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Osaka J. Math.
33 (1996), 207-216

A CONDITION ON LENGTHS OF CONJUGACY CLASSES AND CHARACTER DEGREES

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(Received July 7, 1994)

1. Introduction

In E. Bannai [1], the following condition on finite groups is investigated.

Let G be a finite group, $\mathrm{Irr}(G)=\{\chi_i\}_{1\le i\le k}$ be the set of all irreducible characters of G, and $\mathrm{Cl}(G)=\{C_i\}_{1\le i\le k}$ be the set of all conjugacy classes of G.

Condition. By suitable renumbering i,

$$\chi_i(1)^2=|C_i|, \text{ for } i=1, 2, \cdots, k.$$

We call this condition B-condition ("B" is due to E. Bannai). A few groups satisfying B-condition are known: abelian groups, Suzuki 2-groups $A(n, \theta)$(See [3, VIII.6.7 Example and §7]), ϕ_6, ϕ_{11} in [4], and groups isoclinic to them (For isoclinism, see [2]). In any case, they are nilpotent and their derived lengths are at most 2.

In this paper, we shall construct a family of groups satisfying B-condition. Our groups are, in a sense, generalizations of Suzuki 2-groups. By our examples, we can say that

Theorem. *Derived lengths of groups satisfying B-condition are unbounded.*

2. Construction of groups

Let $F=\mathrm{GF}(2^n)$ be the finite field of order 2^n, and let θ be an automorphism of F. We put, for a positive integer l and $a_1, a_2, \cdots, a_l\in F$,

$$u(a_1, a_2, \cdots, a_l)=\begin{pmatrix} 1 & & & & & \\ a_1 & 1 & & & & \\ a_2 & a_1\theta & 1 & & & \\ a_3 & a_2\theta & a_1\theta^2 & 1 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ a_l & a_{l-1}\theta & a_{l-2}\theta^2 & \cdots & a_1\theta^{l-1} & 1 \end{pmatrix} \in M_{l+1}(F)$$

and

$$A_l(n, \theta)=\{u(a_1, a_2, \cdots, a_l) | a_i \in F\}.$$

The multiplication is defined as a product of two matrices as follows:

$$\begin{aligned} &u(a_1, a_2, \cdots, a_l)u(b_1, b_2, \cdots, b_l) \\ &=u(a_1+b_1,\ a_2+(a_1\theta)b_1+b_2,\ a_3+(a_2\theta)b_1+(a_1\theta^2)b_2+b_3, \\ &\quad \cdots,\ a_l+(a_{l-1}\theta)b_1+\cdots+(a_1\theta^{l-1})b_{l-1}+b_l). \end{aligned}$$

So $A_l(n, \theta)$ becomes a group of order 2^{nl}. If $l=2$, this group is isomorphic to a Suzuki 2-group $A(n, \theta)$ in [3, VIII.6.7 Example and §7].

For $1 \leq i \leq l$, we put

$$G_i=\{u(0, \cdots, 0, a_i, a_{i+1}, \cdots, a_l)\}.$$

Define $\varphi_{l,i-1}: A_l(n, \theta) \longrightarrow A_{i-1}(n, \theta)$ by $\varphi_{l,i-1}(u(a_1, \cdots, a_l))=u(a_1, \cdots, a_{i-1})$. Then $\varphi_{l,i-1}$ is an epimorphism and ker $\varphi_{l,i-1}=G_i$. Thus G_i is a normal subgroup of $A_l(n, \theta)$, $A_l(n, \theta)/G_i \cong A_{i-1}(n, \theta)$, and obviously G_l is in the center of $A_l(n, \theta)$ by the multiplication.

$A_l(n, \theta)$ has important automorphisms. Let $\lambda \in F^{\times}$. We define $\xi_\lambda: A_l(n, \theta) \rightarrow A_l(n, \theta)$ by

$$\xi_\lambda(u(a_1, a_2, \cdots, a_l))=u(\lambda_1 a_1, \lambda_2 a_2, \cdots, \lambda_l a_l)$$

where

$$\begin{aligned} \lambda_1 &= \lambda \\ \lambda_2 &= \lambda(\lambda\theta) \\ \lambda_3 &= \lambda(\lambda\theta)(\lambda\theta^2) \\ &\cdots \\ \lambda_l &= \prod_{i=0}^{l-1}(\lambda\theta^i). \end{aligned}$$

Then this is an automorphism of $A_l(n, \theta)$.

