



Title	Proper Dupin hypersurfaces generated by symmetric submanifolds
Author(s)	Takeuchi, Masaru
Citation	Osaka Journal of Mathematics. 1991, 28(1), p. 153-161
Version Type	VoR
URL	https://doi.org/10.18910/3649
rights	
Note	

The University of Osaka Institutional Knowledge Archive : OUKA

<https://ir.library.osaka-u.ac.jp/>

The University of Osaka

PROPER DUPIN HYPERSURFACES GENERATED BY SYMMETRIC SUBMANIFOLDS

Dedicated to Professor Tadashi Nagano on his sixtieth birthday

MASARU TAKEUCHI

(Received March 6, 1990)

Introduction

A connected oriented hypersurface M of the space form $\bar{M}=E^n$, S^n or H^n is called a *Dupin hypersurface*, if for any curvature submanifold S of M the corresponding principal curvature λ is constant along S . Here by a *curvature submanifold* we mean a connected submanifold S with a smooth function λ on S such that for each point $x \in S$, $\lambda(x)$ is a principal curvature of M at x and $T_x S$ is equal to the principal subspace in $T_x M$ corresponding to $\lambda(x)$. A Dupin hypersurface is said to be *proper*, if all principal curvatures have locally constant multiplicities. A connected oriented hypersurface of \bar{M} is called an *isoparametric hypersurface*, if all principal curvatures are locally constant. Obviously an isoparametric hypersurface is a proper Dupin hypersurface. Another example of a Dupin hypersurface (Pinkall [6]) is an ε -tube M^ε around a symmetric submanifold M of \bar{M} of codimension greater than 1, which is said to be *generated by M* . Recall that a connected submanifold M of \bar{M} is a *symmetric submanifold*, if for each point $x \in M$ there is an involutive isometry σ of \bar{M} leaving M and x invariant such that (-1) -eigenspace of $(\sigma_*)_x$ is equal to $T_x M$. The most simple example is the tube M^ε around a complete totally geodesic submanifold M . This is a complete isoparametric hypersurface with two principal curvatures, which is further *homogeneous* in the sense that the group

$$\text{Aut}(M^\varepsilon) = \{\phi \in I(\bar{M}); \phi(M^\varepsilon) = M^\varepsilon\}$$

acts transitively on M^ε . Here $I(\bar{M})$ denotes the group of isometries of \bar{M} . In this note we will determine all the symmetric submanifolds whose tube is a proper Dupin hypersurface, in the following theorem.

Theorem. *Let M be a non-totally geodesic symmetric submanifold of a space form \bar{M} of codimension greater than 1. Then the tube M^ε around M is a proper Dupin hypersurface if and only if either*

(i) *M is a complete extrinsic sphere of \bar{M} (see Section 2 for definition) of codimen-*

sion greater than 1; or

(ii) M is one of the following symmetric submanifolds of S^n :

(a) the projective plane $P_2(\mathbf{F}) \subset S^{3d+1}$, $d = \dim_{\mathbf{R}} \mathbf{F}$, over $\mathbf{F} = \mathbf{R}, \mathbf{C}$, quaternions \mathbf{H} or octonions \mathbf{O} ;

(b) the complex quadric $Q_3(\mathbf{C}) \subset S^9$;

(c) the Lie quadric $Q^{m+1} \subset S^{2m+1}$, $m \geq 2$;

(d) the unitary symplectic group $Sp(2) \subset S^{15}$.

(Explicit embeddings of these spaces will be given in Section 2.) In case (i), M^{ε} is a Dupin cyclide, i.e., a proper Dupin hypersurface with two principal curvatures, but it is not an isoparametric hypersurface. In case (ii), M^{ε} is a homogeneous isoparametric hypersurface with three or four principal curvatures, and it is an irreducible Dupin hypersurface in the sense of Pinkall [6].

