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THE LIMIT THEOREMS FOR THE SINGLE POINT RANGE OF STRONGLY TRANSIENT RANDOM WALKS

Dedicated to Professor Masatoshi Fukushima on his 60th birthday

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1. Introduction

Let $\{S_n\}_{n=0}^{\infty}$ be a random walk on the d dimensional integer lattice Z^d, that is,

$$S_0=0, \qquad S_n=\sum_{k=1}^{n} X_k,$$

where $\{X_n\}_{n=1}^{\infty}$ is a sequene of independent identically distributed random variables with values in Z^d defined on some probability space $(\Omega,\mathscr{B},P)$. The random walk is called transient if $q>0$ and recurrent otherwise, where q is the probability that the random walk never returns to the origin. An equivalent criterion for transience is the convergence of $\Sigma_{n=0}^{\infty}P(S_n=0)$. When $\Sigma_{n=0}^{\infty}\Sigma_{k=n}^{\infty}P(S_k=0)<\infty$, we call the random walk strongly transient. The random walk is transient if the genuine dimension is greater than or equal to 3, and strongly transient if not less than 5. There are also transient and strongly transient random walks in lower dimensions. If $EX_1=0$ and $E|X_1|^2<\infty$, the random walk will be recurrent when the genuine dimension is equal to 1 or 2, and not strongly transient if it is 3 or 4.

Let R_n be the number of distinct points visited by the random walk before time n. Kesten, Spitzer, and Whitman showed that $R_n/n\rightarrow q$ almost surely for all random walks (see [12]). Jain and Orey [6] proved that for strongly transient random walks, if $q<1$, then Var $R_n\sim\mu^2 n$ for some suitable positive constant μ^2 and $(R_n-qn)/\mu\sqrt{n}$ converges to the standard normal variable in the distribution sense. Moreover, Jain and Pruitt [7] concluded the same statement for genuinely 4 or more dimensional random walks, and in [8] they also established the law of the iterated logarithm in each cases.

The single point range of the random walk, denoted by Q_n, means the number of distinct sites entered once and only once by the random walk before time n. Pitt [11] showed that $EQ_n\sim q^2 n$ and $Q_n/n\rightarrow q^2$ almost surely for all transient random walks. In [3], it was shown that if the random walk has genuine dimension $d \geq 4$ and $q<1$, then there exists a positive constant σ^2 such that $\mathrm{Var}\, Q_n \sim \sigma^2 n$ and the distribution of $(Q_n - q^2 n)/\sigma\sqrt{n}$ is asymptotically equal to the standard normal. Moreover, the law of the iterated logarithm for Q_n is proved in [4]. The case $q=1$ is not of interest since $Q_n = n+1$ almost surely.

We shall consider the central limit theorem and the law of the iterated logarithm for Q_n in the strongly transient case. The limiting behavior of the variance of Q_n and the proof of the central limit theorem are given in Section 3. Section 4 is devoted to the proof of the law of the iterated logarithm.

2. Notation and preliminaries

In this section we will give some notation and one lemma. For $x \in \boldsymbol{Z}^d$, the notation $P_x(\cdot)$ will be used to denote the probability measures of events related to the random walk starting at x. When $x=0$, we will simply use $P(\cdot)$ instead of $P_0(\cdot)$. For $n \geq 0$ and $x,y \in \boldsymbol{Z}^d$, the notation $p^n(x,y)$ means $P_x(S_n = y)$ and note that $p^n(x,y) = p^n(0,y-x)$. For $x \in \boldsymbol{Z}^d$, τ_x will denote the first hitting time of x, *i.e.*

$$\tau_x = \inf\{n \geq 1;\ S_n = x\};$$

if there are no positive integers satisfying $S_n = x$, then $\tau_x = \infty$. The taboo probabilities are defined by

$$p_z^n(x,y) = P_x(S_n = y,\ \tau_z \geq n),\ p_{zw}^n(x,y) = P(S_n = y,\ \tau_z \geq n,\ \tau_w \geq n).$$

For $n \geq 0$ and $x,y \in \boldsymbol{Z}^d$, let

$$G_n(x,y) = \sum_{k=0}^{n} p^k(x,y).$$

For transient random walks, there exists $\lim_{n\to\infty} G_n(x,y)$, which denoted by $G(x,y)$, and this limit is bounded by $G(0,0) = q^{-1} < \infty$. It is trivial that $P_0(\tau_x < \infty) \leq G(0,x)$.

We will use r_n for $P_0(n < \tau_0 < \infty)$, f_n for $p_0^n(0,0)$, and $u_n = p^n(0,0)$. For $n \geq 0$, let

$$t_n = \sum_{j=n+1}^{\infty} r_j.$$

If the random walk is strongly transient, we have that $t_0 < \infty$.

The following lemma plays important roll in calculating the probability estimates of some quantities related the random walk.

Lemma 2.1. *For strongly transient random walks, we have*

$$\sum_{x \in \boldsymbol{Z}^d} G(0,x)G(x,0) = \sum_{i=0}^{\infty} \sum_{j=i}^{\infty} p^j(0,0) = \sum_{j=0}^{\infty} (j+1) p^j(0,0) < \infty.$$

Proof. By the definition of the function G, we have

$$\sum_x G(0,x)G(x,0)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_x p^i(0,x)p^j(x,0).$$

Noting that $\Sigma_x p^i(0,x)p^j(x,0)=p^{i+j}(0,0)$, the right hand side is

$$\sum_{i=0}^{\infty}\sum_{j=i}^{\infty}u_j=\sum_{j=0}^{\infty}\sum_{i=0}^{j}u_j.$$

This means the assertion of Lemma 2.1. ■

For the sake of simplicity of notation, in what follows, we shall use the following convention. If $\{a_n\}$ and $\{b_n\}$ $(b_n>0)$ are sequences of real numbers, then $a_n=o(b_n)$ means $a_n/b_n\to 0$; $a_n=O(b_n)$ means a_n/b_n remains bounded; $a_n\sim b_n$ means $a_n/b_n\to 1$, as $n\to\infty$. C_1, $C_2,\cdots C_{11}$ will denote suitable finite positive real constants.

3. The central limit theorem for Q_n

For the genuinely 2 dimensional, aperiodic random walks with mean 0 and finite variance, Hamana [5] showed that $\operatorname{Var}Q_n\sim Ln^2/(\log n)^6$ for some suitable constant L and $(Q_n-EQ_n)/\sqrt{\operatorname{Var}Q_n}$ converges to the self-intersection local time of a planar Brownian motion multiplied by some constant in the distribution sense. Our goal in this section is to establish the central limit theorem for the single point range of strongly transient random walks, especially, 2 dimensional ones.

