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ON MULTIPLY TRANSITIVE GROUPS II

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Let G be a 4-fold transitive group on $\Omega = \{1, 2, \dots, n\}$ and $H = G_{1,2,3,4}$ the subgroup of G consisting of all the elements leaving the four letters 1, 2, 3 and 4 invariant.

Jordan ([2]) proved in 1872 that if $H=1$ then G must be one of the following groups: S_4 , S_5 , A_6 or M_{11} , where M_{11} is the Mathieu group of degree 11. The Jordan's theorem was extended by M. Hall ([1], Theorem 5.8.1) in the following way: If H is of odd order then G must be one of the following groups: S_4 , S_5 , A_6 , A_7 or M_{11} .

In this paper, we shall treat the case in which H is of even order and prove the following two theorems.

Theorem 1. *Let G and H be as above, P a Sylow 2-subgroup of H and let $\Delta = \{1, 2, 3, 4, \dots\}$ be the totality of the letters left invariant by P . If H is of even order, P is an elementary abelian group and transitive on $\Omega - \Delta$, then G must be one of the following groups: S_6 , A_8 or M_{23} .*

Theorem 2. *Let G , H , P and Δ be as in Theorem 1. If H is of even order, P is a normal subgroup of H and transitive on $\Omega - \Delta$, then G must be one of the following groups: S_6 , A_8 , M_{12} or M_{23} .*

In the proofs of these theorems a central involution of P is important. From the theorem of M. Hall it follows that the number of letters in Δ is not greater than 11. On the other hand, by the transitivity of P on $\Omega - \Delta$, the set of all letters left invariant by a central involution of P coincides with Δ . This shows that a central involution of P leaves only small number of letters invariant. This property of a central involution is essential in the following arguments.

DEFINITIONS AND NOTATIONS. A permutation a is called *semi-regular* if there is no letter left invariant by a . A permutation group S is called *regular* if S is transitive and each element of S , which is different from 1, is semi-regular.

For a set X let $|X|$ denote the number of elements of X . For a set S of permutations on Ω , the totality of the letters left invariant by

S is denoted by $I(S)$. If a subset Δ of Ω is a fixed block of S , i.e. if $\Delta^S = \Delta$, then the restriction of S on Δ is a set of permutations on Δ . We denote it by S^Δ . If S is a permutation group then we have a natural homomorphism $S \rightarrow S^\Delta$. The kernel of this homomorphism consists of all the elements of S leaving each letter in Δ invariant. We shall denote it by S_Δ .

1. Proof of Theorem 1

In this and the next sections, we assume that G is a 4-fold transitive group on $\Omega = \{1, 2, \dots, n\}$ and $H = G_{1,2,3,4}$ is of even order. We denote a Sylow 2-subgroup of H by P , $I(P)$ by Δ , and the normalizer of P in G by N .

Now it is known that a 4-fold transitive group of degree less than 35 is one of the four Mathieu groups $M_{11}, M_{12}, M_{23}, M_{24}$ or a symmetric or alternating group ([1], p.80). Therefore our theorems are true for $n < 35$. (For the Mathieu groups, see [5].) In the following, we assume that

(*) n is not less than 35

and we shall show that there are no groups satisfying the assumptions of our theorems.

The following lemma is an easy consequence of the theorem of M. Hall.

Lemma 1. *The restriction N^Δ of N on Δ is one of the following groups: S_4, S_5, A_6, A_7 or M_{11} .*

Proof. By a theorem of Witt ([5]; [1], Theorem 5.7.1) N^Δ is 4-fold transitive. Since $(N^\Delta)_{1,2,3,4} \cong (N \cap H)/N_\Delta$ and $N_\Delta \supset P$, $(N^\Delta)_{1,2,3,4}$ is of odd order. Then the lemma follows from the theorem of M. Hall.

