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Author(s)	Hanaki, Akihide
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A CONDITION ON LENGTHS OF CONJUGACY CLASSES AND CHARACTER DEGREES

Akihide HANAKI

1. Introduction

In E. Bannai [1], the following condition on finite groups is investigated.

Let G be a finite group, $\mathrm{Irr}(G) = \{\chi_i\}_{1 \le i \le k}$ be the set of all irreducible characters of G, and $\mathrm{Cl}(G) = \{C_i\}_{1 \le i \le k}$ be the set of all conjugacy classes of G.

Condition. By suitable renumbering i,

$$\chi_i(1)^2 = |C_i|, \text{ for } i = 1, 2, \cdots, k.$$

We call this condition B-condition ("B" is due to E. Bannai). A few groups satisfying B-condition are known: abelian groups, Suzuki 2-groups $A(n, \theta)$(See [3, VIII.6.7 Example and §7]), ϕ_6, ϕ_{11} in [4], and groups isoclinic to them (For isoclinism, see [2]). In any case, they are nilpotent and their derived lengths are at most 2.

In this paper, we shall construct a family of groups satisfying B-condition. Our groups are, in a sense, generalizations of Suzuki 2-groups. By our examples, we can say that

Theorem. *Derived lengths of groups satisfying B-condition are unbounded.*

2. Construction of groups

Let $F = \mathrm{GF}(2^n)$ be the finite field of order 2^n, and let θ be an automorphism of F. We put, for a positive integer l and $a_1, a_2, \cdots, a_l \in F$,

$$u(a_1, a_2, \cdots, a_l) = \begin{pmatrix} 1 & & & & & \\ a_1 & 1 & & & & \\ a_2 & a_1\theta & 1 & & & \\ a_3 & a_2\theta & a_1\theta^2 & 1 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ a_l & a_{l-1}\theta & a_{l-2}\theta^2 & \cdots & a_1\theta^{l-1} & 1 \end{pmatrix} \in M_{l+1}(F)$$

and

$$A_l(n, \theta) = \{u(a_1, a_2, \cdots, a_l) | a_i \in F\}.$$

The multiplication is defined as a product of two matrices as follows:

$$u(a_1, a_2, \cdots, a_l)u(b_1, b_2, \cdots, b_l)$$
$$= u(a_1 + b_1, a_2 + (a_1\theta)b_1 + b_2, a_3 + (a_2\theta)b_1 + (a_1\theta^2)b_2 + b_3,$$
$$\cdots, a_l + (a_{l-1}\theta)b_1 + \cdots + (a_1\theta^{l-1})b_{l-1} + b_l).$$

So $A_l(n, \theta)$ becomes a group of order 2^{nl}. If $l=2$, this group is isomorphic to a Suzuki 2-group $A(n, \theta)$ in [3, VIII.6.7 Example and §7].

For $1 \le i \le l$, we put

$$G_i = \{u(0, \cdots, 0, a_i, a_{i+1}, \cdots, a_l)\}.$$

Define $\varphi_{l,i-1} : A_l(n, \theta) \longrightarrow A_{i-1}(n, \theta)$ by $\varphi_{l,i-1}(u(a_1, \cdots, a_l)) = u(a_1, \cdots, a_{i-1})$. Then $\varphi_{l,i-1}$ is an epimorphism and ker $\varphi_{l,i-1} = G_i$. Thus G_i is a normal subgroup of $A_l(n, \theta)$, $A_l(n, \theta)/G_i \cong A_{i-1}(n, \theta)$, and obviously G_l is in the center of $A_l(n, \theta)$ by the multiplication.

$A_l(n, \theta)$ has important automorphisms. Let $\lambda \in F^{\times}$. We define $\xi_\lambda : A_l(n, \theta) \rightarrow A_l(n, \theta)$ by

$$\xi_\lambda(u(a_1, a_2, \cdots, a_l)) = u(\lambda_1 a_1, \lambda_2 a_2, \cdots, \lambda_l a_l)$$

where

$$\lambda_1 = \lambda$$
$$\lambda_2 = \lambda(\lambda\theta)$$
$$\lambda_3 = \lambda(\lambda\theta)(\lambda\theta^2)$$
$$\cdots$$
$$\lambda_l = \prod_{i=0}^{l-1}(\lambda\theta^i).$$

Then this is an automorphism of $A_l(n, \theta)$.