Now we introduce several indicator random variables. For $0\le i<j$, let

$$Z_i^j=\begin{cases}1 & \text{if } S_i\neq S_\alpha \text{ for } i<\alpha\le j,\\ 0 & \text{otherwise,}\end{cases}$$

$$Y_j^i=\begin{cases}1 & \text{if } S_j\neq S_\alpha \text{ for } i\le\alpha<j,\\ 0 & \text{otherwise,}\end{cases}$$

and $Z_n^n=Y_n^n=1$ for $n\ge 0$. Then we have $Q_n=\Sigma_{j=0}^n Y_j^0 Z_j^n$. In order to obtain the variance of Q_n, we need to define other indicator random variables. Let X_0, X_{-n} $(n=1,2,\cdots)$ be independent copies of X_1 and for $l<m$, define

$$X(l,m)=\sum_{k=l+1}^{m}X_k.$$

Using this notation, we can express

$$Z_i^j=\begin{cases}1 & \text{if } X(i,\alpha)\neq 0 \text{ for } i<\alpha\le j,\\ 0 & \text{otherwise,}\end{cases}$$

$$Y_j^i = \begin{cases} 1 & \text{if } X(\alpha,j) \neq 0 \text{ for } i \leq \alpha < j, \\ 0 & \text{otherwise.} \end{cases}$$

For each integer j, let

$$Z_j = \begin{cases} 1 & \text{if } X(j,\alpha) \neq 0 \text{ for } \alpha > j, \\ 0 & \text{otherwise,} \end{cases}$$

$$Y_j = \begin{cases} 1 & \text{if } X(\alpha,j) \neq 0 \text{ for } \alpha < j, \\ 0 & \text{otherwise.} \end{cases}$$

and for $i<j$, $W_i^j = Z_i^j - Z_i$ and $V_j^i = Y_j^i - Y_j$. Note that W_i^j and V_j^i are also indicator random variables. Let $\Lambda_n = \Sigma_{j=0}^n Y_j Z_j$. We aim to approximate Q_n by Λ_n, and the following lemma assure the approximation of Q_n.

Lemma 3.1. *For the strongly transient random walk,*

$$E\left|\sum_{j=0}^n W_j^n\right|^2 = o(n), \qquad E\left|\sum_{j=0}^n V_j^0\right|^2 = o(n).$$

Proof. We have

$$\begin{aligned} E\left|\sum_{j=0}^n W_j^n\right|^2 &= \sum_{j=0}^n EW_j^n + 2\sum_{j=1}^n \sum_{i=0}^{j-1} EW_i^n W_j^n \\ &\leq 2\sum_{j=0}^n EW_j^n + 2\sum_{j=1}^{n-1} \sum_{i=0}^{j-1} EW_i^n W_j^n. \end{aligned} \tag{3.1}$$

For $0 \leq j < n$, $EW_j^n = r_{n-j}$ and $EW_n^n = 1-q$. Then we obtain the first term of (3.1) is dominated by $2(t_0+1)$. For $0 \leq i < j < n$,

$$EW_i^n W_j^n = \sum_{x \neq 0} p_0^{j-i}(0,x) P_x(n-j < \tau_0, \tau_x < \infty).$$

Neglecting the event $\{n-j < \tau_x < \infty\}$,

$$\sum_{i=0}^{j-1} EW_i^n W_j^n \leq \sum_{x \neq 0} G(0,x) P_x(n-j < \tau_0 < \infty).$$

Note that for $m \geq 0$,

$$P_x(m < \tau_y < \infty) \leq \sum_{k=m+1}^{\infty} p^k(x,y) = G(x,y) - G_m(x,y). \tag{3.2}$$

Then the bound of the second term of (3.1) is

$$2\sum_{j=1}^{n-1}\sum_x G(0,x)\{G(x,0)-G_j(x,0)\}. \tag{3.3}$$

Since $G_j(x,0)\to G(x,0)$ as j tends to infinity, by Lemma 2.1,

$$\lim_{n\to\infty}\sum_x G(0,x)\{G(x,0)-G_j(x,0)\}=0,$$

where the dominated convergence theorem have been applied. Hence the order of (3.3) is $o(n)$. The estimate of the part involving V's can be shown in the same fashion by relabeling the indices of X. This completes the proof the lemma. ∎

In repetition of the argument used in calculating $\operatorname{Var}\Lambda_n$ in the case $d=3$ and $\Sigma G(0,x)^2 G(x,0)<\infty$, one sees that Lemma 3.1 gives the asymptotic behavior of the variance of the single point range of strongly transient random walks (see Lemma 3.6 and 3.7, and Theorem 3.8 in [5]). Then we can derive the following theorem.

Theorem 3.2. *For strongly transient random walks, if $q<1$, then there exists a positive constant σ^2 such that* $\operatorname{Var} Q_n \sim \sigma^2 n$.

To prove the central limit theorem for Q_n, we need to define four more indicator random variables. For $n\geq 0$ and $0\leq j<l$, let

$$Z_{j,l}^n=\begin{cases}1 & \text{if } S_j\neq S_\alpha \text{ for } j<\alpha<l,\ l<\alpha\leq n, \\ & \text{and } S_j=S_l, \\ 0 & \text{otherwise,}\end{cases}\qquad \text{if } l<n,$$

$$Z_{j,n}^n=\begin{cases}1 & \text{if } S_j\neq S_\alpha \text{ for } j<\alpha<n, \text{and } S_j=S_n, \\ 0 & \text{otherwise,}\end{cases}\qquad \text{if } l=n,$$

$$Z_{j,l}=\begin{cases}1 & \text{if } S_j\neq S_\alpha \text{ for } j<\alpha<l, \alpha>l, \text{and } S_j=S_l, \\ 0 & \text{otherwise,}\end{cases}$$

$$W_{j,l}^n=Z_{j,l}^n-Z_{j,l}.$$

The random variable $\Sigma_{j=0}^{n-1}\Sigma_{l=j+1}^{n}Z_{j,l}^n$, denoted by $R_n^{(2)}$, is equal to the number of distinct points visited at least twice by random walks up to time n. It is clear that $Q_n=R_n-R_n^{(2)}$. Let $\Gamma_n=\Sigma_{j=0}^n Z_j$. Noting that $R_n=\Sigma_{j=0}^n Z_j^n$, the estimate

$$E|R_n-\Gamma_n|^2=o(n) \tag{3.4}$$

was established in Lemma 3.1. Then we try to obtain the similar assertion about $R_n^{(2)}$. The following two lemmas assure that $R_n^{(2)}$ is approximately a partial sum of a stationary sequence.