Lemma 2. *An element c of the center of P , which is different from 1, is semi-regular on $\Gamma = \Omega - \Delta$, i.e. $I(c) = \Delta$ if P is transitive on Γ .*

Proof. Let c be a central element of P . Clearly $\Delta \subset I(c)$. Since $\Gamma \cap I(c)$ is a fixed block of P , $\Gamma \cap I(c) = \phi$ or Γ . If $\Gamma \cap I(c) = \Gamma$ then $c = 1$. Therefore $\Gamma \cap I(c) = \phi$, i.e. $I(c) \subset \Omega - \Gamma = \Delta$. Thus we have $\Delta = I(c)$ and c is semi-regular on Γ .

Proof of Theorem 1. We assume that P is an elementary abelian group and transitive on $\Gamma = \Omega - \Delta$. Then by Lemma 2 P is regular on Γ .

Since G is 4-fold transitive, there is an involution

$$a = (1)(2)(3, 4) \dots$$

in G which is conjugate to an involution of P . Then $|I(a)| = |\Delta|$. An involutive automorphism α of H is induced by a and then α fixes at least one Sylow 2-subgroup of H . We may assume that α fixes P . Let P_0 be the subgroup of P consisting of all the elements left invariant by α . Since $|I(a)| = |\Delta|$ and a takes the letter 3 in Δ to the letter 4, there is a letter i in $\Gamma \cap I(a)$. By the regularity of P on Γ , for any j in Γ there exists only one element b in P such that $b = (i, j) \dots$ and then $j \in I(a)$ if and only if $b \in P_0$. Therefore $|P_0| = |\Gamma \cap I(a)| \leq |I(a)| - 2 = |\Delta| - 2$. By Lemma in [3] we have $|P| \leq |P_0|^2 \leq (|\Delta| - 2)^2$. Now by Lemma 1 $|\Delta|$ is one of the following number: 4, 5, 6, 7 or 11. If $|\Delta| \leq 7$ then $|P| = |\Gamma| \leq 25$. Therefore $n = |\Delta| + |\Gamma| \leq 32$. This contradicts the assumption (*). Therefore we may assume $|\Delta| = 11$. Then $|\Gamma| = |P| \leq 9^2 = 81$. Since $|P|$ is a power of 2, $|\Gamma| = 64, 32, 16$ or 8. For $|\Gamma| = 16$ or 8, $n < 35$. Therefore we may treat the two cases in which $|\Gamma| = 64$, $n = 75$ and $|\Gamma| = 32$, $n = 43$.

Since $|N^\Delta|$ is divisible by 11 and $n < 11^2$ in our cases, there is an element x of order 11 in N such that x^Δ is a cycle of length 11. If x is a cycle of length 11, then by a theorem of Jordan ([4]) G must be a symmetric or alternating group. Therefore by the assumption of the theorem G must be S_6 or A_6 . This contradicts the assumption (*). Thus x must have at least two cycles of length 11. Let σ be the automorphism of P induced by x and let P^* be the subgroup of P consisting of all the elements left invariant by σ . In our cases $n \not\equiv 0 \pmod{11}$, therefore $I(x) \neq \phi$. Now, since $I(x) \subset \Gamma$ and P is regular on Γ , we have $|I(x)| = |P^*|$ by the same argument as is used above for α . Therefore $|I(x)|$ is a power of 2 and n must be of the form $11k + 2^l$ with $k \geq 2$. But this is impossible for $n = 75$ or 43.

2. Proof of Theorem 2

Let G, H, P, N and Δ be as in §1 and assume (*).

We assume here that P is a normal subgroup of H and transitive on $\Gamma = \Omega - \Delta$. By Lemma 1 we have one of the following cases:

CASE I. $N^\Delta = M_{11}, |\Delta| = 11,$

CASE II. $N^\Delta = A_7, |\Delta| = 7,$

CASE III. $N^\Delta = A_6, |\Delta| = 6,$

CASE IV. $N^\Delta = S_5, |\Delta| = 5,$

CASE V. $N^\Delta = S_4, |\Delta| = 4.$

We now treat these cases separately.

