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AN ANALYTIC APPROACH TO THE EXTENDED DIRICHLET SPACE

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1. Introduction

Since Silverstein [7], several authors had succesfully replaced the symmetry of the Dirichlet form by "nearly symmetry" in applying the "energy" methods to Markov processes,the Potential Theory of such processes being quite amenable. Very signifiant in this direction is the article [3] of Fitzsimmons, which deals with a right Markov process such that the sector condition holds. The purpose of this paper is to improve some basic results in the frame given by a resolvent of kernels. Our first aim is to give the representation of the extended Dirichlet space [th. 2.3], extending the similar result of Fukushima for the symmetric case [4, th. 1.5.3]. Our second result [th. 2.6] shows the invariance of extended Dirichlet space under a special case of "time change", in connection with formulae 6.2.22 and 6.2.23 from [5].

2. The extended Dirichlet space

Let $\mathcal{U} = (U_\alpha)_{\alpha>0}$ be a submarkovian resolvent of positive kernels on a measurable space $(E, \mathcal{E})$, and ξ is a fixed excessive measure on E. We denote by $b(E)$ (resp. $p(E)$) the sets of bounded (resp. positive) measurable functions.

It is well known that $\mathcal{U}$ induces in the obvious way a resolvent of α-contractions on the Banach space $L^2 = L^2(\xi)$, still denoted by $\mathcal{U}$. We assume that $\mathcal{U}$ is *L^2-regular*, that is $\lim_{\alpha\to\infty} \alpha U_\alpha f = f$ (in L^2-sense) for any $f \in L^2$, or equivalently the space $U_\alpha(L^2)$ is dense in L^2 for an $\alpha > 0$ (or for all $\alpha > 0$). We denote by A the generator of $\mathcal{U}$ on the Banach space L^2, and by $D(A)$ its dense domain. We recall that if α is (arbitrary) fixed, then $D(A) = U_\alpha(L^2)$, and we have the basic formula $AU_\alpha = \alpha U_\alpha - I$. The bilinear form a on $D(A) \times D(A)$ defined by

$$a(f, g) = \langle f, -Ag\rangle$$

(where $\langle\ \rangle$ denotes the usual acalar product on L^2) is called the *energy form*, and the seminorm $e(f) = a(f, f)^{1/2}$ (one sees that a is positive) is called the *energy norm*. Our second assumption is the *sector condition*, that is there exists a positive constant

M such that

$$|a(f,g)| \leq Me(f)e(g) \quad f,\ g \in D(A). \tag{S}$$

Denoting by $U = U_0$ the initial kernel of $\mathcal{U}$, our third assumption is that $\mathcal{U}$ is *transient*, that is there exists a function $0 < f \leq 1$ such that Uf is bounded. It follows then the inequality (see [2, 2.3])

$$\langle |u|, g\rangle \leq Me(u)\langle g, Ug\rangle^{1/2} \tag{F}$$

for any $g \in L^2_+$ such that $\langle g, Ug\rangle < \infty$, and for any $u \in D(A)$. This implies in particular that e is a true norm (we can pick $g > 0$ as above). We denote by $L^0(\xi) = L^0$ the vector space of classes of measurable functions finite ξ a.s., with respect to the equivalence relation $u \sim v \Longleftrightarrow \xi\{u \neq v\} = 0$. Following [1, XIII, 55, 56, a)] we define the (*extended*) *Dirichlet space* $\mathcal{D}$ as the set of all elements $f \in L^0$ such that there exists a Cauchy sequence (f_n) in $(D(A), e)$ such that $f_n \to f$ ξ a.s. It is shown that the energy norm e extends to $\mathcal{D}$, $D(A)$ is dense in $\mathcal{D}$, and $(\mathcal{D}, e)$ is a Hilbert space isomorphic to the abstract completion of $(D(A), e)$.

Thanks to (S), the energy form extends to a bounded bilinear form on $\mathcal{D}$, still denoted by a. Also, we shall consider in passing, the traditional Dirichlet space $\mathcal{D}_0$, that is the abstract completion of $D(A)$ with respect to the norm e_β associated with the bilinear form