For simplify our argument, throughout this paper, we assume that θ is the Frobenius automorphism of F, $\theta: x \rightarrow x^2$. Then $\lambda_i=\lambda^{2^i-1}$. We also assume that λ is a generator of $F^{\times}$ and is fixed. Then λ_i generates $F^{\times}$ if and only if $(2^n-1, 2^i-1)=1$. But it is easy to check that $(2^n-1, 2^i-1)=1$ if and only if $(n, i)=1$. In this case, $\langle \xi_\lambda \rangle$ permutes $G_i/G_{i+1}-G_{i+1}$ transitively. If $l<n_0$, where n_0 is the smallest prime divisor of n, then this holds for any i.

Our main result is

Theorem 2.1. *Let θ be the Frobenius automorphism of* $\mathrm{GF}(2^n)$. *Assume that $l<n_0$, where n_0 is the smallest prime divisor of n. Then $A_l(n, \theta)$ satisfies B-condition.*

In particular, if n is a prime and $l<n$ then $A_l(n, \theta)$ satisfies B-condition.

The second part of this theorem is obviously holds by the first part, and the first part is proved by the next theorem.

We put

$$q_i(G)=\#\{C\in \mathrm{Cl}(G)||C|=2^i\}$$
$$r_i(G)=\#\{\chi\in \mathrm{Irr}(G)|\chi(1)=2^i\}.$$

Then B-condition holds for a 2-group G if and only if

$$q_{2i}(G)=r_i(G), \text{ for any } i\geq 0.$$

Theorem 2.2. *Put $G=A_l(n, \theta)$. Assume that θ is the Frobenius automorphism, and $l<n_0$, where n_0 is the smallest prime divisor of n. Then*

(a) $q_0(G)=2^n$, $q_{m(n-1)}(G)=2^m(2^n-1)$ *for* $1\leq m\leq l-1$, *and* $q_i(G)=0$ *for the other* $i>0$.

(b) $r_0(G)=2^n$, $r_{m(n-1)/2}(G)=2^m(2^n-1)$ *for* $1\leq m\leq l-1$, *and* $r_i(G)=0$ *for the other* $i>0$.

REMARK. If $l\geq n_0$ there exist groups which does not satisfy B-condition. For example, $A_2(2, \theta)$, $A_3(3, \theta)$, and $A_4(3, \theta)$, θ the Frobenius automorphism, do not satisfy B-condition.

It is known that $A_2(n, \theta)$ satisfies B-condition when θ is an arbitrary odd order automorphism of $\mathrm{GF}(2^n)$. For odd characteristic finite fields, we can define groups similar to $A_l(n, \theta)$, and they satisfy B-condition if $l=2$ and the order of θ is odd (This is my work and unpublished). This is a general case of ϕ_{11} in [4].

3. Conjugacy classes

In this section, we shall prove Theorem 2.2 (a). In this and later sections, we assume that $l<n_0$, where n_0 is the smallest prime divisor of n. If $l=1$ then $A_l(n, \theta)$ is abelian so we assume $l\geq 2$. Note that n is odd.

Theorem 3.1. *The following is a complete set of representatives of conjugacy classes of $A_l(n, \theta)$.*

$\{\xi_\lambda^j u(e_1, e_2, \cdots, e_l)|0\leq j<2^n-1,\ e_i=0 \text{ or } 1 \text{ and at least one } e_i=1\}\cup\{u(0, \cdots, 0)\}$

When $e_1=\cdots=e_{i-1}=0$ *and* $e_i=1$, *the order of the centralizer of* $\xi_\lambda^j u(e_1, e_2, \cdots, e_l)$ *is* 2^{ni+l-i}.

To prove this, we need two lemmas.

Lemma 3.2. *The order of the centralizer of* $u(0, \cdots, 0, e_i, \cdots, e_l)$, $e_i=1$ *and* $e_j=1$ *or* 0 *for* $j>i$, *is* 2^{ni+l-i}.

