

Title	A condition on lengths of conjugacy classes and character degrees
Author(s)	Hanaki, Akihide
Citation	Osaka Journal of Mathematics. 33(1) P.207-P.216
Issue Date	1996
Text Version	publisher
URL	https://doi.org/10.18910/3672
DOI	10.18910/3672
rights	
Note	

Osaka University Knowledge Archive : OUKA

<https://ir.library.osaka-u.ac.jp/>

Osaka University

A CONDITION ON LENGTHS OF CONJUGACY CLASSES AND CHARACTER DEGREES

AKIHIDE HANAKI

(Received July 7, 1994)

1. Introduction

In E. Bannai [1], the following condition on finite groups is investigated.

Let G be a finite group, $\text{Irr}(G) = \{\chi_i\}_{1 \leq i \leq k}$ be the set of all irreducible characters of G , and $\text{Cl}(G) = \{C_i\}_{1 \leq i \leq k}$ be the set of all conjugacy classes of G .

Condition. By suitable renumbering i ,

$$\chi_i(1)^2 = |C_i|, \text{ for } i=1, 2, \dots, k.$$

We call this condition B-condition ("B" is due to E. Bannai). A few groups satisfying B-condition are known: abelian groups, Suzuki 2-groups $A(n, \theta)$ (See [3, VIII.6.7 Example and §7]), ϕ_6, ϕ_{11} in [4], and groups isoclinic to them (For isoclinism, see [2]). In any case, they are nilpotent and their derived lengths are at most 2.

In this paper, we shall construct a family of groups satisfying B-condition. Our groups are, in a sense, generalizations of Suzuki 2-groups. By our examples, we can say that

Theorem. *Derived lengths of groups satisfying B-condition are unbounded.*

2. Construction of groups

Let $F = \text{GF}(2^n)$ be the finite field of order 2^n , and let θ be an automorphism of F . We put, for a positive integer l and $a_1, a_2, \dots, a_l \in F$,

Our main result is

Theorem 2.1. *Let θ be the Frobenius automorphism of $\text{GF}(2^n)$. Assume that $l < n_0$, where n_0 is the smallest prime divisor of n . Then $A_l(n, \theta)$ satisfies B-condition.*

In particular, if n is a prime and $l < n$ then $A_l(n, \theta)$ satisfies B-condition.

The second part of this theorem is obviously holds by the first part, and the first part is proved by the next theorem.

We put

$$q_i(G) = \#\{C \in \text{Cl}(G) \mid |C| = 2^i\}$$

$$r_i(G) = \#\{\chi \in \text{Irr}(G) \mid \chi(1) = 2^i\}.$$

Then B-condition holds for a 2-group G if and only if

$$q_{2^i}(G) = r_i(G), \text{ for any } i \geq 0.$$

Theorem 2.2. *Put $G = A_l(n, \theta)$. Assume that θ is the Frobenius automorphism, and $l < n_0$, where n_0 is the smallest prime divisor of n . Then*

- (a) $q_0(G) = 2^n$, $q_m(n-1)(G) = 2^m(2^n - 1)$ for $1 \leq m \leq l-1$, and $q_i(G) = 0$ for the other $i > 0$.
- (b) $r_0(G) = 2^n$, $r_{m(n-1)/2}(G) = 2^m(2^n - 1)$ for $1 \leq m \leq l-1$, and $r_i(G) = 0$ for the other $i > 0$.

REMARK. If $l \geq n_0$ there exist groups which does not satisfy B-condition. For example, $A_2(2, \theta)$, $A_3(3, \theta)$, and $A_4(3, \theta)$, θ the Frobenius automorphism, do not satisfy B-condition.

It is known that $A_2(n, \theta)$ satisfies B-condition when θ is an arbitrary odd order automorphism of $\text{GF}(2^n)$. For odd characteristic finite fields, we can define groups similar to $A_l(n, \theta)$, and they satisfy B-condition if $l=2$ and the order of θ is odd (This is my work and unpublished). This is a general case of ϕ_{11} in [4].

3. Conjugacy classes

In this section, we shall prove Theorem 2.2 (a). In this and later sections, we assume that $l < n_0$, where n_0 is the smallest prime divisor of n . If $l=1$ then $A_l(n, \theta)$ is abelian so we assume $l \geq 2$. Note that n is odd.