1. Principal curvatures of tubes

Let M be a connected submanifold of a space form \bar{M} of codimension $q > 1$, NM and $U(NM)$ the normal bundle and the unit normal bundle of M , respectively. Denote by A_{ξ} the shape operator of M . Suppose that the map $f^{\varepsilon}: U(NM) \rightarrow \bar{M}$, $\varepsilon > 0$, defined by

$$f^{\varepsilon}(u) = \text{Exp}(\varepsilon u) \quad \text{for } u \in U(NM)$$

is an embedding, and set $M^{\varepsilon} = f^{\varepsilon}(U(NM)) \subset \bar{M}$. Then (cf. Cecil-Ryan [1]) we have the following

Lemma 1.1. *Let $\lambda_1, \dots, \lambda_p$ be the eigenvalues of A_u , $u \in U(NM)$, with multiplicities m_1, \dots, m_p , respectively. Then the principal curvatures of M^{ε} at $f^{\varepsilon}(u)$ with respect to the outward unit normal are given as follows.*

$$\begin{aligned} & \frac{\lambda_i}{1 - \lambda_i \varepsilon}, 1 \leq i \leq p, \quad \text{and} \quad -\frac{1}{\varepsilon} \quad \text{for } \bar{M} = E^n, \\ & \frac{\sin \varepsilon + \lambda_i \cos \varepsilon}{\cos \varepsilon - \lambda_i \sin \varepsilon}, 1 \leq i \leq p, \quad \text{and} \quad -\cot \varepsilon \quad \text{for } \bar{M} = S^n, \\ & \frac{-\sinh \varepsilon + \lambda_i \cosh \varepsilon}{\cosh \varepsilon - \lambda_i \sinh \varepsilon}, 1 \leq i \leq p, \quad \text{and} \quad -\coth \varepsilon \quad \text{for } \bar{M} = H^n, \end{aligned}$$

with multiplicities $m_1, \dots, m_p, q-1$, respectively.

Corollary 1.2. *Suppose that M^{ε} is a proper Dupin hypersurface. Then, for each point $x \in M$, the number of eigenvalues of A_{ξ} , $\xi \in N_x M - \{0\}$, is a constant independent of ξ .*

In what follows in this section, let T and N be finite dimensional real vector spaces with inner product $\langle \cdot, \cdot \rangle$, and $A: N \ni \xi \mapsto A_{\xi} \in \text{Sym}(T)$ a linear map

from N to the space $\text{Sym}(T)$ of symmetric endomorphisms of T satisfying

- (1.1) the number $\nu(\xi)$ of eigenvalues of A_ξ , $\xi \in N - \{0\}$, is a constant p independent of ξ .

Lemma 1.3. *Assume that N is an orthogonal sum:*

$$N = N_1 \oplus N_2 \quad \text{with} \quad N_1 \neq \{0\}, \dim N_2 = 1,$$

and there are a linear map $A^{(1)}: N_1 \rightarrow \text{Sym}(T)$ and a vector $\eta_2 \in N_2$ such that

$$A_{\xi_1 + \xi_2} = A_{\xi_1}^{(1)} + \langle \xi_2, \eta_2 \rangle I \quad \text{for any} \quad \xi_1 \in N_1, \xi_2 \in N_2.$$

Then there exists a vector $\eta_1 \in N_1$ such that

$$A_{\xi_1}^{(1)} = \langle \xi_1, \eta_1 \rangle I \quad \text{for any} \quad \xi_1 \in N_1.$$

Proof. For any $\xi_2 \in N_2$, $\xi_2 \neq 0$, we have $A_{\xi_2} = \langle \xi_2, \eta_2 \rangle I$. Thus one has $p=1$. Hence, for any $\xi_1 \in N_1$, $\xi_1 \neq 0$, $A_{\xi_1}^{(1)} = A_{\xi_1}$ is a scalar operator on T . Now the linearity of $A^{(1)}$ implies the existence of η_1 above. q.e.d.

Lemma 1.4. *Assume that N is an orthogonal sum as in Lemma 1.3, and also T is an orthogonal sum:*

$$T = T_1 \oplus T_2 \quad \text{with} \quad T_1 \neq \{0\}, T_2 \neq \{0\}.$$

Furthermore assume that there are a linear map $A^{(1)}: N_1 \rightarrow \text{Sym}(T_1)$ and different vectors $\eta_2, \eta'_2 \in N_2$ such that

$$A_{\xi_1 + \xi_2} = (A_{\xi_1}^{(1)} + \langle \xi_2, \eta_2 \rangle I_{T_1}) \oplus \langle \xi_2, \eta'_2 \rangle I_{T_2} \quad \text{for any} \quad \xi_1 \in N_1, \xi_2 \in N_2.$$

Then $A^{(1)} = 0$.