CASE I. $N^\Delta = M_{11}$, $|\Delta| = 11$.

Since $N_{1,2,3,4} = N \cap H = H$ and $(N^\Delta)_{1,2,3,4} = (N_{1,2,3,4})^\Delta = 1$, we have $I(H) = \Delta$.

Now, since G is 4-fold transitive, there is an involution $a = (1, 2)(3, 4)\cdots$ which is conjugate to a central involution of P . Then $|I(a)| = 11$ by Lemma 2, $a \in N$ and a^Δ is an involution of M_{11} . Therefore $|I(a^\Delta)| = |I(a) \cap I(H)| = 3$ (see [5]). For any subset $\{i_1, i_2\}$ of $I(a)$ consisting of two letters, $G_{1,2,i_1,i_2}$ is normalized by a . From the same reason as above we have $|I(a) \cap I(G_{1,2,i_1,i_2})| = 3$. Let $I(a) \cap I(G_{1,2,i_1,i_2}) = \{i_1, i_2, i_3\}$. Then $G_{1,2,i_1,i_2} = G_{1,2,i_1,i_3} = G_{1,2,i_2,i_3}$. Now consider the mapping $\varphi: \{i_1, i_2\} \rightarrow G_{1,2,i_1,i_2}$ from the set of all the subsets of $I(a)$ consisting of two letters into the set of subgroups of G . Then each $\varphi^{-1}(G_{1,2,i_1,i_2})$ consists of three subsets, since if $G_{1,2,i_1,i_2} = G_{1,2,j_1,j_2}$ for some $\{j_1, j_2\} \subset I(a)$ then $\{j_1, j_2\} \subset I(a) \cap I(G_{1,2,i_1,i_2}) = \{i_1, i_2, i_3\}$. Therefore we have ${}_{11}C_2 = 55 \equiv 0 \pmod{3}$. This is a contradiction.

CASE II. $N^\Delta = A_7$, $|\Delta| = 7$.

Let $a = (1, 2)\cdots$ be an involution which is conjugate to a central involution of P . Then $|I(a)| = 7$.

For a subset $\{i_1, i_2\}$ of $I(a)$ consisting of two letters, let P' be a Sylow 2-subgroup of $G_{1,2,i_1,i_2}$ and let $\Delta' = I(P')$. Then a normalizes P' and $a^{\Delta'}$ is an even permutation. Therefore we have

$$a = (1, 2)(i_1)(i_2)(i_3)(k, l)\cdots,$$

where $\Delta' = \{1, 2, i_1, i_2, i_3, k, l\}$. P' is then the common Sylow 2-subgroup of $G_{1,2,i_\mu,i_\nu}$ and $G_{1,2,k,l}$. Thus a subset $\{i_1, i_2\}$ of $I(a)$ determines uniquely a 2-cycle (k, l) of a . If a subset $\{j_1, j_2\}$ of $I(a)$ determines the same 2-cycle (k, l) , then the Sylow 2-subgroup P' of $G_{1,2,k,l}$ is contained in $G_{1,2,j_1,j_2}$. Therefore $\{j_1, j_2\} \subset I(P') \cap I(a) = \{i_1, i_2, i_3\}$. Thus just three subsets $\{i_\mu, i_\nu\}$ of $I(P') \cap I(a)$ determine the same 2-cycle (k, l) of a .

Now suppose that a 2-cycle (k, l) of a other than $(1, 2)$ is given. Let P'' be a Sylow 2-subgroup of $G_{1,2,k,l}$ and let $\Delta'' = I(P'') = \{1, 2, i_1, i_2, i_3, k, l\}$. Then a normalizes $G_{1,2,k,l}$ and $a^{\Delta''}$ is an even permutation. Therefore $\{i_1, i_2, i_3\} \subset I(a)$. Since $P'' \subset G_{1,2,i_1,i_2}$, $\{i_1, i_2\}$ determines (k, l) in the above sense.

