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GROWTH PROPERTIES OF HYPERPLANE INTEGRALS OF SOBOLEV FUNCTIONS IN A HALF SPACE

Dedicated to Professor Masayuki Ito on the occasion of his sixtieth birthday

TETSU SHIMOMURA

(Received January 21, 2000)

1. Introduction

Let $D \subset \mathbb{R}^n$ ($n \geq 2$) denote the half space

$$D = \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}^1 : x_n > 0\}$$

and set

$$S = \partial D;$$

we sometimes identify $x' \in \mathbb{R}^{n-1}$ with $(x', 0) \in S$. We define the hyperplane integral $S_q(u)$ over $S$ by

$$S_q(u) = \left( \int_S |u(x')|^q dx' \right)^{1/q}$$

for a measurable function $u$ on $S$ and $q > 0$.

Set

$$U_r(x') = u(x', r) - \sum_{k=0}^{m-1} \frac{r^k}{k!} \left[ \frac{\partial}{\partial x_n} \right]^k u(x', 0)$$

for quasicontinuous Sobolev functions $u$ on $D$, where the vertical limits

$$\left( \frac{\partial}{\partial x_n} \right)^k u(x', 0) = \lim_{x_n \to 0} \left( \frac{\partial}{\partial x_n} \right)^k u(x', x_n)$$

exist for almost every $x' = (x', 0) \in \partial D$ and $0 \leq k \leq m - 1$ (see [8, Theorem 2.4, Chapter 8]).

Our main aim in this note is to study the existence of limits of $S_q(U_r)$ at $r = 0$. More precisely, we show (in Theorem 3.1 below) that

$$\lim_{r \to 0} r^{-\omega} S_q(U_r) = 0$$
for some \( \omega > 0 \).

Consider the Dirichlet problem for polyharmonic equation

\[ \Delta^m u(x) = 0 \]

with the boundary conditions

\[ \left( \frac{\partial}{\partial x_n} \right)^k u(x', 0) = f_k(x') \quad (k = 0, 1, \ldots, m - 1). \]

We show (in Corollary 3.1 below) that if \( 1 < p \leq q < \infty, \frac{n}{p} - \frac{n - 1}{q} < 1 \) and \( u \in W^{m,p}(D) \) is a solution of the Dirichlet problem with \( f_k(x') = (\partial/\partial x_n)^k u(x', 0) \) for \( 0 \leq k \leq m - 1 \), then

\[ \lim_{r \to 0} r^{n/p-(n-1)/q} S_q(u_r) = 0, \]

where \( U_r(x') = u(x', r) - \sum_{k=0}^{m-1} \frac{r^k}{k!} f_k(x') \).

To prove our results, we apply the integral representation in [6, 8]. For this purpose, we are concerned with \( K \)-potentials \( U_K f \) defined by

\[ U_K f(x) = \int K(x - y) f(y) dy \]

for functions \( f \) on \( \mathbb{R}^n \) satisfying weighted \( L^p \) condition:

\[ \int_{\mathbb{R}^n} |f(y)|^p |y_n|^{\beta} dy < \infty. \]

In connection with our integral representation, \( K(x) \) is of the form \( x^\lambda |x|^{-n} \) for a multi-index \( \lambda \) with length \( m \). Our basic fact is stated as follows (see Theorem 2.1 below):

\[ \lim_{r \to 0} r^{n/p-(n-1)/q} S_q(u_r) = 0, \]

where \( u_r(x') = U_K f(x', r) - \sum_{k=0}^{m-1} \frac{r^k}{k!} \left( \frac{\partial^k}{\partial x_n^k} U_K f \right)(x') \).

In the final section, we give growth estimates of higher differences of Sobolev functions.

For related results, see Gardiner [2], Stoll [14, 15, 16] and Mizuta [5, 6, 9]. We also refer the reader to Mizuta-Shimomura [10, 11] concerning monotone functions as a generalization of harmonic functions.
2. Hyperplane integrals of potentials

For a multi-index \( \lambda \) and \( l > 0 \), set

\[
K(x) = \frac{x^\lambda}{|x|^l}.
\]

We define the \( K \)-potential \( U_K f \) by

\[
U_K f(x) = \int_{\mathbb{R}^n} K(x-y) f(y) dy
\]

for a measurable function \( f \) on \( \mathbb{R}^n \) satisfying

\[
\int_{\mathbb{R}^n} (1 + |y|)^{|\lambda| - l} |f(y)| dy < \infty
\]

and

\[
\int_{\mathbb{R}^n} |f(y)|^p |y_n|^\beta dy < \infty, \quad y = (y_1, \ldots, y_n).
\]

In particular, \( K \) is the Riesz \( \alpha \)-kernel when \( \lambda = 0 \) and \( l = n - \alpha \). In this case, \( U_K f \) is written as \( U_\alpha f \) with \( \alpha = |\lambda| - l + n > 0 \). Note here that (2.1) is equivalent to the condition that

\[
U_\alpha |f| \neq \infty.
\]

Throughout this paper, let \( M \) denote various constants independent of the variables in question.