Proof. For any $\xi_2 \in N_2$, $\xi_2 \neq 0$, we have

$$A_{\xi_2} = \langle \xi_2, \eta_2 \rangle I_{T_1} \oplus \langle \xi_2, \eta'_2 \rangle I_{T_2},$$

with $\langle \xi_2, \eta_2 \rangle \neq \langle \xi_2, \eta'_2 \rangle$, and hence $p=2$. We fix an arbitrary $\xi_1 \in N_1$, $\xi_1 \neq 0$.

First we assume that the eigenvalues $\lambda_1, \dots, \lambda_k, k \geq 1$, of $A_{\xi_1}^{(1)}$ are all nonzero. Then, for $\xi = \alpha \xi_1 + \xi_2$ with $\xi_2 \in N_2$, $\xi_2 \neq 0$, and sufficiently small nonzero $\alpha \in \mathbf{R}$, the numbers $\alpha \lambda_1 + \langle \xi_2, \eta_2 \rangle, \dots, \alpha \lambda_k + \langle \xi_2, \eta_2 \rangle, \langle \xi_2, \eta'_2 \rangle$ are different each other, and hence $\nu(\xi) = k + 1$. Thus, by (1.1) we get $k=1$, i.e., $A_{\xi_1}^{(1)} = \lambda_1 I_{T_1}$, $\lambda_1 \neq 0$. Take $\xi_2 \in N_2$, $\xi_2 \neq 0$, and $\beta \in \mathbf{R}$ with

$$\beta \lambda_1 + \langle \xi_2, \eta_2 \rangle = \langle \xi_2, \eta'_2 \rangle.$$

Then, for $\xi = \beta \xi_1 + \xi_2 \neq 0$, we have $A_\xi = \langle \xi_2, \eta'_2 \rangle I$, and hence $\nu(\xi) = 1$. This is a contradiction to $p=2$.

We next assume that $A_{\xi_1}^{(1)}$ has eigenvalue 0, together with possible nonzero

eigenvalues $\lambda_1, \dots, \lambda_k, k \geq 0$. Then, for $\xi = \alpha\xi_1 + \xi_2$ with $\xi_2 \in N_2, \xi_2 \neq 0$, and sufficiently small $\alpha \neq 0$, one has $\nu(\xi) = k + 2$. Thus, by (1.1) we get $k = 0$, i.e., $A_{\xi_1}^{(1)} = 0$.

Since $\xi_1 \in N_1, \xi_1 \neq 0$, is arbitrary, we obtain $A^{(1)} = 0$. q.e.d.

Lemma 1.5. *Assume that both N and T have orthogonal decompositions:*

$$\begin{aligned} N &= N_1 \oplus N_2 && \text{with } N_1 \neq \{0\}, N_2 \neq \{0\}, \\ T &= T_1 \oplus T_2 && \text{with } T_1 \neq \{0\}, T_2 \neq \{0\}, \end{aligned}$$

and there are linear maps $A^{(1)}: N_1 \rightarrow \text{Sym}(T_1)$ and $A^{(2)}: N_2 \rightarrow \text{Sym}(T_2)$ such that

$$A_{\xi_1 + \xi_2} = A_{\xi_1}^{(1)} \oplus A_{\xi_2}^{(2)} \quad \text{for any } \xi_1 \in N_1, \xi_2 \in N_2.$$

Then $A = 0$.

Proof. We fix arbitrary $\xi_1 \in N_1, \xi_1 \neq 0$, and $\xi_2 \in N_2, \xi_2 \neq 0$.