Proof. Let $u(a_1, a_2, \cdots, a_l)$ centralize $u(0, \cdots, 0, e_i, \cdots, e_l)$. Then by direct calculation (note that θ acts trivially on e_j),

$$\begin{aligned} e_i a_1(a_1^{2^i-1}+1)&=0 \\ e_i a_2(a_2^{2^i-1}+1)&=e_{i+1}a_1(a_1^{2^{i+1}-1}+1) \\ \cdots\quad&\cdots \\ e_i a_{l-i}(a_{l-i}^{2^i-1}+1)&=e_{l-1}a_1(a_1^{2^{l-1}-1}+1)+\cdots+e_{i+1}a_{l-i-1}(a_{l-i-1}^{2^{i+1}-1}+1) \end{aligned}$$

By our assumption, the map $x \to x^{2^i-1}$ is a bijection from F to F, so the first equation say that $a_1=0$ or 1. Hence the right hand side of the second equation is 0, and thus $a_2=0$ or 1. We can continue this argument until a_{l-i}. Thus the order of the centralizer of $u(0, \cdots, 0, e_i, \cdots, e_l)$ is $2^{l-i}\cdot 2^{ni}=2^{ni+l-i}$. The proof is complete. □

Let tr be the trace map from $GF(2^n)$ to $GF(2)$: $\mathrm{tr}(x)=\sum_{i=0}^{n-1} x\theta^i$. The next holds.

Lemma 3.3. $\xi_\lambda^j u(e_1, e_2, \cdots, e_l)$, $e_i=0$ *or* 1, *and* $\xi_\lambda^k u(f_1, f_2, \cdots, f_l)$, $f_i=0$ *or* 1, *are conjugate if and only if* $j=k$ *and* $e_i=f_i$, *for all* i.

Proof. Assume $\xi_\lambda^j u(e_1, e_2, \cdots, e_l)$, $e_i=0$ or 1, and $\xi_\lambda^k u(f_1, f_2, \cdots, f_l)$, $f_i=0$ or 1, are conjugate in $A_l(n, \theta)$. If $e_1=\cdots=e_{i-1}=0$ and $e_i=1$, then obviously $f_1=\cdots=f_{i-1}=0$ and $f_i=1$, and $j=k$, since G_i/G_{i+1} is in the center of G_1/G_{i+1}. So we may assume that $j=k=0$. Then there exists $u(a_1, \cdots, a_l)$ such that $u(e_1, \cdots, e_l)u(a_1, \cdots, a_l)=u(a_1, \cdots, a_l)u(f_1, \cdots, f_l)$. Obviously $e_1=f_1$. Suppose that $e_i=f_i$, for $i<m$. Then by direct calculation,

$$\begin{aligned} e_m+f_m&=e_{m-1}a_1+e_{m-2}a_2+\cdots+e_1a_{m-1} \\ &\quad+f_{m-1}(a_1\theta^{m-1})+f_{m-2}(a_2\theta^{m-2})+\cdots+f_1(a_{m-1}\theta) \\ &=e_{m-1}(a_1+a_1\theta^{m-1})+e_{m-2}(a_2+a_2\theta^{m-2})+\cdots+e_1(a_{m-1}+a_{m-1}\theta). \end{aligned}$$

The right hand side of this equation is in the kernel of tr, since $e_i=0$ or 1. But the left hand side is 0 or 1. So $e_m=f_m$. Thus the proof is complete. □

Now Theorem 3.1 is easily shown.

$$\xi_\lambda^j u(e_1, e_2, \cdots, e_l), \text{ for } 0 \leq j < 2^n - 1,\ e_i = 0 \text{ or } 1$$

are in distinct conjugacy classes each other. Consider the lengths of these classes. The sum of their lengths is

$$\begin{aligned} &1+1\cdot(2^n-1)+2^{n-1}\cdot 2(2^n-1)+2^{2(n-1)}\cdot 2^2(2^n-1) \\ &\quad +\cdots+2^{(l-1)(n-1)}\cdot 2^{l-1}(2^n-1) \\ &=2^{nl}=|A_l(n, \theta)|. \end{aligned}$$

Thus they are representatives of conjugacy classes of $A_l(n, \theta)$. Theorem 3.1 and also Theorem 2.2 (a) are proved.