For simplify our argument, throughout this paper, we assume that θ is the Frobenius automorphism of F, $\theta : x \rightarrow x^2$. Then $\lambda_i = \lambda^{2^i-1}$. We also assume that λ is a generator of $F^{\times}$ and is fixed. Then λ_i generates $F^{\times}$ if and only if $(2^n-1, 2^i-1) = 1$. But it is easy to check that $(2^n-1, 2^i-1) = 1$ if and only if $(n, i) = 1$. In this case, $\langle \xi_\lambda \rangle$ permutes $G_i/G_{i+1} - G_{i+1}$ transitively. If $l < n_0$, where n_0 is the smallest prime divisor of n, then this holds for any i.

Our main result is

Theorem 2.1. *Let θ be the Frobenius automorphism of $GF(2^n)$. Assume that $l < n_0$, where n_0 is the smallest prime divisor of n. Then $A_l(n, \theta)$ satisfies B-condition.*

In particular, if n is a prime and $l < n$ then $A_l(n, \theta)$ satisfies B-condition.

The second part of this theorem is obviously holds by the first part, and the first part is proved by the next theorem.

We put

$$q_i(G) = \#\{C \in \mathrm{Cl}(G) | |C| = 2^i\}$$
$$r_i(G) = \#\{\chi \in \mathrm{Irr}(G) | \chi(1) = 2^i\}.$$

Then B-condition holds for a 2-group G if and only if

$$q_{2i}(G) = r_i(G), \text{ for any } i \geq 0.$$

Theorem 2.2. *Put $G = A_l(n, \theta)$. Assume that θ is the Frobenius automorphism, and $l < n_0$, where n_0 is the smallest prime divisor of n. Then*

(a) $q_0(G) = 2^n$, $q_{m(n-1)}(G) = 2^m(2^n - 1)$ for $1 \leq m \leq l-1$, and $q_i(G) = 0$ for the other $i > 0$.

(b) $r_0(G) = 2^n$, $r_{m(n-1)/2}(G) = 2^m(2^n - 1)$ for $1 \leq m \leq l-1$, and $r_i(G) = 0$ for the other $i > 0$.

REMARK. If $l \geq n_0$ there exist groups which does not satisfy B-condition. For example, $A_2(2, \theta)$, $A_3(3, \theta)$, and $A_4(3, \theta)$, θ the Frobenius automorphism, do not satisfy B-condition.

It is known that $A_2(n, \theta)$ satisfies B-condition when θ is an arbitrary odd order automorphism of $GF(2^n)$. For odd characteristic finite fields, we can define groups similar to $A_l(n, \theta)$, and they satisfy B-condition if $l = 2$ and the order of θ is odd (This is my work and unpublished). This is a general case of ϕ_{11} in [4].

3. Conjugacy classes

In this section, we shall prove Theorem 2.2 (a). In this and later sections, we assume that $l < n_0$, where n_0 is the smallest prime divisor of n. If $l = 1$ then $A_l(n, \theta)$ is abelian so we assume $l \geq 2$. Note that n is odd.

Theorem 3.1. *The following is a complete set of representatives of conjugacy classes of $A_l(n, \theta)$.*

$$\{\xi_\lambda^j u(e_1, e_2, \cdots, e_l) | 0 \leq j < 2^n - 1, e_i = 0 \text{ or } 1 \text{ and at least one } e_i = 1\} \cup \{u(0, \cdots, 0)\}$$

When $e_1 = \cdots = e_{i-1} = 0$ and $e_i = 1$, the order of the centralizer of $\xi_\lambda^j u(e_1, e_2, \cdots, e_l)$ is 2^{ni+l-i}.

To prove this, we need two lemmas.