For a nonnegative integer \( m \), consider

\[
K_m(x, y) = K(x-y) - \sum_{k=0}^{m} \frac{x_n^k}{k!} \left[ \left( \frac{\partial}{\partial x_n} \right)^k K \right] (x'-y),
\]

where \( x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \); we sometimes identify \( x' \) with \( (x', 0) \).

**Lemma 2.1.** Let \( m \) be a nonnegative integer such that \( |\lambda| - l < m + 1 \).

1. If \( |x' - y| \geq x_n/2 > 0 \) and \( |x - y| \geq x_n/2 > 0 \), then

\[
|K_m(x, y)| \leq Mx_n^{m+1}|x' - y|^{|\lambda| - l - m - 1}.
\]

2. If \( |x - y| < x_n/2 \), then

\[
|K_m(x, y)| \leq M(x_n^{|\lambda| - l} + |x - y|^{|\lambda| - l}).
\]

3. If \( |x' - y| < x_n/2 \), then

\[
|K_m(x, y)| \leq M(x_n^{|\lambda| - l} + x_n^m|x' - y|^{|\lambda| - l - m}).
\]
Proof. If \(|x' - y| > 2x_n\), then by Taylor’s theorem, we obtain

\[
|K_m(x, y)| \leq M \frac{x_{n+1}^m}{(m+1)!} |(x', \theta x_n) - y|^{|\lambda| - l - m - 1} (0 < \theta < 1)
\]

\[
\leq Mx_{n+1}^m |x' - y|^{|\lambda| - l - m - 1},
\]

If \(x_n/2 < |x' - y| < 2x_n\) and \(|x - y| \geq x_n/2 > 0\), then

\[
|K_m(x, y)| \leq |K(x - y)| + \sum_{k=0}^{m} \frac{x_n^k}{k!} \left( \left( \frac{\partial}{\partial x_n} \right)^k K \right) (x' - y)
\]

\[
\leq Mx_n^{|\lambda| - l} + M \sum_{k=0}^{m} \frac{x_n^k}{k!} |x' - y|^{|\lambda| - l - k}
\]

\[
\leq Mx_n^{|\lambda| - l}
\]

\[
\leq Mx_{n+1}^m |x' - y|^{|\lambda| - l - m - 1},
\]

so that (1) is proved.

If \(|x' - y| < x_n/2\), then \(x_n/2 < |x - y| < 3x_n/2\), so that

\[
|K_m(x, y)| \leq |K(x - y)| + \sum_{k=0}^{m} \frac{x_n^k}{k!} \left( \left( \frac{\partial}{\partial x_n} \right)^k K \right) (x' - y)
\]

\[
\leq Mx_n^{|\lambda| - l} + M \sum_{k=0}^{m} \frac{x_n^k}{k!} |x' - y|^{|\lambda| - l - k}
\]

\[
\leq M(x_n^{|\lambda| - l} + x_n^m |x' - y|^{|\lambda| - l - m}),
\]

which proves (3).

Finally, if \(|x - y| < x_n/2\), then \(x_n/2 < |x' - y| \leq x_n + |x - y| < 3x_n/2\), so that

\[
|K_m(x, y)| \leq |K(x - y)| + \sum_{k=0}^{m} \frac{x_n^k}{k!} \left( \left( \frac{\partial}{\partial x_n} \right)^k K \right) (x' - y)
\]

\[
\leq |x - y|^{|\lambda| - l} + M|x' - y|^{|\lambda| - l}
\]

\[
\leq M(x_n^{|\lambda| - l} + |x - y|^{|\lambda| - l}),
\]

which proves (2). Thus the present lemma is established. \(\square\)

For a point \(x \in \mathbb{R}^n\) and \(r > 0\), we denote by \(B(x, r)\) the open ball with center at \(x\) and radius \(r\).
**Lemma 2.2** (cf. [9, Lemma 3.2]). Let $\beta > -1$, $q > 0$ and $|\lambda| - l + n/q > 0$. Let $m$ be a nonnegative integer such that

$$m < |\lambda| - l + \frac{n + \beta}{q} < m + 1.$$ 

Then

$$\left( \int_{E_3} |K_m(x, y)|^q |y_n|^{\beta} dy \right)^{1/q} \leq M x_n^{\lambda |l + m q + \beta/q}$$

for all $x = (x', x_n) \in D$.