4. Irreducible characters

In this section, we shall prove Theorem 2.2 (b). We need many lemmas to prove this.

We put $G=A_l(n, \theta)$. Recall that

$$G_i=\{u(0, \cdots, 0, a_i, a_{i+1}, \cdots, a_l)\}$$

and $G/G_i \cong A_{i-1}(n, \theta)$.

Lemma 4.1. *$C_G(G_i)=G_{l-i+1}$. Especially, G_i is abelian if and only if $i \geq (l+1)/2$.*

Proof. This holds by direct calculations. □

Lemma 4.2. *Let G_N be abelian, and let $\varphi \in \mathrm{Irr}(G_N)$ such that $\ker\varphi \ngeq G_l$. Then*

$$|I_G(\varphi)|=2^{nN+l-N},$$

where $I_G(\varphi)$ is the stabilizer of φ in G.

To show this, we may assume that φ_{G_l} is $u(0, \cdots, 0, a_l) \to (-1)^{\mathrm{tr}(a_l)}$ since $\mathrm{Irr}(G_l) -\{1_{G_l}\}$ is transitively permuted by $\langle \xi_\lambda \rangle$, and note that any character of G_l is invariant in G since G_l is in the center of G. This lemma will be shown later.

Let G_N be abelian. Then $\varphi \in \mathrm{Irr}(G_N)$ can be regarded as a homomorphism from G_N to $F_2=\mathrm{GF}(2)$, since G_N is an elementary abelian 2-group. Thus φ can be regarded as a sum of homomorphisms from G_i/G_{i+1} to F_2, $i=N, N+1, \cdots, l$. Note that G_i/G_{i+1} is isomorphic to $F=\mathrm{GF}(2^n)$ as an additive group.

Lemma 4.3. *Define $\Phi: F \longrightarrow \mathrm{Hom}_{F_2}(F, F_2)$ by $\phi(a)(x)=\mathrm{tr}(ax)$. Then ϕ is an isomorphism as abelian groups.*

Proof. Put $K=\ker \mathrm{tr}$. If $aK=bK$ implies $a=b$ then the proof is complete. Thus we shall show $aK=K$ implies $a=1$.

If $a\neq 1$ then a induces a permutation on K. Obviously $C_K(a)=\{0\}$, and the lengths of $\langle a\rangle$-orbits are the order of a. But by our assumption, $(|K|-1, o(a))=1$. This is a contradiction. The proof is complete. □

By this lemma, any $\varphi\in\mathrm{Irr}(G_N)$ has a form $\varphi: u(0, \cdots, 0, x_N, \cdots, x_l)\rightarrow(-1)^{\sum_{i=N}^{l}\mathrm{tr}(a_i x_i)}$ for some $a_1, \cdots, a_l\in F$, and we can denote this by $\varphi(a_N, \cdots, a_l)$.

Lemma 4.4. *Define* $\rho_{i,j}: F\times F\longrightarrow F$ *by* $\rho_{i,j}(a, b)=(a\theta^j)b+a(b\theta^i)$. *Then, for* $i>0$, $j>0$, *and* $0<i+j\leq l$, *there exits an integer* m *such that* $(2^i-1)m\equiv 2^{i+j}-1(\mathrm{mod}\ 2^n-1)$, *and*

$$\rho_{i,j}(a, F)=a^m\ker \mathrm{tr}.$$

Especially, $\rho_{i,j}(a, F)=\rho_{i,j}(b, F)$ *if and only if* $a=b$.

Proof. Since $i\leq l<n_0$, 2^i-1 is coprime to 2^n-1. Thus such m exits.

Obviously $\rho_{i,j}$ is bilinear. If $(a\theta^j)b+a(b\theta^i)=0$ then $a^{2^j}b+ab^{2^j}=0$, and so $b=0$ or $a^{(2^j-1)/(2^i-1)}$. Hence $|\rho_{i,j}(a, F)|=2^{n-1}$ for $a\neq 0$.

