $\xi_2\in N_2$, $\xi_2\neq 0$, and sufficiently small nonzero $\alpha\in\boldsymbol{R}$, the numbers $\alpha\lambda_1+\langle\xi_2,\eta_2\rangle,\cdots,\alpha\lambda_k+\langle\xi_2,\eta_2\rangle,\langle\xi_2,\eta_2'\rangle$ are different each other, and hence $\nu(\xi)=k+1$. Thus, by (1.1) we get $k=1$, i.e., $A^{(1)}_{\xi_1}=\lambda_1 I_{T_1}$, $\lambda_1\neq 0$. Take $\xi_2\in N_2$, $\xi_2\neq 0$, and $\beta\in\boldsymbol{R}$ with

$$\beta\lambda_1+\langle\xi_2,\eta_2\rangle=\langle\xi_2,\eta_2'\rangle.$$

Then, for $\xi=\beta\xi_1+\xi_2\neq 0$, we have $A_\xi=\langle\xi_2,\eta_2'\rangle I$, and hence $\nu(\xi)=1$. This is a contradiction to $p=2$.

We next assume that $A^{(1)}_{\xi_1}$ has eigenvalue 0, together with possible nonzero

eigenvalues $\lambda_1, \cdots, \lambda_k$, $k \geq 0$. Then, for $\xi = \alpha\xi_1 + \xi_2$ with $\xi_2 \in N_2$, $\xi_2 \neq 0$, and sufficiently small $\alpha \neq 0$, one has $\nu(\xi) = k+2$. Thus, by (1.1) we get $k=0$, i.e., $A^{(1)}_{\xi_1} = 0$.
Since $\xi_1 \in N_1$, $\xi_1 \neq 0$, is arbitrary, we obtain $A^{(1)} = 0$. q.e.d.

Lemma 1.5. *Assume that both N and T have orthogonal decompositions*:

$$N = N_1 \oplus N_2 \qquad \text{with} \quad N_1 \neq \{0\}, \, N_2 \neq \{0\},$$
$$T = T_1 \oplus T_2 \qquad \text{with} \quad T_1 \neq \{0\}, \, T_2 \neq \{0\},$$

and there are linear maps $A^{(1)}$: $N_1 \to \mathrm{Sym}(T_1)$ and $A^{(2)}$: $N_2 \to \mathrm{Sym}(T_2)$ such that

$$A_{\xi_1+\xi_2} = A^{(1)}_{\xi_1} \oplus A^{(2)}_{\xi_2} \qquad \text{for any} \quad \xi_1 \in N_1, \, \xi_2 \in N_2.$$

Then $A=0$.

Proof. We fix arbitrary $\xi_1 \in N_1$, $\xi_1 \neq 0$, and $\xi_2 \in N_2$, $\xi_2 \neq 0$.
Case (a): Both $A^{(1)}_{\xi_1}$ and $A^{(2)}_{\xi_2}$ have only nonzero eigenvalues $\lambda_1, \cdots, \lambda_k$, $k \geq 1$, and $\mu_1, \cdots, \mu_l$, $l \geq 1$, respectively. Then, for $\xi = \xi_1 + \alpha\xi_2$ with sufficiently small $\alpha \neq 0$, one has $\nu(\xi) = k+l$. On the other hand, one has $\nu(\xi_1) = k+1$. Thus, by (1.1) we get $l=1$. In the same way we get $k=1$. It follows that $p=2$ and $A^{(1)}_{\xi_1} = \lambda_1 I_{T_1}$, $A^{(2)}_{\xi_2} = \mu_1 I_{T_2}$ with λ_1, $\mu_1 \neq 0$. Now, for $\xi = \mu_1\xi_1 + \lambda_1\xi_2$, we get $A_\xi = (\lambda_1\mu_1)I$. This is a contradiction to $p=2$.
Case (b): One of the $A^{(i)}_{\xi_i}$, say $A^{(1)}_{\xi_1}$, has only nonzero eigenvalues $\lambda_1, \cdots, \lambda_k$, $k \geq 1$, and the other $A^{(2)}_{\xi_2}$ has eigenvalue 0 together with possible nonzero eigenvalues $\mu_1, \cdots, \mu_l$, $l \geq 0$. Then, for $\xi = \alpha\xi_1 + \xi_2$ with sufficiently small $\alpha \neq 0$, one has $\nu(\xi) = k+l+1$. Together with $\nu(\xi_2) = l+1$, we get $k=0$. This is a contradiction to $k \geq 1$.
Case (c): Both $A^{(1)}_{\xi_1}$ and $A^{(2)}_{\xi_2}$ have eigenvalue 0, together with possible nonzero eigenvalues $\lambda_1, \cdots, \lambda_k$, $k \geq 0$, and $\mu_1, \cdots, \mu_l$, $l \geq 0$, respectively. Then, for $\xi = \xi_1 + \alpha\xi_2$ with sufficiently small $\alpha \neq 0$, one has $\nu(\xi) = k+l+1$. Together with $\nu(\xi_1) = k+1$, $\nu(\xi_2) = l+1$, we get $k=l=0$, i.e., $A^{(1)}_{\xi_1} = 0$ and $A^{(2)}_{\xi_2} = 0$.
Thus we conclude that $A=0$. q.e.d.