Thus we have that the number of 2-cycles of a other than $(1, 2)$ is equal to $\frac{1}{3} {}_7C_2 = 7$. Hence $n = 2 + 7 + 2 \times 7 = 23$. This contradicts the assumption (*).

CASE III. $N^\Delta = A_6$, $|\Delta| = 6$.

Since $N_{1,2,3,4} = H$ and $(N^\Delta)_{1,2,3,4} = (N_{1,2,3,4})^\Delta = 1$, we have $I(H) = \Delta$.

Let $a = (1, 2)\cdots$ be an involution which is conjugate to a central

involution of P . Then $|I(a)|=6$. For a subset $\{i_1, i_2\}$ of $I(a)$ consisting of two letters let P' be a Sylow 2-subgroup of $G_{1,2,i_1,i_2}$ and $\Delta'=I(P')=\{1, 2, i_1, i_2, k, l\}$. Then a normalizes P' and $a^{\Delta'}$ is an even permutation. Therefore we have

$$a^{\Delta'} = (1, 2)(i_1)(i_2)(k, l).$$

Since $I(G_{1,2,i_1,i_2})=\Delta'$ we have $G_{1,2,i_1,i_2}=G_{1,2,k,l}$. In this way, any given subset $\{i_1, i_2\}$ of $I(a)$ determines uniquely a 2-cycle (k, l) of a in such a way that $G_{1,2,i_1,i_2}=G_{1,2,k,l}$. Conversely for any given 2-cycle (k, l) of a other than $(1, 2)$, there is a subset $\{i_1, i_2\}$ in $I(a)$ such as $G_{1,2,k,l}=G_{1,2,i_1,i_2}$, and then $I(G_{1,2,k,l})=I(G_{1,2,i_1,i_2})=\{1, 2, i_1, i_2, k, l\}$. Thus we have that the number of 2-cycles of a other than $(1, 2)$ is equal to ${}_6C_2=15$ and hence $n=2+6+2\times 15=38$.

Now consider the mapping $\varphi: i \rightarrow G_{1,2,3,i}$ from $\{4, 5, \dots, 38\}$ to the set of subgroups $\{G_{1,2,3,i}\}$. Since $G_{1,2,3,j}=G_{1,2,3,i}$ if and only if $j \in I(G_{1,2,3,i})$ and $|I(G_{1,2,3,i})|=6$, $\varphi^{-1}(G_{1,2,3,i})$ consists of three letters. Therefore we have $35 \equiv 0 \pmod{3}$. This is a contradiction.

CASE IV. $N^\Delta = S_5, |\Delta|=5$.

As in CASE III we have $\Delta=I(H)$. Let $a=(1, 2)\dots$ be an involution which is conjugate to a central involution of P . Then $|I(a)|=5$.

For any two letters i, j in $I(a)$, let $\Delta'=I(G_{1,2,i,j})=\{1, 2, i, j, k\}$. Then a normalizes $G_{1,2,i,j}$ and hence $\Delta'^a=\Delta'$ and $k \in I(a)$. Clearly $G_{1,2,i,j}=G_{1,2,j,k}=G_{1,2,k,i}$. On the other hand, if $G_{1,2,i',j'}=G_{1,2,i,j}$ for some $\{i', j'\} \subset I(a)$ then $\{i', j'\} \subset I(G_{1,2,i,j})$.

Now consider the mapping $\varphi: \{i, j\} \rightarrow G_{1,2,i,j}$ from the set of all the subsets $\{i, j\}$ of $I(a)$ consisting of two letters into the set of subgroups of G . As is shown above, $\varphi^{-1}(G_{1,2,i,j})$ ($i, j \in I(a)$) consists of three subsets. Therefore we have ${}_5C_2=10 \equiv 0 \pmod{3}$. This is a contradiction.

