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ON SEMI-SIMPLE ABELIAN GATEGORIES

Dedicated to Professor Keizo Asano for his 60th birthday

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Let $\mathfrak{A}$ be an abelian category. We can define an ideal $\mathfrak{C}$ in $\mathfrak{A}$ similarly to the ring (cf. [3]). Especially, Kelly has defined the Jacobson radical of $\mathfrak{A}$ in [7], and we call $\mathfrak{A}$ semi-simple if its radical is equal to zero.

In the first section of this note we shall show that $\mathfrak{A}$ is semi-simple if and only if $\mathfrak{A}$ is completely reducible under a condition that $\mathfrak{A}$ is artinian or noetherian, which is defferent from Theorem 1 in [9], and give a characterization of completely reducible C_3-abelian category (see [10], p. 82 for the definition).

In the section 2, we shall consider a C_3-abelian category. For every artinian projective object P we show that any idempotent subobject of P (see the definition in §2) contains a direct summand of P, which is a well known theorem in the case where P is equal to a ring, and show that $\mathfrak{A}$ is equivalent to the category of the right modules over an artinian ring if and only if $\mathfrak{A}$ contains a projective artinian generator, (cf. [8]).

Finally, we shall apply this argument to the case of module and show that the endomorphism ring of an artinian projective module is also artinian.

1 Semi-simple categories

Let $\mathfrak{A}$ be an additive category. We call $\mathfrak{A}$ *semi-simple*, if the ring $[A, A]$ is semi-simple in the sense of Jacobson for every object A in $\mathfrak{A}$, (cf. [7] and [9]).

Lemma 1.1 *Let $\mathfrak{A}$ be an additive semi-simple category with coproduct. If $\alpha: M\to N$ is not zero, then there exists $\beta: N\to M$ (resp. $\beta': N\to M$) such that $\beta\alpha\neq 0$ (resp. $\alpha\beta'\neq 0$). If $[M, N]=0$, $[N, M]=0$.*

Proof. We assume $[N, M]\alpha=0$. Put $P=M\oplus N$ and $R=[P, P]$. Then $R\begin{pmatrix}0 & 0\\ \alpha & 0\end{pmatrix}$ is a nilpotent left ideal, and hence $a=0$, which proves the lemma.

We call $\mathfrak{A}$ *completely reducible* if every object is a directsum of minimal objects and $\mathfrak{A}$ is called *artinian* (resp. *noetherian*) if every object in $\mathfrak{A}$ is artinian (resp. noetherian).

Theorem 1.2 *Let $\mathfrak{A}$ be an artinian or noetherian abelian category. Then A is completely reducible if and only if $\mathfrak{A}$ is semi-simple.*

Proof. If $\mathfrak{A}$ is completely reducible and artinian or noetherian, then $[M, M]$ is a semi-simple ring and hence, $\mathfrak{A}$ is semi-simple. Next, we assume that $\mathfrak{A}$ is noetherian and semi-simple. Let N be a maximal subobject of M and $\alpha: M \rightarrow M/N$ a natural epimorphism. Since $\alpha \neq 0$, there exists $\beta \in [M/N, M]$ such that $\alpha\beta \neq 0$ by (1.1). However, M/N is a minimal object and hence, $\alpha\beta$ is isomorphic and we may assume that $\alpha\beta = 1_{M/N}$. Therefore, $M = \mathrm{Im}\,\beta \oplus \mathrm{Ker}\,\alpha = \mathrm{Im}\,\beta \oplus N$ and $\mathrm{Im}\,\beta$ is minimal. Repeating this argument to N, finally we obtain $M = \sum \oplus N_i$; N_i is minimal, since M is noetherian. We assume that $\mathfrak{A}$ is artinian and semi-simple. Let N be a minimal subobject of M and α the inclusion of N into M. Then there exists $\beta \in [M, N]$ such that $\beta\alpha \neq 0$. Hence, $M = N \oplus \mathrm{Ker}\,\beta$. From the same reason as above, M is completely reducible.

We note from the above proof that every minimal subobject is a direct summand if $\mathfrak{A}$ is semi-simple.

Lemma 1.3 *Let $\mathfrak{A}$ be a semi-simple abelian category. If $[M, M]$ is a division ring for some $M \in \mathfrak{A}$, then M is minimal.*

Proof. Let N be a proper subobject of M. Then $[M, N] = 0$. Therefore, $[N, M] = 0$ by (1.1). Hence, M is minimal.