Lemma 3.3. *For strongly transient random walks, we have*

$$E\left|\sum_{j=0}^{n-1}\sum_{l=j+1}^{n} W_{j,l}^{n}\right|^2 = o(n).$$

Proof. We can calculate this expectation analogously to Lemma 4.1 in [5]. However, the method of estimating is slightly deferent.

$$\begin{aligned} E\left|\sum_{j=0}^{n-1}\sum_{l=j+1}^{n} W_{j,l}^{n}\right|^2 &= \sum_{j=0}^{n-1}\sum_{l=j+1}^{n} EW_{j,l}^{n} \\ &\quad + 2\sum_{j=1}^{n-1}\sum_{i=0}^{j-1}\sum_{l=j+1}^{n}\sum_{h=l+1}^{n} EW_{j,l}^{n}W_{i,h}^{n} \\ &\quad + 2\sum_{j=1}^{n-1}\sum_{i=0}^{j-1}\sum_{l=j+1}^{n}\sum_{h=j+1}^{l-1} EW_{j,l}^{n}W_{i,h}^{n} \\ &\quad + 2\sum_{j=1}^{n-1}\sum_{i=0}^{j-1}\sum_{l=j+1}^{n}\sum_{h=i+1}^{j-1} EW_{j,l}^{n}W_{i,h}^{n} \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}. \end{aligned}$$

For $0 \le j < l \le n$, it is clear that $EW_{j,l}^{n} = f_{l-j} r_{n-l}$. Hence we have $\mathrm{I} = O(1)$. Since, for $0 \le i < j < l < h \le n$,

$$EW_{j,l}^{n}W_{i,h}^{n} \le \sum_{x} p^{j-i}(0,x) p_{x}^{l-j}(x,x) p^{h-l}(x,0) P_0(n-h < \tau_0, \tau_x < \infty),$$

we obtain by neglecting the event $\{n-h < \tau_x < \infty\}$,

$$\mathrm{II} \le \sum_{l=2}^{n-1}\sum_{j=1}^{l-1}\sum_{h=l+1}^{n} f_{l-j} r_{n-h} \sum_{x} p^{h-l}(x,0) G(0,x) = O(1).$$

The third term is dominated by

$$\sum_{l=3}^{n}\sum_{h=2}^{l-1}\sum_{x} p^{l-h}(0,x) G(x,0) G(0,x) r_{n-l} = O(1)$$

and the bound of the fourth term is

$$\sum_{l=3}^{n}\sum_{x} G(0,x)\{G(x,0) - G_l(x,0)\} G(x,0) + \sum_{l=3}^{n} r_{n-l} \sum_{x} G(0,x) G(x,0).$$

The first part is $o(n)$ by the same method as in obtaining the order of (3.3) and the second part is $O(1)$ by Lemma 2.1. ■

Lemma 3.4. *For strongly transient random walks,*

$$E\left|\sum_{j=0}^{n-1}\sum_{l=n+1}^{\infty} Z_{j,l}\right|^2 = o(n).$$

Proof. We can prove the assertion similarly to Lemma 4.2 in [5]. Namely, we have

$$\begin{aligned} E\left|\sum_{j=0}^{n-1}\sum_{l=n+1}^{\infty} Z_{j,l}\right|^2 &= \sum_{j=0}^{n-1}\sum_{l=n+1}^{\infty} EZ_{j,l} \\ &\quad + 2\sum_{j=1}^{n-1}\sum_{i=0}^{j-1}\sum_{l=n+1}^{\infty}\sum_{h=l+1}^{\infty} EZ_{j,l}Z_{i,h} \\ &\quad + 2\sum_{j=1}^{n-1}\sum_{i=0}^{j-1}\sum_{h=n+1}^{\infty}\sum_{l=h+1}^{\infty} EZ_{j,l}Z_{i,h} \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned}$$

The limiting behavior of each term can be easily obtained. We have that the first term I is bounded and the second term II is no larger than

$$\sum_{j=1}^{n-1}\sum_{i=0}^{j-1} r_{n-j}\sum_x p^{j-i}(0,x)G(x,0) = O(1).$$

The third term III is bounded by

$$\sum_{j=0}^{n-1}\sum_x G(0,x)\{G(x,0) - G_{n-j}(x,0)\}G(0,x) = o(n).$$

Therefore we can conclude the assertion. ■

For $n \geq 0$, define $\Gamma_n^{(2)} = \Sigma_{j=0}^{n-1}\Sigma_{l=j+1}^{\infty} Z_{j,l}$. Combining Lemma 3.3 and Lemma 3.4, we have

$$E|R_n^{(2)} - \Gamma_n^{(2)}|^2 = o(n). \tag{3.5}$$

We are now ready to show the central limit theorem. The main tool is the Lindeberg theorem for triangular arrays.

Theorem 3.5. *For strongly transient random walks, if $q < 1$, then $(Q_n - q^2 n)/\sigma\sqrt{n}$ converges to the standard normal variable in the distribution sense.*

Proof. Let $\Psi_n=\Gamma_n-\Gamma_n^{(2)}$. We conclude that $E|Q_n-\Psi_n|^2=o(n)$ by (3.4) and (3.5). Thus $(Q_n-\Psi_n)/\sqrt{n}\to 0$ in probability as n tends to infinity, and hence it suffeces to consider $(\Psi_n-q^2n)/\sigma\sqrt{n}$.