$$a_\beta(f,g) = a(f,g) + \beta\langle f, g\rangle \qquad f,\ g \in D(A),$$

$\beta > 0$ (arbitrary) fixed. (The transience of $\mathcal{U}$ is no longer necessary). One can see that $\mathcal{D}_0$ is isomorphic to the extended Dirichlet space asociated with the resolvent $\mathcal{U}^\beta = (U_{\beta+\alpha})_{\alpha>0}$. We remark that $\mathcal{D}$ (and $\mathcal{D}_0$) depends only on the resolvent $\mathcal{U}$ (of pseudokernels) on L^2, and we identify two resolvents which induce the same resolvent on L^2. So, we can speak of versions for a resolvent on L^2. According to this, we extend a little our frame, until other specification. Let ξ be a σ-finite measure on a measurable space $(E, \mathcal{E})$. A (regular) resolvent family $(U_\alpha)_{\alpha>0}$ of (nonsymmetric) α-contractions on $L^2(\xi)$ is called *submarkovian* if $0 \leq f \leq 1$ a.s. $\Rightarrow 0 \leq \alpha U_\alpha f \leq 1$ a.s. for any $\alpha > 0$, and *transient* if there exists $g \in L^0(\xi)$, $g > 0$, such that $\langle Ug, g\rangle_\xi < \infty$, where U is obviously defined by $Uf = \lim_{\alpha\to 0} U_\alpha f$, for any $f \geq 0$. In this case, an elementary adaptation of arguments from [1, XIII, 56, b)] shows that $\lim_{\alpha\to 0} \alpha U_\alpha f = 0$ for any $f \in L^2(\xi)$, which enables to extend (without specification) to this frame both [2, 2.3] and [2, 3.1], together all results quoted from [1], which still hold in this frame. See also [6]. We say that ξ is *excessive* if $\xi \circ \alpha U_\alpha \leq \xi$, for any $\alpha > 0$, where U_α are considered as pseudokernels.

Finally, we say that $\mathcal{D}_0$ is *transient* if there exists $g \in L^1$, $g > 0$ and bounded, such that

$$\langle |u|, g\rangle_\xi \leq e(u) \qquad u \in \mathcal{D}_0. \tag{T}$$

Any such function is called a *reference function* of the Dirichlet space $\mathcal{D}_0$.

Proposition 2.1. *The traditional Dirichlet space $\mathcal{D}_0$ is transient iff the resolvent $\mathcal{U}$ is transient.*

Proof. The implication "$\Longleftarrow$" is obvious, using a normalization in (F). Conversely, if $\mathcal{D}_0$ is transient and g is a reference function, we remark first that $g \in L^2$. Hence $U_\alpha g \in D(A)$ for any $\alpha > 0$ and we have the relations (take $u = U_\alpha g$ in (T)):

$$\langle U_\alpha g, g\rangle^2 \le a(U_\alpha g, U_\alpha g) = \langle U_\alpha g, g - \alpha U_\alpha g\rangle \le \langle U_\alpha g, g\rangle$$
$$\text{and hence } \langle Ug, g\rangle = \lim_{\alpha\to 0}\langle U_\alpha g, g\rangle \le 1. \qquad \square$$

We recall that a function $T\colon R \to R$ is called a *normal contraction* if $T(O) = O$ and $|T(x) - T(y)| \le |x - y|$ for any $x,\ y \in R$. If $\mathcal{F}$ is set of functions on E, we say that T operates on $\mathcal{F}$ if $T \circ u \in \mathcal{F}$ for any $u \in \mathcal{F}$. If moreover γ is a bilinear form on $\mathcal{F} \times \mathcal{F}$ we say that T *operates on* γ if we have

$$\text{(C)} \qquad \begin{aligned} &\gamma(u + T\circ u, u - T\circ u) \ge 0 \quad u \in \mathcal{F} \\ &\gamma(u - T\circ u, u + T\circ u) \ge 0. \end{aligned}$$

The unit contraction is defined by $T(x) = x^+ \wedge 1$. Let now ξ be a σ-finite measure on a measurable space $(E, \mathcal{E})$.

DEFINITION 2.2. A couple $(\mathcal{D}, a)$ is called an (extended) transient Dirichlet space (with reference measure ξ) if:

1) $\mathcal{D}$ is a Hilbert space whose norm e comes from a given bounded (nonsymmetric) bilinear form a on $\mathcal{D} \times \mathcal{D}$, that is $e(u) = a(u, u)^{1/2}$, and there exists a constant $M > 0$ such that

$$a(u, v) \le Me(u)e(v) \quad u,\ v \in \mathcal{D}.$$

2) There exists a ξ integrable bounded fuction $g > 0$ on E such that $\mathcal{D} \subset L^1(g\cdot\xi)(\subset L^0(\xi))$ and

$$\langle |u|, g\rangle_\xi \le e(u) \qquad u \in \mathcal{D}.$$

3) $\mathcal{D} \cap L^2(\xi)$ is dense in both $L^2(\xi)$ and $\mathcal{D}$.
4) The unit contraction operates on a.