Lemma 3.2. *The order of the centralizer of $u(0, \cdots, 0, e_i, \cdots, e_l)$, $e_i = 1$ and $e_j = 1$ or 0 for $j > i$, is 2^{ni+l-i}.*

Proof. Let $u(a_1, a_2, \cdots, a_l)$ centralize $u(0, \cdots, 0, e_i, \cdots, e_l)$. Then by direct calculation (note that θ acts trivially on e_j),

$$e_i a_1 (a_1^{2^i-1} + 1) = 0$$
$$e_i a_2 (a_2^{2^i-1} + 1) = e_{i+1} a_1 (a_1^{2^{i+1}-1} + 1)$$
$$\cdots \quad \cdots$$
$$e_i a_{l-i} (a_{l-i}^{2^i-1} + 1) = e_{l-1} a_1 (a_1^{2^{l-1}-1} + 1) + \cdots + e_{i+1} a_{l-i-1} (a_{l-i-1}^{2^{i+1}-1} + 1)$$

By our assumption, the map $x \to x^{2^i-1}$ is a bijection from F to F, so the first equation say that $a_1 = 0$ or 1. Hence the right hand side of the second equation is 0, and thus $a_2 = 0$ or 1. We can continue this argument until a_{l-i}. Thus the order of the centralizer of $u(0, \cdots, 0, e_i, \cdots, e_l)$ is $2^{l-i} \cdot 2^{ni} = 2^{ni+l-i}$. The proof is complete. $\square$

Let tr be the trace map from $\mathrm{GF}(2^n)$ to $\mathrm{GF}(2)$: $\mathrm{tr}(x) = \sum_{i=0}^{n-1} x\theta^i$. The next holds.

Lemma 3.3. *$\xi_\lambda^j u(e_1, e_2, \cdots, e_l)$, $e_i = 0$ or 1, and $\xi_\lambda^k u(f_1, f_2, \cdots, f_l)$, $f_i = 0$ or 1, are conjugate if and only if $j = k$ and $e_i = f_i$, for all i.*

Proof. Assume $\xi_\lambda^j u(e_1, e_2, \cdots, e_l)$, $e_i = 0$ or 1, and $\xi_\lambda^k u(f_1, f_2, \cdots, f_l)$, $f_i = 0$ or 1, are conjugate in $A_l(n, \theta)$. If $e_1 = \cdots = e_{i-1} = 0$ and $e_i = 1$, then obviously $f_1 = \cdots = f_{i-1} = 0$ and $f_i = 1$, and $j = k$, since G_i/G_{i+1} is in the center of G_1/G_{i+1}. So we may assume that $j = k = 0$. Then there exists $u(a_1, \cdots, a_l)$ such that $u(e_1, \cdots, e_l) u(a_1, \cdots, a_l) = u(a_1, \cdots, a_l) u(f_1, \cdots, f_l)$. Obviously $e_1 = f_1$. Suppose that $e_i = f_i$, for $i < m$. Then by direct calculation,

$$e_m + f_m = e_{m-1} a_1 + e_{m-2} a_2 + \cdots + e_1 a_{m-1}$$
$$+ f_{m-1}(a_1 \theta^{m-1}) + f_{m-2}(a_2 \theta^{m-2}) + \cdots + f_1(a_{m-1}\theta)$$
$$= e_{m-1}(a_1 + a_1 \theta^{m-1}) + e_{m-2}(a_2 + a_2 \theta^{m-2}) + \cdots + e_1(a_{m-1} + a_{m-1}\theta).$$

The right hand side of this equation is in the kernel of tr, since $e_i = 0$ or 1. But the left hand side is 0 or 1. So $e_m = f_m$. Thus the proof is complete. $\square$

Now Theorem 3.1 is easily shown.

$$\xi_\lambda^j u(e_1, e_2, \cdots, e_l), \text{ for } 0 \le j < 2^n - 1, e_i = 0 \text{ or } 1$$

are in distinct conjugacy classes each other. Consider the lengths of these classes. The sum of their lengths is

$$1 + 1 \cdot (2^n - 1) + 2^{n-1} \cdot 2(2^n - 1) + 2^{2(n-1)} \cdot 2^2(2^n - 1)$$
$$+ \cdots + 2^{(l-1)(n-1)} \cdot 2^{l-1}(2^n - 1)$$
$$= 2^{nl} = |A_l(n, \theta)|.$$

Thus they are representatives of conjugacy classes of $A_l(n, \theta)$. Theorem 3.1 and also Theorem 2.2 (a) are proved.