2. Proof of Theorem

We first explain some terminologies. The Riemannian metric of $\bar{M}$ will be denoted by $\langle \cdot, \cdot \rangle$. A connected submanifold M of $\bar{M}$ is called an *extrinsic sphere*, if the mean curvature normal η of M is nonzero and parallel (with respect to the normal conncetion in NM), and moreover each shape oparator A_ξ is the scalar operator $\langle \xi, \eta \rangle I$. A submanifold of a space form $\bar{M}$ is said to be *strongly full*, if it is full in $\bar{M}$, and further it is not contained in any extrinsic sphere of $\bar{M}$ of codimension 1.

Let now M be a symmetric submanifold as in Theorem, and suppose that M^ε is a proper Dupin hypersurface.

First we assume that M is not full in $\bar{M}$. Then there exists a complete totally geodesic hypersurface $\bar{M}^{n-1}$ of $\bar{M}$ with $M\subset\bar{M}^{n-1}$. Applying Lemma 1.3 to the shape operator $A^{(1)}$ of $M\subset\bar{M}^{n-1}$ and $\eta_2=0$, we see that $A^{(1)}_{\xi_1}=\langle\xi_1, \eta_1\rangle I$ for any normal vector ξ_1 to $M\subset\bar{M}^{n-1}$. Here η_1 is the mean curvature normal of $M\subset\bar{M}^{n-1}$, which is parallel since the second fundamental form of $M\subset\bar{M}$ is parallel (cf. Naitoh-Takeuchi [4]). Thus M is a complete totally geodesic submanifold or a complete extrinsic sphere of $\bar{M}$. Since the first case is excluded from the assumption, we obtain the case (i) in Theorem. In this case, the principal curvatures of M^ε at $f^\varepsilon(u)$, $u\in U(NM)$, are calculated by Lemma 1.1 as follows.

$$\frac{\langle u, \eta\rangle}{1-\langle u, \eta\rangle\varepsilon} \quad \text{and} \quad -\frac{1}{\varepsilon} \qquad \text{for } \bar{M}=E^n,$$

$$\frac{\sin\varepsilon+\langle u, \eta\rangle\cos\varepsilon}{\cos\varepsilon-\langle u, \eta\rangle\sin\varepsilon} \quad \text{and} \quad -\cot\varepsilon \qquad \text{for } \bar{M}=S^n,$$

$$\frac{-\sinh\varepsilon+\langle u, \eta\rangle\cosh\varepsilon}{\cosh\varepsilon-\langle u, \eta\rangle\sinh\varepsilon} \quad \text{and} \quad -\coth\varepsilon \qquad \text{for } \bar{M}=H^n,$$

where η is the nonzero mean curvature normal of $M\subset\bar{M}$. Thus M^ε is a non-isoparametric Dupin cyclide in $\bar{M}$.

