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<th>Title</th>
<th>A condition on lengths of conjugacy classes and character degrees</th>
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1. Introduction

In E. Bannai [1], the following condition on finite groups is investigated. Let $G$ be a finite group, $\text{Irr}(G) = \{\chi_i\}_{1 \leq i \leq k}$ be the set of all irreducible characters of $G$, and $\text{Cl}(G) = \{C_i\}_{1 \leq i \leq k}$ be the set of all conjugacy classes of $G$.

Condition. By suitable renumbering $i$,

$$\chi_i(1)^2 = |C_i|, \text{ for } i = 1, 2, \cdots, k.$$ 

We call this condition B-condition ("B" is due to E. Bannai). A few groups satisfying B-condition are known: abelian groups, Suzuki 2-groups $A(n, \theta)$ (See [3, VIII.6.7 Example and §7]), $\phi_6, \phi_{11}$ in [4], and groups isoclinic to them (For isoclinism, see [2]). In any case, they are nilpotent and their derived lengths are at most 2.

In this paper, we shall construct a family of groups satisfying B-condition. Our groups are, in a sense, generalizations of Suzuki 2-groups. By our examples, we can say that

**Theorem.** Derived lengths of groups satisfying B-condition are unbounded.

2. Construction of groups

Let $F = \text{GF}(2^n)$ be the finite field of order $2^n$, and let $\theta$ be an automorphism of $F$. We put, for a positive integer $l$ and $a_1, a_2, \cdots, a_l \in F$, 

A CONDITION ON LENGTHS OF CONJUGACY CLASSES AND CHARACTER DEGREES

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\[
\begin{pmatrix}
1 & 1 & 1 \\
\alpha_1 & \alpha_1 \theta & \alpha_1 \theta^2 \\
\alpha_2 & \alpha_2 \theta & \alpha_2 \theta^2 \\
\vdots & \vdots & \vdots \\
\alpha_l & \alpha_l \theta & \alpha_l \theta^2 \\
\end{pmatrix}
\in M_{l+1}(F)
\]

and

\[
A_l(n, \theta) = \{u(\alpha_1, \alpha_2, \ldots, \alpha_l) | \alpha_i \in F\}.
\]

The multiplication is defined as a product of two matrices as follows:

\[
u(\alpha_1, \alpha_2, \ldots, \alpha_l)u(\beta_1, \beta_2, \ldots, \beta_l) = u(\alpha_1 + \beta_1, \alpha_2 + (\alpha_1 \theta)\beta_1 + \beta_2, \ldots, \alpha_l + (\alpha_{l-1} \theta)\beta_{l-1} + (\alpha_l \theta^{-1})\beta_l).
\]

So \(A_l(n, \theta)\) becomes a group of order \(2^n\). If \(l = 2\), this group is isomorphic to a Suzuki 2-group \(A(n, \theta)\) in [3, VIII.6.7 Example and §7].

For \(1 \leq i \leq l\), we put

\[
G_i = \{u(0, \ldots, 0, \alpha_i, \alpha_{i+1}, \ldots, \alpha_l)\}.
\]

Define \(\varphi_{i,i-1}: A_l(n, \theta) \to A_{l-1}(n, \theta)\) by \(\varphi_{i,i-1}(u(\alpha_1, \ldots, \alpha_i)) = u(\alpha_1, \ldots, \alpha_{i-1})\).

Then \(\varphi_{i,i-1}\) is an epimorphism and \(\ker \varphi_{i,i-1} = G_i\). Thus \(G_i\) is a normal subgroup of \(A_l(n, \theta)\), \(A_l(n, \theta)/G_i \cong A_{l-1}(n, \theta)\), and obviously \(G_i\) is in the center of \(A_l(n, \theta)\) by the multiplication.