On the other hand,

$$\begin{aligned}(a\theta^j)b+a(b\theta^i)&=a^m(a^{2^j-m}b+(a^{2^j-m}b)^{2^i})\\ &\in a^m\ker \mathrm{tr}.\end{aligned}$$

Thus $\rho_{i,j}(a, F)=a^m\ker \mathrm{tr}$.

The last part of the lemma clearly holds by the same way as the proof of Lemma 4.3 and $(m, 2^m-1)=1$.

The proof is complete. □

In general, conjugates of elements in $A_l(n, \theta)$ is very complicated. So we prepare the easy cases.

Lemma 4.5. (a)

$$\begin{aligned}&u(0, \cdots, 0, a_i, 0, \cdots, 0)u(0, \cdots, 0, x_m, 0, \cdots, 0)u(0, \cdots, 0, a_i, 0, \cdots, 0)^{-1}\\ &=u(0, \cdots, 0, x_m, 0, \cdots, 0,(x_m\theta^i+x_m)_{m+i}, 0, \cdots, 0,(x_m\theta^{2i}+x_m)_{m+2i}, \cdots).\end{aligned}$$

(b) *When* $i+j=l$,

$$\begin{aligned}&u(0, \cdots, 0, g_i, 0, \cdots, 0)u(0, \cdots, 0, x_j, 0, \cdots, 0)u(0, \cdots, 0, g_i, 0, \cdots, 0)^{-1}\\ &=u(0, \cdots, 0, x_j, 0, \cdots, 0,(x_j\theta^i)g_i+x_j(g_i\theta^j)).\end{aligned}$$

Proof. Note that

$$u(0, \cdots, 0, g_i, 0, \cdots, 0)^{-1}$$

$$=u(0, \cdots, 0, g_i, 0, \cdots, 0, ((g_i\theta^i)g_i)_{2i}, 0, \cdots, 0, ((g_i\theta^{2i})(g_i\theta^i)g_i)_{3i}, \cdots).$$

So the results follow by direct calculations. □

We define a subgroup H of G by

$$H=\{u(a_1, a_2, \cdots, a_l) | a_i=1 \text{ or } 0\}.$$

Obviously this is a subgroup of G and abelian. H is generated by $u(0, \cdots, 0, 1_i, 0, \cdots, 0), 1 \leq i \leq l$, and has the order 2^l.

Lemma 4.6. *Assume that G_N is abelian. For $\varphi=\varphi(a_N, \cdots, a_l)$, $a_i=1$ or 0,*

$$I_G(\varphi)=HG_{l-N+1}.$$

Proof. $I_G(\varphi) \geq G_{l-N+1}$ since $C_G(G_N)=G_{l-N+1}$ holds by Lemma 4.1, and $I_G(\varphi) \geq H$ by Lemma 4.5 (a). So $I_G(\varphi) \geq HG_{l-N+1}$.

If $N=l$ the result holds obviously. Assume the result holds for $\varphi_{G_{N+1}}=\varphi(a_{N+1}, \cdots, a_l)$. Then $I_G(\varphi) \leq G_G(\varphi_{G_{N+1}})=HG_{l-N}$. Let $g \in I_G(\varphi)$. We can write $g=hg'$, $h \in H$, $g' \in G_{l-N}$. Then $g' \in I_G(\varphi)$. Since $G_{l-N+1} \leq I_G(\varphi)$, we may assume that $g'=u(0, \cdots, 0, g_{l-N}, 0, \cdots, 0)$. Consider the action of g' on $u(0, \cdots, 0, x_N, 0, \cdots, 0)$. By Lemma 4.5 (b), $(x_N\theta^{l-N})g_{l-N}+x_N(g_{l-N}\theta^N)$ must be in ker tr for any $x_N \in F$. But by Lemma 4.4,

$$\{(x_N\theta^{l-N})g_{l-N}+x_N(g_{l-N}\theta^N) | x_N \in F\}=\rho_{l-N,N}(g_{l-N}, F)=g_{l-N}^m \ker \operatorname{tr},$$

where $(2^N-1)m \equiv 2^l-1 (\operatorname{mod} 2^n-1)$. Thus if $g_{l-N} \neq 0$, g_{l-N} must be 1 by Lemma 4.4 and $\rho_{l-N,N}(1, F)=\ker \operatorname{tr}$. So $g_{l-N}=1$ or 0. Now $I_G(\varphi) \leq HG_{l-N+1}$ and the result follows. □