Theorem 3.1. *The following is a complete set of representatives of conjugacy classes of $A_l(n, \theta)$.*

$$\{\xi^j u(e_1, e_2, \dots, e_l) \mid 0 \leq j < 2^n - 1, e_i = 0 \text{ or } 1 \text{ and at least one } e_i = 1\} \cup \{u(0, \dots, 0)\}$$

When $e_1 = \cdots = e_{i-1} = 0$ and $e_i = 1$, the order of the centralizer of $\xi^j u(e_1, e_2, \dots, e_i)$ is 2^{ni+l-i} .

To prove this, we need two lemmas.

Lemma 3.2. *The order of the centralizer of $u(0, \dots, 0, e_i, \dots, e_i)$, $e_i = 1$ and $e_j = 1$ or 0 for $j > i$, is 2^{ni+l-i} .*

Proof. Let $u(a_1, a_2, \dots, a_l)$ centralize $u(0, \dots, 0, e_i, \dots, e_i)$. Then by direct calculation (note that θ acts trivially on e_j),

$$\begin{aligned} e_i a_1 (a_1^{2^i-1} + 1) &= 0 \\ e_i a_2 (a_2^{2^i-1} + 1) &= e_{i+1} a_1 (a_1^{2^{i+1}-1} + 1) \\ &\quad \dots \quad \dots \\ e_i a_{l-i} (a_{l-i}^{2^i-1} + 1) &= e_{l-1} a_1 (a_1^{2^{l-1}-1} + 1) + \cdots + e_{i+1} a_{l-i-1} (a_{l-i-1}^{2^{i+1}-1} + 1) \end{aligned}$$

By our assumption, the map $x \rightarrow x^{2^i-1}$ is a bijection from F to F , so the first equation says that $a_1 = 0$ or 1 . Hence the right hand side of the second equation is 0 , and thus $a_2 = 0$ or 1 . We can continue this argument until a_{l-i} . Thus the order of the centralizer of $u(0, \dots, 0, e_i, \dots, e_i)$ is $2^{l-i} \cdot 2^{ni} = 2^{ni+l-i}$. The proof is complete. \square

Let tr be the trace map from $\text{GF}(2^n)$ to $\text{GF}(2)$: $\text{tr}(x) = \sum_{i=0}^{n-1} x\theta^i$. The next holds.

Lemma 3.3. *$\xi^j u(e_1, e_2, \dots, e_i)$, $e_i = 0$ or 1 , and $\xi^k u(f_1, f_2, \dots, f_i)$, $f_i = 0$ or 1 , are conjugate if and only if $j = k$ and $e_i = f_i$, for all i .*

Proof. Assume $\xi^j u(e_1, e_2, \dots, e_i)$, $e_i = 0$ or 1 , and $\xi^k u(f_1, f_2, \dots, f_i)$, $f_i = 0$ or 1 , are conjugate in $A_l(n, \theta)$. If $e_1 = \cdots = e_{i-1} = 0$ and $e_i = 1$, then obviously $f_1 = \cdots = f_{i-1} = 0$ and $f_i = 1$, and $j = k$, since G_i/G_{i+1} is in the center of G_1/G_{i+1} . So we may assume that $j = k = 0$. Then there exists $u(a_1, \dots, a_l)$ such that $u(e_1, \dots, e_i)u(a_1, \dots, a_l) = u(a_1, \dots, a_l)u(f_1, \dots, f_i)$. Obviously $e_1 = f_1$. Suppose that $e_i = f_i$, for $i < m$. Then by direct calculation,

$$\begin{aligned} e_m + f_m &= e_{m-1} a_1 + e_{m-2} a_2 + \cdots + e_1 a_{m-1} \\ &\quad + f_{m-1} (a_1 \theta^{m-1}) + f_{m-2} (a_2 \theta^{m-2}) + \cdots + f_1 (a_{m-1} \theta) \\ &= e_{m-1} (a_1 + a_1 \theta^{m-1}) + e_{m-2} (a_2 + a_2 \theta^{m-2}) + \cdots + e_1 (a_{m-1} + a_{m-1} \theta). \end{aligned}$$

The right hand side of this equation is in the kernel of tr , since $e_i = 0$ or 1 . But the left hand side is 0 or 1 . So $e_m = f_m$. Thus the proof is complete. \square

Now Theorem 3.1 is easily shown.