Theorem 2.3. *For any (extended) transient Dirichlet space (with reference measure ξ) $(\mathcal{D}, a)$, there exists a transient regular submarkovian resolvent of α-contractions $\mathcal{U} = (U_\alpha)_{\alpha>0}$ on $L^2(\xi)$ for which ξ is excessive, $\mathcal{U}$ satisfies the sector condition, such* *that* $(\mathcal{D}, a)$ *is the* (*extended*) *Dirichlet space associated with* $\mathcal{U}$ *and* ξ. *Moreover* $\mathcal{U}$ *is unique subject to the relation*

$$a(u, Uf) = \langle u, f\rangle_\xi \qquad u \in \mathcal{D}, \tag{I}$$

valid for any $f \in L^0_+(\xi)$ *such that* $\langle f, Uf\rangle_\xi < \infty$.

Proof. Let us consider the vector space $\mathcal{G} = \mathcal{D} \cap L^2$ (we abbrev. L^2 for $L^2(\xi)$) which is dense in L^2 from hypothesis 3). For any $\alpha > 0$, we consider the bilinear form a_α on $\mathcal{G} \times \mathcal{G}$ defined by

$$a_\alpha(u, v) = a(u, v) + \alpha\langle u, v\rangle_\xi \quad u, \ v \in \mathcal{G}.$$

One can see that $\mathcal{G}$ is complete with respect to the norm e_α associated with a_α. Since the unit contraction also operates on the restriction of a to $\mathcal{G} \times \mathcal{G}$, it follows from [1, XIII, 53] that there exists a regular submarkovian resolvent of α-contractions $\mathcal{U} = (U_\alpha)_{\alpha>0}$ on $L^2(\xi)$, for which ξ is excessive and $\mathcal{U}$ satisfies the sector condition, its (traditional) Dirichlet space $\mathcal{D}_0$ coincides to $\mathcal{G}$ and $a|\mathcal{G} \times \mathcal{G}$ is its energy form. So, we can consider the extended Dirichlet space $\widetilde{\mathcal{D}}$ associated to $\mathcal{U}$. $\widetilde{\mathcal{D}}$ is isomorphic to $\mathcal{D}$ as the abstract completion of $D(A)$, since $D(A)$ is dense in $\mathcal{G}$ (with respect to norm e) and $\mathcal{G}$ is dense in $\mathcal{D}$ from hypothesis 3), $\mathcal{D}$ being complete as Hilbert space from hypothesis. In order to show that actually $\mathcal{D} = \widetilde{\mathcal{D}}$ as subspaces of L^0, let $u \in \mathcal{D}$, and a sequence $(u_n) \subset D(A)$ such that $e(u_n - u) \to 0$. Using hypothesis 2) we can pick a subsequence (u_{n_k}) of (u_n) such that $u_{n_k} \to u$ ξ a.s. and so $u \in \widetilde{\mathcal{D}}$ (definition of $\widetilde{\mathcal{D}}$). Conversely, if $u \in \widetilde{\mathcal{D}}$, let $(u_n) \subset D(A)$ such that (u_n) is e-Cauchy and $u_n \to u$ ξ a.s. Since $\mathcal{D}$ is complete, there exists $u' \in \mathcal{D}$ such that $e(u_n - u') \to 0$, and using again 2), we conclude that $u = u' \in \mathcal{D}$.

Using one more time hypothesis 2), it follows from Proposition 2.1 that $\mathcal{U}$ is transient.

As to the relation (I), it holds from [2, 3.1], where it is also shown that $Uf \in \mathcal{D}$ for such f. Let $\mathcal{U}^1 = (U^1_\alpha)_{\alpha>0}$ and $\mathcal{U}^2 = (U^2_\alpha)_{\alpha>0}$ be two resolvents as above for which the initial kernels U^1 and U^2 satisfy (I). We can choose $g > 0$ such that $\langle g, U^1 g\rangle < \infty$, $\langle g, U^2 g\rangle < \infty$. We have then

$$a(u, U^1 f) = a(u, U^2 f) \quad u \in \mathcal{D},$$

for any $0 \le f \le k \cdot g$, where $k \in N$. Taking $u = U^1 f - U^2 f$ we see that $U^1 f = U^2 f$ as elements of L^0, and then from the resolvent equation it follows that $U^1_\alpha f = U^2_\alpha f$ for any $\alpha > 0$. Since the set $\{f \in L^2 : 0 \le |f| \le k \cdot g; k \in N\}$ is dense in $L^2(\xi)$, it follows that $\mathcal{U}^1 = \mathcal{U}^2$. □