**Proof.** For fixed $x \in D$, consider the sets

$$E_1 = B \left( x, \frac{x_n}{2} \right), \quad E_2 = B \left( x', \frac{x_n}{2} \right), \quad E_3 = \mathbb{R}^n - (E_1 \cup E_2).$$

Since $|\lambda| - l + (n + \beta)/q - m - 1 < 0$, applying the polar coordinates about $x'$, we have by Lemma 2.1(1)

$$\left( \int_{E_3} |K_m(x, y)|^q |y_n|^{\beta} dy \right)^{1/q} \leq M x_n^{\lambda + m + 1} \left( \int_{E_3} |x' - y|^{|\lambda| - l + m - 1 + \beta} |y_n|^{\beta} dy \right)^{1/q} \leq M x_n^{\lambda + m + 1} \left( \int_{x_n/2}^{\infty} r^{|\lambda| - l + m - 1 + \beta} r^{-1} dr \right)^{1/q} = M x_n^{\lambda |l + m q + \beta/q}.$$ 

Similarly, since $|\lambda| - l + n/q > 0$, we have by Lemma 2.1(2)

$$\left( \int_{E_1} |K_m(x, y)|^q |y_n|^{\beta} dy \right)^{1/q} \leq M x_n^{\lambda - l + n} \left( \int_{E_1} (x_n^{\lambda|l - l - m q|} |x' - y|^{\lambda|l - l - m q|} dy \right)^{1/q} = M x_n^{\lambda |l + m q + \beta/q}.$$ 

Finally, since $|\lambda| - l + (n + \beta)/q - m > 0$, we obtain by Lemma 2.1(3)

$$\left( \int_{E_2} |K_m(x, y)|^q |y_n|^{\beta} dy \right)^{1/q} \leq M \left( \int_{E_2} (x_n^{\lambda|l - l + m q|} |x' - y|^{\lambda|l - l + m q|} |y_n|^{\beta} dy \right)^{1/q} \leq M x_n^{\lambda |l + m q + \beta/q} + M x_n^{m} \left( \int_{x_n/2}^{\infty} r^{|\lambda| - l + m q + \beta} r^{-1} dr \right)^{1/q} = M x_n^{\lambda |l + m q + \beta/q}.$$ 

The required inequality now follows. □
Lemma 2.3 (cf. [9, Lemma 3.4]). Let $q > 0$ and $m$ be a nonnegative integer such that
\[ m < |\lambda| - l + \frac{n-1}{q} < m + 1. \]

If $x = (x', x_n) \in \mathbb{D}$ and $y = (y', y_n) \in \mathbb{R}^n$, then
\[
\left( \int_{\mathbb{R}^{n-1}} |K_m(x, y)|^q dx' \right)^{1/q} \leq Mx_n^{m+1} (x_n + |y_n|)^{|\lambda| - l - m - 1 + (n-1)/q}.
\]

Proof. Let $x = (x', x_n) \in \mathbb{D}$ and $y = (y', y_n) \in \mathbb{R}^n$. If $|y_n| \geq 2x_n$, then, since $|\lambda| - l - m - 1 + (n-1)/q < 0$, we have by Lemma 2.1(1)
\[
\left( \int_{\mathbb{R}^{n-1}} |K_m(x, y)|^q dx' \right)^{1/q} \leq \frac{Mx_n^{m+1} \left( \int_{\mathbb{R}^{n-1}} |x' - y'|^{(|\lambda| - l - m - 1)q} dx' \right)^{1/q}}{r_{y_n}^2 + y_n^2} \leq \frac{Mx_n^{m+1} |y_n|^{(|\lambda| - l - m - 1 + n-1)/q}}{r_{y_n}^2 + y_n^2}.
\]