Let $m=[n^{1/3}]$ and for $i=1,2,\cdots,m^2$,

$$A_i=\sum_{j=(i-1)m}^{im-1} Z_j^{im}, \qquad A_i^{(2)}=\sum_{j=(i-1)m}^{im-1}\sum_{l=j+1}^{im} Z_{j,l}^{im},$$

$$U_i=\sum_{j=(i-1)m}^{im-1} W_j^{im}, \qquad U_i^{(2)}=\sum_{j=(i-1)m}^{im-1}\sum_{l=j+1}^{im} W_{j,l}^{im},$$

$$V_i^{(2)}=\sum_{j=(i-1)m}^{im-1}\sum_{l=im+1}^{\infty} Z_{j,l}.$$

Then we have

$$\Gamma_{m^3}-qm^3=\sum_{i=1}^{m^2}(A_i-EA_i)-\sum_{i=1}^{m^2}(U_i-EU_i)+(Z_{m^3}-q),$$

$$\Gamma_{m^3}^{(2)}-q(1-q)m^3=\sum_{i=1}^{m^2}(A_i^{(2)}-EA_i^{(2)}) -\sum_{i=1}^{m^2}(U_i^{(2)}-EU_i^{(2)})+\sum_{i=1}^{m^2}(V_i^{(2)}-EV_i^{(2)}).$$

Firstly we show that both A part and $A^{(2)}$ part are dominant comparing with the other parts. Recalling the argument used in Theorem 3 in [7] and in Theorem 4.1 in [9], we can derive that

$$\text{Var}\left(\sum_{i=1}^{m^2} U_i\right)=o(m^3). \tag{3.6}$$

We next estimate $\text{Var}(\Sigma U_i^{(2)})$ and $\text{Var}(\Sigma V_i^{(2)})$. However, the way of calculation is similar to that used in Theorem 4.4 in [9].

$$\begin{aligned}
&\text{Var}\left(\sum_{i=1}^{m^2} U_i^{(2)}\right)\\
&=\sum_{i=1}^{m^2}\sum_{j=(i-1)m}^{im-1}\sum_{s=(i-1)m}^{im-1}\sum_{l=j+1}^{im}\sum_{h=s+1}^{im}\text{Cov}(W_{j,l}^{im},W_{s,h}^{im})\\
&\quad+2\sum_{j=2}^{m^2}\sum_{i=1}^{j-1}\sum_{r=(j-1)m}^{jm-1}\sum_{s=(i-1)m}^{im-1}\sum_{l=r+1}^{jm}\sum_{h=s+1}^{im}\text{Cov}(W_{r,l}^{jm},W_{s,h}^{im})\\
&=\text{I}+\text{II}.
\end{aligned}$$

Dominating each $\text{Cov}(W_{j,l}^{im},W_{s,h}^{im})$ in I by $EW_{j,l}^{im}W_{s,h}^{im}$, we can estimate the term I in the same way as Lemma 3.3, and also have $\text{I}\le o(m^3)$. Now we estimate the term

II. The bound of $\mathrm{Cov}(W_{r,l}^{jm}, W_{s,h}^{im})$ is

$$f_{h-s}\sum_{k=1}^{l-r}\sum_{x}p^{r-h}(0,x)p^{k}(x,0)p^{l-r-k}(0,x)r_{jm-l}$$

$$\to f_{h-s}f_{l-r}\sum_{x\neq 0}p_0^{r-h}(0,x)P_x(\tau_0<\tau_x,\ jm-l<\tau_x<\infty).$$

For details, see the proof of Theorem 4.4 in [9]. Then the first part of the term II is dominated by

$$\sum_{j=2}^{m^2}\sum_{l=(j-1)m+1}^{jm}\sum_{r=(j-1)m}^{l-1}\sum_{k=1}^{l-r}\sum_{x}G(0,x)p^{k}(x,0)p^{l-r-k}(0,x)r_{jm-l}$$

$$\leq q^{-1}\sum_{j=2}^{m^2}\sum_{l=(j-1)m+1}^{jm}\sum_{r=(j-1)m}^{l-1}(l-r)u_{l-r}r_{jm-l}.$$

By Lemma 2.1, it is of order m^2. The remaining part is no larger than

$$\sum_{j=2}^{m^2}\sum_{l=(j-1)m+1}^{jm}\sum_{x}G(0,x)P_x(\tau_0<\tau_x,\ jm-l<\tau_x<\infty)=o(m^3).$$

Here we have applied the argument used in Theorem 4.1 in [9] again. Thus

$$\mathrm{Var}\left(\sum_{i=1}^{m^2}U_i^{(2)}\right)=o(m^3). \tag{3.7}$$

We shall show the $V^{(2)}$ part has the same estimate as the $U^{(2)}$ one.

$$\mathrm{Var}\left(\sum_{i=1}^{m^2}V_i^{(2)}\right)$$
$$=\sum_{i=1}^{m^2}\sum_{j=(i-1)m}^{im-1}\sum_{s=(i-1)m}^{im-1}\sum_{l=im+1}^{\infty}\sum_{h=im+1}^{\infty}\mathrm{Cov}(Z_{j,l},Z_{s,h})$$
$$+2\sum_{j=2}^{m^2}\sum_{i=1}^{j-1}\sum_{r=(j-1)m}^{jm-1}\sum_{s=(i-1)m}^{im-1}\sum_{l=jm+1}^{\infty}\mathrm{Cov}(Z_{r,l},Z_{s,r})$$
$$+2\sum_{j=2}^{m^2}\sum_{i=1}^{j-1}\sum_{r=(j-1)m}^{jm-1}\sum_{s=(i-1)m}^{im-1}\sum_{l=jm+1}^{\infty}\mathrm{Cov}(Z_{r,l},Z_{s,l})$$
$$+2\sum_{j=2}^{m^2}\sum_{i=1}^{j-1}\sum_{r=(j-1)m}^{jm-1}\sum_{s=(i-1)m}^{im-1}\sum_{l=jm+1}^{\infty}\sum_{h=im}^{r-1}\mathrm{Cov}(Z_{r,l},Z_{s,h})$$
$$+2\sum_{j=2}^{m^2}\sum_{i=1}^{j-1}\sum_{r=(j-1)m}^{jm-1}\sum_{s=(i-1)m}^{im-1}\sum_{l=jm+1}^{\infty}\sum_{h=r+1}^{l-1}\mathrm{Cov}(Z_{r,l},Z_{s,h})$$

$$+2\sum_{j=2}^{m^2}\sum_{i=1}^{j-1}\sum_{r=(j-1)m}^{jm-1}\sum_{s=(i-1)m}^{im-1}\sum_{l=jm+1}^{\infty}\sum_{h=l+1}^{\infty}\mathrm{Cov}(Z_{r,l},Z_{s,h})$$

$$=\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}+\mathrm{VI}.$$

The first three terms can be estimated in the same manner as the $U^{(2)}$ part by choosing the order of the summation carefully and each term is bounded by $o(m^3)$. the method of estimating the term IV is similar to that used in Theorem 4.4 in [5]. In fact, for $s<h<r<l$,

$$\mathrm{Cov}(Z_{r,l},Z_{s,h})\le f_{h-s}f_{l-r}\sum_{x}p^{r-h}(0,x)G(x,0).$$

For fixed s, we sum over h in (s,r), and take the summation on i. Then

$$\mathrm{IV}\le 2\sum_{j=2}^{m^2}\sum_{r=(j-1)m}^{jm-1}\sum_{s=0}^{(j-1)m}\sum_{h=s+1}^{r-1}f_{h-s}r_{jm-r}\sum_{x}p^{r-h}(0,x)G(x,0).$$