$$\xi^j u(e_1, e_2, \dots, e_l), \text{ for } 0 \leq j < 2^n - 1, e_i = 0 \text{ or } 1$$

are in distinct conjugacy classes each other. Consider the lengths of these classes. The sum of their lengths is

$$\begin{aligned} & 1 + 1 \cdot (2^n - 1) + 2^{n-1} \cdot 2(2^n - 1) + 2^{2(n-1)} \cdot 2^2(2^n - 1) \\ & + \dots + 2^{(l-1)(n-1)} \cdot 2^{l-1}(2^n - 1) \\ & = 2^{nl} = |A_l(n, \theta)|. \end{aligned}$$

Thus they are representatives of conjugacy classes of $A_l(n, \theta)$. Theorem 3.1 and also Theorem 2.2 (a) are proved.

4. Irreducible characters

In this section, we shall prove Theorem 2.2 (b). We need many lemmas to prove this.

We put $G = A_l(n, \theta)$. Recall that

$$G_i = \{u(0, \dots, 0, a_i, a_{i+1}, \dots, a_l)\}$$

and $G/G_i \cong A_{i-1}(n, \theta)$.

Lemma 4.1. $C_G(G_i) = G_{l-i+1}$. Especially, G_i is abelian if and only if $i \geq (l+1)/2$.

Proof. This holds by direct calculations. \square

Lemma 4.2. Let G_N be abelian, and let $\varphi \in \text{Irr}(G_N)$ such that $\ker \varphi \not\cong G_l$. Then

$$|I_G(\varphi)| = 2^{nN+l-N},$$

where $I_G(\varphi)$ is the stabilizer of φ in G .

To show this, we may assume that φ_{G_l} is $u(0, \dots, 0, a_l) \rightarrow (-1)^{\text{tr}(a_l)}$ since $\text{Irr}(G_l) - \{1_{G_l}\}$ is transitively permuted by $\langle \xi_\lambda \rangle$, and note that any character of G_l is invariant in G since G_l is in the center of G . This lemma will be shown later.

Let G_N be abelian. Then $\varphi \in \text{Irr}(G_N)$ can be regarded as a homomorphism from G_N to $F_2 = \text{GF}(2)$, since G_N is an elementary abelian 2-group. Thus φ can be regarded as a sum of homomorphisms from G_i/G_{i+1} to F_2 , $i = N, N+1, \dots, l$. Note that G_i/G_{i+1} is isomorphic to $F = \text{GF}(2^n)$ as an additive group.

Lemma 4.3. Define $\Phi : F \rightarrow \text{Hom}_{F_2}(F, F_2)$ by $\phi(a)(x) = \text{tr}(ax)$. Then ϕ is an isomorphism as abelian groups.

Proof. Put $K = \ker \text{tr}$. If $aK = bK$ implies $a = b$ then the proof is complete. Thus we shall show $aK = K$ implies $a = 1$.

If $a \neq 1$ then a induces a permutation on K . Obviously $C_K(a) = \{0\}$, and the lengths of $\langle a \rangle$ -orbits are the order of a . But by our assumption, $(|K| - 1, o(a)) = 1$. This is a contradiction. The proof is complete. \square

By this lemma, any $\varphi \in \text{Irr}(G_N)$ has a form $\varphi: u(0, \dots, 0, x_N, \dots, x_l) \rightarrow (-1)^{\sum_{i=1}^l \text{tr}(a_i x_i)}$ for some $a_1, \dots, a_l \in F$, and we can denote this by $\varphi(a_N, \dots, a_l)$.

Lemma 4.4. Define $\rho_{i,j}: F \times F \rightarrow F$ by $\rho_{i,j}(a, b) = (a\theta^j)b + a(b\theta^i)$. Then, for $i > 0, j > 0$, and $0 < i + j \leq l$, there exists an integer m such that $(2^i - 1)m \equiv 2^{i+j} - 1 \pmod{2^n - 1}$, and

$$\rho_{i,j}(a, F) = a^m \ker \text{tr}.$$

Epecially, $\rho_{i,j}(a, F) = \rho_{i,j}(b, F)$ if and only if $a = b$.

Proof. Since $i \leq l < n_0$, $2^i - 1$ is coprime to $2^n - 1$. Thus such m exists.