If $|y_n| < 2x_n$, then we have as in the proof of Lemma 2.2
\[
\left( \int_{\mathbb{R}^{n-1}} |K_m(x, y)|^q dx' \right)^{1/q} \leq M \left( \int_{\{x', y \in E_1\}} (x_n^{\lambda| - l + |x - y|^{\lambda| - l - m q} dx') \right)^{1/q} + M \left( \int_{\{x', y \in E_1\}} (x_n^{\lambda| - l + x_n^m |x' - y|^{\lambda| - l - m q} dx') \right)^{1/q}
\]
\[
+ Mx_n^{m+1} \left( \int_{\{x', y \in E_1\}} |x' - y|^{(|\lambda| - l - m - 1)q} dx' \right)^{1/q} \leq Mx_n^{\lambda| - l + (n-1)/q} + M \left( \int_{B(y', x_n/2)} (x_n + |x' - y'|^{\lambda| - l - m)q} dx' \right)^{1/q}
\]
\[
+ Mx_n^{m+1} \left( \int_{B(y', x_n/2)} |x' - y'|^{(|\lambda| - l - m - 1)q} dx' \right)^{1/q} + Mx_n^{m+1} \left( \int_{\mathbb{R}^{n-1}} (x_n + |x' - y'|^{(|\lambda| - l - m - 1)q} dx' \right)^{1/q}
\]
\[
= Mx_n^{\lambda| - l + (n-1)/q}.
\]

Therefore the required inequality now follows. \qed
Lemma 2.4 (cf. [1, Theorem 13.5], [8, Sections 6.5 and 8.2]). Let $\alpha = |\lambda| - I + n$, $p > 1$, $\alpha p > 1$, $\alpha p > 1 + \beta$ and $-1 < \beta < p - 1$. If $f$ is a measurable function on $\mathbb{R}^n$ satisfying (2.2) and (2.3), then $U_K f$ has the (ACL) property; in particular, $U_K f(x', x_n)$ is absolutely continuous on $\mathbb{R}$ for almost every $x' \in \mathbb{R}^{n-1}$. Moreover, in case $m$ is a positive integer such that $(\alpha - m)p > 1$ and $(\alpha - m)p > 1 + \beta$,

$$\left( \frac{\partial}{\partial x_n} \right)^m U_K f(x', x_n) = \int \left( \frac{\partial}{\partial x_n} \right)^m K(x - y) f(y) dy$$

is absolutely continuous on $\mathbb{R}$ for almost every $x' \in \mathbb{R}^{n-1}$.

Theorem 2.1 (cf. [5, Theorem 2.1] and [9, Theorem 2.1]). Let $\alpha = |\lambda| - I + n$ satisfy $m + 1/p < \alpha < m + n$. Let $1 < p \leq q < \infty$, $-1 < \beta < p - 1$ and

$$\frac{n - \alpha p}{p(n - \alpha)} < \frac{n - 1}{q(n - \alpha + m)} \quad \text{when } n - \alpha > 0.$$

Further suppose $m < \omega < m + 1$, where $\omega = (n - 1)/q - (n - \alpha p + \beta)/p$. If $f$ is a nonnegative measurable function on $\mathbb{R}^n$ satisfying (2.2) and (2.3), then

$$\lim_{r \to 0} r^{-\omega} S_f(u_r) = 0,$$

where $u_r(x') = U_K f(x', r) - \sum_{k=0}^m (r^k/k!) (\partial/\partial x_n)^k U_K f(x', 0)$.

Proof. Under the assumptions on $p$, $\alpha$, $\beta$, $q$ and $m$ in Theorem 2.1, we can take $(\delta, \gamma)$ such that

1. $\beta < \gamma < p(n - \alpha + m + 1)\delta + \beta - \frac{p(n - 1)}{q}$,
2. $p(n - \alpha + m + 1)\delta + (\alpha - m - 1)p - n < \gamma < p(n - \alpha + m)\delta + (\alpha - m)p - n$,
3. $\beta < \gamma < p - 1$, $0 < \delta < 1$,
4. $\delta p(n - \alpha) > n - \alpha p$

and

$$\frac{n - 1}{q(n - \alpha + m + 1)} < \delta < \frac{n - 1}{q(n - \alpha + m)}$$

