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Author(s)	Harada, Manabu
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ON SEMI-SIMPLE ABELIAN CATEGORIES

Dedicated to Professor Keizo Asano for his 60th birthday

MANABU HARADA

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Let \mathfrak{A} be an abelian category. We can define an ideal \mathfrak{C} in \mathfrak{A} similarly to the ring (cf. [3]). Especially, Kelly has defined the Jacobson radical of \mathfrak{A} in [7], and we call \mathfrak{A} semi-simple if its radical is equal to zero.

In the first section of this note we shall show that \mathfrak{A} is semi-simple if and only if \mathfrak{A} is completely reducible under a condition that \mathfrak{A} is artinian or noetherian, which is different from Theorem 1 in [9], and give a characterization of completely reducible C_3 -abelian category (see [10], p. 82 for the definition).

In the section 2, we shall consider a C_3 -abelian category. For every artinian projective object P we show that any idempotent subobject of P (see the definition in §2) contains a direct summand of P , which is a well known theorem in the case where P is equal to a ring, and show that \mathfrak{A} is equivalent to the category of the right modules over an artinian ring if and only if \mathfrak{A} contains a projective artinian generator, (cf. [8]).

Finally, we shall apply this argument to the case of module and show that the endomorphism ring of an artinian projective module is also artinian.

1 Semi-simple categories

Let \mathfrak{A} be an additive category. We call \mathfrak{A} *semi-simple*, if the ring $[A, A]$ is semi-simple in the sense of Jacobson for every object A in \mathfrak{A} , (cf. [7] and [9]).

Lemma 1.1 *Let \mathfrak{A} be an additive semi-simple category with coproduct. If $\alpha: M \rightarrow N$ is not zero, then there exists $\beta: N \rightarrow M$ (resp. $\beta': N \rightarrow M$) such that $\beta\alpha \neq 0$ (resp. $\alpha\beta' \neq 0$). If $[M, N]=0$, $[N, M]=0$.*

Proof. We assume $[N, M]\alpha=0$. Put $P=M \oplus N$ and $R=[P, P]$. Then $R \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix}$ is a nilpotent left ideal, and hence $a=0$, which proves the lemma.

We call \mathfrak{A} *completely reducible* if every object is a directsum of minimal objects and \mathfrak{A} is called *artinian* (resp. *noetherian*) if every object in \mathfrak{A} is artinian (resp. noetherian).

Theorem 1.2 *Let \mathfrak{A} be an artinian or noetherian abelian category. Then \mathfrak{A} is completely reducible if and only if \mathfrak{A} is semi-simple.*

Proof. If \mathfrak{A} is completely reducible and artinian or noetherian, then $[M, M]$ is a semi-simple ring and hence, \mathfrak{A} is semi-simple. Next, we assume that \mathfrak{A} is noetherian and semi-simple. Let N be a maximal subobject of M and $\alpha: M \rightarrow M/N$ a natural epimorphism. Since $\alpha \neq 0$, there exists $\beta \in [M/N, M]$ such that $\alpha\beta \neq 0$ by (1.1). However, M/N is a minimal object and hence, $\alpha\beta$ is isomorphic and we may assume that $\alpha\beta = 1_{M/N}$. Therefore, $M = \text{Im } \beta \oplus \text{Ker } \alpha = \text{Im } \beta \oplus N$ and $\text{Im } \beta$ is minimal. Repeating this argument to N , finally we obtain $M = \sum \oplus N_i$; N_i is minimal, since M is noetherian. We assume that \mathfrak{A} is artinian and semi-simple. Let N be a minimal subobject of M and α the inclusion of N into M . Then there exists $\beta \in [M, N]$ such that $\beta\alpha \neq 0$. Hence, $M = N \oplus \text{Ker } \beta$. From the same reason as above, M is completely reducible.

We note from the above proof that every minimal subobject is a direct summand if \mathfrak{A} is semi-simple.

Lemma 1.3 *Let \mathfrak{A} be a semi-simple abelian category. If $[M, M]$ is a division ring for some $M \in \mathfrak{A}$, then M is minimal.*

Proof. Let N be a proper subobject of M . Then $[M, N] = 0$. Therefore, $[N, M] = 0$ by (1.1). Hence, M is minimal.