CASE V. $N^\Delta = S_4, |\Delta|=4$.

The proof in this case is a little complicated. Since P is transitive on $\Gamma=\{5, 6, \dots, n\}$, H is also transitive on Γ and hence G is 5-fold transitive. Let $K=G_{1,2,3,4,5}$. Then $Q=K \cap P$ is a normal Sylow 2-subgroup of K . Let N' be the normalizer of Q in G , and let $\Delta'=I(Q)$.

We first remark that a central involution c of P is contained in N' and semi-regular on $\Delta'-\Delta$, therefore $|\Delta'|$ must be even.

By the same argument as in the proof of Lemma 1, it is shown that $(N')^{\Delta'}$ must be one of the following groups: S_5, S_6, A_7, A_8 or M_{12} . Thus, from the remark above, we have one of the following cases:

CASE (V.A). $N^\Delta = S_4, (N')^{\Delta'} = M_{12}, |\Delta|=4, |\Delta'|=12,$

CASE (V.B.). $N^\Delta = S_4$, $(N')^{\Delta'} = A_8$, $|\Delta| = 4$, $|\Delta'| = 8$,

CASE (V.C.). $N^\Delta = S_4$, $(N')^{\Delta'} = S_6$, $|\Delta| = 4$, $|\Delta'| = 6$.

We shall treat three cases separately.

CASE (V.A.). $N^\Delta = S_4$, $(N')^{\Delta'} = M_{12}$, $|\Delta| = 4$, $|\Delta'| = 12$.

Let $\Delta' = \{1, 2, \dots, 12\}$. Since G is 5-fold transitive, there is an involution $a = (1, 2)(3, 4)(5) \dots$ which is conjugate to a central involution of P . Then $|I(a)| = 4$. Now $a \in N'$ and $a^{\Delta'}$ is an involution in M_{12} . Therefore we may assume that

$$a = (1, 2)(3, 4)(5)(6)(7)(8)(9, 10)(11, 12) \dots$$

and then $I(a) = \{5, 6, 7, 8\}$. Hence $I(a) \subset I(Q)$, where Q is a Sylow 2-subgroup of $G_{1,2,3,4,5}$. From the same reason, for any 2-cycle (k, l) of a other than $(1, 2)$, we have $I(a) \subset I(Q')$, where Q' is a Sylow 2-subgroup of $G_{1,2,5,k,l}$. Assume that $k, l > 12$. Since $\{6, 7\} \subset I(a) \subset I(Q')$, $Q' \subset G_{1,2,5,6,7}$. Therefore Q and Q' are both normal Sylow 2-subgroup of $G_{1,2,5,6,7}$ and hence $Q = Q'$. On the other hand, k, l are in $I(Q')$ but not in $I(Q)$. This is a contradiction. Hence we have $n = 12$, which contradicts the assumption (*).

CASE (V.B.). $N^\Delta = S_4$, $(N')^{\Delta'} = A_8$, $|\Delta| = 4$, $|\Delta'| = 8$.

Let $\Delta' = \{1, 2, \dots, 8\}$ and let $a = (1, 2)(3, 4)(5) \dots$ be an involution which is conjugate to a central involution of P . Then $|I(a)| = 4$. Now $a \in N'$ and $a^{\Delta'}$ is an even permutation. Therefore we have

$$a = (1, 2)(3, 4)(5)(6)(7)(8) \dots$$

and $I(a) = \{5, 6, 7, 8\} \subset I(Q)$. Then by the same argument as in CASE (V.A.) we have a contradiction.

CASE (V.C.). $N^\Delta = S_4$, $(N')^{\Delta'} = S_6$, $|\Delta| = 4$, $|\Delta'| = 6$.