(if $\alpha \geq n$, then (2.7) clearly holds). Set $a = (1 - \delta)p'$ and $b = -\gamma p'/p$, where $p' = p/(p - 1)$. Then, by (2.6), we have

$$b > -1.$$  
In case $\alpha \geq n$, we clearly find

$$\alpha - n + \frac{n}{a} > 0,$$
and in case \( \alpha < n \), (2.10) also holds by (2.7). Further, (2.5) implies
\[
(2.11) \quad m < \alpha - n + \frac{n+b}{a} < m + 1.
\]
By the fact that \( m + 1/p < \alpha \), we have
\[
(2.12) \quad \alpha p > 1.
\]
Since \( \omega > m \), we have
\[
(2.13) \quad (\alpha - m)p > 1 + \beta.
\]
By (2.12), (2.13) and Lemma 2.4, we first note that
\[
\begin{align*}
u_{\gamma}(x') &= U_K f(x) - \sum_{k=0}^{m} \frac{x_n^k}{k!} \left( \frac{\partial}{\partial x_n} \right)^k U_K f(x', 0) \\
&= \int K_m(x, y) f(y) dy.
\end{align*}
\]
Using Hölder’s inequality, we have
\[
|\nu_{\gamma}(x')| \leq \left( \int \left| K_m(x, y) \right|^\alpha |y_n|^{q_0} dy \right)^{(1-\delta)/\alpha} \left( \int \left| K_m(x, y) \right|^\delta p f(y) |y_n|^{q} dy \right)^{1/p}.
\]
By (2.9)–(2.11) and Lemma 2.2, we have
\[
|\nu_{\gamma}(x')| \leq M_{n}^{\alpha - n(1-\delta)+\alpha'p'/\gamma} \left( \int \left| K_m(x, y) \right|^\delta p f(y) |y_n|^{q} dy \right)^{1/p}.
\]
In view of Minkowski’s inequality for integral we have
\[
S_{q}(\nu_{\gamma}) \leq M_{n}^{\alpha - n(1-\delta)+\alpha'/\gamma} \times \left\{ \int \left( \int_{\mathbb{R}^{n-1}} \left| K_m(x, y) \right|^\delta q dx' \right)^{p/q} f(y) |y_n|^{q} dy \right\}^{1/p}.
\]
Here, noting (2.8), we have by Lemma 2.3
\[
\left( \int_{\mathbb{R}^{n-1}} \left| K_m(x, y) \right|^\delta q dx' \right)^{p/q} \leq M_{n}^{m+1}(x_n + |y_n|)^{\alpha - n - m - 1 + (n-1)/\delta q} \delta p.
\]
Consequently
\[ S_q(u_r) \leq M r^{(\alpha-n)1-\gamma/p+\gamma/p+\delta} \times \left\{ \int (r + |y_n|)^{\alpha-n-1+(n-1)/\delta} f(y)^p |y_n|^{\beta} dy \right\}^{1/p} . \]

Consider the function
\[ k(r, y_n) = r^{(\alpha-n-p+\beta)\gamma/p+(n-1)/\delta} \times (r + |y_n|)^{\alpha-n-1+(n-1)/\delta} |y_n|^{\beta} . \]

Then
\[ r^{-\omega} S_q(u_r) \leq M \left\{ \int k(r, y_n) f(y)^p |y_n|^{\beta} dy \right\}^{1/p} , \]

where \( \omega = (n-1)/q - (n - \alpha p + \beta)/p \). It follows from (2.4) that
\[ r^{-\omega} r^{(\alpha-n)(1-\delta)n/p'-\gamma/p+\delta} = r^{(\alpha-n+\delta+\gamma)/p-(n-1)/q} \to 0 \]
as \( r \to 0 \). If \( r < |y_n| \), then
\[ k(r, y_n) \leq M \left( \frac{r}{|y_n|} \right)^{(n-\alpha+\delta+\gamma-p(n-1)/q} \leq M; \]

if \( |y_n| \leq r \), then
\[ k(r, y_n) \leq M \left( \frac{|y_n|}{r} \right)^{-\gamma-\beta} \leq M. \]

Hence Lebesgue’s dominated convergence theorem implies that
\[ \lim_{r \to 0} r^{-\omega} S_q(u_r) = 0. \]

Now the proof of Theorem 2.1 is completed. \( \square \)