Changing the order of summing on h and s after widening the summation on s. Hence we have that

$$\mathrm{IV}\le C_1\sum_{j=2}^{m^2}\sum_{r=(j-1)m}^{jm-1}r_{jm-r}=O(m^2).$$

The estimate of the term V is quite easy. Indeed, for $s<r<h<l$,

$$EZ_{r,l}Z_{s,h}\le\sum_{x}p^{r-s}(0,x)p^{h-r}(x,0)p^{l-h}(0,x).$$

Therefore, summing first on i and then on l after changing the order of the summation on l and h,

$$\mathrm{V}\le 2\sum_{j=2}^{m^2}\sum_{r=(j-1)m}^{jm-1}\sum_{s=0}^{(j-1)m}\sum_{x}p^{r-s}(0,x)G(x,0)G(0,x)$$

$$\le 2\sum_{j=2}^{m^2}\sum_{r=(j-1)m}^{jm-1}\sum_{x}\{G(0,x)-G_{r-(j-1)m}(0,x)\}G(x,0)G(0,x)$$

$$=o(m^3).$$

We will estimate the term VI. Since for $s<r<l<h$,

$$EZ_{r,l}Z_{s,h}\le\sum_{x\neq 0}p_0^{r-s}(0,x)p_{0x}^{l-r}(x,x)p_{0x}^{h-l}(x,0)P_x(\tau_0,\tau_x=\infty),$$

dominating $P_x(\tau_0,\tau_x=\infty)$ by 1, we have (see [10] page 272).

Theorem 4.1. *For strongly transient random walks, if* $q<1$, *then*

$$\limsup_{n\to\infty}\frac{Q_n-q^2n}{\sqrt{2\sigma^2 n\log\log n}}=1 \quad \text{a.s.}$$

and the lim inf *of this sequence is* -1 *almost surely.*

Proof. The sequence $\{n_i\}$ is formed by taking for $k=3,4,\cdots$ all integers in $[2^{2k},2^{2k+2})$ of the form $2^{2k}+j\xi_k$ with j a non-negative integer and $\xi_k=[2^k(\log k)^{-3/4}]+1$. The number of members of the sequence which are in $[2^{2k},2^{2k+2})$ will be at most $3\cdot 2^k(\log k)^{3/4}$. Since if $n_i\le n<n_{i+1}$, then

$$|Q_n-EQ_n-Q_{n_i}+EQ_{n_i}|\le 2(n-n_i)=o(n_i^{1/2}),$$

it is enough to show the statement of this theorem along the subsequence $\{n_i\}$. We also need to see that EQ_n can be replaced by q^2n but this is valid since

$$EQ_n=\sum_{j=0}^{n}(q+r_j)(q+r_{n-j})=q^2n+O(1)$$

for strongly transient random walks.

Recall the definitions of indicator random variables Z_i^j, Y_j^i introduced in Section 3. We define two more ones. For $0\le a\le b<c$, let $W_a^{b,c}=Z_a^b-Z_a^c$ and $V_c^{b,a}=Y_c^b-Y_c^a$. Using these indicators, we have

$$\begin{aligned}Q_{n_i}-Z_0^{n_i}=&\sum_{j=0}^{i-1}\sum_{h=n_j+1}^{n_{j+1}}Y_h^{n_j}Z_h^{n_{j+1}}\\&-\sum_{j=0}^{i-1}\sum_{h=n_j+1}^{n_{j+1}}W_h^{n_{j+1},n_i}Y_h^{n_j}\\&-\sum_{j=0}^{i-1}\sum_{h=n_j+1}^{n_{j+1}}V_h^{n_j,0}Z_h^{n_{j+1}}\\&+\sum_{j=0}^{i-1}\sum_{h=n_j+1}^{n_{j+1}}W_h^{n_{j+1},n_i}V_h^{n_j,0}\\=&\,\mathrm{I}-\mathrm{II}-\mathrm{III}-\mathrm{IV}.\end{aligned}$$

Firstly we will show that the term I is dominant comparing with the other terms and next check the first term satisfies the Kolmogorov condition.

If $2^{2k}\le n_j<2^{2k+2}$, then we define $\eta_k=[2^k(\log k)^{-1/4}]$ and $\rho_j=\eta_{k(j)}$, where $k(j)=[\log n_j/\log 4]$. The second term is divided into two parts, that is,

$$\sum_{h=l+1}^{\infty} EZ_{r,l}Z_{s,h} \leq f_{l-r} \sum_{x \neq 0} p_0^{r-s}(0,x)P_x(\tau_0 < \infty, \tau_0 < \tau_x).$$

Hence, neglecting the event $\{\tau_0 < \tau_x\}$,

$$\mathrm{VI} \leq 2\sum_{j=2}^{m^2} \sum_{r=(j-1)m}^{jm-1} \sum_{s=0}^{(j-1)m} r_{jm-r} \sum_x p^{r-s}(0,x)G(x,0) = O(m^2).$$

Accordingly we obtain that

$$\mathrm{Var}\left(\sum_{i=1}^{m^2} V_i^{(2)}\right) = o(m^3). \tag{3.8}$$

From now on, we show the central limit theorem. For the sake of convenience, put $\Delta_i = A_i - A_i^{(2)}$. By (3.6), (3.7), and (3.8), we have

$$\mathrm{Var}\left[\sum_{i=1}^{m^2} (U_i^{(2)} - V_i^{(2)} - U_i)\right] = o(m^3) = o(n),$$

which yields that for any $\varepsilon > 0$,

$$P\left[\left|(\Psi_{m^3} - q^2 m^3) - \sum_{i=1}^{m^2} (\Delta_i - E\Delta_i)\right| > \varepsilon\sqrt{n}\right] = o(1).$$