Case (a): Both $A_{\xi_1}^{(1)}$ and $A_{\xi_2}^{(2)}$ have only nonzero eigenvalues $\lambda_1, \dots, \lambda_k, k \geq 1$, and $\mu_1, \dots, \mu_l, l \geq 1$, respectively. Then, for $\xi = \xi_1 + \alpha\xi_2$ with sufficiently small $\alpha \neq 0$, one has $\nu(\xi) = k + l$. On the other hand, one has $\nu(\xi_1) = k + 1$. Thus, by (1.1) we get $l = 1$. In the same way we get $k = 1$. It follows that $p = 2$ and $A_{\xi_1}^{(1)} = \lambda_1 I_{T_1}, A_{\xi_2}^{(2)} = \mu_1 I_{T_2}$ with $\lambda_1, \mu_1 \neq 0$. Now, for $\xi = \mu_1 \xi_1 + \lambda_1 \xi_2$, we get $A_{\xi} = (\lambda_1 \mu_1) I$. This is a contradiction to $p = 2$.

Case (b): One of the $A_{\xi_i}^{(i)}$, say $A_{\xi_1}^{(1)}$, has only nonzero eigenvalues $\lambda_1, \dots, \lambda_k, k \geq 1$, and the other $A_{\xi_2}^{(2)}$ has eigenvalue 0 together with possible nonzero eigenvalues $\mu_1, \dots, \mu_l, l \geq 0$. Then, for $\xi = \alpha\xi_1 + \xi_2$ with sufficiently small $\alpha \neq 0$, one has $\nu(\xi) = k + l + 1$. Together with $\nu(\xi_2) = l + 1$, we get $k = 0$. This is a contradiction to $k \geq 1$.

Case (c): Both $A_{\xi_1}^{(1)}$ and $A_{\xi_2}^{(2)}$ have eigenvalue 0, together with possible nonzero eigenvalues $\lambda_1, \dots, \lambda_k, k \geq 0$, and $\mu_1, \dots, \mu_l, l \geq 0$, respectively. Then, for $\xi = \xi_1 + \alpha\xi_2$ with sufficiently small $\alpha \neq 0$, one has $\nu(\xi) = k + l + 1$. Together with $\nu(\xi_1) = k + 1, \nu(\xi_2) = l + 1$, we get $k = l = 0$, i.e., $A_{\xi_1}^{(1)} = 0$ and $A_{\xi_2}^{(2)} = 0$.

Thus we conclude that $A = 0$. q.e.d.

2. Proof of Theorem

We first explain some terminologies. The Riemannian metric of \bar{M} will be denoted by $\langle \cdot, \cdot \rangle$. A connected submanifold M of \bar{M} is called an *extrinsic sphere*, if the mean curvature normal η of M is nonzero and parallel (with respect to the normal connection in NM), and moreover each shape operator A_{ξ} is the scalar operator $\langle \xi, \eta \rangle I$. A submanifold of a space form \bar{M} is said to be *strongly full*, if it is full in \bar{M} , and further it is not contained in any extrinsic sphere of \bar{M} of codimension 1.

Let now M be a symmetric submanifold as in Theorem, and suppose that M^* is a proper Dupin hypersurface.

First we assume that M is not full in \bar{M} . Then there exists a complete totally geodesic hypersurface \bar{M}^{n-1} of \bar{M} with $M \subset \bar{M}^{n-1}$. Applying Lemma 1.3 to the shape operator $A^{(1)}$ of $M \subset \bar{M}^{n-1}$ and $\eta_2=0$, we see that $A_{\xi_1}^{(1)} = \langle \xi_1, \eta_1 \rangle I$ for any normal vector ξ_1 to $M \subset \bar{M}^{n-1}$. Here η_1 is the mean curvature normal of $M \subset \bar{M}^{n-1}$, which is parallel since the second fundamental form of $M \subset \bar{M}$ is parallel (cf. Naitoh-Takeuchi [4]). Thus M is a complete totally geodesic submanifold or a complete extrinsic sphere of \bar{M} . Since the first case is excluded from the assumption, we obtain the case (i) in Theorem. In this case, the principal curvatures of M^e at $f^e(u)$, $u \in U(NM)$, are calculated by Lemma 1.1 as follows.

$$\begin{aligned} & \frac{\langle u, \eta \rangle}{1 - \langle u, \eta \rangle \varepsilon} \quad \text{and} \quad -\frac{1}{\varepsilon} \quad \text{for } \bar{M} = E^n, \\ & \frac{\sin \varepsilon + \langle u, \eta \rangle \cos \varepsilon}{\cos \varepsilon - \langle u, \eta \rangle \sin \varepsilon} \quad \text{and} \quad -\cot \varepsilon \quad \text{for } \bar{M} = S^n, \\ & \frac{-\sinh \varepsilon + \langle u, \eta \rangle \cosh \varepsilon}{\cosh \varepsilon - \langle u, \eta \rangle \sinh \varepsilon} \quad \text{and} \quad -\coth \varepsilon \quad \text{for } \bar{M} = H^n, \end{aligned}$$

where η is the nonzero mean curvature normal of $M \subset \bar{M}$. Thus M^e is a non-isoparametric Dupin cyclide in \bar{M} .