Next we assume that M is full, but not strongly full. Then there exists a complete extrinsic sphere $\bar{M}^{n-1}$ of $\bar{M}$ of codimension 1 such that M is a strongly full submanifold in $\bar{M}^{n-1}$. Applying Lemma 1.3 to the shape operator $A^{(1)}$ of $M\subset\bar{M}^{n-1}$ and the mean curvature normal η_2 of $\bar{M}^{n-1}\subset\bar{M}$, we see that M is a totally geodesic submanifold or an extrinsic sphere of $\bar{M}^{n-1}$. This is a contradiction to that M is strongly full in $\bar{M}^{n-1}$.

Thus it remains to determine M in the case where M is a strongly full symmetric submanifold of $\bar{M}$. We will use the classification of such submanifolds in Takeuchi [10] (see also Naitoh-Takeuchi [4]).

(I) Case $\bar{M}=E^n$: One has $M=E^{n_1}\times M'\subset E^{n_1}\times S^{n_2}(r)\subset E^{n_1}\times E^{n_2+1}=E^n$, n_1, $n_2\geq 1$, $n_1+n_2=n-1$, where M' is a symmetric submanifold of the hypersphere $S^{n_2}(r)$ with radius $r>0$ in E^{n_2+1} such that $M'\subset E^{n_2+1}$ is substantial. Applying Lemma 1.4 to the shape operator $A^{(1)}$ of $M'\subset S^{n_2}(r)$, we see that M' is totally geodesic in $S^{n_2}(r)$. This is a contradiction to that $M'\subset E^{n_2+1}$ is substantial.

(II) Case $\bar{M}=H^n$: We regard H^n as

$$H^n=\{(x_i)\in \boldsymbol{R}^{n+1};\ -x_1^2+x_2^2+\cdots+x_{n+1}^2=-1,\ x_1>0\}.$$

Then $M=H^{n_1}(r_1)\times M'\subset H^{n_1}(r_1)\times S^{n_2}(r_2)\subset H^n$, $n_1, n_2\geq 1$, $n_1+n_2=n-1$, $r_1, r_2>0$, $r_1^2-r_2^2=1$, where

$$H^{n_1}(r_1)=\{(x_i)\in \boldsymbol{R}^{n_1+1};\ -x_1^2+x_2^2+\cdots+x_{n_1+1}^2=-r_1^2,\ x_1>0\},$$

and M' is a symmetric submanifold of $S^{n_2}(r_2)\subset\boldsymbol{R}^{n_2+1}$ such that $M'\subset\boldsymbol{R}^{n_2+1}$ is substantial. In the same way as in (I), we see that M' is totally geodesic in $S^{n_2}(r_2)$, which leads to a contradiction.

(III) Case $\bar{M}=S^n$: In this case, M is a symmetric R-space and the inclusion $M\subset S^n$ is induced from the substantial standard embedding $M\subset \boldsymbol{R}^{n+1}$ (Ferus [2]). If M is a reducible symmetric R-space, one has $M=M_1\times M_2\subset S^{n_1}(r_1)\times S^{n_2}(r_2)\subset S^n$ with $n_1, n_2\geq 1$, $n_1+n_2=n-1$, $r_1, r_2>0$, $r_1^2+r_2^2=1$. Let one of the M_i, say M_1, be equal to $S^{n_i}(r_i)$. Then, applying Lemma 1.4 to the shape operator $A^{(1)}$ of $M_2\subset S^{n_2}(r_2)$, we see that M_2 is totally geodesic in $S^{n_2}(r_2)$. This is a contradiction to that $M\subset \boldsymbol{R}^{n+1}$ is substantial. Otherwise, one has $\dim M_1<n_1$ and $\dim M_2<n_2$. Since the shape operator of $M\subset S^{n_1}(r_1)\times S^{n_2}(r_2)$ also satisfies (1.1), we can apply Lemma 1.5 to the shape operators $A^{(i)}$ of $M_i\subset S^{n_i}(r_i)$ to see that both M_i are totally geodesic in $S^{n_i}(r_i)$. This is also a contradiction to that $M\subset \boldsymbol{R}^{n+1}$ is substantial.