4. Irreducible characters

In this section, we shall prove Theorem 2.2 (b). We need many lemmas to prove this.

We put $G = A_l(n, \theta)$. Recall that

$$G_i = \{u(0, \cdots, 0, a_i, a_{i+1}, \cdots, a_l)\}$$

and $G/G_i \cong A_{i-1}(n, \theta)$.

Lemma 4.1. $C_G(G_i) = G_{l-i+1}$. *Especially, G_i is abelian if and only if $i \ge (l+1)/2$.*

Proof. This holds by direct calculations. $\square$

Lemma 4.2. *Let G_N be abelian, and let $\varphi \in \mathrm{Irr}(G_N)$ such that $\ker\varphi \not\ge G_l$. Then*

$$|I_G(\varphi)| = 2^{nN + l - N},$$

where $I_G(\varphi)$ is the stabilizer of φ in G.

To show this, we may assume that φ_{G_l} is $u(0, \cdots, 0, a_l) \to (-1)^{\mathrm{tr}(a_l)}$ since $\mathrm{Irr}(G_l) - \{1_{G_l}\}$ is transitively permuted by $\langle \xi_\lambda \rangle$, and note that any character of G_l is invariant in G since G_l is in the center of G. This lemma will be shown later.

Let G_N be abelian. Then $\varphi \in \mathrm{Irr}(G_N)$ can be regarded as a homomorphism from G_N to $F_2 = \mathrm{GF}(2)$, since G_N is an elementary abelian 2-group. Thus φ can be regarded as a sum of homomorphisms from G_i/G_{i+1} to F_2, $i = N, N+1, \cdots, l$. Note that G_i/G_{i+1} is isomorphic to $F = \mathrm{GF}(2^n)$ as an additive group.

Lemma 4.3. *Define $\Phi : F \longrightarrow \mathrm{Hom}_{F_2}(F, F_2)$ by $\phi(a)(x) = \mathrm{tr}(ax)$. Then ϕ is an isomorphism as abelian groups.*

Proof.   Put $K=\ker$ tr. If $aK=bK$ implies $a=b$ then the proof is complete. Thus we shall show $aK=K$ implies $a=1$.

If $a\neq 1$ then a induces a permutation on K. Obviously $C_K(a)=\{0\}$, and the lengths of $\langle a\rangle$-orbits are the order of a. But by our assumption, $(|K|-1, o(a))=1$. This is a contradiction. The proof is complete. $\square$

By this lemma, any $\varphi\in\mathrm{Irr}(G_N)$ has a form $\varphi: u(0, \cdots, 0, x_N, \cdots, x_l)\mapsto(-1)^{\Sigma_i \text{-}_N\text{tr}(a_i x_i)}$ for some $a_1, \cdots, a_l\in F$, and we can denote this by $\varphi(a_N, \cdots, a_l)$.

Lemma 4.4.   *Define* $\rho_{i,j}: F\times F\longrightarrow F$ *by* $\rho_{i,j}(a, b)=(a\theta^j)b+a(b\theta^i)$. *Then, for* $i>0$, $j>0$, *and* $0<i+j\leq l$, *there exits an integer* m *such that* $(2^i-1)m\equiv 2^{i+j}-1\pmod{2^n-1}$, *and*

$$\rho_{i,j}(a, F)=a^m\ker\text{tr}.$$

Especially, $\rho_{i,j}(a, F)=\rho_{i,j}(b, F)$ *if and only if* $a=b$.

Proof.   Since $i\leq l<n_0$, 2^i-1 is coprime to 2^n-1. Thus such m exits.

Obviously $\rho_{i,j}$ is bilinear. If $(a\theta^j)b+a(b\theta^i)=0$ then $a^{2^j}b+ab^{2^i}=0$, and so $b=0$ or $a^{(2^j-1)/(2^i-1)}$. Hence $|\rho_{i,j}(a, F)|=2^{n-1}$ for $a\neq 0$.

On the other hand,

$$(a\theta^j)b+a(b\theta^i)=a^m(a^{2^j-m}b+(a^{2^j-m}b)^{2^i})$$
$$\in a^m\ker\text{tr}.$$

Thus $\rho_{i,j}(a, F)=a^m\ker$ tr.