We shall give a characterization of a special completely reducible abelian category.

Theorem 1.4 *Let \mathfrak{A} be a C_3 -abelian category. Then the following statements are equivalent.*

- 1) \mathfrak{A} is completely reducible.
- 2) $[M, M]$ is a product of closed primitive rings¹⁾ for every object M in \mathfrak{A} .
- 3) $[M, M]$ is a product of primitive rings with non-zero socle.²⁾

Furthermore, \mathfrak{A} is equivalent to the category of right modules over a semi-simple artinian ring if and only if $[M, M]$ is a directsum of finite many of primitive rings with non-zero socle for every object M in \mathfrak{A} .

Proof. We note first that every minimal object N in a C_3 -category is small, since if $N \subseteq \sum_{\alpha_i \in I} \oplus M_{\alpha_i}$, then $N = N \cap (\sum \oplus M_{\alpha_i}) = \cap (N \cap \sum_{\alpha_i \in J} \oplus M_{\alpha_i})$, where J runs through all finite set of I , and hence, $N \subseteq \sum_{i \in J'} \oplus M_{\alpha_i}$ for some J' .

1) \rightarrow 2) If \mathfrak{A} is completely reducible, then $M = \sum \oplus M_{\alpha}$, and $M_{\alpha} = \sum \oplus M_{\alpha\beta}$, where $M_{\alpha\beta}$'s are minimal objects such that $M_{\alpha\beta} \approx M_{\alpha\beta'}$ and $M_{\alpha\beta} \approx M_{\alpha'\beta'}$ if $\alpha \neq \alpha'$. Then $[M, M] = \Pi[M_{\alpha}, M]$. On the other hand $[M_{\alpha}, M_{\beta}] = 0$ if

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- 1) The ring of all linear transformation of a vector space over a division ring.
 - 2) See [6] for the definition.

$\alpha \neq \beta$. Hence, $[M, M] = \Pi[M_\alpha, M_\alpha]$. Since $M_{\alpha\beta}$ is small, we can prove by using matrices that $[M_\alpha, M_\alpha]$ is isomorphic to the ring of row finite matrices over a division ring.

2) \rightarrow 3) It is clear.

3) \rightarrow 1). We assume that $[M, M] = \Pi R_\alpha$ and R_α is a primitive ring with non-zero socle. Let e be a primitive idempotent in R_α , then $[eM, eM] = eR_\alpha e$ and $eR_\alpha e$ is a division ring, (cf. [6], p. 68). Hence, eM is a minimal object in M from (1.3). Conversely if N is a minimal object of M , then N is a direct summand of M from the remark after (1.2). Hence $N = fM$ for some primitive idempotent f . Since $f \in \Pi R_\alpha$, $f \in R_\alpha$ for some α . Hence, the representative class of all minimal subobjects in M is a set. Therefore, we can take the sum $S(M)$ of all minimal subobjects in M . If $M \neq S(M)$, then there exists a subobject M_1 of M such that $M_1 \supseteq S(M)$ and $M_1/S(M)$ is minimal from the above, (replace M by $M/S(M)$). Then $M_1 \rightarrow M_1/S(M)$ splits from the proof of (1.2), which is a contradiction. Hence, $M = S(M)$. It is clear that M is completely reducible. Furthermore, we assume that $[M, M]$ is a directsum of finite many of primitive rings for every $M \in \mathfrak{A}$. If there was a infinite set of non-isomorphic minimal objects M_i of \mathfrak{A} , then $[\sum \oplus M_i, \sum \oplus M_i]$ was a product of infinite many of division rings. Hence \mathfrak{A} contains a finite set F' of minimal objects of \mathfrak{A} such that every minimal object is isomorphic to some object in F' . Put $U = \sum_{i \in F'} \oplus M_i$, then U is a small generator. Since every object is projective, it is equivalent to the right modules over $R = [U, U]$ and R is artinian semi-simple.

2 Abelian category with projective generator

In the structure of an artinian ring R , the following theorem is very important:

no nilpotent one sided ideal contains a non-zero idempotent.