We shall give a characterization of a special completely reducible abelian category.

Theorem 1.4 *Let $\mathfrak{A}$ be a C_3-abelian category. Then the following statements are equivalent.*

1) *$\mathfrak{A}$ is completely reducible.*

2) *$[M, M]$ is a product of closed primitive rings*[1] *for every object M in $\mathfrak{A}$.*

3) *$[M, M]$ is a product of primitive rings with non-zero socle.*[2]

Furthermore, $\mathfrak{A}$ is equivalent to the category of right modules over a semi-simple artinian ring if and only if $[M, M]$ is a directsum of finite many of primitive rings with non-zero socle for every object M in $\mathfrak{A}$.

Proof. We note first that every minimal object N in a C_3-category is small, since if $N \subseteq \sum_{\alpha \in I} \oplus M_\alpha$, then $N = N \cap (\sum \oplus M_\alpha) = \bigcap_J (N \cap \sum_{\alpha_i \in J} \oplus M_{\alpha_i})$, where J runs through all finite set of I, and hence, $N \subseteq \sum_{J'} \oplus M_{\alpha i}$ for some J'.

1)$\rightarrow$2) If $\mathfrak{A}$ is completely reducible, then $M = \sum \oplus M_\alpha$, and $M_\alpha = \sum \oplus M_{\alpha\beta}$, where $M_{\alpha\beta}$'s are minimal objects such that $M_{\alpha\beta} \approx M_{\alpha\beta'}$ and $M_{\alpha\beta} \not\approx M_{\alpha'\beta'}$ if $\alpha \neq \alpha'$. Then $[M, M] = \Pi[M_\alpha, M]$. On the other hand $[M_\alpha, M_\beta] = 0$ if

1) The ring of all linear transformation of a vector space over a division ring.

2) See [6] for the definition.

$\alpha \neq \beta$. Hence, $[M, M] = \Pi[M_\alpha, M_\alpha]$. Since $M_{\alpha\beta}$ is small, we can prove by using matrices that $[M_\alpha, M_\alpha]$ is isomorphic to the ring of row finite matrices over a division ring.

2) $\to$ 3) It is clear.

3) $\to$ 1). We assume that $[M, M] = \Pi R_\alpha$ and R_α is a primitive ring with non-zero socle. Let e be a primitive idempotent in R_α, then $[eM, eM] = eR_\alpha e$ and $eR_\alpha e$ is a division ring, (cf. [6], p. 68). Hence, eM is a minimal object in M from (1.3). Conversely if N is a minimal object of M, then N is a direct summand of M from the remark after (1.2). Hence $N = fM$ for some primitive idempotent f. Since $f \in \Pi R_\alpha$, $f \in R_\alpha$ for some α. Hence, the representative class of all minimal subobjects in M is a set. Therefore, we can take the sum $S(M)$ of all minimal subobjects in M. If $M \neq S(M)$, then there exists a subobject M_1 of M such that $M_1 \supseteq S(M)$ and $M_1/S(M)$ is minimal from the above, (replace M by $M/S(M)$). Then $M_1 \to M_1/S(M)$ splits from the proof of (1.2), which is a contradiction. Hence, $M = S(M)$. It is clear that M is completely reducible. Furthermore, we assume that $[M, M]$ is a directsum of finite many of primitive rings for every $M \in \mathfrak{A}$. If there was a infinite set of non-isomorphic minimal objects M_i of $\mathfrak{A}$, then $[\sum \oplus M_i, \sum \oplus M_i]$ was a product of infinite many of division rings. Hence $\mathfrak{A}$ contains a finite set F' of minimal objects of $\mathfrak{A}$ such that every minimal object is isomorphic to some object in F'. Put $U = \sum_{i \in F'} \oplus M_i$, then U is a small generator. Since every object is projective, it is equivalent to the right modules over $R = [U, U]$ and R is artinian semi-simple.

2 Abelian category with projective generator

In the structure of an artinian ring R, the following theorem is very important:

no nilpotent one sided ideal contains a non-zero idempotent.