Thus it remains to consider an irreducible symmetric R-space M. For this we recall the construction of the standard embedding of M (cf. Ferus [2], Takeuchi [10], [11]). Let

$$\mathfrak{g}=\mathfrak{g}_{-1}+\mathfrak{g}_0+\mathfrak{g}_1, \quad [\mathfrak{g}_p, \mathfrak{g}_q]\subset\mathfrak{g}_{p+q}$$

be a simple symmetric graded Lie algebra over $\boldsymbol{R}$, with a Cartan involution τ satisfying $\tau\mathfrak{g}_p=\mathfrak{g}_{-p}$, $-1\leq p\leq 1$. The characteristic element $e\in\mathfrak{g}_0$ is the unique element with

$$\mathfrak{g}_p=\{x\in\mathfrak{g};\, [e, x]=px\}, \quad -1\leq p\leq 1.$$

Let

$$\mathfrak{g}=\mathfrak{k}+\mathfrak{p}, \quad \mathfrak{g}_0=\mathfrak{k}_0+\mathfrak{p}_0 \qquad \text{with } \ e\in\mathfrak{p}_0$$

be the Cartan decompositions associated to τ. We denote by K the compact connected subgroup of $GL(\mathfrak{p})$ generated by $\mathrm{ad}_{\mathfrak{p}}\mathfrak{k}$, and set

$$K_0=\{k\in K;\, k\cdot e=e\}.$$

Then we have identifications: $\mathfrak{k}=\mathrm{Lie}\, K$ and $\mathfrak{k}_0=\mathrm{Lie}\, K_0$. Making use of the Killing form B of $\mathfrak{g}$, we define a K-invariant inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{p}$ by

$$\langle x, y\rangle=\frac{1}{2\dim\mathfrak{g}_{-1}}B(x, y) \qquad \text{for } \ x, y\in\mathfrak{p},$$

to identify $\mathfrak{p}$ with the euclidean space $\boldsymbol{R}^{n+1}$, $n=\dim\mathfrak{p}-1$. Then e is in the unit sphere S^n of $\boldsymbol{R}^{n+1}$, and

$$M=K/K_0=K\cdot e$$

gives the required embedding. Let $\mathfrak{a}$ be a maximal abelian subalgebra in $\mathfrak{p}$ including e, and set $r=\dim\mathfrak{a}$. Then one has $\mathfrak{a}\subset\mathfrak{p}_0$. Let $W=N_K(\mathfrak{a})/Z_K(\mathfrak{a})\subset O(\mathfrak{a})$

be the Weyl group of $\mathfrak{g}$, where

$$N_K(\mathfrak{a}) = \{k \in K;\ k \cdot \mathfrak{a} = \mathfrak{a}\},$$
$$Z_K(\mathfrak{a}) = \{k \in K;\ k \cdot h = h \text{ for any } h \in \mathfrak{a}\}.$$

We define g to be a half of the cardinality $\sharp W$ of W. Denote by $\Sigma \subset \mathfrak{a}$ the root system of $\mathfrak{g}$ relative to $\mathfrak{a}$, and set

$$\Sigma_1 = \{\gamma \in \Sigma;\ \langle \gamma, e \rangle = 1\}.$$

Let $\mathfrak{p}_1$ be the orthogonal complement to $\mathfrak{p}_0$ in $\mathfrak{p}$. Then one has

$$T_e M = \mathfrak{p}_1 = \sum_{\gamma \in \Sigma_1} \oplus \mathfrak{p}^\gamma,$$

where $\mathfrak{p}^\gamma$ is the subspace of $\mathfrak{p}$ defined by

$$\mathfrak{p}^\gamma = \{x \in \mathfrak{p};\ [h, [h, x]] = \langle h, \gamma \rangle^2 x \text{ for any } h \in \mathfrak{a}\}.$$

Thus the normal space $N_e M$ to $M \subset S^n$ at e is given by

$$N_e M = \mathfrak{a}_0 \oplus \mathfrak{q}_0,$$

where $\mathfrak{q}_0$ and $\mathfrak{a}_0$ are the orthogonal complement to $\mathfrak{a}$ in $\mathfrak{p}_0$ and the one to $\boldsymbol{R}e$ in $\mathfrak{a}$, respectively. The shape operator A of $M \subset S^n$ at e can be calculated by the same way as in Takagi-Takahashi [8] to get

$$A_h x = -\langle h, \gamma \rangle x \qquad \text{for} \quad h \in \mathfrak{a}_0,\ x \in \mathfrak{p}^\gamma,\ \gamma \in \Sigma_1. \tag{2.1}$$