3. Sobolev functions

For an open set \( G \subset \mathbb{R}^n \), we denote by \( BL_m(L^p_{\text{loc}}(G)) \) the Beppo Levi space
\[ BL_m(L^p_{\text{loc}}(G)) = \{ u \in L^p_{\text{loc}}(G) : D^\beta u \in L^p_{\text{loc}}(G) \quad (|\beta| = m) \} \]
(see [8, Chapter 6]). Set \( K_\lambda(x) = |x|^\lambda \) and
\[ \tilde{K}_\lambda(x, y) = \begin{cases} K_\lambda(x-y), & y \in B(0, 1), \\ K_\lambda(x-y) - \sum_{|\mu| \leq m-1} \frac{x^\mu}{\mu!} \left[ \frac{\partial}{\partial x} \right]^\mu K_\lambda(-y), & y \in \mathbb{R}^n - B(0, 1). \end{cases} \]
In view of [8, Theorem 7.2, Chapter 6], each \( u \in BL_m(L^p_{\text{loc}}(D)) \) satisfying

\[
\int_D |\nabla_m u(x)|^p \lambda_n^\beta dx < \infty
\]

has an \((m, p)\)-quasicontinuous representative \( \tilde{u} \), where

\[
|\nabla_m u(x)| = \left( \sum_{|\mu|=m} |D^\mu u(x)|^2 \right)^{1/2},
\]

\(1 < p < \infty\) and \(-1 < \beta < p - 1\). Moreover, \( \tilde{u} \) is given by

\[
\tilde{u}(x) = \sum_{|\lambda|=m} \alpha_\lambda \int K_{\lambda,m}(x, y) D^\lambda \overline{u}(y) dy + P(x),
\]

where \( \overline{u} \) is an extension of \( u \) to \( \mathbb{R}^n \), \( P(x) \) is a polynomial of degree at most \( m - 1 \).

Note further from Lemma 2.4 that for each \( k \) with \( 0 \leq k \leq m - 1 \) and for almost every \( x' \in \mathbb{R}^{n-1} \),

\[
\left( \frac{\partial}{\partial x_n} \right)^k \int K_{\lambda,m}(x, y) D^\lambda \overline{u}(y) dy = \int \left( \frac{\partial}{\partial x_n} \right)^k \tilde{K}_{\lambda,m}(x, y) D^\lambda \overline{u}(y) dy
\]

holds for \( x_n \in \mathbb{R} \), where \( x = (x', x_n) \).

Since \( Q(x) - \sum_{k=0}^{m-1} (x_n^k / k!) [(\partial / \partial x_n)^k Q](x') = 0 \) for any polynomial \( Q \) of degree at most \( m - 1 \), we have

\[
U(x) \equiv \tilde{u}(x) - \sum_{k=0}^{m-1} \frac{x_n^k}{k!} \left( \frac{\partial}{\partial x_n} \right)^k \tilde{u}(x')
\]

\[
= \sum_{|\lambda|=m} \alpha_\lambda \int K_{\lambda,m}(x, y) D^\lambda \overline{u}(y) dy = \tilde{u}(x) - P(x)
\]

for \( x \in D \), where \( K_{\lambda,m}(x, y) = K_{\lambda}(x - y) - \sum_{k=0}^{m-1} (x_n^k / k!) [(\partial / \partial x_n)^k K_{\lambda}](x' - y) \).

Theorem 2.1 now gives the following result.

**Theorem 3.1.** Let \( 1 < p \leq q < \infty \),

\[
\frac{n - mp}{p(n - m)} < \frac{1}{q} \quad \text{when } n - m > 0
\]

and

\[
\frac{n - p + \beta}{p(n - 1)} < \frac{1}{q} < \frac{n + \beta}{p(n - 1)}.
\]

If \( u \in BL_m(L^p_{\text{loc}}(D)) \) satisfying (3.1) for \(-1 < \beta < p - 1\) is \((m, p)\)-quasicontinuous on \( D \), then

\[
\lim_{r \to 0} r^{(n - mp \beta)/(p(n - 1)) q} S_q(U_r) = 0,
\]
where \( U_r(x') = u(x', r) - \sum_{k=0}^{m-1} (r^k/k!)(\partial/\partial x_n)^ku(x', 0) \).

Consider the Dirichlet problem for polyharmonic equation:
\[
\Delta^m u(x) = 0
\]
with the boundary conditions
\[
\left( \frac{\partial}{\partial x_n} \right)^k u(x', 0) = f_k(x') \quad (k = 0, 1, \ldots, m - 1).
\]

We denote by \( W^{m,p}(G) \) the Sobolev space
\[
W^{m,p}(G) = \{ u \in L^p(G) : D^\lambda u \in L^p(G) \quad (|\lambda| \leq m) \}
\]
(see Stein [13, Chapter 6]). If \( u \in W^{m,p}(\mathbb{D}) \) is a solution of the Dirichlet problem, then the vertical limit \( (\partial/\partial x_n)^ku(x', 0) \) exists for almost every \( x' = (x', 0) \in \partial\mathbb{D} \) and \( 0 \leq k \leq m - 1 \) (see [6], [7]).