We consider, in this section, this property in a cocomplete abelian category \mathfrak{A} . Let A be a object in \mathfrak{A} and $R = [A, A]$. For any subset S in R , we can define a morphism

$$\varphi: \sum_{\lambda \in S} \oplus A_\lambda \rightarrow A, \varphi|_{A_\lambda} (= A) = \lambda.$$

We denote $\text{Im } \varphi$ by SA , then it is clear that $SA = \bigcup_{\lambda \in S} \text{Im } \lambda$. It is clear from the definition that $(SS')A = S(S'A)$ for any subset S, S' in R , where $S(S'A) = \bigcup_{\lambda} \text{Im } (\lambda|_{S'A})$.

Furthermore, for any subobject B of A , $r_B = [A, B]$ is a right ideal in R . We call r_B a *right ideal of a subobject B* . If $B = rA = r^2A$ for some right ideal r in R , then we call B *idempotent* and r *quasi-idempotent*. In this case $B \supseteq r_B A \supseteq r_B^2 A \supseteq r^2 A = B$ since $r_B \supseteq r$ and hence, $B = r_B A = r_B^2 A$. If r_B is nilpotent, we call B *nilpotent*.

Proposition 2.1 *Every minimal subobject B of A is either a direct summand of A or nilpotent.*

Proof. We assume $\mathfrak{r}_B^2 \neq 0$. Then there exist x, y in \mathfrak{r}_B such that $xy \neq 0$. Since B is minimal, $B = xA = xyA$. We consider a morphism $x: A \rightarrow xA = B$. Since yA is minimal, $x|_{yA}$ is isomorphic. Hence, $A = \text{Ker } x \oplus yA = \text{Ker } x \oplus B$.

DEFINITION. Let A be an object in \mathfrak{A} . If for any subobject B of A and the following diagram with row exact

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \longrightarrow & 0 \\ & & \uparrow g & & \\ & & A & & \end{array}$$

there exists $h: A \rightarrow A$ such that $fh = g$, then A is called *semi-projective*.

Every projective object is semi-projective.

Proposition 2.2 *Let A be an object in \mathfrak{A} . Then A is semi-projective if and only if every principal right ideal of R is an ideal of subobject, where $R = [A, A]$.*

Proof. We put $\mathfrak{r} = [A, xA]$ for $x \in R$. For $r \in \mathfrak{r}$ we have

$$\begin{array}{ccccc} A & \xrightarrow{x} & xA & \longrightarrow & 0 \\ & & \uparrow r & & \\ & & A & & \end{array}$$

If A is semi-projective, there exists y in R such that $r = xy \in xR$. Hence, xR is of a subobject. The converse is clear.

Proposition 2.3. *Let A be a semi-projective object in \mathfrak{A} . If A is artinian, then every non-zero quasi-idempotent right ideal in $R = [A, A]$ contains a non-zero idempotent, (every non-zero idempotent subobject contains a direct summand of A).*

Proof. Let \mathfrak{b} be a quasi-idempotent right ideal and B be a minimal one among idempotent subobjects in A such that $B = \mathfrak{a}A = \mathfrak{a}^2A$ and $\mathfrak{b} \supseteq \mathfrak{a}$; say $B = \mathfrak{a}A = \mathfrak{a}^2A \neq 0$. Since \mathfrak{a} is not nilpotent, there exists $x \in \mathfrak{a}$ such that $x\mathfrak{a} \neq 0$. Now we take a minimal one among $x'A$, where $x' \in \mathfrak{a}$ and $x'\mathfrak{a} \neq 0$; say $x'A$. Since $xx\mathfrak{a}A = x\mathfrak{a}A \neq 0$, there exists $y \in x\mathfrak{a} \subseteq \mathfrak{a}$ such that $y\mathfrak{a} \neq 0$ and $yA \subseteq x\mathfrak{a}A \subseteq xA$. Therefore, $yA = xA$. From the assumption and (2.2), we obtain $x = xa$ for some $a \in \mathfrak{a}$. $0 \neq x = xa^2 = \dots = xa^n = \dots$, and hence a is not nilpotent and $x(a - a^2) = 0$. We put $n = a - a^2$. If $n = 0$, a is idempotent. We assume $n \neq 0$. Put $\mathfrak{r} = \{z \in \mathfrak{a}, xz = 0\}$. Then $\mathfrak{a} \supseteq \mathfrak{r}$ and $\mathfrak{a}A \neq \mathfrak{r}A$. Therefore, we know from the minimality of $\mathfrak{a}A$ that \mathfrak{r} is nilpotent, since $\mathfrak{r}A \supseteq \mathfrak{r}^2A \supseteq \dots$ and hence n is

nilpotent. By using the same argument in the case of ring, we can prove that \mathfrak{a} contains a non-zero idempotent, (see [2], p. 160).