Obviously $\rho_{i,j}$ is bilinear. If $(a\theta^j)b + a(b\theta^i) = 0$ then $a^{2^j}b + ab^{2^j} = 0$, and so $b = 0$ or $a^{(2^j-1)/(2^i-1)}$. Hence $|\rho_{i,j}(a, F)| = 2^{n-1}$ for $a \neq 0$.

On the other hand,

$$\begin{aligned} (a\theta^j)b + a(b\theta^i) &= a^m(a^{2^j-m}b + (a^{2^j-m}b)^{2^i}) \\ &\in a^m \ker \text{tr}. \end{aligned}$$

Thus $\rho_{i,j}(a, F) = a^m \ker \text{tr}$.

The last part of the lemma clearly holds by the same way as the proof of Lemma 4.3 and $(m, 2^m - 1) = 1$.

The proof is complete. \square

In general, conjugates of elements in $A_l(n, \theta)$ is very complicated. So we prepare the easy cases.

Lemma 4.5. (a)

$$\begin{aligned} &u(0, \dots, 0, a_i, 0, \dots, 0)u(0, \dots, 0, x_m, 0, \dots, 0)u(0, \dots, 0, a_i, 0, \dots, 0)^{-1} \\ &= u(0, \dots, 0, x_m, 0, \dots, 0, (x_m\theta^i + x_m)_{m+i}, 0, \dots, 0, (x_m\theta^{2i} + x_m)_{m+2i}, \dots). \end{aligned}$$

(b) When $i + j = l$,

$$\begin{aligned} &u(0, \dots, 0, g_i, 0, \dots, 0)u(0, \dots, 0, x_j, 0, \dots, 0)u(0, \dots, 0, g_i, 0, \dots, 0)^{-1} \\ &= u(0, \dots, 0, x_j, 0, \dots, 0, (x_j\theta^i)g_i + x_j(g_i\theta^j)). \end{aligned}$$

Proof. Note that

$$u(0, \dots, 0, g_i, 0, \dots, 0)^{-1}$$

$$= u(0, \dots, 0, g_i, 0, \dots, 0, ((g_i\theta^i)g_i)_{2i}, 0, \dots, 0, ((g_i\theta^{2i})(g_i\theta^i)g_i)_{3i}, \dots).$$

So the results follow by direct calculations. \square

We define a subgroup H of G by

$$H = \{u(a_1, a_2, \dots, a_l) \mid a_i = 1 \text{ or } 0\}.$$

Obviously this is a subgroup of G and abelian. H is generated by $u(0, \dots, 0, 1_i, 0, \dots, 0), 1 \leq i \leq l$, and has the order 2^l .

Lemma 4.6. *Assume that G_N is abelian. For $\varphi = \varphi(a_N, \dots, a_l)$, $a_i = 1$ or 0 ,*

$$I_G(\varphi) = HG_{l-N+1}.$$

Proof. $I_G(\varphi) \geq G_{l-N+1}$ since $C_G(G_N) = G_{l-N+1}$ holds by Lemma 4.1, and $I_G(\varphi) \geq H$ by Lemma 4.5 (a). So $I_G(\varphi) \geq HG_{l-N+1}$.

If $N = l$ the result holds obviously. Assume the result holds for $\varphi_{G_{N+1}} = \varphi(a_{N+1}, \dots, a_l)$. Then $I_G(\varphi) \leq G_C(\varphi_{G_{N+1}}) = HG_{l-N}$. Let $g \in I_G(\varphi)$. We can write $g = hg'$, $h \in H$, $g' \in G_{l-N}$. Then $g' \in I_G(\varphi)$. Since $G_{l-N+1} \leq I_G(\varphi)$, we may assume that $g' = u(0, \dots, 0, g_{l-N}, 0, \dots, 0)$. Consider the action of g' on $u(0, \dots, 0, x_N, 0, \dots, 0)$. By Lemma 4.5 (b), $(x_N\theta^{l-N})g_{l-N} + x_N(g_{l-N}\theta^N)$ must be in $\ker \text{tr}$ for any $x_N \in F$. But by Lemma 4.4,

$$\{(x_N\theta^{l-N})g_{l-N} + x_N(g_{l-N}\theta^N) \mid x_N \in F\} = \rho_{l-N,N}(g_{l-N}, F) = g_{l-N}^m \ker \text{tr},$$

where $(2^N - 1)m \equiv 2^l - 1 \pmod{2^n - 1}$. Thus if $g_{l-N} \neq 0$, g_{l-N} must be 1 by Lemma 4.4 and $\rho_{l-N,N}(1, F) = \ker \text{tr}$. So $g_{l-N} = 1$ or 0 . Now $I_G(\varphi) \leq HG_{l-N+1}$ and the result follows. \square