Lemma 4.7. *Assume that G_N is abelian. If $\varphi(a_N, \cdots, a_l)^g=\varphi(b_N, \cdots, b_l)$, $a_i=1$ or 0, $b_i=1$ or 0, then $a_i=b_i$ for all i.*

Proof. Suppose that $a_m \neq b_m$ and $a_i=b_i$ for all $i>m$. We may assume that $a_m=1$ and $b_m=0$. $g=u(g_1, \cdots, g_l)$ fixes $\varphi(a_{m+1}, \cdots, a_l)$ so $g \in HG_{l-m}$ by Lemma 4.6. Consider the action of g on $\varphi(a_m, \cdots, a_l)$. Since HG_{l-m+1} stabilizes $\varphi(a_m, \cdots, a_l)$, we may assume that $g=u(0, \cdots, 0, g_{l-m}, 0, \cdots, 0)$. Then

$$\begin{aligned}
&\varphi(a_m, \cdots, a_l)^g(u(0, \cdots, 0, a_m, 0, \cdots, 0)) \\
&=\varphi(a_m, \cdots, a_l)(u(0, \cdots, 0, a_m, 0, \cdots, 0)^{g^{-1}}) \\
&=\varphi(a_m, \cdots, a_l)(u(0, \cdots, 0, a_m, 0, \cdots, 0, g_{l-m}\theta^m+g_{l-m})) \\
&=1
\end{aligned}$$

by Lemma 4.5 (b). But $\varphi(b_m, \cdots, b_l)(u(0, \cdots, 0, 1_m, 0, \cdots, 0))=0$. This is a contradiction, and the proof is complete. □

Now we can prove Lemma 4.2.

Proof of Lemma 4.2. Let G_N be abelian. Then the number of irreducible characters of G_N whose kernels do not contain G_l is $2^{(l-N)n}(2^n-1)$. Lemma 4.6 says that $\varphi(a_N, \cdots, a_l)$, $a_i=1$ or 0, are in distinct G-orbits, and Lemma 4.6 says that their orbits have lengths $2^{(l-N)(n-1)}$. Also $\varphi(a_N, \cdots, a_l)^{\varepsilon i}$ are in distinct G-orbits. Thus they are complete representatives of G-orbits. Now the result follows from Lemma 4.6. □

Using Lemma 4.2, we can prove Theorem 2.2 (b) by induction on l. We separate the cases l as odd from l as even.

Lemma 4.8. *Assume that Theorem* 2.2 *holds for* $A_{l-1}(n, \theta)$ *and* l *is odd. Then Theorem* 2.2 *holds for* $A_l(n, \theta)$.

Proof. Put $N=(l+1)/2$, then G_N is abelian. Let χ be an irreducible character of G such that $\ker\chi \not\geq G_l$. Let φ be an irreducible character of G_N such that $(\chi_{G_N}, \varphi)\neq 0$ and $\ker\varphi \not\geq G_l$. By Lemma 4.2, $|G: I_G(\varphi)|=2^{(n-1)(l-1)/2}$. So $\chi(1) \geq 2^{(n-1)(l-1)/2}$. The number of such χ is $2^{l-1}(2^n-1)$ by the number of conjugacy classes. Conider that

$$\begin{aligned}|G|=\sum_{\chi\in\mathrm{Irr}(G)}\chi(1)^2=\sum_{\ker\chi\geq G_l}\chi(1)^2+\sum_{\ker\chi\not\geq G_l}\chi(1)^2=2^{n(l-1)}+\sum_{\ker\chi\not\geq G_l}\chi(1)^2\\ \geq 2^{n(l-1)}+2^{(n-1)(l-1)}\cdot 2^{l-1}(2^n-1)=2^{nl}=|G|.\end{aligned}$$

Thus $\chi(1)=2^{(n-1)(l-1)/2}$. The proof is complete. □

Lemma 4.9. *Assume that Theorem* 2.2 *holds for* $A_{l-1}(n, \theta)$ *and* l *is even. Then Theorem* 2.2 *holds for* $A_l(n, \theta)$.