Proposition 2.4 *Let P be an artinian semi-projective object in \mathfrak{A} . Then $[P, P]$ is a semi-primary ring.*

Proof. Since P is artinian, P is a directsum of finite many of directly indecomposable object P_i . It is clear that P_i is also semi-projective. First we assume that P is directly indecomposable. Let $R=[P, P]$ and N the radical of R . Since P is artinian, there exists n such that $N^n P = N^{n+1} P$. Hence, N is nilpotent by (2.3). Let \mathfrak{r} be a right ideal containing N . Then $\mathfrak{r} = N$ or \mathfrak{r}^n is quasi-idempotent for some n . Therefore, \mathfrak{r} contains an idempotent e if $\mathfrak{r} \neq N$ and hence $P = eP \oplus (1-e)P$. Which means $e=1$, since P is directly indecomposable. Therefore, R/N is a division ring, and R is semi-primary. Next, we assume $P = \sum \oplus P_i$, where P_i 's are directly indecomposable and $P_i \approx P_j$ if $i \neq j$. We put $R_{ij} = [P_j, P_i]$ and denote the radical of R_{ii} by N_i . Then $R = (R_{ij})$. If we put

$$N = \begin{pmatrix} N_1 R_{12} & \cdots & R_{1n} \\ R_{21} N_2 & \cdots & R_{2n} \\ \cdots & \cdots & \cdots \\ R_{n1} & \cdots & N_{nn} \end{pmatrix},$$

then by using the usual argument in the endomorphism ring of indecomposable modules (cf. [1], p. 23), we can prove that N is nilpotent, since N_i 's are nilpotent and R_{ii}/N_i are division rings. In general, we assume that $P = \sum \oplus P_{ij}$, where P_{ij} are directly indecomposable and $P_{ij} \approx P_{ik'}$, $P_{ij} \approx P_{i'k}$ if $i \neq i'$. We put $P_0 = \sum_i \oplus P_{ii}$ and $R_0 = [P_0, P_0]$. Then R_0 is a basic ring of R . Hence, R is semiprimary from the second argument.

Theorem 2.5 *Let \mathfrak{A} be a C_3 -abelian category with projective artinian generator U . Then \mathfrak{A} is equivalent to the category of right R -modules, where $R = [U, U]$ is a right artinian ring, (cf. [11]).*

Proof. U is a semi-primary generator from (2.4). We can define a function φ of \mathfrak{A} such that $M/\varphi(M)$ is completely reducible for every object M in \mathfrak{A} and $\varphi^n(U) = 0$ for some n by [5], Theorem 7 and Lemma 5. Since U is artinian, U is noetherian. Therefore, U is small, (cf. [10], p. 83, 1.6).

We shall consider an analogous proposition to (2.4) for noetherian objects.

Lemma 2.6 *Let P be a projective object, then every finitely generated right ideal \mathfrak{r} in $R = [P, P]$ is the ideal of subobject of P , namely $\mathfrak{r} = [P, \mathfrak{r}P]$. Furthermore, if P is small, then every right ideal of R is of subobject (cf. [11]).*

Proof. We assume that $\mathfrak{r} = \sum_{i=1}^n x_i R$. Then, we have a diagram with row

exact for any $x \in [P, \mathfrak{r}P]$

$$\begin{array}{c} \sum_{i'} \oplus P_i \xrightarrow{\varphi} \mathfrak{r}P \longrightarrow 0 \\ \quad \quad \quad \nwarrow h \quad \quad \quad \uparrow x \\ \quad \quad \quad P \end{array}$$

where $P_i = P$ and $\varphi|P_i = x_i$. Since $x = \varphi h = \varphi \sum i_j p_j h = \sum x_i p_i h$, $x \in \mathfrak{r}$ since $p_i h \in R$. If P is small, we can replace $\sum_i P_i$ by $\sum_{\lambda \in \mathfrak{r}'} P$ for any right ideal \mathfrak{r}' .