We consider, in this section, this property in a cocomplete abelian category $\mathfrak{A}$. Let A be a object in $\mathfrak{A}$ and $R = [A, A]$. For any subset S in R, we can define a morphism

$$\varphi\colon \sum_{\lambda \in S} \oplus A_\lambda \to A, \ \varphi \mid A_\lambda (= A) = \lambda\ .$$

We denote $\operatorname{Im}\varphi$ by SA, then it is clear that $SA = \bigcup_{\lambda \in S} \operatorname{Im}\lambda$. It is clear from the definition that $(SS')A = S(S'A)$ for any subset S, S' in R, where $S(S'A) = \bigcup_{\lambda} \operatorname{Im}(\lambda \mid S'A)$.

Furthermore, for any subobject B of A, $\mathfrak{r}_B = [A, B]$ is a right ideal in R. We call $\mathfrak{r}_B$ a *right ideal of a subobject* B. If $B = \mathfrak{r}A = \mathfrak{r}^2 A$ for some right ideal $\mathfrak{r}$ in R, then we call B *idempotent* and $\mathfrak{r}$ *quasi-idempotent*. In this case $B \supseteq \mathfrak{r}_B A \supseteq \mathfrak{r}_B^2 A \supseteq \mathfrak{r}^2 A = B$ since $\mathfrak{r}_B \supseteq \mathfrak{r}$ and hence, $B = \mathfrak{r}_B A = \mathfrak{r}_B^2 A$. If $\mathfrak{r}_B$ is nilpotent, we call B *nilpotent*.

Proposition 2.1 *Every minimal subobject B of A is either a direct summand of A or nilpotent.*

Proof. We assume $\mathfrak{r}_B^2 \neq 0$. Then there exist x, y in $\mathfrak{r}_B$ such that $xy \neq 0$. Since B is minimal, $B = xA = xyA$. We consider a morphism $x: A \to xA = B$. Since yA is minimal, $x \mid yA$ is isomorphic. Hence, $A = \operatorname{Ker} x \oplus yA = \operatorname{Ker} x \oplus B$.

DEFINITION. Let A be an object in $\mathfrak{A}$. If for any subobject B of A and the following diagram with row exact

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r] & 0 \\ & A \arrow[u, "g"'] & \end{tikzcd}$$

there exists $h: A \to A$ such that $fh = g$, then A is called *semi-projective*.

Every projective object is semi-projective.

Proposition 2.2 *Let A be an object in A. Then A is semi-projective if and only if every principal right ideal of R is an ideal of subobject, where $R = [A, A]$.*

Proof. We put $\mathfrak{r} = [A, xA]$ for $x \in R$. For $r \in \mathfrak{r}$ we have

$$\begin{tikzcd} A \arrow[r, "x"] & xA \arrow[r] & 0\,. \\ & A \arrow[u, "r"'] & \end{tikzcd}$$

If A is semi-projective, there exists y in R such that $r = xy \in xR$. Hence, xR is of a subobject. The converse is clear.

Proposition 2.3. *Let A be a semi-projective object in $\mathfrak{A}$. If A is artinian, then every non-zero quasi-idempotent right ideal in $R = [A, A]$ contains a non-zero idempotent, (every non-zero idempotent subobject contains a direct summand of A).*

Proof. Let $\mathfrak{b}$ be a quasi-idempotent right ideal and B be a minimal one among idempotent subobjects in A such that $B = \mathfrak{a}A = \mathfrak{a}^2 A$ and $\mathfrak{b} \supseteq \mathfrak{a}$; say $B = \mathfrak{a}A = \mathfrak{a}^2\ A \neq 0$. Since $\mathfrak{a}$ is not nilpotent, there exists $x \in \mathfrak{a}$ such that $x\mathfrak{a} \neq 0$. Now we take a minimal one among $x'A$, where $x' \in \mathfrak{a}$ and $x'\mathfrak{a} \neq 0$; say xA. Since $x\mathfrak{a}\mathfrak{a}A = x\mathfrak{a}A \neq 0$, there exists $y \in x\mathfrak{a} \subseteq \mathfrak{a}$ such that $y\mathfrak{a} \neq 0$ and $yA \subseteq x\mathfrak{a}A \subseteq xA$. Therefore, $yA = xA$. From the assumption and (2.2), we obtain $x = xa$ for some $a \in \mathfrak{a}$. $0 \neq x = xa^2 = \cdots = xa^n = \cdots$, and hence a is not nilpotent and $x(a - a^2) = 0$. We put $n = a - a^2$. If $n = 0$, a is idempotent. We assume $n \neq 0$. Put $\mathfrak{r} = \{z \mid \in \mathfrak{a},\ xz = 0\}$. Then $\mathfrak{a} \supseteq \mathfrak{r}$ and $\mathfrak{a}A \neq \mathfrak{r}A$. Therefore, we know from the minimality of $\mathfrak{a}A$ that $\mathfrak{r}$ is nilpotent, since $\mathfrak{r}A \supseteq \mathfrak{r}^2 A \supseteq \cdots$ and hence n is nilpotent. By using the same argumet in the case of ring, we can prove that $\mathfrak{a}$ contains a non-zero idempotent, (see [2], p. 160).