Next we assume that M is full, but not strongly full. Then there exists a complete extrinsic sphere \bar{M}^{n-1} of \bar{M} of codimension 1 such that M is a strongly full submanifold in \bar{M}^{n-1} . Applying Lemma 1.3 to the shape operator $A^{(1)}$ of $M \subset \bar{M}^{n-1}$ and the mean curvature normal η_2 of $\bar{M}^{n-1} \subset \bar{M}$, we see that M is a totally geodesic submanifold or an extrinsic sphere of \bar{M}^{n-1} . This is a contradiction to that M is strongly full in \bar{M}^{n-1} .

Thus it remains to determine M in the case where M is a strongly full symmetric submanifold of \bar{M} . We will use the classification of such submanifolds in Takeuchi [10] (see also Naitoh-Takeuchi [4]).

(I) Case $\bar{M} = E^n$: One has $M = E^{n_1} \times M' \subset E^{n_1} \times S^{n_2}(r) \subset E^{n_1} \times E^{n_2+1} = E^n$, $n_1, n_2 \geq 1$, $n_1 + n_2 = n - 1$, where M' is a symmetric submanifold of the hypersphere $S^{n_2}(r)$ with radius $r > 0$ in E^{n_2+1} such that $M' \subset E^{n_2+1}$ is substantial. Applying Lemma 1.4 to the shape operator $A^{(1)}$ of $M' \subset S^{n_2}(r)$, we see that M' is totally geodesic in $S^{n_2}(r)$. This is a contradiction to that $M' \subset E^{n_2+1}$ is substantial.

(II) Case $\bar{M} = H^n$: We regard H^n as

$$H^n = \{(x_i) \in \mathbf{R}^{n+1}; -x_1^2 + x_2^2 + \dots + x_{n+1}^2 = -1, x_1 > 0\}.$$

Then $M = H^{n_1}(r_1) \times M' \subset H^{n_1}(r_1) \times S^{n_2}(r_2) \subset H^n$, $n_1, n_2 \geq 1$, $n_1 + n_2 = n - 1$, $r_1, r_2 > 0$, $r_1^2 - r_2^2 = 1$, where

$$H^{n_1}(r_1) = \{(x_i) \in \mathbf{R}^{n_1+1}; -x_1^2 + x_2^2 + \dots + x_{n_1+1}^2 = -r_1^2, x_1 > 0\},$$

and M' is a symmetric submanifold of $S^{n_2}(r_2) \subset \mathbf{R}^{n_2+1}$ such that $M' \subset \mathbf{R}^{n_2+1}$ is

substantial. In the same way as in (I), we see that M' is totally geodesic in $S^{n_2}(r_2)$, which leads to a contradiction.

(III) Case $\bar{M}=S^n$: In this case, M is a symmetric R -space and the inclusion $M \subset S^n$ is induced from the substantial standard embedding $M \subset \mathbf{R}^{n+1}$ (Ferus [2]). If M is a reducible symmetric R -space, one has $M=M_1 \times M_2 \subset S^{n_1}(r_1) \times S^{n_2}(r_2) \subset S^n$ with $n_1, n_2 \geq 1, n_1+n_2=n-1, r_1, r_2 > 0, r_1^2+r_2^2=1$. Let one of the M_i , say M_1 , be equal to $S^{n_i}(r_i)$. Then, applying Lemma 1.4 to the shape operator $A^{(1)}$ of $M_2 \subset S^{n_2}(r_2)$, we see that M_2 is totally geodesic in $S^{n_2}(r_2)$. This is a contradiction to that $M \subset \mathbf{R}^{n+1}$ is substantial. Otherwise, one has $\dim M_1 < n_1$ and $\dim M_2 < n_2$. Since the shape operator of $M \subset S^{n_1}(r_1) \times S^{n_2}(r_2)$ also satisfies (1.1), we can apply Lemma 1.5 to the shape operators $A^{(i)}$ of $M_i \subset S^{n_i}(r_i)$ to see that both M_i are totally geodesic in $S^{n_i}(r_i)$. This is also a contradiction to that $M \subset \mathbf{R}^{n+1}$ is substantial.