Proposition 2.7 *Let P be a projective noetherian object in a C_3 -category \mathfrak{A} . Then $[P, P]$ is a right noetherian ring.*

Proof. If P is noetherian, then P is small, and hence $[P, \mathfrak{r}P] = \mathfrak{r}$ for every right ideal \mathfrak{r} in $R = [P, P]$ from (2.6). Therefore, R is right noetherian.

Finally we shall give an application of (2.4) for the case of modules.

Theorem 2.8 *Let R be a ring. If M is a non-zero projective and artinian right R -module, then $\text{Hom}_R(M, M)$ is a right artinian ring and M is a directsum of finite many of right principal ideals of R which is generated by an idempotent.*

Proof. First, we assume that M is directly indecomposable. Since M is R -projective, $M = M\tau(M)$ and $\tau(M)^2 = \tau(M)$, where $\tau(M)$ is the trace ideal of M . We put $S = \text{Hom}_R(M, M)$. Then S is a semi-primary ring with radical N_S such that S/N_S is a division ring by (2.4). We define $\mu: M \otimes_R \text{Hom}_R(M, R) \rightarrow S$ by setting $\mu(m \otimes f)m' = mf(m')$. If $\text{Im } \mu \neq S$, then $\text{Im } \mu \subseteq N_S$ and hence, $\text{Im } \mu$ is nilpotent. For any element $s = \mu(m_1 \otimes f_1)\mu(m_2 \otimes f_2) \cdots \mu(m_n \otimes f_n)$ in $(\text{Im } \mu)^n$, we have $sm = m_1 f_1(m_2) \cdots f_{n-1}(m_n) f_n(m)$ for $m \in M$. Therefore, if $(\text{Im } \mu)^n = 0$, $M\tau(M)^n = 0$, which is a contradiction. Hence, M is finitely generated projective R -module. Next, we put $\bar{M} = M/MN$, where N is the radical of R . Then \bar{M} is $\bar{R} = R/N$ -projective and $\text{Hom}_{\bar{R}}(\bar{M}, \bar{M}) = S/N_S = \bar{S}$. Since \bar{M} is finitely generated projective \bar{R} -module, there exist $f_i \in \text{Hom}_{\bar{R}}(\bar{M}, \bar{R})$ and $\bar{m}_i \in \bar{M}$ such that $\bar{m} = \sum \bar{m}_i f_i(\bar{m})$ for every $\bar{m} \in \bar{M}$. If f_i is not monomorphic, then for any element $x \neq 0$ in $\text{Ker } f_i$ and any $y \in \bar{M}$, $\mu(y \otimes f_i) = z \in \bar{S}$ is not monomorphic, since $zx = zf_i(x)$. However, \bar{S} is a division ring and hence, $\mu(y \otimes f_i) = 0$ for every $y \in \bar{M}$. This means $\bar{M}f_i(\bar{M}) = 0$. Therefore, there exists some f_j such that f_j is monomorphic. Hence, we may assume that \bar{M} is a right ideal of \bar{R} . Since R is semi-simple, $\bar{M}^2 \neq 0$. Hence, there exists $\bar{m} \in \bar{M}$ such that $\bar{m}\bar{M} \neq 0$. The natural homomorphism φ of \bar{M} to $\bar{m}\bar{M}$ defined by $\varphi(x) = mx$ is not zero. Hence, φ is isomorphic and $\bar{M} = \bar{m}\bar{M} = \bar{m}\bar{R}$, since \bar{S} is a division ring. Therefore, $M = mR$. Furthermore, M is R -projective, $M \approx Re$ for some idempotent e in R . Next, we assume $M = \sum \oplus M_i$, where M_i are

all directly indecomposable. Since M is a finitely generated R -module from the above, M is small. Hence S is right artinian from the proof of (2.7)

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