We also see that every function in \( W^{m,p}(\mathbb{D}) \) can be extended to a function in \( W^{m,p}(\mathbb{R}^n) \) (see Stein [13, Theorem 5, Chapter 6]). Hence Theorem 3.1 gives the following result.

**Corollary 3.1.** Let \( 1 < p \leq q < \infty \) and
\[
(0 <) \; \frac{n}{p} - \frac{n-1}{q} < 1.
\]
If \( u \in W^{m,p}(\mathbb{D}) \) is a solution of the Dirichlet problem with \( f_k(x') = (\partial/\partial x_n)^ku(x', 0) \) for \( 0 \leq k \leq m - 1 \), then
\[
\lim_{r \to 0} r^{n/p-(n-1)/q-m} S_q(U_r) = 0,
\]
where \( U_r(x') = u(x', r) - \sum_{k=0}^{m-1} (r^k/k!)f_k(x') \).

**4. Higher differences**

For \( r > 0 \) and a function \( u \), we define the first difference
\[
\Delta_r u(t) = \Delta_1^1 u(t) = u(t + r) - u(t)
\]
and the \( m \)-th difference
\[
\Delta_r^m u(t) = \Delta_{\partial r}^{m-1} (\Delta_r u(\cdot))(t),
\]
It is easy to see that
\[ \Delta_t^m u(t) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} u(t + kr). \]

As in Section 2, we consider
\[ K(x) = \frac{x}{|x|^p} \]
and define
\[ u_r(x') = \Delta_t^m U_K f(x', \cdot)(0) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} U_K f(x', kr). \]

**Theorem 4.1.** Let \( \alpha = \lfloor \lambda \rfloor - l + n \), \( 1 < p \leq q < \infty \), \( \beta < p - 1 \) and
\[ \frac{n - \alpha p}{p(n - 1)} < \frac{1}{q} \quad \text{(when } n - \alpha > 0 \text{).} \]

Further suppose \( 0 < \omega < m \), where \( \omega = (n - 1)/q - (n - \alpha p + \beta)/p \). If \( f \) is a nonnegative measurable function on \( \mathbb{R}^n \) satisfying (2.2) and (2.3), then
\[ \lim_{r \to 0} r^{-\omega} S_q(u_r) = 0, \]
where \( u_r(x') = \Delta_t^m U_K f(x', \cdot)(0) \).

To prove this, we have only to prepare the following two lemmas instead of Lemmas 2.2 and 2.3.

**Lemma 4.1.** Let \( \beta > -1 \), \( q > 0 \) and \( |\lambda| - l + n/q > 0 \). Let \( m \) be a positive integer such that
\[ 0 < |\lambda| - l + \frac{n + \beta}{q} < m. \]

Then
\[ \left( \int |K_m^*(x, y)|^q |y_n|^\beta dy \right)^{1/q} \leq M_{\lambda} |\lambda| - l + n + \beta/q \]
for all \( x = (x', x_n) \in \mathbf{D} \), where \( K_m^*(x, y) = \Delta_t^m K(x' - y', \cdot - y_n)(0) \) for \( x = (x', x_n) \in \mathbf{D} \) and \( y = (y', y_n) \in \mathbb{R}^q \).
Proof. For $x = (x', x_n) \in \mathbf{D}$, write
\[
\left( \int |K_m^*(x, y)|^q |y_n|^2 \, dy \right)^{1/q} = U'(x_n) + U''(x_n),
\]
where
\[
U'(x_n) = \left( \int_{\{y = (y', y_n) : |x' - y| \geq (m + 2)x_n\}} |K_m^*(x, y)|^q |y_n|^2 \, dy \right)^{1/q},
\]
\[
U''(x_n) = \left( \int_{\{y = (y', y_n) : |x' - y| \leq (m + 2)x_n\}} |K_m^*(x, y)|^q |y_n|^2 \, dy \right)^{1/q}.
\]
If $|x' - y| \geq (m + 2)x_n$, then we obtain by Taylor's theorem,
\[
(4.1) \quad |K_m^*(x, y)| \leq M_x^m |x' - y|^{\lambda - l - m}.
\]
Since $|\lambda| - l - m + (n + \beta)/q < 0$, applying the polar coordinates about $x'$, we have
\[
|U'(x_n)| \leq M_x^m \left( \int_{\{y = (y', y_n) : |x' - y| \geq (m + 2)x_n\}} |x' - y|^{(\lambda - l - m)q} |y_n|^2 \, dy \right)^{1/q}
= M_x^m \left( \int_{(m + 2)x_n}^{\infty} r^{(\lambda - l - m)q + \beta} r^{-n-1} \, dr \right)^{1/q}
= M_x^{|\lambda| - l + (n + \beta)/q}.
\]
On the other hand, since $|\lambda| - l + n/q > 0$ and $|\lambda| - l + (n + \beta)/q > 0$, we have by Lemma 2.2
\[
|U''(x_n)| \leq M \sum_{k=0}^{m} \left( \int_{\{y = (y', y_n) : |x' - y| \leq (m + 2)x_n\}} |x' - y + kx_n e^{(\lambda - l - m)q} |y_n|^2 \, dy \right)^{1/q}
\leq M_x^{|\lambda| - l + (n + \beta)/q},
\]
where $e = (0, \ldots, 0, 1)$. \hfill \Box