REMARK 2.4. The converse of above result is also true, that is the extended Dirichlet space associated to such a resolvent fulfills the four axioms. The first three are obvious, and are well known. Also the unit contraction operates on the restriction of a to $\mathcal{D}_0 \times \mathcal{D}_0$ (see [1, XIII, 51] where our $\mathcal{D}_0$ is denoted by $\mathcal{D}$), and the principle of contractions on $\mathcal{D}_0$ holds. If $u \in \mathcal{D}$, take $u_n \in \mathcal{D}_0$ such that $e(u_n - u) \to 0$ and $u_n \to u$ ξ a.s. Since $T \circ u_n \to T \circ u$ ξ a.s., and the sequence $e(T \circ u_n)(\leq e(u_n))$ is bounded, if follows that $T \circ u_n$ converges weakly to $T \circ u$ in $\mathcal{D}$ (see [2, 2.5]). Hence $a(u_n, T \circ u_n)$ (resp. $a(T \circ u_n, u_n)) \to a(u, T \circ u)$ (resp. $a(T \circ u, u)$), and $e(T \circ u) \leq \liminf_n e(T \circ u_n))$, which show that

$$a(u + T \circ u, u - T \circ u) \geq 0$$
$$a(u - T \circ u, u + T \circ u) \geq 0.$$

Corollary 2.5. $\mathcal{D}_0 = \mathcal{D} \cap L^2$.

Proof. The initial kernel U fulfils the relation (I) in the statement of theorem. On the other hand we have produced in the proof of theorem a transient resolvent $\mathcal{U}'$ whose associated traditional (resp. extended) Dirichlet space is $\mathcal{D} \cap L^2$ (resp. $\mathcal{D}$) and its initial kernel U' fulfils (I). Hence $\mathcal{U} = \mathcal{U}'$ by the uniqueness assertion and so $\mathcal{D}_0 = \mathcal{D} \cap L^2$. □

Suppose that $(E, \mathcal{E})$ is a Radon measurable space. Using a general regularization procedure (see [1, XIII, 43]) it follows that $\mathcal{U}$ comes from a transient resolvent of kernels on $(E, \mathcal{E})$ as in our initial frame. We return now to this frame, so that the measurable space $(E, \mathcal{E})$ is *arbitrary*, and $\mathcal{U}$ is a transient resolvent of kernels.

We suppose from now on that $U1 > 0$. We choose a function $g \in L^1$, $1 \geq g > 0$, such that Ug is bounded and moreover there exist $\alpha > 0$ and $c > 0$ such that $U_\alpha g \leq c \cdot g$. To see that such a function exists, we start with $f \in L^1$, $0 < f \leq 1$, Uf bounded, we pick $0 < \beta$, and we put $g = \beta U_\beta f$. Then $g \in L^1$ since ξ is excessive, and for any $\alpha > \beta$ we can take $c = (\alpha - \beta)^{-1}$. Let now $\mathcal{V} = (V_\alpha)_{\alpha > 0}$ be the submarkovian resolvent of kernels on $(E, \mathcal{E})$ whose initial kernel is $V(\cdot) = U(g \cdot)$ (Hunt's transform).

We consider the measure $\eta = g \cdot \xi$ which is *finite* and $\mathcal{V}$-excessive.

Theorem 2.6. *The resolvent $\mathcal{V}$ considered on $\mathcal{L}^2(\eta)$ is regular, it satisfies the sector condition, and its extended Dirichlet space coincides with the extended Dirichlet space of $\mathcal{U}$ on $L^2(\xi)$.*

Proof. We fix $\alpha > 0$. For the first statement, we have to show that $V_\alpha(L^2(\eta))$ is dense in $L^2(\eta)$. Since $V = V_0$ is a bounded kernel, it follows from resolvent equation that $V_\alpha(b(E)) = V(b(E))$.

Using the definition of V, we have the identity

$$V(b(E)) = \{U(f) : \exists n \in N \text{ such that } |f| \leq ng\}. \tag{1}$$

We look at the equation

$$Uh = U_\alpha h + \alpha U U_\alpha h$$

and we remark that if $|h| \le ng$ for some $n \in N$, the same being true for $|U_\alpha h|$ by the choice of g, it follows that

$$U_\alpha h = U(h - \alpha U_\alpha h) \in \{U(f) : \exists n \in N \text{ such that } |f| \le ng\} \tag{2}$$

But the resolvent $\mathcal{U}$ is regular on $L^2(\xi)$, that is the set $U_\alpha(L^2(\xi))$ is dense in $L^2(\xi)$, and by truncation the same is true for the set $\{U_\alpha(h) : \exists n \in N \text{ such that } |h| \le ng\}$.