\(A_l(n, \theta)\) has important automorphisms. Let \(\lambda \in F^x\). We define \(\xi_\lambda: A_l(n, \theta) \to A_l(n, \theta)\) by

\[
\xi_\lambda(u(\alpha_1, \alpha_2, \ldots, \alpha_l)) = u(\lambda_1 \alpha_1, \lambda_2 \alpha_2, \ldots, \lambda_l \alpha_l)
\]

where

\[
\lambda_1 = \lambda \\
\lambda_2 = \lambda(\lambda \theta) \\
\lambda_3 = \lambda(\lambda \theta)(\lambda \theta^2) \\
\vdots \\
\lambda_l = \prod_{i=0}^{l-1}(\lambda \theta^i).
\]

Then this is an automorphism of \(A_l(n, \theta)\).

For simplify our argument, throughout this paper, we assume that \(\theta\) is the Frobenius automorphism of \(F\), \(\theta: x \to x^2\). Then \(\lambda_i = \lambda^{2^{i-1}}\). We also assume that \(\lambda\) is a generator of \(F^x\) and is fixed. Then \(\lambda_i\) generates \(F^x\) if and only if \((2^n-1, 2^i-1) = 1\). But it is easy to check that \((2^n-1, 2^i-1) = 1\) if and only if \((n, i) = 1\). In this case, \(\langle \xi_\lambda \rangle \) permutes \(G_i/G_{i+1} - G_{i+1}\) transitively. If \(l < n_0\), where \(n_0\) is the smallest prime divisor of \(n\), then this holds for any \(i\).
Our main result is

**Theorem 2.1.** Let \( \theta \) be the Frobenius automorphism of \( GF(2^n) \). Assume that \( l < n_0 \), where \( n_0 \) is the smallest prime divisor of \( n \). Then \( A_l(n, \theta) \) satisfies B-condition.

In particular, if \( n \) is a prime and \( l < n \) then \( A_l(n, \theta) \) satisfies B-condition.

The second part of this theorem is obviously holds by the first part, and the first part is proved by the next theorem.

We put

\[
q_l(G) = \#\{C \subseteq Cl(G) | |C| = 2^l\}
\]
\[
r_l(G) = \#\{\chi \in \text{Irr}(G) | \chi(1) = 2^l\}.
\]

Then B-condition holds for a 2-group \( G \) if and only if

\[
q_{2^l}(G) = r_l(G), \text{ for any } i \geq 0.
\]

**Theorem 2.2.** Put \( G = A_l(n, \theta) \). Assume that \( \theta \) is the Frobenius automorphism, and \( l < n_0 \), where \( n_0 \) is the smallest prime divisor of \( n \). Then

(a) \( q_0(G) = 2^n \), \( q_m(n-1)(G) = 2^m(2^n - 1) \) for \( 1 \leq m \leq l-1 \), and \( q_l(G) = 0 \) for the other \( i > 0 \).

(b) \( r_0(G) = 2^n \), \( r_m(n-1)(G) = 2^m(2^n - 1) \) for \( 1 \leq m \leq l-1 \), and \( r_l(G) = 0 \) for the other \( i > 0 \).

**Remark.** If \( l \geq n_0 \) there exist groups which does not satisfy B-condition. For example, \( A_2(2, \theta) \), \( A_3(3, \theta) \), and \( A_4(3, \theta) \), \( \theta \) the Frobenius automorphism, do not satisfy B-condition.

It is known that \( A_2(n, \theta) \) satisfies B-condition when \( \theta \) is an arbitrary odd order automorphism of \( GF(2^n) \). For odd characteristic finite fields, we can define groups similar to \( A_l(n, \theta) \), and they satisfy B-condition if \( l = 2 \) and the order of \( \theta \) is odd (This is my work and unpublished). This is a general case of \( \phi_{11} \) in [4].

3. **Conjugacy classes**

In this section, we shall prove Theorem 2.2 (a). In this and later sections, we assume that \( l < n_0 \), where \( n_0 \) is the smallest prime divisor of \( n \). If \( l = 1 \) then \( A_l(n, \theta) \) is abelian so we assume \( l \geq 2 \). Note that \( n \) is odd.