The last part of the lemma clearly holds by the same way as the proof of Lemma 4.3 and $(m, 2^m-1)=1$.

The proof is complete. $\square$

In general, conjugates of elements in $A_l(n, \theta)$ is very complicated. So we prepare the easy cases.

Lemma 4.5.   (a)
$$u(0, \cdots, 0, a_i, 0, \cdots, 0)u(0, \cdots, 0, x_m, 0, \cdots, 0)u(0, \cdots, 0, a_i, 0, \cdots, 0)^{-1}$$
$$=u(0, \cdots, 0, x_m, 0, \cdots, 0, (x_m\theta^i+x_m)_{m+i}, 0, \cdots, 0, (x_m\theta^{2i}+x_m)_{m+2i}, \cdots).$$

(b)   *When* $i+j=l$,
$$u(0, \cdots, 0, g_i, 0, \cdots, 0)u(0, \cdots, 0, x_j, 0, \cdots, 0)u(0, \cdots, 0, g_i, 0, \cdots, 0)^{-1}$$
$$=u(0, \cdots, 0, x_j, 0, \cdots, 0, (x_j\theta^i)g_i+x_j(g_i\theta^j)).$$

Proof.   Note that
$$u(0, \cdots, 0, g_i, 0, \cdots, 0)^{-1}$$

$$= u(0, \cdots, 0, g_i, 0, \cdots, 0, ((g_i\theta^i)g_i)_{2i}, 0, \cdots, 0, ((g_i\theta^{2i})(g_i\theta^i)g_i)_{3i}, \cdots).$$

So the results follow by direct calculations. $\square$

We define a subgroup H of G by

$$H = \{u(a_1, a_2, \cdots, a_l) | a_i = 1 \text{ or } 0\}.$$

Obviously this is a subgroup of G and abelian. H is generated by $u(0, \cdots, 0, 1_i, 0, \cdots, 0), 1 \leq i \leq l$, and has the order 2^l.

Lemma 4.6. *Assume that G_N is abelian. For $\varphi = \varphi(a_N, \cdots, a_l)$, $a_i = 1$ or 0,*

$$I_G(\varphi) = HG_{l-N+1}.$$

Proof. $I_G(\varphi) \geq G_{l-N+1}$ since $C_G(G_N) = G_{l-N+1}$ holds by Lemma 4.1, and $I_G(\varphi) \geq H$ by Lemma 4.5 (a). So $I_G(\varphi) \geq HG_{l-N+1}$.

If $N = l$ the result holds obviously. Assume the result holds for $\varphi_{G_{N+1}} = \varphi(a_{N+1}, \cdots, a_l)$. Then $I_G(\varphi) \leq G_G(\varphi_{G_{N+1}}) = HG_{l-N}$. Let $g \in I_G(\varphi)$. We can write $g = hg'$, $h \in H$, $g' \in G_{l-N}$. Then $g' \in I_G(\varphi)$. Since $G_{l-N+1} \leq I_G(\varphi)$, we may assume that $g' = u(0, \cdots, 0, g_{l-N}, 0, \cdots, 0)$. Consider the action of g' on $u(0, \cdots, 0, x_N, 0, \cdots, 0)$. By Lemma 4.5 (b), $(x_N\theta^{l-N})g_{l-N} + x_N(g_{l-N}\theta^N)$ must be in ker tr for any $x_N \in F$. But by Lemma 4.4,

$$\{(x_N\theta^{l-N})g_{l-N} + x_N(g_{l-N}\theta^N) | x_N \in F\} = \rho_{l-N,N}(g_{l-N}, F) = g_{l-N}^m \ker \text{tr},$$

where $(2^N - 1)m \equiv 2^l - 1 (\text{mod } 2^n - 1)$. Thus if $g_{l-N} \neq 0$, g_{l-N} must be 1 by Lemma 4.4 and $\rho_{l-N,N}(1, F) = \ker \text{tr}$. So $g_{l-N} = 1$ or 0. Now $I_G(\varphi) \leq HG_{l-N+1}$ and the result follows. $\square$

Lemma 4.7. *Assume that G_N is abelian. If $\varphi(a_N, \cdots, a_l)^g = \varphi(b_N, \cdots, b_l)$, $a_i = 1$ or 0, $b_i = 1$ or 0, then $a_i = b_i$ for all i.*