Proposition 2.4 *Let P be an artinian semi-projective object in $\mathfrak{A}$. Then $[P, P]$ is a semi-primary ring.*

Proof. Since P is artinian, P is a directsum of finite many of directly indecomposable object P_i. It is clear that P_i is also semi-projective. First we assume that P is directly indecomposable. Let $R=[P, P]$ and N the radical of R. Since P is artinian, there exists n such that $N^nP=N^{n+1}P$. Hence, N is nilpotent by (2.3). Let $\mathfrak{r}$ be a right ideal containing N. Then $\mathfrak{r}=N$ or $\mathfrak{r}^n$ is quasi-idempotent for some n. Therefore, $\mathfrak{r}$ contains an idempotent e if $\mathfrak{r}\neq N$ and hence $P=eP\oplus(1-e)P$. Which means $e=1$, since P is directly indecomposable. Therefore, R/N is a division ring, and R is semi-primary. Next, we assume $P=\sum\oplus P_i$, where P_i's are directly indecomposable and $P_i\not\approx P_j$ if $i\neq j$. We put $R_{ij}=[P_j, P_i]$ and denote the radical of R_{ii} by N_i. Then $R=(R_{ij})$. If we put

$$N=\begin{pmatrix} N_1R_{12}\cdots R_{1n} \\ R_{21}N_2\cdots R_{2n} \\ \cdots\cdots\cdots\cdots\cdots \\ R_{n1}\quad\cdots N_{nn} \end{pmatrix},$$

then by using the usual argument in the endomorphism ring of indecomposable modules (cf. [1], p. 23), we can prove that N is nilpotent, since N_i's are nilpotent and R_{ii}/N_i are division rings. In general, we assume that $P=\sum\oplus P_{ij}$, where P_{ij} are directly indecomposable and $P_{ij}\approx P_{ik'}$, $P_{ij}\not\approx P_{i'k}$ if $i\neq i'$. We put $P_0=\sum_i\oplus P_{i1}$ and $R_0=[P_0, P_0]$. Then R_0 is a basic ring of R. Hence, R is semiprimary from the second argument.

Theorem 2.5 *Let $\mathfrak{A}$ be a C_3-abelian category with projective artinian generator U. Then $\mathfrak{A}$ is equivalent to the categry of right R-modules, where $R=[U, U]$ is a right artinian ring*, (cf. [11]).

Proof. U is a semi-primary generator from (2.4). We can define a function φ of $\mathfrak{A}$ such that $M/\varphi(M)$ is completely reducible for every object M in $\mathfrak{A}$ and $\varphi^n(U)=0$ for some n by [5], Theorem 7 and Lemma 5. Since U is artinian, U is noetherian. Therefore, U is small, (cf. [10], p. 83, 1.6).

We shall consider an analogous proposition to (2.4) for noetherian objects.

Lemma 2.6 *Let P be a projective object, then every finitely generated right ideal $\mathfrak{r}$ in $R=[P, P]$ is the ideal of subobject of P, namely $\mathfrak{r}=[P, \mathfrak{r}P]$. Furthermore, if P is small, then every right ideal of R is of subobject* (cf. [11]).

Proof. We assume that $\mathfrak{r}=\sum_{i=1}^{n} x_iR$. Then, we have a diagram with row exact for any $x\in[P, \mathfrak{r}P]$

$$\begin{tikzcd} \sum_{i'}\oplus P_i \arrow[r, "\varphi"] & \mathfrak{r}P \arrow[r] & 0 \\ & P \arrow[u, "x"'] \arrow[ul, dashed, "h"] & \end{tikzcd}$$

where $P_i=P$ and $\varphi|P_i=x_i$. Since $x=\varphi h=\varphi\sum i_j p_j h=\sum x_i p_i h$, $x\in\mathfrak{r}$ since $p_i h\in R$. If P is small, we can replace $\sum_i P_i$ by $\sum_{\lambda\in\mathfrak{r}'} P$ for any right ideal $\mathfrak{r}'$.