**Theorem 3.1.** The following is a complete set of representatives of conjugacy classes of \( A_l(n, \theta) \).

\[
\{ \xi | u(e_1, e_2, \ldots, e_l) | 0 \leq j < 2^n - 1, e_i = 0 \text{ or } 1 \text{ and at least one } e_i = 1 \} \cup \{ u(0, \ldots, 0) \}
\]
When \( e_1 = \cdots = e_{i-1} = 0 \) and \( e_i = 1 \), the order of the centralizer of \( \xi^i u(e_1, e_2, \ldots, e_i) \) is \( 2^{n+i-l} \).

To prove this, we need two lemmas.

**Lemma 3.2.** The order of the centralizer of \( u(0, \ldots, 0, e_i, \ldots, e_i) \), \( e_1 = 1 \) and \( e_j = 1 \) or \( 0 \) for \( j > i \), is \( 2^{n+i-l} \).

**Proof.** Let \( u(a_1, a_2, \ldots, a_i) \) centralize \( u(0, \ldots, 0, e_i, \ldots, e_i) \). Then by direct calculation (note that \( \theta \) acts trivially on \( e_i \)),

\[
e_i a_1 (a_i^{2^{i-1}} + 1) = 0
\]

\[
e_i a_2 (a_i^{2^{i-1}} + 1) = e_{i+1} a_1 (a_i^{2^{i-1}} + 1)
\]

\[
\vdots
\]

\[
e_i a_{i-1} (a_i^{2^{i-1}} + 1) = e_{i-1} a_1 (a_i^{2^{i-1}} + 1) + \ldots + e_{i+1} a_{i-1-1} (a_i^{2^{i-1}} + 1)
\]

By our assumption, the map \( x \mapsto x^{2^{i-1}} \) is a bijection from \( F \) to \( F \), so the first equation says that \( a_1 = 0 \) or \( 1 \). Hence the right hand side of the second equation is 0, and thus \( a_2 = 0 \) or \( 1 \). We can continue this argument until \( a_{i-1} \). Thus the order of the centralizer of \( u(0, \ldots, 0, e_i, \ldots, e_i) \) is \( 2^{n+i-l} \). The proof is complete. \( \Box \)

Let \( \text{tr} \) be the trace map from \( GF(2^n) \) to \( GF(2) \) : \( \text{tr}(x) = \sum_{i=0}^{n-1} x \theta^i \). The next holds.

**Lemma 3.3.** \( \xi^i u(e_1, e_2, \ldots, e_i), e_i = 0 \) or \( 1 \), and \( \xi^i u(f_1, f_2, \ldots, f_i), f_i = 0 \) or \( 1 \), are conjugate if and only if \( j = k \) and \( e_i = f_i \), for all \( i \).

**Proof.** Assume \( \xi^i u(e_1, e_2, \ldots, e_i), e_i = 0 \) or \( 1 \), and \( \xi^i u(f_1, f_2, \ldots, f_i), f_i = 0 \) or \( 1 \), are conjugate in \( A_i(n, \theta) \). If \( e_1 = \cdots = e_{i-1} = 0 \) and \( e_i = 1 \), then obviously \( f_1 = \cdots = f_{i-1} = 0 \) and \( f_i = 1 \), and \( j = k \), since \( G_i/G_{i+1} \) is in the center of \( G_i/G_{i+1} \). So we may assume that \( j = k = 0 \). Then there exists \( u(a_1, \ldots, a_i) \) such that \( u(e_1, \ldots, e_i) u(a_1, \ldots, a_i) = u(a_1, \ldots, a_i) u(f_1, \ldots, f_i) \). Obviously \( e_i = f_i \). Suppose that \( e_i = f_i \), for \( i < m \). Then by direct calculation,

\[
\begin{align*}
\sum \sum_{i=0}^{m-1} a_i a_i^{-i} + \sum_{i=0}^{m-1} a_i a_i^{-i} a_i + \cdots + \sum_{i=0}^{m-1} a_i a_i^{-i} a_i a_i = e_m - f_m
\end{align*}
\]

The right hand side of this equation is in the kernel of \( \text{tr} \), since \( e_i = 0 \) or \( 1 \). But the left hand side is \( 0 \) or \( 1 \). So \( e_m = f_m \). Thus the proof is complete. \( \Box \)

Now Theorem 3.1 is easily shown.
are in distinct conjugacy classes each other. Consider the lengths of these classes. The sum of their lengths is
\[1 + 1 \cdot (2^n - 1) + 2^{n-1} \cdot 2(2^n - 1) + 2^{n-2} \cdot 2^2(2^n - 1) + \ldots + 2^{(i-1)(n-1)} \cdot 2^i(2^n - 1) = 2^n = |A_l(n, \theta)|.\]
Thus they are representatives of conjugacy classes of $A_l(n, \theta)$. Theorem 3.1 and also Theorem 2.2 (a) are proved.