Proof. Suppose that $a_m \neq b_m$ and $a_i = b_i$ for all $i > m$. We may assume that $a_m = 1$ and $b_m = 0$. $g = u(g_1, \cdots, g_l)$ fixes $\varphi(a_{m+1}, \cdots, a_l)$ so $g \in HG_{l-m}$ by Lemma 4.6. Consider the action of g on $\varphi(a_m, \cdots, a_l)$. Since HG_{l-m+1} stabilizes $\varphi(a_m, \cdots, a_l)$, we may assume that $g = u(0, \cdots, 0, g_{l-m}, 0, \cdots, 0)$. Then

$$\begin{aligned}
&\varphi(a_m, \cdots, a_l)^g(u(0, \cdots, 0, a_m, 0, \cdots, 0)) \\
&= \varphi(a_m, \cdots, a_l)(u(0, \cdots, 0, a_m, 0, \cdots, 0)^{g^{-1}}) \\
&= \varphi(a_m, \cdots, a_l)(u(0, \cdots, 0, a_m, 0, \cdots, 0, g_{l-m}\theta^m + g_{l-m})) \\
&= 1
\end{aligned}$$

by Lemma 4.5 (b). But $\varphi(b_m, \cdots, b_l)(u(0, \cdots, 0, 1_m, 0, \cdots, 0)) = 0$. This is a contradiction, and the proof is complete. $\square$

Now we can prove Lemma 4.2.

Proof of Lemma 4.2.   Let G_N be abelian.   Then the number of irreducible characters of G_N whose kernels do not contain G_l is $2^{(l-N)n}(2^n-1)$.   Lemma 4.6 says that $\varphi(a_N, \cdots, a_l)$, $a_i=1$ or 0, are in distinct G-orbits, and Lemma 4.6 says that their orbits have lengths $2^{(l-N)(n-1)}$.   Also $\varphi(a_N, \cdots, a_l)^{\epsilon_i}$ are in distinct G-orbits. Thus they are complete representatives of G-orbits.   Now the result follows from Lemma 4.6. $\square$

Using Lemma 4.2, we can prove Theorem 2.2 (b) by induction on l.   We separate the cases l as odd from l as even.

Lemma 4.8.   *Assume that Theorem 2.2 holds for $A_{l-1}(n, \theta)$ and l is odd. Then Theorem 2.2 holds for $A_l(n, \theta)$.*

Proof.   Put $N=(l+1)/2$, then G_N is abelian.   Let χ be an irreducible character of G such that $\ker\chi\not\geq G_l$.   Let φ be an irreducible character of G_N such that $(\chi_{G_N}, \varphi)\neq 0$ and $\ker\varphi\not\geq G_l$.   By Lemma 4.2, $|G : I_G(\varphi)|=2^{(n-1)(l-1)/2}$.   So $\chi(1) \geq 2^{(n-1)(l-1)/2}$.   The number of such χ is $2^{l-1}(2^n-1)$ by the number of conjugacy classes.   Conider that

$$|G|=\sum_{\chi\in\mathrm{Irr}(G)}\chi(1)^2=\sum_{\ker\chi\geq G_l}\chi(1)^2+\sum_{\ker\chi\not\geq G_l}\chi(1)^2=2^{n(l-1)}+\sum_{\ker\chi\not\geq G_l}\chi(1)^2$$
$$\geq 2^{n(l-1)}+2^{(n-1)(l-1)}\cdot 2^{l-1}(2^n-1)=2^{nl}=|G|.$$

Thus $\chi(1)=2^{(n-1)(l-1)/2}$.   The proof is complete. $\square$

Lemma 4.9.   *Assume that Theorem 2.2 holds for $A_{l-1}(n, \theta)$ and l is even. Then Theorem 2.2 holds for $A_l(n, \theta)$.*