Proof. Put $N=l/2$. Note that G_N is not abelian. Let χ be an irreducible character of G such that $\ker\chi \not\geq G_l$. Let φ be an irreducible character of G_N such that $(\chi_{G_N}, \varphi)\neq 0$ and $\ker\varphi \not\geq G_l$. G_{N+1} is in the center of G_N. So $\varphi_{G_{N+1}}$ is homogeneous. Let ψ be the homogeneous constituent of $\varphi_{G_{N+1}}$. Then

$$|I_G(\varphi)|\leq|I_G(\psi)|=2^{(l/2-1)+n(l/2+1)}.$$

Consider the structure of G_N. Put

$$\begin{aligned}A&=\{u(0, \cdots, 0, a_{N+1}, a_{N+2}, \cdots, a_{l-1}, 0)\}\\ B&=\{u(0, \cdots, 0, a_N, 0, \cdots, 0, a_l)\}.\end{aligned}$$

Then obviously $G_N=A\times B$ and A is in the center of G_N. Since φ_{G_l} is homogeneous, $|G_l\cap\ker\varphi|=2^{n-1}$. Put $K=G_l\cap\ker\varphi$. The commutator map $B/G_l\times B/G_l \longrightarrow G_l$ can be regarded as $\rho_{N,N}$ in Lemma 4.4:

$$[u(0, \cdots, a_N, 0, \cdot, 0), u(0, \cdots, b_N, 0, \cdot, 0)]=u(0, \cdots, 0, a_N(b_N\theta^N)+(a_N\theta^N)b_N).$$

Thus Lemma 4.4 says that there exists the unique non zero a_N such that $u(0, \cdots, a_N, 0, \cdots, 0)$ is in the center of B/K. Clearly $\mathrm{D}(B/K)=\Phi(B/K)=G_l/K$ and its order is 2. Thus B/K is isomorphic to a central product of an extraspecial group of order 2^n and an abelian group of order 4 (it is not so hard to check that the center of B/K is cyclic of order 4 but this is not necessary for our argument). It is well known that an irreducible character degree of an extraspecial group of order p^{2r+1} is 1 or p^r. So $\varphi(1)=2^{(n-1)/2}$. Now

$$\begin{aligned}\chi(1)&\geq|G: I_G(\varphi)|\cdot\varphi(1)\\&\geq 2^{nl-(l/2-1)-n(l/2+1)}\cdot 2^{(n-1)/2}\\&=2^{(n-1)l-1)/2}\end{aligned}$$

By the same argument as Lemma 4.8, the result follows. □

Now Theorem 2.2 (b) is proved and $A_l(n, \theta)$, $l<n_0$, satisfies B-condition.

5. Derived lengths

In this section, we consider the derived length of $A_l(n, \theta)$. The next holds.

Theomem 5.1. *If* $2^{d-1}\leq l<2^d$, *then the derivd length of* $A_l(n, \theta)$ *is* d.

This theorem is an easy consequence from the following lemma.

Lemma 5.2. $[G_i, G_j]=G_{i+j}$, *where* $G_m=1$ *if* $m>l$.

Proof. Obviously $[G_i, G_j]\leq G_{i+j}$. Suppose that $i+j\leq l$, otherwise $[G_i, G_j]\geq G_{i+j}=1$ holds. Then $[G_i, G_j]\geq G_l$ holds since G_l-1 is transitively permuted by $\langle\xi_\lambda\rangle$. Inductively $[G_i, G_j]/G_m\geq G_{m-1}/G_m$ for $i+j\leq m\leq l$. Thus $[G_i, G_j]\geq G_{i+j}$ and so $[G_i, G_j]=G_{i+j}$. □

Theorem 5.1 holds obviously from this lemma. We can also see that the nilpotency class of $A_l(n, \theta)$ is l.

Now Theorem in introduction holds from Theorem 2.1 and Theorem 5.1.

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