4. Irreducible characters

In this section, we shall prove Theorem 2.2 (b). We need many lemmas to prove this.

We put $G = A_l(n, \theta)$. Recall that
\[G_i = \{u(0, \ldots, 0, a_i, a_{i+1}, \ldots, a_l)\}\]
and $G/G_i \cong A_{l-1}(n, \theta)$.

**Lemma 4.1.** $C_G(G_i) = G_{l-i+1}$. Especially, $G_i$ is abelian if and only if $i \geq (l + 1)/2$.

**Proof.** This holds by direct calculations. \(\square\)

**Lemma 4.2.** Let $G_N$ be abelian, and let $\phi \in \text{Irr}(G_N)$ such that $\text{ker}\phi \supseteq G_i$. Then
\[|I_G(\phi)| = 2^{n+1-N},\]
where $I_G(\phi)$ is the stabilizer of $\phi$ in $G$.

To show this, we may assume that $\phi_{G_i}$ is $u(0, \ldots, 0, a_i) \mapsto (-1)^{tr(a_i)}$ since $\text{Irr}(G_i) - \{1_{G_i}\}$ is transitively permuted by $\langle \xi_i \rangle$, and note that any character of $G_i$ is invariant in $G$ since $G_i$ is in the center of $G$. This lemma will be shown later.

Let $G_N$ be abelian. Then $\phi \in \text{Irr}(G_N)$ can be regarded as a homomorphism from $G_N$ to $F_2 = \text{GF}(2)$, since $G_N$ is an elementary abelian 2-group. Thus $\phi$ can be regarded as a sum of homomorphisms from $G_i/G_{i+1}$ to $F_2$, $i = N, N + 1, \ldots, l$. Note that $G_i/G_{i+1}$ is isomorphic to $F = \text{GF}(2^n)$ as an additive group.

**Lemma 4.3.** Define $\Phi : F \longrightarrow \text{Hom}_{F_2}(F, F_2)$ by $\phi(a)(x) = \text{tr}(ax)$. Then $\phi$ is an isomorphism as abelian groups.
Proof. Put $K = \ker \text{tr}$. If $aK = bK$ implies $a = b$ then the proof is complete. Thus we shall show $aK = K$ implies $a = 1$.

If $a \neq 1$ then $a$ induces a permutation on $K$. Obviously $C_K(a) = \{0\}$, and the lengths of $\langle a \rangle$-orbits are the order of $a$. But by our assumption, $([K] - 1, o(a)) = 1$. This is a contradiction. The proof is complete. \[\square\]

By this lemma, any $\varphi \in \text{Irr}(G_N)$ has a form $\varphi : u(0, \ldots, 0, x_N, \ldots, x_l) \rightarrow (-1)^{\sum_{o(i \neq a_j)}}$ for some $a_i, \ldots, a_l \in F$, and we can denote this by $\varphi(a_N, \ldots, a_l)$.

**Lemma 4.4.** Define $\rho_{i,j} : F \times F \rightarrow F$ by $\rho_{i,j}(a, b) = (a \theta^i) b + a(b \theta^j)$. Then, for $i > 0, j > 0$, and $0 < i + j \leq l$, there exists an integer $m$ such that $(2^i - 1)m = 2^{i+j} - 1 (\text{mod} 2^n - 1)$, and

$$\rho_{i,j}(a, F) = a^m \ker \text{tr}.$$  

Especially, $\rho_{i,j}(a, F) = \rho_{i,j}(b, F)$ if and only if $a = b$.