Lemma 4.7. *Assume that G_N is abelian. If $\varphi(a_N, \dots, a_l)^g = \varphi(b_N, \dots, b_l)$, $a_i = 1$ or 0 , $b_i = 1$ or 0 , then $a_i = b_i$ for all i .*

Proof. Suppose that $a_m \neq b_m$ and $a_i = b_i$ for all $i > m$. We may assume that $a_m = 1$ and $b_m = 0$. $g = u(g_1, \dots, g_l)$ fixes $\varphi(a_{m+1}, \dots, a_l)$ so $g \in HG_{l-m}$ by Lemma 4.6. Consider the action of g on $\varphi(a_m, \dots, a_l)$. Since HG_{l-m+1} stabilizes $\varphi(a_m, \dots, a_l)$, we may assume that $g = u(0, \dots, 0, g_{l-m}, 0, \dots, 0)$. Then

$$\begin{aligned} & \varphi(a_m, \dots, a_l)^g(u(0, \dots, 0, a_m, 0, \dots, 0)) \\ &= \varphi(a_m, \dots, a_l)(u(0, \dots, 0, a_m, 0, \dots, 0)^{g^{-1}}) \\ &= \varphi(a_m, \dots, a_l)(u(0, \dots, 0, a_m, 0, \dots, 0, g_{l-m}\theta^m + g_{l-m})) \\ &= 1 \end{aligned}$$

by Lemma 4.5 (b). But $\varphi(b_m, \dots, b_l)(u(0, \dots, 0, 1_m, 0, \dots, 0)) = 0$. This is a contradiction, and the proof is complete. \square

Now we can prove Lemma 4.2.

Proof of Lemma 4.2. Let G_N be abelian. Then the number of irreducible characters of G_N whose kernels do not contain G_l is $2^{(l-N)n}(2^n-1)$. Lemma 4.6 says that $\varphi(a_N, \dots, a_l)$, $a_i=1$ or 0 , are in distinct G -orbits, and Lemma 4.6 says that their orbits have lengths $2^{(l-N)(n-1)}$. Also $\varphi(a_N, \dots, a_l)^{\varepsilon^k}$ are in distinct G -orbits. Thus they are complete representatives of G -orbits. Now the result follows from Lemma 4.6. \square

Using Lemma 4.2, we can prove Theorem 2.2 (b) by induction on l . We separate the cases l as odd from l as even.

Lemma 4.8. *Assume that Theorem 2.2 holds for $A_{l-1}(n, \theta)$ and l is odd. Then Theorem 2.2 holds for $A_l(n, \theta)$.*

Proof. Put $N=(l+1)/2$, then G_N is abelian. Let χ be an irreducible character of G such that $\ker \chi \not\supseteq G_l$. Let φ be an irreducible character of G_N such that $(\chi_{G_N}, \varphi) \neq 0$ and $\ker \varphi \not\supseteq G_l$. By Lemma 4.2, $|G : I_G(\varphi)| = 2^{(n-1)(l-1)/2}$. So $\chi(1) \geq 2^{(n-1)(l-1)/2}$. The number of such χ is $2^{l-1}(2^n-1)$ by the number of conjugacy classes. Consider that

$$\begin{aligned} |G| &= \sum_{\chi \in \text{Irr}(G)} \chi(1)^2 = \sum_{\ker \chi \supseteq G_l} \chi(1)^2 + \sum_{\ker \chi \not\supseteq G_l} \chi(1)^2 = 2^{n(l-1)} + \sum_{\ker \chi \not\supseteq G_l} \chi(1)^2 \\ &\geq 2^{n(l-1)} + 2^{(n-1)(l-1)} \cdot 2^{l-1}(2^n-1) = 2^{nl} = |G|. \end{aligned}$$

Thus $\chi(1) = 2^{(n-1)(l-1)/2}$. The proof is complete. \square

Lemma 4.9. *Assume that Theorem 2.2 holds for $A_{l-1}(n, \theta)$ and l is even. Then Theorem 2.2 holds for $A_l(n, \theta)$.*