Now we come back to our problem. If $r=1$, one has $M=S^n$. This case is excluded because of $\operatorname{codim} M>1$. If $r \geq 2$, one has $\sharp\Sigma_1>1$, since $\sharp\Sigma_1=1$ would imply $r=1$. Therefore, if we denote the orthogonal projection $\mathfrak{a} \to \mathfrak{a}_0$ by ϖ, we have $\sharp\varpi(\Sigma_1)>1$, noting that $\varpi(\gamma)=\gamma-e$ for each $\gamma \in \Sigma_1$. It follows that if $r \geq 3$ there exist $h, h' \in \mathfrak{a}_0 - \{0\}$ such that

$$\sharp\{-\langle h, \gamma \rangle;\ \gamma \in \Sigma_1\} \neq \sharp\{-\langle h', \gamma \rangle;\ \gamma \in \Sigma_1\}.$$

This is a contradiction to Corollary 1.2 by virtue of (2.1). Thus we must have $r=2$. In this case, by the classification of irreducible symmetric R-spaces (Kobayashi-Nagano [3], Takeuchi [9]) we see that only the following four cases are possible.

(a) Case $g=3$: $M=P_2(\boldsymbol{F})$, the projective plane over $\boldsymbol{F}=\boldsymbol{R}, \boldsymbol{C}, \boldsymbol{H}$ or $\boldsymbol{O}$, and the standard embedding $P_2(\boldsymbol{F}) \subset \boldsymbol{R}^{3d+2}$ is the generalized Veronese embedding (Tai [7]).

Case $g=4$:

(b) M is the complex quadric of complex dimension 3:

$$Q_3(\boldsymbol{C}) = \{[z] \in P_4(\boldsymbol{C});\ {}^t z z = 0\},$$

and $\mathfrak{p}$ is identified with the space $A_5(\boldsymbol{R})$ of real alternating 5×5 matrices with inner product:

$$\langle X, Y\rangle = -\frac{1}{2}\operatorname{tr}(XY) \qquad \text{for} \quad X, Y\in A_5(\boldsymbol{R}).$$

Any $[z]\in Q_3(\boldsymbol{C})$ can be written as

$$z = x+\sqrt{-1}y \qquad \text{with} \quad x, y\in S^4\subset \boldsymbol{R}^5,\ \langle x, y\rangle = 0.$$

The map $[z]\mapsto x^ty-y^tx$ is the standard embedding.

(c) M is the Lie quadric of dimension $m+1$, $m\geq 2$:

$$Q^{m+1} = \{[z]\in P_{m+2}(\boldsymbol{R});\ -z_1^2-z_2^2+z_3^2+\cdots+z_{m+3}^2 = 0\},$$

and $\mathfrak{p}$ is identified with the space $M_{m+1,2}(\boldsymbol{R})$ of real $(m+1)\times 2$ matrices with inner product:

$$\langle X, Y\rangle = \operatorname{tr}({}^tXY) \qquad \text{for} \quad X, Y\in M_{m+1,2}(\boldsymbol{R}).$$

Any $[z]\in Q^{m+1}$ can be written as

$$z = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{with} \quad x\in S^1\subset \boldsymbol{R}^2,\ y\in S^m\subset \boldsymbol{R}^{m+1}.$$

The map $[z]\mapsto y^tx$ is the standard embedding.

(d) M is the uuitary symplectic group of degree 2:

$$Sp(2) = \{z\in M_2(\boldsymbol{H});\ {}^t\bar{z}z = 1_2\},$$

$M_2(\boldsymbol{H})$ being the space of quaternion 2×2 matrices, and $\mathfrak{p}$ is identified with $M_2(\boldsymbol{H})$ with inner product:

$$\langle X, Y\rangle = \frac{1}{2}\operatorname{Re}\operatorname{tr}({}^t\bar{X}Y) \qquad \text{for} \quad X, Y\in M_2(\boldsymbol{H}).$$

The inclusion $Sp(2)\subset M_2(\boldsymbol{H})$ is the standard embedding.