Proof. Since $i \leq l < n_0$, $2^i - 1$ is coprime to $2^n - 1$. Thus such $m$ exists. Obviously $\rho_{i,j}$ is bilinear. If $(a \theta^i) b + a(b \theta^j) = 0$ then $a^{2^i} b + a^{2^j} = 0$, and so $b = 0$ or $a^{(2^i - 1)/(2^j - 1)}$. Hence $|\rho_{i,j}(a, F)| = 2^{n-1}$ for $a \neq 0$.

On the other hand,

$$(a \theta^i) b + a(b \theta^j) = a^m(a^{2^i - m} b + (a^{2^j - m} b)^{2^i})$$  

$\in a^m \ker \text{tr}.$

Thus $\rho_{i,j}(a, F) = a^m \ker \text{tr}$.

The last part of the lemma clearly holds by the same way as the proof of Lemma 4.3 and $(m, 2^n - 1) = 1$.

The proof is complete. \[\square\]

In general, conjugates of elements in $A_i(n, \theta)$ is very complicated. So we prepare the easy cases.

**Lemma 4.5.** (a)

$$u(0, \ldots, 0, a_i, 0, \ldots, 0)u(0, \ldots, 0, x_m, 0, \ldots, 0)u(0, \ldots, 0, a_i, 0, \ldots, 0)^{-1}$$

$$= u(0, \ldots, 0, x_m, 0, \ldots, 0)(x_m \theta^i + x_m)_{i+1}, 0, \ldots, 0, (x_m \theta^{2i} + x_m)_{m+2i}, \ldots).$$

(b) When $i + j = l$,

$$u(0, \ldots, 0, g_i, 0, \ldots, 0)u(0, \ldots, 0, x_j, 0, \ldots, 0)u(0, \ldots, 0, g_i, 0, \ldots, 0)^{-1}$$

$$= u(0, \ldots, 0, x_j, 0, \ldots, 0)(x_j \theta^i g_i + x_j(g_i \theta^j)).$$

Proof. Note that

$$u(0, \ldots, 0, g_i, 0, \ldots, 0)^{-1}$$
We define a subgroup $H$ of $G$ by

$$H = \{ u(a_1, a_2, \ldots, a_l) | a_i = 1 \text{ or } 0 \}.$$ 

Obviously this is a subgroup of $G$ and abelian. $H$ is generated by $u(0, \ldots, 0, 1, 0, \ldots, 0), 1 \leq i \leq l$, and has the order $2^l$.

**Lemma 4.6.** Assume that $G_N$ is abelian. For $\varphi = \varphi(a_N, \ldots, a_1), a_i = 1$ or $0$,

$$I_G(\varphi) = HG_{l-N+1}.$$ 

**Proof.** $I_G(\varphi) \supseteq G_{l-N+1}$ since $C_G(G_N) = G_{l-N}$ holds by Lemma 4.1, and $I_G(\varphi) \supseteq H$ by Lemma 4.5 (a). So $I_G(\varphi) \supseteq HG_{l-N+1}$.

If $N = l$ the result holds obviously. Assume the result holds for $\varphi_{G_{n+1}} = \varphi(a_{n+1}, \ldots, a_1).$ Then $I_G(\varphi) \supseteq G_G(\varphi_{G_{n+1}}) = HG_{l-N}$. Let $g \in I_G(\varphi)$. We can write $g = h^g, h \in H, g' \in G_{l-N}$. Then $g' \in I_G(\varphi)$. Since $G_{l-N+1} \subseteq I_G(\varphi)$, we may assume that $g' = u(0, \ldots, 0, g_{l-N}, 0, \ldots, 0)$. Consider the action of $g'$ on $u(0, \ldots, 0, x_N, 0, \ldots, 0)$. By Lemma 4.5 (b), $(x_N\theta_N^{l-N})g_{l-N} + x_N(g_{l-N}\theta_N)$ must be in ker tr for any $x_N \in F$. But by Lemma 4.4,

$$\{(x_N\theta_N^{l-N})g_{l-N} + x_N(g_{l-N}\theta_N) | x_N \in F\} = \rho_{l-N,N}(g_{l-N}, F) = g_{l-N}^n\text{ker tr},$$

where $(2^n-1)m \equiv 2^l-1 \text{mod } 2^n-1$. Thus if $g_{l-N} \neq 0, g_{l-N}$ must be $1$ by Lemma 4.4 and $\rho_{l-N,N}(1, F) = \text{ker tr}$. So $g_{l-N} = 1$ or $0$. Now $I_G(\varphi) \leq HG_{l-N+1}$ and the result follows. \( \square \)