Lemma 4.2. Let $q > 0$ and $m$ be a positive integer such that
\[
0 < |\lambda| - l + \frac{n - 1}{q} < m.
\]
If $x = (x', x_n) \in \mathbf{D}$ and $y = (y', y_n) \in \mathbb{R}^n$, then
\[
\left( \int_{\mathbb{R}^{n-1}} |K_m^*(x, y)|^q \, dx' \right)^{1/q} \leq M_x^m (|x_n| + |y_n|)^{|\lambda| - l - m + (n - 1)/q}.
\]
Proof. Let \( x = (x', x_n) \in D \) and \( y = (y', y_n) \in \mathbb{R}^n \). If \( |y_n| \geq (m+2)x_n \), then, since \(|\lambda| - l - m + (n-1)/q < 0\), we have by (4.1)

\[
\left( \int_{\mathbb{R}^n} |K_m^*(x, y)|^q \, dx' \right)^{1/q} \leq Mx_n^m \left( \int_{\mathbb{R}^n} |x' - y|^{(|\lambda| - l - m)q} \, dx' \right)^{1/q}
\]

\[
= Mx_n^m |y_n|^{(|\lambda| - l - m + (n-1)/q)}.
\]

If \( |y_n| < (m+2)x_n \), then we have by (4.1) and Lemma 2.3

\[
\left( \int_{\mathbb{R}^n} |K_m^*(x, y)|^q \, dx' \right)^{1/q} \leq Mx_n^m \left( \int_{\{x' : |x' - y| \geq 2(m+2)x_n\}} |x' - y|^{(|\lambda| - l - m)q} \, dx' \right)^{1/q}
\]

\[
+ M \sum_{k=0}^m \left( \int_{\{x' : |x' - y| \leq 2(m+2)x_n\}} |x' - y + kx_n e_i|^{(|\lambda| - l - m)q} \, dx' \right)^{1/q}
\]

\[
\leq Mx_n^{|\lambda| - l + (n-1)/q}.
\]

Therefore the required inequality now follows. \( \square \)

**Theorem 4.2.** Let \( 1 < p \leq q < \infty \),

\[
\frac{n - mp}{p(n-1)} < \frac{1}{q} \quad \text{when } n - m > 0
\]

and

\[
\frac{n - mp + \beta}{p(n-1)} < \frac{1}{q} < \frac{n + \beta}{p(n-1)}.
\]

If \( u \in BL_m(L^p_{\text{loc}}(D)) \) satisfying (3.1) for \(-1 < \beta < p - 1\) is \((m, p)\)-quasicontinuous on \(D\), then

\[
\lim_{r \to 0} r^{(n - mp + \beta)/p - (n-1)/q} S_q(U_r) = 0,
\]

where \( U_r(x') = \Delta_m^p u(x', \cdot)(0) \) for \( r > 0 \).

In fact, since \( \Delta_m^p Q = 0 \) for any polynomial \( Q \) of degree at most \( m - 1 \), we have

\[
U(x) = \Delta_m^p u(x', \cdot)(0) = \sum_{|\lambda| = m} a_\lambda \int K_{\lambda, m}(x, y) D^{\lambda} \tilde{u}(y) \, dy,
\]

where \( K_{\lambda, m}(x, y) = \Delta_m^p K_\lambda(x' - y', -y_n)(0) \) with \( K_\lambda(x) = x^\lambda |x|^{-n} \). Now we can apply Theorem 4.1 to obtain the present result.
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References