Proposition 2.7 *Let P be a projective noetherian object in a C_3-category $\mathfrak{A}$. Then $[P, P]$ is a right noetherian ring.*

Proof. If P is noetherian, then P is small, and hence $[P, \mathfrak{r}P]=\mathfrak{r}$ for every right ideal $\mathfrak{r}$ in $R=[P, P]$ from (2.6). Therefore, R is right noetherian.

Finally we shall give an application of (2.4) for the case of modules.

Theorem 2.8 *Let R be a ring. If M is a non-zero projective and artinian right R-module, then $Hom_R(M, M)$ is a right artinian ring and M is a directsum of finite many of right principal ideals of R which is generated by an idempotent.*

Proof. First, we assume that M is directly indecomposable. Since M is R-projective, $M=M\tau(M)$ and $\tau(M)^2=\tau(M)$, where $\tau(M)$ is the trace ideal of M. We put $S=\mathrm{Hom}_R(M, M)$. Then S is a semi-primary ring with radical N_S such that S/N_S is a division ring by (2.4). We define μ: $M\otimes_R\mathrm{Hom}_R(M, R)\to S$ by setting $\mu(m\otimes f)m'=mf(m')$. If $\mathrm{Im}\,\mu\neq S$, then $\mathrm{Im}\,\mu\subseteq N_S$ and hence, $\mathrm{Im}\,\mu$ is nilpotent. For any element $s=\mu(m_1\otimes f_1)\mu(m_2\otimes f_2)\cdots\mu(m_n\otimes f_n)$ in $(\mathrm{Im}\,\mu)^n$, we have $sm=m_1f_1(m_2)\cdots f_{n-1}(m_n)f_n(m)$ for $m\in M$. Therefore, if $(\mathrm{Im}\,\mu)^m=0$, $M\tau(M)^m=0$, which is a contradiction. Hence, M is finitely generated projective R-module. Next, we put $\bar{M}=M/MN$, where N is the radical of R. Then $\bar{M}$ is $\bar{R}=R/N$-projective and $\mathrm{Hom}_{\bar{R}}(\bar{M}, \bar{M})=S/N_S=\bar{S}$. Since $\bar{M}$ is finitely generated projective $\bar{R}$-module, there exist $f_i\in\mathrm{Hom}_{\bar{R}}(\bar{M}, \bar{R})$ and $\bar{m}_i\in\bar{M}$ such that $\bar{m}=\sum\bar{m}_if_i(\bar{m})$ for every $\bar{m}\in\bar{M}$. If f_i is not monomorphic, then for any element $x\neq 0$ in $\mathrm{Ker}\,f_i$ and any $y\in\bar{M}$, $\mu(y\otimes f_i)=z\in\bar{S}$ is not monomorphic, since $zx=zf_i(x)$. However, $\bar{S}$ is a division ring and hence, $\mu(y\otimes f_i)=0$ for every $y\in\bar{M}$. This means $\bar{M}f_i(\bar{M})=0$. Therefore, there exists some f_j such that f_j is monomorphic. Hence, we may assume that $\bar{M}$ is a right ideal of $\bar{R}$. Since R is semi-simple, $\bar{M}^2\neq 0$. Hence, there exists $\bar{m}\in\bar{M}$ such that $\bar{m}\bar{M}\neq 0$. The natural homomorphism φ of $\bar{M}$ to $\bar{m}\bar{M}$ defined by $\varphi(x)=mx$ is not zero. Hence, φ is isomorphic and $\bar{M}=\bar{m}\bar{M}=\bar{m}\bar{R}$, since $\bar{S}$ is a division ring. Therefore, $M=mR$. Furthermore, M is R-projective, $M\approx Re$ for some idempotent e in R. Next, we assume $M=\sum\oplus M_i$, where M_i are all directly indecomposable. Since M is a finitely generated R-module from the above, M is small. Hence S is right artinian from the proof of (2.7)

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