**Lemma 4.7.** Assume that $G_N$ is abelian. If $\varphi(a_N, \ldots, a_i) = \varphi(b_N, \ldots, b_i), a_i = 1$ or $0, b_i = 1$ or $0$, then $a_i = b_i$ for all $i$.

**Proof.** Suppose that $a_m \neq b_m$ and $a_i = b_i$ for all $i > m$. We may assume that $a_m = 1$ and $b_m = 0$. $g = u(g_1, \ldots, g_l)$ fixes $\varphi(a_{m+1}, \ldots, a_i)$ so $g \in HG_{l-m}$ by Lemma 4.6. Consider the action of $g$ on $\varphi(a_m, \ldots, a_i)$. Since $HG_{l-m+1}$ stabilizes $\varphi(a_m, \ldots, a_i)$, we may assume that $g = u(0, \ldots, 0, g_{l-m}, 0, \ldots, 0)$. Then

$$\varphi(a_m, \ldots, a_i) = \varphi(a_m, \ldots, a_i) = \varphi(a_m, \ldots, a_i)(u(0, \ldots, 0, a_m, 0, \ldots, 0))$$

$$= \varphi(a_m, \ldots, a_i)(u(0, \ldots, 0, a_m, 0, \ldots, 0, g_{l-m}\theta^m + g_{l-m}))$$

$$= 1$$

by Lemma 4.5 (b). But $\varphi(b_m, \ldots, b_i)(u(0, \ldots, 0, 1_m, 0, \ldots, 0)) = 0$. This is a contradiction, and the proof is complete. \( \square \)
Now we can prove Lemma 4.2.

Proof of Lemma 4.2. Let $G_N$ be abelian. Then the number of irreducible characters of $G_N$ whose kernels do not contain $G_1$ is $2^{l-(n-1)}(2^n-1)$. Lemma 4.6 says that $\varphi(a_n, \ldots, a_l)$, $a_i=1$ or 0, are in distinct $G$-orbits, and Lemma 4.6 says that their orbits have lengths $2^{l-(n-1)}$. Also $\varphi(a_n, \ldots, a_l)^{G_1}$ are in distinct $G$-orbits. Thus they are complete representatives of $G$-orbits. Now the result follows from Lemma 4.6. □

Using Lemma 4.2, we can prove Theorem 2.2 (b) by induction on $l$. We separate the cases $l$ as odd from $l$ as even.

**Lemma 4.8.** Assume that Theorem 2.2 holds for $A_l(n, \theta)$ and $l$ is odd. Then Theorem 2.2 holds for $A_l(n, \theta)$.

Proof. Put $N=(l+1)/2$, then $G_N$ is abelian. Let $\chi$ be an irreducible character of $G$ such that $\ker \chi \ni G_1$. Let $\varphi$ be an irreducible character of $G_N$ such that $(\chi_{G_N}, \varphi) \neq 0$ and $\ker \varphi \ni G_1$. By Lemma 4.2, $|G : I_G(\varphi)|=2^{(n-1)(l-1)/2}$. So $\chi(1) \geq 2^{(n-1)(l-1)/2}$. The number of such $\chi$ is $2^{l-1}(2^n-1)$ by the number of conjugacy classes. Consider that

$$|G| = \sum_{\chi \in \text{Irr}(G)} \chi(1)^2 = \sum_{\chi \in \text{Irr}(G)} \chi(1)^2 + \sum_{\ker \chi \ni G_1} \chi(1)^2 = 2^n + \sum_{\ker \chi \ni G_1} \chi(1)^2 \geq 2^n + 2^{(n-1)(l-1)/2} \cdot 2^{l-1}(2^n-1) = 2^n = |G|.$$

Thus $\chi(1) = 2^{(n-1)(l-1)/2}$. The proof is complete. □

**Lemma 4.9.** Assume that Theorem 2.2 holds for $A_l(n, \theta)$ and $l$ is even. Then Theorem 2.2 holds for $A_l(n, \theta)$.