In these cases, any tube around M is obtained as M^ε with $0<\varepsilon<\pi/g$, and each M^ε is a homogeneous isoparametric hypersurface of S^n with g principal curvatures. In order to show this, first note that K acts on $U(NM)$ transitively. In fact, since the semisimple part of $\mathfrak{g}_0$ has rank 1, K_0 acts on the unit sphere in $\mathfrak{a}_0\oplus\mathfrak{q}_0=N_eM$ transitively. We choose a unit vector $f\in\mathfrak{a}_0$, and thus $f\in U_e(NM)$. Then the stabilizer Z_0 of f in K is given by

$$Z_0 = Z_{K_0}(f) = Z_K(\mathfrak{a}).$$

Now for each $\varepsilon\in\boldsymbol{R}$ the map f^ε: $U(NM)\rightarrow S^n$ is K-equivariant, and hence $M^\varepsilon=f^\varepsilon(U(NM))$ is the K-orbit in S^n through

$$h^\varepsilon = (\cos\varepsilon)e+(\sin\varepsilon)f.$$

Note that h^ε is W-regular if and only if $\varepsilon\notin(\pi/g)\boldsymbol{Z}$. It follows that $M^\varepsilon=M^{\varepsilon'}$ if and only if h^ε and $h^{\varepsilon'}$ are W-conjugate, and that f^ε is an embedding if and only if M^ε is a regular K-orbit in S^n, which is the same as that h^ε is W-regular. Moreover, any regular K-orbit is a homogeneous isoparametric hypersurface in S^n with g principal curvatures (Takagi-Takahashi [8], Ozeki-Takeuchi [5]). These imply our claim.

It is known (Pinkall [6]) that an isoparametric hypersurface $\tilde{M}\subset S^n$ is an irreducible Dupin hypersurface, if $\mathrm{Aut}(\tilde{M})\subset O(n+1)$ acts irreducibly on $\boldsymbol{R}^{n+1}$. But, $\mathrm{Aut}(M^\varepsilon)$ for our tube M^ε acts irreducibly on $\boldsymbol{R}^{n+1}$, because the subgroup K of $\mathrm{Aut}(M^\varepsilon)$ acts on $\mathfrak{p}$ irreducibly by virtue of simplicity of $\mathfrak{g}$. Thus we get the last assertion in Theorem.

We finally note that a Dupin cyclide as in case (i) is always a reducible Dupin hypersurface.

References

[1] T.E. Cecil-P.J. Ryan: Tight and taut immersions of manifolds, Res. Notes Math. 107, Pitman, London, 1985.

[2] D. Ferus: *Immersions with parallel second fundamental form,* Math. Z. **140** (1974), 87–93.

[3] S. Kobayashi-T. Nagano: *On filtered Lie algebras and geometric structures I,* J. Math. Mech. **13** (1964), 875–907.

[4] H. Naitoh-M. Takeuchi: *Symmetric submanifolds of symmetric spaces,* Suguku Exp. **2** (1989), 157–188.

[5] H. Ozeki-M. Takeuchi: *On some types of isoparametric hypersurfaces in spheres II,* Tohoku Math. J. **28** (1975), 7–55.

[6] U. Pinkall: *Dupin hypersurfaces,* Math. Ann. **270** (1985), 429–440.

[7] S.S. Tai: *On minimum imbeddings of compact symmetric spaces of rank one,* J. Diff. Geom. **2** (1968), 55–66.

[8] R. Takagi-T. Takahashi: *On the principal curvatures of homogeneous hypersurfaces in a sphere,* Differential Geometry, in honor of K. Yano, Kinokuniya, Tokyo, 1972, 469–481.

[9] M. Takeuchi: *Cell decompositions and Morse equalities on certain symmetric spaces,* J. Fac. Sci. Univ. Tokyo. Ser. I, **12** (1965), 81–192.

[10] M. Takeuchi: *Parallel submanifolds of space forms,* Manifolds and Lie Groups, in honor of Y. Matsushima, Birkhäuser, Boston, 1981, 429–447.

[11] M. Takeuchi: *Basic transformations of symmetric R-spaces,* Osaka J. Math. **25** (1988), 259–297.

Department of Mathematics
College of General Education
Osaka University
Toyonaka, Osaka 560
Japan