Thus it remains to consider an irreducible symmetric R -space M . For this we recall the construction of the standard embedding of M (cf. Ferus [2], Takeuchi [10], [11]). Let

$$\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1, \quad [\mathfrak{g}_p, \mathfrak{g}_q] \subset \mathfrak{g}_{p+q}$$

be a simple symmetric graded Lie algebra over \mathbf{R} , with a Cartan involution τ satisfying $\tau \mathfrak{g}_p = \mathfrak{g}_{-p}, -1 \leq p \leq 1$. The characteristic element $e \in \mathfrak{g}_0$ is the unique element with

$$\mathfrak{g}_p = \{x \in \mathfrak{g}; [e, x] = px\}, \quad -1 \leq p \leq 1.$$

Let

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}, \quad \mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0 \quad \text{with } e \in \mathfrak{p}_0$$

be the Cartan decompositions associated to τ . We denote by K the compact connected subgroup of $GL(\mathfrak{p})$ generated by $\text{ad}_{\mathfrak{p}} \mathfrak{k}$, and set

$$K_0 = \{k \in K; k \cdot e = e\}.$$

Then we have identifications: $\mathfrak{k} = \text{Lie } K$ and $\mathfrak{k}_0 = \text{Lie } K_0$. Making use of the Killing form B of \mathfrak{g} , we define a K -invariant inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{p} by

$$\langle x, y \rangle = \frac{1}{2 \dim \mathfrak{g}_{-1}} B(x, y) \quad \text{for } x, y \in \mathfrak{p},$$

to identify \mathfrak{p} with the euclidean space $\mathbf{R}^{n+1}, n = \dim \mathfrak{p} - 1$. Then e is in the unit sphere S^n of \mathbf{R}^{n+1} , and

$$M = K/K_0 = K \cdot e$$

gives the required embedding. Let \mathfrak{a} be a maximal abelian subalgebra in \mathfrak{p} including e , and set $r = \dim \mathfrak{a}$. Then one has $\mathfrak{a} \subset \mathfrak{p}_0$. Let $W = N_K(\mathfrak{a})/Z_K(\mathfrak{a}) \subset O(\mathfrak{a})$

be the Weyl group of \mathfrak{g} , where

$$N_K(\mathfrak{a}) = \{k \in K; k \cdot \mathfrak{a} = \mathfrak{a}\},$$

$$Z_K(\mathfrak{a}) = \{k \in K; k \cdot h = h \text{ for any } h \in \mathfrak{a}\}.$$

We define g to be a half of the cardinality $\#\Sigma$ of Σ . Denote by $\Sigma \subset \mathfrak{a}$ the root system of \mathfrak{g} relative to \mathfrak{a} , and set

$$\Sigma_1 = \{\gamma \in \Sigma; \langle \gamma, e \rangle = 1\}.$$

Let \mathfrak{p}_1 be the orthogonal complement to \mathfrak{p}_0 in \mathfrak{p} . Then one has

$$T_e M = \mathfrak{p}_1 = \sum_{\gamma \in \Sigma_1} \mathfrak{p}^\gamma,$$

where \mathfrak{p}^γ is the subspace of \mathfrak{p} defined by

$$\mathfrak{p}^\gamma = \{x \in \mathfrak{p}; [h, [h, x]] = \langle h, \gamma \rangle^2 x \text{ for any } h \in \mathfrak{a}\}.$$