Proof. Put $N=l/2$. Note that $G_N$ is not abelian. Let $\chi$ be an irreducible character of $G$ such that $\ker \chi \ni G_1$. Let $\varphi$ be an irreducible character of $G_N$ such that $(\chi_{G_N}, \varphi) \neq 0$ and $\ker \varphi \ni G_1$. $G_{N+1}$ is in the center of $G_N$. So $\varphi_{G_{N+1}}$ is homogeneous. Let $\psi$ be the homogeneous constituent of $\varphi_{G_{N+1}}$. Then

$$|I_G(\varphi)| \leq |I_G(\psi)| = 2^{(l/2-1) + n(l/2+1)}.$$

Consider the structure of $G_N$. Put

$$A = \{u(0, \ldots, 0, a_{n+1}, a_{n+2}, \ldots, a_{l-1}, 0)\}$$

$$B = \{u(0, \ldots, 0, a_n, 0, \ldots, 0, a_l)\}.$$

Then obviously $G_N = A \times B$ and $A$ is in the center of $G_N$. Since $\varphi_{G_1}$ is homogeneous, $|G_1 \cap \ker \varphi| = 2^{n-1}$. Put $K = G_1 \cap \ker \varphi$. The commutator map $B/G_1 \rightarrow G_1$ can be regarded as $\rho_{N,N}$ in Lemma 4.4:
[u(0, \cdots, a_N, 0, \cdots, 0), u(0, \cdots, b_N, 0, \cdots, 0)] = u(0, \cdots, 0, a_N(b_N\theta^N)+(a_N\theta^N)b_N).

Thus Lemma 4.4 says that there exists the unique non zero $a_N$ such that $u(0, \cdots, a_N, 0, \cdots, 0)$ is in the center of $B/K$. Clearly $D(B/K) = \Phi(B/K) = G_1/K$ and its order is 2. Thus $B/K$ is isomorphic to a central product of an extraspecial group of order $2^n$ and an abelian group of order 4 (it is not so hard to check that the center of $B/K$ is cyclic of order 4 but this is not necessary for our argument). It is well known that an irreducible character degree of an extraspecial group of order $p^{2r+1}$ is 1 or $p^r$. So $\varphi(1) = 2^{(n-1)/2}$. Now

$$\chi(1) \geq |G : I_G(\varphi)| \cdot \varphi(1) \geq 2^{(n-1)/2} \cdot 2^{(n-1)/2} = 2^{(n-1)/2}$$

By the same argument as Lemma 4.8, the result follows. □

Now Theorem 2.2 (b) is proved and $A_l(n, \theta)$, $l < n_0$, satisfies B-condition.

5. Derived lengths

In this section, we consider the derived length of $A_l(n, \theta)$. The next holds.

Theorem 5.1. If $2^d - 1 < l < 2^d$, then the derived length of $A_l(n, \theta)$ is $d$.

This theorem is an easy consequence from the following lemma.

Lemma 5.2. $[G_i, G_j] = G_{i+j}$, where $G_m = 1$ if $m > l$.

Proof. Obviously $[G_i, G_j] \leq G_{i+j}$. Suppose that $i+j \leq l$, otherwise $[G_i, G_j] \geq G_{i+j} = 1$ holds. Then $[G_i, G_j] \geq G_i$ holds since $G_i - 1$ is transitively permuted by $\langle \xi \rangle$. Inductively $[G_i, G_j]/G_{m} \geq G_{m-1}/G_{m}$ for $i+j \leq m \leq l$. Thus $[G_i, G_j] \geq G_{i+j}$ and so $[G_i, G_j] = G_{i+j}$. □

Theorem 5.1 holds obviously from this lemma. We can also see that the nilpotency class of $A_l(n, \theta)$ is $l$.

Now Theorem in introduction holds from Theorem 2.1 and Theorem 5.1.

References


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