We need to show the residual part $(\Psi_n - \Psi_{m^3} - q^2 n + q^2 m^3)/\sqrt{n}$ converges to 0 in probability. However, it is valid since $\mathrm{Var}(\Psi_n - \Psi_{m^3}) = O(n^{2/3})$. It still remains to check the Lindeberg-Feller condition. Δ_i are independent and their each distribution coincides with that of $Q_m - 1$, and $\Delta_i \leq m$ for each i. Thus

$$\mathrm{Var}\left(\sum_{i=1}^{m^2} \frac{\Delta_i - E\Delta_i}{\sigma\sqrt{n}}\right) = \frac{m^2 \mathrm{Var} Q_m}{\sigma^2 n} \sim 1,$$

and since $|\Delta_i - E\Delta_i| \leq m$,

$$\sum_{i=1}^{m^2} E\left[\left(\frac{\Delta_i - E\Delta_i}{\sqrt{n}}\right)^2 ; |\Delta_i - E\Delta_i| > \varepsilon\sqrt{n}\right] \leq \frac{m^4}{\varepsilon^2 n^2} \mathrm{Var} Q_m,$$

which is of order $n^{-1/3}$. This completes the proof of the theorem. ∎

4. The law of the iterated logarithm for Q_n

In this section we will give a proof of the law of the iterated logarithm for the single point range of strongly transient random walks. The main tool is the Kolmogorov-Cantelli condition for a sequence of independent random variables

$$\mathrm{II}=\sum_{j=0}^{i-1}\sum_{h=n_j+1}^{n_{j+1}} W_h^{n_{j+1}+\rho_j,n_i} Y_h^{n_j}+\sum_{j=0}^{i-1}\sum_{h=n_j+1}^{n_{j+1}} W_h^{n_{j+1},n_{j+1}+\rho_j} Y_h^{n_j}.$$

For the sake of convenience, we put

$$\alpha_j=\sum_{h=n_j+1}^{n_{j+1}} W_h^{n_{j+1}+\rho_j,n_i} Y_h^{n_j},\quad \beta_j=\sum_{h=n_j+1}^{n_{j+1}} W_h^{n_{j+1},n_{j+1}+\rho_j} Y_h^{n_j},$$

and then have that $\mathrm{II}=\Sigma_{j=0}^{i-1}(\alpha_j+\beta_j)$.

We first consider the part involving α's. For $2^{2k}\le n_i<2^{2k+2}$,

$$E\left(\sum_{j=0}^{i-1}\alpha_j\right)\le\sum_{j=0}^{i-1}\sum_{h=n_j+1}^{n_{j+1}} r_{n_{j+1}+\rho_j-h}=\sum_{j=0}^{i-1}\sum_{h=0}^{n_{j+1}-n_j-1} r_{h+\rho_j}.$$

Then we have that

$$\begin{aligned}E\left(\sum_{j=0}^{i-1}\alpha_j\right)&\le\sum_{l=3}^{k}\sum_{m:2^{2l}\le n_m<2^{2l+2}}\sum_{h=0}^{n_{m+1}-n_m-1} r_{h+\eta_l}\\&\le\sum_{l=3}^{k}\sum_{m:2^{2l}\le n_m<2^{2l+2}}(n_{m+1}-n_m)r_{\eta_l}\\&\le C_2\sum_{l=3}^{k}2^{2l}r_{\eta_l}.\end{aligned}$$

By the definition of the sequence $\{n_i\}$, we note that there are no positive integers m satisfying $2^{2l}\le n_m$ for $l=0,1,2$. On the other hand,

$$P\left[\left|\sum_{j=0}^{i-1}(\alpha_j-E\alpha_j)\right|>\varepsilon 2^k(\log k)^{\frac12}\right]\le\varepsilon^{-1}2^{-k}(\log k)^{-\frac12}E\left|\sum_{j=0}^{i-1}(\alpha_j-E\alpha_j)\right|.$$

Noting $\alpha_j\ge 0$ for every $j\ge 0$, we have the expectation of the right hand side is bounded by $2E\Sigma\alpha_j$. Now form a subsequence $\{n_{\nu_m}\}$ by taking every $[2^k(\log k)^{1/2}]$th member of $\{n_i\}$ in $[2^{2k},2^{2k+2})$, and then there will be $O\{(\log k)^{1/4}\}$ members of the subsequence in this interval. Then we obtain

$$\begin{aligned}&\sum_{m=1}^{\infty}P\left[\left|\sum_{j=0}^{\nu_m-1}(\alpha_j-E\alpha_j)\right|>\varepsilon(n_{\nu_m}\log\log n_{\nu_m})^{\frac12}\right]\\&\le C_3\sum_{k=3}^{\infty}2^{-k}(\log k)^{-\frac12}\sum_{l=3}^{k}2^{2l}r_{\eta_l}\times(\log k)^{\frac14}\\&\le C_4\sum_{l=3}^{\infty}2^{l}(\log l)^{-\frac14}r_{\eta_l}.\end{aligned}$$

Since $r_n \leq \Sigma_{h=n+1}^{\infty} u_h$, applying Lemma 2.1, the last summation is dominated by

$$\sum_{l=3}^{\infty} 2^l (\log l)^{-\frac{1}{4}} \sum_{m=l}^{\infty} \sum_{\eta=\eta_m+1}^{\eta_{m+1}} u_\eta \leq C_5 \sum_{m=3}^{\infty} 2^m (\log m)^{-\frac{1}{4}} \sum_{\eta=\eta_m}^{\eta_{m+1}} u_\eta$$

$$\leq C_5 \sum_{\eta=1}^{\infty} \eta u_\eta < \infty.$$

This means that $\Sigma(\alpha_j - E\alpha_j)$ divided by $(n_i \log\log n_i)^{1/2}$ tends to zero along the subsequence $\{n_{\nu_m}\}$. If $2^{2k} \leq n_{\nu_m} \leq n_i < n_{\nu_{m+1}} \leq 2^{2k+2}$, then

$$\sum_{j=0}^{\nu_m - 1} (\alpha_j - E\alpha_j) - \sum_{j=\nu_m}^{i-1} E\alpha_j \leq \sum_{j=0}^{i-1} (\alpha_j - E\alpha_j) \leq \sum_{j=0}^{\nu_{m+1}-1} (\alpha_j - E\alpha_j) + \sum_{j=i}^{\nu_{m+1}-1} E\alpha_j,$$

which implies that we only need a estimate of $E\alpha_j$ to see that convergence to zero along the original sequence $\{n_i\}$. For $n_j \leq 2^{2k+2}$,

$$E\alpha_j \leq \sum_{h=0}^{n_{j+1}-n_j} r_{h+\rho_j} \leq \sum_{h=\eta_{k(j)}}^{\infty} r_h = o(1)$$

as $j \to \infty$. Moreover, both $\nu_{m+1} - i$ and $i - \nu_m$ are less than or equal to $2^k (\log k)^{1/2}$. Thus

$$\sum_{j=\nu_m}^{i-1} E\alpha_j, \quad \sum_{j=i}^{\nu_{m+1}-1} E\alpha_j = o\{2^k (\log k)^{\frac{1}{2}}\}.$$