Since $K = N'_{1,2,3,4,5}$ and $(N')^{\Delta'}_{1,2,3,4,5} = (N'_{1,2,3,4,5})^{\Delta'} = 1$, we have $K^{\Delta'} = 1$, $\Delta' = I(K)$.

Let C be the center of P . By Lemma 2 C is semi-regular on $\Omega - \Delta = \{5, 6, \dots, n\}$. Therefore $C \cap K = 1$. On the other hand, $N'_{1,2,3,4} \supset CK$ and $N'_{1,2,3,4}/K = (N')^{\Delta'}_{1,2,3,4}$ is of order 2. Therefore the order of C is 2 and P has the unique central involution c . Then c is also the unique central involution of H , since P is a normal Sylow 2-subgroup of H .

Let c_i ($i \geq 4$) be the unique central involution of $G_{1,2,3,i}$. Then $I(c_i) = \{1, 2, 3, i\}$. Let X be the subgroup of $G_{1,2,3}$ generated by $\{c = c_4, c_5, \dots, c_n\}$. Then X is a normal subgroup of $G_{1,2,3}$. In the following we shall show that X is a Frobenius group as a permutation group on $\{4, 5, \dots, n\}$. If this has been done, the Frobenius kernel of X is a

normal subgroup which is regular on $\{4, 5, \dots, n\}$. Then by [3], Theorem 2 G must be one of the following groups: S_5, S_6, S_7, A_7 or M_{12} . This contradicts the assumption (*).

We first remark that if $c_i = (1)(2)(3)(i)(k, l) \dots$ then $G_{1,2,3,k,l} = G_{1,2,3,i,k} = G_{1,2,3,i,l}$. Since c_i normalizes $G_{1,2,3,k,l}$ and $|I(G_{1,2,3,k,l})| = 6$, a letter j in $I(G_{1,2,3,k,l})$, which is different from $1, 2, 3, k, l$, is left invariant by c_i . Therefore $j = i$ and $I(G_{1,2,3,k,l}) = \{1, 2, 3, i, k, l\}$. The remark above follows now easily.

(a) We first show that the order of $c_i c_j$ is 3 and $I(c_i c_j) = \{1, 2, 3\}$ if $i \neq j$.

Let

$$\begin{aligned} c_i &= (1)(2)(3)(i)(j, j') \dots, \\ c_j &= (1)(2)(3)(j)(i, i') \dots. \end{aligned}$$

Then from the remark above $I(G_{1,2,3,i,j}) = \{1, 2, 3, i, j, j'\} = \{1, 2, 3, i, j, i'\}$. Therefore we have $i' = j'$ and

$$c_i c_j = (1)(2)(3)(i, i', j) \dots.$$

Now $(c_i c_j)^{-1} c_j (c_i c_j) = (1)(2)(3)(i)(i', j) \dots$ and it is the central involution of $G_{1,2,3,i}$. Hence $c_i = (c_i c_j)^{-1} c_j (c_i c_j)$, $(c_i c_j)^3 = 1$. Thus $c_i c_j$ is of order 3.

Now suppose that a letter $k > 3$ is left invariant by $c_i c_j$. Then $k^{c_i} = k^{c_j}$, and c_i and c_j have a 2-cycle (k, l) ($l = k^{c_i} = k^{c_j}$) in common. But by the remark above we have $I(G_{1,2,3,k,l}) = \{1, 2, 3, k, l, i\} = \{1, 2, 3, k, l, j\}$, which is a contradiction. Thus we have $I(c_i c_j) = \{1, 2, 3\}$.

(b) We next show that $c_i c_j c_k$ is the central involution of $G_{1,2,3,i}$, i.e. $c_i c_j c_k = c_l$ for some l . If two of $\{i, j, k\}$ are the same, this is clear. We assume that i, j, k are all different. For the simplicity, let $i = 4, j = 5$.