Thus the normal space $N_e M$ to $M \subset S^n$ at e is given by

$$N_e M = \mathfrak{a}_0 \oplus \mathfrak{q}_0,$$

where \mathfrak{q}_0 and \mathfrak{a}_0 are the orthogonal complement to \mathfrak{a} in \mathfrak{p}_0 and the one to $\mathbf{R}e$ in \mathfrak{a} , respectively. The shape operator A of $M \subset S^n$ at e can be calculated by the same way as in Takagi-Takahashi [8] to get

$$(2.1) \quad A_h x = -\langle h, \gamma \rangle x \quad \text{for } h \in \mathfrak{a}_0, x \in \mathfrak{p}^\gamma, \gamma \in \Sigma_1.$$

Now we come back to our problem. If $r=1$, one has $M=S^n$. This case is excluded because of $\text{codim } M > 1$. If $r \geq 2$, one has $\#\Sigma_1 > 1$, since $\#\Sigma_1=1$ would imply $r=1$. Therefore, if we denote the orthogonal projection $\mathfrak{a} \rightarrow \mathfrak{a}_0$ by ϖ , we have $\#\varpi(\Sigma_1) > 1$, noting that $\varpi(\gamma) = \gamma - e$ for each $\gamma \in \Sigma_1$. It follows that if $r \geq 3$ there exist $h, h' \in \mathfrak{a}_0 - \{0\}$ such that

$$\#\{-\langle h, \gamma \rangle; \gamma \in \Sigma_1\} \neq \#\{-\langle h', \gamma \rangle; \gamma \in \Sigma_1\}.$$

This is a contradiction to Corollary 1.2 by virtue of (2.1). Thus we must have $r=2$. In this case, by the classification of irreducible symmetric R -spaces (Kobayashi-Nagano [3], Takeuchi [9]) we see that only the following four cases are possible.

(a) Case $g=3$: $M=P_2(\mathbf{F})$, the projective plane over $\mathbf{F}=\mathbf{R}, \mathbf{C}, \mathbf{H}$ or \mathbf{O} , and the standard embedding $P_2(\mathbf{F}) \subset \mathbf{R}^{3d+2}$ is the generalized Veronese embedding (Tai [7]).

Case $g=4$:

(b) M is the complex quadric of complex dimension 3:

$$Q_3(\mathbf{C}) = \{[z] \in P_4(\mathbf{C}); {}^t z z = 0\},$$

and \mathfrak{p} is identified with the space $A_5(\mathbf{R})$ of real alternating 5×5 matrices with inner product:

$$\langle X, Y \rangle = -\frac{1}{2} \operatorname{tr}(XY) \quad \text{for } X, Y \in A_5(\mathbf{R}).$$

Any $[z] \in Q_3(\mathbf{C})$ can be written as

$$z = x + \sqrt{-1}y \quad \text{with } x, y \in S^4 \subset \mathbf{R}^5, \langle x, y \rangle = 0.$$

The map $[z] \mapsto x^t y - y^t x$ is the standard embedding.

(c) M is the Lie quadric of dimension $m+1$, $m \geq 2$:

$$Q^{m+1} = \{[z] \in P_{m+2}(\mathbf{R}); -z_1^2 - z_2^2 + z_3^2 + \cdots + z_{m+3}^2 = 0\},$$

and \mathfrak{p} is identified with the space $M_{m+1,2}(\mathbf{R})$ of real $(m+1) \times 2$ matrices with inner product:

$$\langle X, Y \rangle = \operatorname{tr}({}^tXY) \quad \text{for } X, Y \in M_{m+1,2}(\mathbf{R}).$$

Any $[z] \in Q^{m+1}$ can be written as

$$z = \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{with } x \in S^1 \subset \mathbf{R}^2, y \in S^m \subset \mathbf{R}^{m+1}.$$

The map $[z] \mapsto y^t x$ is the standard embedding.

(d) M is the unitary symplectic group of degree 2:

$$Sp(2) = \{z \in M_2(\mathbf{H}); {}^t\bar{z}z = 1_2\},$$

$M_2(\mathbf{H})$ being the space of quaternion 2×2 matrices, and \mathfrak{p} is identified with $M_2(\mathbf{H})$ with inner product:

$$\langle X, Y \rangle = \frac{1}{2} \operatorname{Re} \operatorname{tr}({}^t\bar{X}Y) \quad \text{for } X, Y \in M_2(\mathbf{H}).$$

The inclusion $Sp(2) \subset M_2(\mathbf{H})$ is the standard embedding.