Next we estimate the part involving β's and have

$$\mathrm{Var}\left(\sum_{j=0}^{i-1} \beta_j\right) = \sum_{j=0}^{i-1} \mathrm{Var}\beta_j + 2 \sum_{j=1}^{i-1} \sum_{l=0}^{j-1} \mathrm{Cov}(\beta_j, \beta_l). \tag{4.1}$$

For $0 \leq j < i$, the variance of β_j is

$$\sum_{h=n_j+1}^{n_{j+1}} \mathrm{Var}(W_h^{n_{j+1}, n_{j+1}+\rho_j} Y_h^{n_j}) \tag{4.2}$$
$$+ 2 \sum_{m=n_j+1}^{n_{j+1}-1} \sum_{h=m+1}^{n_{j+1}} \mathrm{Cov}(W_h^{n_{j+1}, n_{j+1}+\rho_j} Y_h^{n_j}, W_m^{n_{j+1}, n_{j+1}+\rho_j} Y_m^{n_j}).$$

The bound of the first part of (4.2) is

$$\sum_{h=n_j+1}^{n_{j+1}} E W_h^{n_{j+1}, n_{j+1}+\rho_j} \leq \sum_{h=n_j+1}^{n_{j+1}} r_{n_{j+1}-h} = O(1)$$

and that of the second part of (4.2) is

$$2\sum_{m=n_j+1}^{n_{j+1}-1}\sum_{h=m+1}^{n_{j+1}} EW_m^{n_{j+1},n_{j+1}+\rho_j} \leq 2\sum_{m=n_j+1}^{n_{j+1}-1}(n_{j+1}-m)r_{n_{j+1}-m}$$

$$\leq 2\sum_{m=1}^{\rho_j} mr_m,$$

where we use $n_{j+1}-n_j \leq \rho_j$. Therefore we obtain that the first term of (4.1) is no larger than $3i\Sigma_{m=1}^{\rho_i} mr_m$. The second term of (4.1) is equal to

$$2\sum_{j=1}^{i-1}\sum_{l=0}^{j-1}\sum_{h=n_j+1}^{n_{j+1}}\sum_{m=n_l+1}^{n_{l+1}} \operatorname{Cov}(W_h^{n_{j+1},n_{j+1}+\rho_j}Y_h^{n_j}, W_m^{n_{l+1},n_{l+1}+\rho_l}Y_m^{n_l}).$$

To estimate it, we split it up into several parts. For $0 \leq l < j < i$, we consider the three cases:

(i) $n_{l+1}+\rho_l \leq n_j$ (ii) $n_j < n_{l+1}+\rho_l \leq h$ (iii) $h < n_{l+1}+\rho_l$.

In Case (i), $W_h^{n_{j+1},n_{j+1}+\rho_j}Y_h^{n_j}$ and $W_m^{n_{l+1},n_{l+1}+\rho_l}Y_m^{n_l}$ are independent. Therefore the covariance and also the summation are equal to zero. In Case (ii), we have that

$$W_m^{n_{l+1},n_{l+1}+\rho_l} = W_m^{n_{l+1},n_j} + W_m^{n_j,n_{l+1}+\rho_l}$$

and $W_m^{n_{l+1},n_j}Y_m^{n_l}$ is independent of $W_h^{n_{j+1},n_{j+1}+\rho_j}Y_h^{n_j}$. Hence

$$\begin{aligned}
&\operatorname{Cov}(W_h^{n_{j+1},n_{j+1}+\rho_j}Y_h^{n_j}, W_m^{n_{l+1},n_{l+1}+\rho_l}Y_m^{n_l} \\
&\quad = \operatorname{Cov}(W_h^{n_{j+1},n_{j+1}+\rho_j}Y_h^{n_j}, W_m^{n_j,n_{l+1}+\rho_l}Y_m^{n_l} \\
&\quad \leq EW_h^{n_{j+1},n_{j+1}+\rho_j}W_m^{n_j,n_{l+1}+\rho_l} \\
&\quad \leq r_{n_{j+1}-h}r_{n_j-m}.
\end{aligned}$$

Then, summing first on l, the multiple sum is bounded by

$$\sum_{j=1}^{i-1}\sum_{h=n_j+1}^{n_{j+1}}\sum_{m=1}^{n_j} r_{n_{j+1}-h}r_{n_j-m}.$$

By strong transience, it is of order i. In Case (iii), the covariance is equal to

$$\begin{aligned}
\text{(4.3)} \qquad &\operatorname{Cov}(W_h^{n_{j+1},n_{j+1}+\rho_j}Y_h^{n_j}, W_m^{n_{l+1},n_j}Y_m^{n_l}) \\
&+\operatorname{Cov}(W_h^{n_{j+1},n_{j+1}+\rho_j}Y_h^{n_j}, W_m^{n_j,h}Y_m^{n_l}) \\
&+\operatorname{Cov}(W_h^{n_{j+1},n_{j+1}+\rho_j}Y_h^{n_j}, W_m^{h,n_{l+1}+\rho_l}Y_m^{n_l}).
\end{aligned}$$

We immediately have that the first part of (4.3) is zero by independence, and can easily show that the summation of the second part is of order i by the same argument as in Case (ii). The last part of (4.3) is no larger than $EW_m^{h,n_{l+1}+\rho_l}\leq r_{h-m}$. Hence the summation is bounded by

$$\sum_{l=0}^{i-1}\sum_{m=n_l+1}^{n_{l+1}}\sum_{h=n_{l+1}+1}^{n_{l+1}+\rho_l} r_{h-m}\leq\sum_{l=0}^{i-1}\sum_{m=0}^{n_{l+1}-n_l}\sum_{h=m}^{m+\rho_l} r_h$$
$$\leq C_6\sum_{l=2}^{i-1}\sum_{m=0}^{\rho_l}\sum_{h=m}^{2\rho_l} r_h$$
$$\leq C_7 i\sum_{h=1}^{2\rho_i} hr_h.$$

Combining with these estimates, we can conclude that

$$\mathrm{Var}\left(\sum_{j=0}^{i-1}\beta_j\right)=O\left(i\sum_{m=1}^{2\rho_i} mr_m\right).$$

Then we can show that $\Sigma\beta_j$ is negligible along the same line as in [8]. Therefore we can conclude that $\mathrm{II}-E\mathrm{II}=o\{(n_i\log\log n_i)^{1/2}\}$ almost surely and can prove that $\mathrm{III}-E\mathrm{III}=o\{(n_i\log\log n_i)^{1/2}\}$ a.s. in the same technique as in estimating II by splitting III into two parts:

$$\mathrm{III}=\sum_{j=0}^{i-1}\sum_{h=n_j+1}^{n_{j+1}} V_h^{n_j,n_j-\rho_j}Z_h^{n_{j+1}}+\sum_{j=0}^{i-1}\sum_{h=n_j+1}^{n_{j+1}} V_h^{n_j-\rho_j,0}Z_h^{n_{j+1}}.$$

In this case, V's and Z's play the same roles as W's and Y's respectively by considering the time-reversed random walk.