From (1) and (2) it follows that $V(b(E) \cap L^2(\xi)$ is dense in $L^2(\xi)$, and since $\eta \le \xi$ and $L^2(\xi)$ is dense in $L^2(\eta)$ as subset, it follows easily that $V(b(E)) = V_\alpha(b(E))$ (and hence $V_\alpha(L^2(\eta)))$ is dense in $L^2(\eta)$.

Next, we may consider the Dirichlet form a defined on $D(A')XD(A')$, where A' denotes the generator of the resolvent $\mathcal{V}$ on $L^2(\eta)$, and in fact $D(A') = V_\alpha(L^2(\eta))$. Using the basic formulas it is easy to see that $V(b(E)) = V_\alpha(b(E))$ is dense in $D(A')$ with respect to the seminorm e' associated with a', since $b(E)$ is dense in $L^2(\eta)$.

We claim that the following formula is valid (we don't know yet that the sector condition is fulfilled by $\mathcal{V}$, which would imply it by [2, 3.1]):

$$a'(u, Vf) = \langle u, f\rangle_\eta \quad u \in D(A'), \ \ f \in b(E). \tag{3}$$

Indeed, from the resolvent equation, we have:

$$\begin{aligned} a'(u, Vf) &= a'(u, V_\alpha f + \alpha V_\alpha V f) \\ &= \langle u, f\rangle_\eta - \alpha\langle u, V_\alpha f\rangle_\eta + \alpha\langle u, Vf\rangle_\eta - \alpha^2\langle u, V_\alpha V f\rangle_\eta = \langle u, f\rangle_\eta. \end{aligned}$$

Now, if we put $u = Vh (h \in b(E))$, and we make use of [2, 3.1] for $\mathcal{U}$, it follows the relations

$$a'(Vh, Vf) = \langle Vh, f\rangle_\eta = \langle U(gh), gf\rangle_\xi = a(U(gh), U(gf)) \tag{4}$$

In particular, the above formula insures us that $\mathcal{V}$ satisfies the sector condition (with the same constant as for $\mathcal{U}$), and we may consider the extended Dirichlet space $\mathcal{D}'$. Finally, since the spaces $V(b(E))$ and $\{U(gh); f \in b(E)\}$ are dense in $\mathcal{D}'$ and resp. in $\mathcal{D}$, and moreover they coincide, it follows from (4) and definition of extended Dirichlet space) that $\mathcal{D}' = \mathcal{D}$, $a' = a$. □

From [1, XIII, 60], we know that the operator $V_\alpha : \mathcal{D}'_0 \to \mathcal{D}'$ (the composite of the restriction of the α-contraction V_α on $L^2(\eta)$ to $\mathcal{D}'_0$ and the injection of $\mathcal{D}'_0$ in $\mathcal{D}'$) extends to a continuous operator $\widetilde{V}_\alpha : \mathcal{D}' \to \mathcal{D}'$.

Corollary 2.7. *$\widetilde{V}_\alpha u = V_\alpha u$ for any $u \in \mathcal{D}_+$, where V_α is considered here as pseudokernel.*

Proof. Let $u_n \in D(A') \subset L^2(\eta)$ converging to u in $\mathcal{D}$. Since from [1, XIII, 60] the sequence $u_n - \alpha V_\alpha u_n$ is converging to $u - \alpha \widetilde{V}_\alpha u$ in $\mathcal{D}$, it follows in particular that the sequence $V_\alpha u_n$ is Cauchy in $\mathcal{D}$. But $u_n \to u$ in $L^1(\eta)$ (see [2, 2.3] for $f \equiv 1$) and hence

$$\int |V_\alpha u_n - V_\alpha u| \, d\eta \leq \int V_\alpha |u_n - u| \, d\eta \leq \frac{1}{\alpha} \int |u_n - u| \, d\eta \to 0.$$

In particular, there exists a subsequence (n_k) such that $V_\alpha u_{n_k} \to V_\alpha u$ η a.s. Therefore $V_\alpha u \in \mathcal{D}$ and moreover $V_\alpha u = \widetilde{V}_\alpha u$. $\square$

Note. Applying Theorem 2.3 to the form $\hat{a}(u, v) = a(v, u)$, we get a resolvent $\hat{U}$ with the same properties as $\mathcal{U}$, such that $\mathcal{U}$ and $\hat{\mathcal{U}}$ are in duality with respect to ξ.

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