In these cases, any tube around M is obtained as M^ε with $0 < \varepsilon < \pi/g$, and each M^ε is a homogeneous isoparametric hypersurface of S^n with g principal curvatures. In order to show this, first note that K acts on $U(NM)$ transitively. In fact, since the semisimple part of \mathfrak{g}_0 has rank 1, K_0 acts on the unit sphere in $\mathfrak{a}_0 \oplus \mathfrak{q}_0 = N_e M$ transitively. We choose a unit vector $f \in \mathfrak{a}_0$, and thus $f \in U_e(NM)$. Then the stabilizer Z_0 of f in K is given by

$$Z_0 = Z_{K_0}(f) = Z_K(\mathfrak{a}).$$

Now for each $\varepsilon \in \mathbf{R}$ the map $f^\varepsilon: U(NM) \rightarrow S^n$ is K -equivariant, and hence $M^\varepsilon = f^\varepsilon(U(NM))$ is the K -orbit in S^n through

$$h^\varepsilon = (\cos \varepsilon)e + (\sin \varepsilon)f.$$

Note that h^e is W -regular if and only if $\varepsilon \notin (\pi/g)\mathcal{Z}$. It follows that $M^e = M^{e'}$ if and only if h^e and $h^{e'}$ are W -conjugate, and that f^e is an embedding if and only if M^e is a regular K -orbit in S^n , which is the same as that h^e is W -regular. Moreover, any regular K -orbit is a homogeneous isoparametric hypersurface in S^n with g principal curvatures (Takagi-Takahashi [8], Ozeki-Takeuchi [5]). These imply our claim.

It is known (Pinkall [6]) that an isoparametric hypersurface $\tilde{M} \subset S^n$ is an irreducible Dupin hypersurface, if $\text{Aut}(\tilde{M}) \subset O(n+1)$ acts irreducibly on \mathbf{R}^{n+1} . But, $\text{Aut}(M^e)$ for our tube M^e acts irreducibly on \mathbf{R}^{n+1} , because the subgroup K of $\text{Aut}(M^e)$ acts on \mathfrak{p} irreducibly by virtue of simplicity of \mathfrak{g} . Thus we get the last assertion in Theorem.

We finally note that a Dupin cyclide as in case (i) is always a reducible Dupin hypersurface.

References

- [1] T.E. Cecil-P.J. Ryan: *Tight and taut immersions of manifolds*, Res. Notes Math. 107, Pitman, London, 1985.
- [2] D. Ferus: *Immersions with parallel second fundamental form*, Math. Z. **140** (1974), 87–93.
- [3] S. Kobayashi-T. Nagano: *On filtered Lie algebras and geometric structures I*, J. Math. Mech. **13** (1964), 875–907.
- [4] H. Naitoh-M. Takeuchi: *Symmetric submanifolds of symmetric spaces*, Sugaku Exp. **2** (1989), 157–188.
- [5] H. Ozeki-M. Takeuchi: *On some types of isoparametric hypersurfaces in spheres II*, Tohoku Math. J. **28** (1975), 7–55.
- [6] U. Pinkall: *Dupin hypersurfaces*, Math. Ann. **270** (1985), 429–440.
- [7] S.S. Tai: *On minimum imbeddings of compact symmetric spaces of rank one*, J. Diff. Geom. **2** (1968), 55–66.
- [8] R. Takagi-T. Takahashi: *On the principal curvatures of homogeneous hypersurfaces in a sphere*, Differential Geometry, in honor of K. Yano, Kinokuniya, Tokyo, 1972, 469–481.
- [9] M. Takeuchi: *Cell decompositions and Morse equalities on certain symmetric spaces*, J. Fac. Sci. Univ. Tokyo. Ser. I, **12** (1965), 81–192.
- [10] M. Takeuchi: *Parallel submanifolds of space forms*, Manifolds and Lie Groups, in honor of Y. Matsushima, Birkhäuser, Boston, 1981, 429–447.
- [11] M. Takeuchi: *Basic transformations of symmetric R-spaces*, Osaka J. Math. **25** (1988), 259–297.

Department of Mathematics
 College of General Education
 Osaka University
 Toyonaka, Osaka 560
 Japan