Now let

$$c_4 = (1)(2)(3)(4)(5, 6)(7, 8)(9, 10)(11, 12) \dots.$$

Then as is shown in (a)

$$c_5 = (1)(2)(3)(5)(4, 6) \dots.$$

Since $c_4 c_5 c_4 = c_6$ and $(c_4 c_5)^3 = 1$, we have $c_4 c_5 c_6 = c_5$. Therefore we have only to prove the assertion for $k \geq 7$. Let $k = 7$ and assume that c_5 takes 7 to 9. Then $8^{c_5} \neq 10$ and hence $8^{c_5} \geq 11$ since $(c_4 c_5)^3 = 1$. We may assume that c_5 takes 8 to 11. Then $7^{c_5 c_4 c_5} = 10^{c_5} = 7^{c_4 c_5 c_4} = 12$, i.e. c_5 takes 10 to 12. Thus we have

$$c_5 = (1)(2)(3)(5)(4, 6)(7, 9)(8, 11)(10, 12) \dots.$$

Using the same arguments as in (a) and as above for c_4, c_7 and c_5, c_7 , we have easily

$$c_7 = (1)(2)(3)(7)(4, 8)(5, 9)(6, 12)(10, 11) \dots$$

and then

$$c_4 c_5 c_7 = (1)(2)(3)(11)(4, 12)(5, 8)(6, 9)(7, 10) \dots$$

Now we shall show that $c_4 c_5 c_7 = c_{11}$. Since $c_{11} = (c_4 c_5)^{-1} c_7 (c_4 c_5)$, c_{11} and $c_4 c_5$ induce the same permutation on $\Sigma = \{1, 2, \dots, 12\}$. Since $c_4 c_5 c_7$ is in $G_{1,2,3,11}$, it commutes with c_{11} . Hence $(c_4 c_5 c_7) c_{11} = c_{11} (c_4 c_5 c_7)$, $c_4 c_5 c_7 c_{11} c_7 = c_{11} c_4 c_5$. Since $c_4 c_5 c_7 c_{11}$ is in G_{Σ} , it commutes with c_7 . Hence $c_7 c_4 c_5 c_7 c_{11} = c_{11} c_4 c_5$, $(c_7 c_{11})^{-1} (c_4 c_5) (c_7 c_{11}) = c_4 c_5$. Thus $c_4 c_5$ and $c_7 c_{11}$ commute with each other. Therefore we have

$$(c_4 c_5 c_7 c_{11})^3 = (c_4 c_5)^3 (c_7 c_{11})^3 = 1.$$

Now $c_4 c_5 c_7$ commutes with c_{11} , hence $(c_4 c_5 c_7 c_{11})^3 = (c_4 c_5 c_7)^3 c_{11} = 1$ and we have

$$(c_4 c_5 c_7)^3 = c_{11}.$$

This shows that $c_4 c_5 c_7$ is of order 2 or 6 and $I(c_4 c_5 c_7) = \{1, 2, 3, 11\}$. If $c_4 c_5 c_7$ is of order 2, we have $c_4 c_5 c_7 = c_{11}$. We assume that $c_4 c_5 c_7$ is of order 6, and let $a = (c_4 c_5 c_7)^2$. Then $|I(a)| = 4 + 2r$, where r is the number of 2-cycles of $c_4 c_5 c_7$. On the other hand, if (i, j, k) is a 3-cycle of a , then a normalizes $G_{1,i,j,k}$ and hence a commutes with the central involution c of $G_{1,i,j,k}$. Therefore c induces a semi-regular permutation on $I(a) - \{1\}$, and hence $|I(a) - \{1\}| = 3 + 2r$ must be even. This is a contradiction.

(c) From (b), it follows that X consists of the elements c_i 's and $(c_i c_j)$'s, and $I(c_i) = \{1, 2, 3, i\}$, $I(c_i c_j) = \{1, 2, 3\}$ if $i \neq j$. Therefore X is a Frobenius group as a permutation group on $\{4, 5, \dots, n\}$.

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