Proof. Put $N=l/2$. Note that G_N is not abelian. Let χ be an irreducible character of G such that $\ker \chi \not\supseteq G_l$. Let φ be an irreducible character of G_N such that $(\chi_{G_N}, \varphi) \neq 0$ and $\ker \varphi \not\supseteq G_l$. G_{N+1} is in the center of G_N . So $\varphi_{G_{N+1}}$ is homogeneous. Let ψ be the homogeneous constituent of $\varphi_{G_{N+1}}$. Then

$$|I_G(\varphi)| \leq |I_G(\psi)| = 2^{(l/2-1)+n(l/2+1)}.$$

Consider the structure of G_N . Put

$$\begin{aligned} A &= \{u(0, \dots, 0, a_{N+1}, a_{N+2}, \dots, a_{l-1}, 0)\} \\ B &= \{u(0, \dots, 0, a_N, 0, \dots, 0, a_l)\}. \end{aligned}$$

Then obviously $G_N = A \times B$ and A is in the center of G_N . Since φ_{G_l} is homogeneous, $|G_l \cap \ker \varphi| = 2^{n-1}$. Put $K = G_l \cap \ker \varphi$. The commutator map $B/G_l \times B/G_l \rightarrow G_l$ can be regarded as $\rho_{N,N}$ in Lemma 4.4:

$$[u(0, \dots, a_N, 0, \cdot, 0), u(0, \dots, b_N, 0, \cdot, 0)] = u(0, \dots, 0, a_N(b_N\theta^N) + (a_N\theta^N)b_N).$$

Thus Lemma 4.4 says that there exists the unique non zero a_N such that $u(0, \dots, a_N, 0, \cdot, 0)$ is in the center of B/K . Clearly $D(B/K) = \Phi(B/K) = G_l/K$ and its order is 2. Thus B/K is isomorphic to a central product of an extraspecial group of order 2^n and an abelian group of order 4 (it is not so hard to check that the center of B/K is cyclic of order 4 but this is not necessary for our argument). It is well known that an irreducible character degree of an extraspecial group of order p^{2r+1} is 1 or p^r . So $\varphi(1) = 2^{(n-1)/2}$. Now

$$\begin{aligned} \chi(1) &\geq |G : I_G(\varphi)| \cdot \varphi(1) \\ &\geq 2^{nl - (l/2 - 1) - n(l/2 + 1)} \cdot 2^{(n-1)/2} \\ &= 2^{(n-1)l - 1/2} \end{aligned}$$

By the same argument as Lemma 4.8, the result follows. \square

Now Theorem 2.2 (b) is proved and $A_l(n, \theta)$, $l < n_0$, satisfies B-condition.

5. Derived lengths

In this section, we consider the derived length of $A_l(n, \theta)$. The next holds.

Theorem 5.1. *If $2^{d-1} \leq l < 2^d$, then the derived length of $A_l(n, \theta)$ is d .*

This theorem is an easy consequence from the following lemma.

Lemma 5.2. $[G_i, G_j] = G_{i+j}$, where $G_m = 1$ if $m > l$.

Proof. Obviously $[G_i, G_j] \leq G_{i+j}$. Suppose that $i + j \leq l$, otherwise $[G_i, G_j] \geq G_{i+j} = 1$ holds. Then $[G_i, G_j] \geq G_l$ holds since $G_l - 1$ is transitively permuted by $\langle \xi_\lambda \rangle$. Inductively $[G_i, G_j]/G_m \geq G_{m-1}/G_m$ for $i + j \leq m \leq l$. Thus $[G_i, G_j] \geq G_{i+j}$ and so $[G_i, G_j] = G_{i+j}$. \square

Theorem 5.1 holds obviously from this lemma. We can also see that the nilpotency class of $A_l(n, \theta)$ is l .

Now Theorem in introduction holds from Theorem 2.1 and Theorem 5.1.

References

- [1] E. Bannai : *Association schemes and fusion algebras (an introduction)*, J. Alg. Comb. **2** (1993) 327-344.
- [2] P. Hall : *The classification of prime-power groups*, J. Reine Angew. Math. **182** (1940), 130-141.

- [3] B. Huppert and N. Blackburn : Finite Groups II, Berlin-Heidelberg-New York 1982.
- [4] R. James : *The groups of order p^6 (p an odd prime)*, Math. Computation **43** (1980) 613-637.

Faculty of Engineering,
Yamanashi University
Takeda 4, Kofu,
400, Japan