From now on, we will estimate the last part VI. For $0\leq j<i$, let

$$\gamma_j=\sum_{h=n_j+1}^{n_{j+1}} W_h^{n_{j+1},n_i}V_h^{n_j,0}.$$

Then

$$E\left(\sum_{j=0}^{i-1}\gamma_j\right)\leq\sum_{j=0}^{i-1}\sum_{h=n_j+1}^{n_{j+1}} r_{n_{j+1}-h}r_{h-n_j}=\sum_{j=0}^{i-1}\sum_{h=1}^{n_{j+1}-n_j} r_{n_{j+1}-n_j-h}r_h.$$

We divide the summation on h into two parts:

(i) $1\leq h\leq\langle n_{j+1}-n_j\rangle$ (ii) $\langle n_{j+1}-n_j\rangle\leq h\leq n_{j+1}-n_j$,

where $\langle x\rangle$ is the integer part of a real number $x/2$, and easily see that the summation on h of each case is of order $r_{\langle n_{j+1}-n_j\rangle}$ since r_n is non-increasing. Therefore, for $2^{2k}\leq n_i<2^{2k+2}$,

$$E\left(\sum_{j=0}^{i-1}\gamma_j\right)\leq C_8\sum_{l=3}^{k}\sum_{m:2^{2l}\leq n_m<2^{2l+2}} r_{\langle\xi_l\rangle}\leq C_9\sum_{l=3}^{k}2^l(\log l)^{\frac{3}{4}}r_{\langle\xi_l\rangle}.$$

Now we consider a subsequence $\{n_{\mu_m}\}$ by taking every $[2^k(\log k)^{1/4}]$th member of $\{n_i\}$ in $[2^{2k},2^{2k+2})$ and there will be $O\{(\log k)^{1/2}\}$ members of the subsequence in this interval. To see that $\Sigma(\gamma_j-E\gamma_j)$ divided by $(n_i\log\log n_i)^{1/2}$ tends to 0 along the subsequence $\{n_{\mu_m}\}$, it suffices to show the convergence of

$$\sum_{k=3}^{\infty}2^{-k}(\log k)^{-\frac{1}{2}}\sum_{l=3}^{k}2^l(\log l)^{\frac{3}{4}}r_{\langle\xi_l\rangle}\times(\log k)^{\frac{1}{2}}.$$

Indeed, it is bounded by

$$C_{10}\sum_{l=3}^{\infty}(\log l)^{\frac{3}{4}}\sum_{m=l}^{\infty}\sum_{\xi=\langle\xi_m\rangle+1}^{\langle\xi_{m+1}\rangle}u_\xi\leq C_{10}\sum_{m=3}^{\infty}m(\log m)^{\frac{3}{4}}\sum_{\xi=\langle\xi_m\rangle+1}^{\langle\xi_{m+1}\rangle}u_\xi$$

$$\leq C_{11}\sum_{\xi=1}^{\infty}\xi u_\xi<\infty.$$

By the same argument used on $\Sigma\beta_j$, we can obtain that $\mathrm{IV}-E\mathrm{IV}$ is equal to $o\{(n_i\log\log n_i)^{1/2}\}$ almost surely.

It remains to check that the term I satisfies Kolmogorov condition.
For $0\leq j<i$, let

$$\zeta_j=\sum_{h=n_j+1}^{n_{j+1}}Y_h^{n_j}Z_h^{n_{j+1}}$$

Since the ζ_j are mutually independent and the law of ζ_j coincides with $Q_{n_{j+1}-n_j}-Z_0^{n_{j+1}-n_j}$ for each j,

$$\mathrm{Var}\left(\sum_{j=0}^{i-1}\zeta_j\right)\sim\sum_{j=0}^{i-1}\sigma^2(n_{j+1}-n_j)\sim\sigma^2 n_i$$

and for $2^{2k}\leq n_i<2^{2k+2}$,

$$|\zeta_i-E\zeta_i|\leq 2(n_{i+1}-n_i)\leq 2\{2^k(\log k)^{-\frac{3}{4}}+1\}=o\left\{\left(\frac{n_i}{\log\log n_i}\right)^{\frac{1}{2}}\right\}.$$

This completes the proof of Theorem 4.1. ■

References

[1] A. Dvoretzky and P. Erdös: Some problems on random walk in space, Proc. Second Berkeley

Symp. Math. Stat. Probab., University of California Press, Berkeley, 1951, 353–367.
[2] P. Erdös and S.J. Taylor: *Some problems concerning the structure of random walk paths*, Acta. Math. Acad. Sci. Hungar. **11** (1960), 137–162.
[3] Y. Hamana: *On the central limit theorem for the multiple point range of random walk*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **39** (1992), 339–363.
[4] ———: *The law of the iterated logarithm for the single point range of random walk*, Tokyo J. Math. **17** (1994), 171–180.
[5] ———: *The fluctuation results for the single point range of random walks in low dimensions*, to appear in Japan. J. Math., **21**
[6] N.C. Jain and S. Orey: *On the range of random walk*, Israel J. Math. **6** (1968), 373–380.
[7] N.C. Jain and W.E. Pruitt: *The range of transient random walk*, J. Analyse Math. **24** (1971), 369–393.
[8] ———: *The law of the iterated logarithm for the range of random walk*, Ann. Math. Statist. **43** (1972), 1692–1697.
[9] ———: *Further limit theorems for the range of random walk*, J. Analyse Math. **27** (1974), 94–117.
[10] M. Loève: Probability Theory I, 4th ed., Springer, Berlin Heidelberg, 1977.
[11] J.H. Pitt: *Multiple points of transient random walk*, Proc. Amer. Math. Soc. **43** (1974), 195–199.
[12] F. Spitzer: Principles of Random Walk, Springer, Berlin Heidelberg, 1976.

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