Proof.   Put $N=l/2$.   Note that G_N is not abelian.   Let χ be an irreducible character of G such that $\ker \chi\not\geq G_l$.   Let φ be an irreducible character of G_N such that $(\chi_{G_N}, \varphi)\neq 0$ and $\ker \varphi\not\geq G_l$. G_{N+1} is in the center of G_N.   So $\varphi_{G_{N+1}}$ is homogeneous.   Let ψ be the homogeneous constituent of $\varphi_{G_{N+1}}$.   Then

$$|I_G(\varphi)|\leq|I_G(\psi)|=2^{(l/2-1)+n(l/2+1)}.$$

Consider the structure of G_N.   Put

$$A=\{u(0, \cdots, 0, a_{N+1}, a_{N+2}, \cdots, a_{l-1}, 0)\}$$
$$B=\{u(0, \cdots, 0, a_N, 0, \cdots, 0, a_l)\}.$$

Then obviously $G_N=A\times B$ and A is in the center of G_N.   Since φ_{G_l} is homogeneous, $|G_l\cap\ker \varphi|=2^{n-1}$.   Put $K=G_l\cap\ker \varphi$.   The commutator map $B/G_l\times B/G_l\longrightarrow G_l$ can be regarded as $\rho_{N,N}$ in Lemma 4.4:

$$[u(0, \cdots, a_N, 0, \cdot, 0), u(0, \cdots, b_N, 0, \cdot, 0)] = u(0, \cdots, 0, a_N(b_N\theta^N) + (a_N\theta^N)b_N).$$

Thus Lemma 4.4 says that there exists the unique non zero a_N such that $u(0, \cdots, a_N, 0, \cdots, 0)$ is in the center of B/K. Clearly $D(B/K) = \Phi(B/K) = G_l/K$ and its order is 2. Thus B/K is isomorphic to a central product of an extraspecial group of order 2^n and an abelian group of order 4 (it is not so hard to check that the center of B/K is cyclic of order 4 but this is not necessary for our argument). It is well known that an irreducible character degree of an extraspecial group of order p^{2r+1} is 1 or p^r. So $\varphi(1) = 2^{(n-1)/2}$. Now

$$\begin{aligned}
\chi(1) &\geq |G : I_G(\varphi)| \cdot \varphi(1) \\
&\geq 2^{nl-(l/2-1)-n(l/2+1)} \cdot 2^{(n-1)/2} \\
&= 2^{(n-1)l-1)/2}
\end{aligned}$$

By the same argument as Lemma 4.8, the result follows. $\square$

Now Theorem 2.2 (b) is proved and $A_l(n, \theta)$, $l < n_0$, satisfies B-condition.

5. Derived lengths

In this section, we consider the derived length of $A_l(n, \theta)$. The next holds.

Theomem 5.1. *If $2^{d-1} \leq l < 2^d$, then the derivd length of $A_l(n, \theta)$ is d.*

This theorem is an easy consequence from the following lemma.

Lemma 5.2. $[G_i, G_j] = G_{i+j}$, *where $G_m = 1$ if $m > l$.*

Proof. Obviously $[G_i, G_j] \leq G_{i+j}$. Suppose that $i+j \leq l$, otherwise $[G_i, G_j] \geq G_{i+j} = 1$ holds. Then $[G_i, G_j] \geq G_l$ holds since $G_l - 1$ is transitively permuted by $\langle \xi_\lambda \rangle$. Inductively $[G_i, G_j]/G_m \geq G_{m-1}/G_m$ for $i+j \leq m \leq l$. Thus $[G_i, G_j] \geq G_{i+j}$ and so $[G_i, G_j] = G_{i+j}$. $\square$

Theorem 5.1 holds obviously from this lemma. We can also see that the nilpotency class of $A_l(n, \theta)$ is l.

Now Theorem in introduction holds from Theorem 2.1 and Theorem 5.1.

References

[1] E. Bannai : *Association schemes and fusion algebras (an introduction)*, J. Alg. Comb. **2** (1993) 327 –344.

[2] P. Hall : *The classification of prime-power groups*, J. Reine Angew. Math. **182** (1940), 130–141.

[3]   B. Huppert and N. Blackburn : Finite Groups II, Berlin-Heidelberg-New York 1982.
[4]   R. James : *The groups of order p^6 (p an odd prime)*, Math. Computation **43** (1980) 613-637.

Faculty of Engineering,
Yamanashi University
Takeda 4, Kofu,
400, Japan