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Author(s)	Noda, Ryuzaburo
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Osaka University

## ON TRANSITIVE EXTENSIONS OF FINITE PERMUTATION GROUPS

RYUZABURO NODA

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### 1. Introduction

Let  $G$  be a permutation group on a finite set  $\Omega$ . A transitive group  $T$  on  $\Omega \cup \{\infty\}$ , where  $\infty$  denotes an additional point, is said to be a transitive extension of  $G$  if the action on  $\Omega$  of the stabilizer in  $T$  of the point  $\infty$  is permutation isomorphic to that of  $G$  on  $\Omega$ . What permutation groups have transitive extensions is a rather difficult problem. In the present paper we study this problem in the case  $G$  is simply transitive on  $\Omega$ . Firstly we give some necessary condition for a simply transitive group to have a transitive extension, and secondly, making use of it, prove the non-existence of transitive extensions of some classes of simply transitive groups with particular exceptions.

Before stating our results we define some terminology. Let a permutation group  $G$  on a finite set  $\Omega$  act (not necessarily faithfully) on subsets  $\Omega_1$  and  $\Omega_2$  of  $\Omega$ . Then we say that

(\*)  $(G, \Omega_1)$  is similar to  $(G, \Omega_2)$  on  $\Omega$ ,

if there is an element  $x$  in the symmetric group on  $\Omega$  such that

- (i)  $x$  normalizes  $G$ , and
- (ii)  $x$  interchanges the subsets  $\Omega_1$  and  $\Omega_2$  of  $\Omega$ .

Our result is as follows.

**Theorem 1.** *Let  $G$  be a simply transitive group on  $\Omega$  with self-paired orbitals  $\Delta$  and  $\Gamma$  such that*

- (i) *for  $a \in \Omega$   $(G_a, \Delta(a))$  and  $(G_a, \Gamma(a))$  are not similar on  $\Omega - \{a\}$ ,*
- (ii)  *$G$  has no orbital  $\Pi$  different from  $\Delta$  and  $\Gamma$  so that  $(G_a, \Pi(a))$  is similar to  $(G_a, \Delta(a))$  or  $(G_a, \Gamma(a))$  on  $\Omega - \{a\}$ , and*
- (iii)  *$|\Delta(a) \cap \Delta(c)| \neq 0$  for  $a \in \Omega$  and  $c \in \Gamma(a)$ .*

*Assume that  $G$  has a transitive extension. Then either of the following cases occurs:*

- (A). *For  $a \in \Omega$  and  $b \in \Delta(a)$   $G_{ab}$  has a fixed block  $\Lambda$  on  $\Delta(a) - \{b\}$  such that  $\Lambda$  is different from  $\Delta(a) \cap \Delta(b)$  and  $(G_{ab}, \Lambda)$  is similar to  $(G_{ab}, \Delta(a) \cap \Delta(b))$  on  $\Omega - \{a, b\}$ .*

(B). For  $a \in \Omega$  and  $c \in \Gamma(a)$ ,  $G_{ac}$  has a fixed block  $\Lambda$  on  $\Delta(a)$  such that  $\Lambda$  is separated from  $\Delta(a) \cap \Delta(c)$  and  $(G_{ac}, \Lambda)$  is similar to  $(G_{ac}, \Delta(a) \cap \Delta(c))$  on  $\Omega - \{a, c\}$ .

By Theorem 1 we have, for example, the following results.

**Theorem 4.3.** The symmetric group  $S_m$  or the alternating group  $A_m$  on a set  $\Sigma$ ,  $|\Sigma| = m$ , considered as a permutation group of degree  $\binom{m}{r}$  on  $r$ -element subsets of  $\Sigma$  with  $1 < r < m-1$  has no transitive extension except the cases  $(r, m) = (2, 4), (2, 5)$  and  $(2, 6)$ .

**Theorem 4.4.** A subgroup of  $P\Gamma L(m, q)$  containing  $PSL(m, q)$  with  $q > 2$  and  $m > 3$  considered as a permutation group of degree  $\frac{(q^m-1)(q^m-q)\cdots(q^m-q^r)}{(q^{r+1}-1)(q^{r+1}-q)\cdots(q^{r+1}-q^r)}$  on  $r$ -dimensional subspaces of  $PG(m-1, q)$  with  $1 \leq r \leq m-3$  has no transitive extension.

The case  $r=0$  in Theorem 4.4. was treated by H. Zassenhaus [5].

Acknowledgement. Professor E. Bannai has kindly pointed out that the conclusion of Theorem 1 remains valid without the assumptions (i) and (ii) if all orbitals of  $G$  are self-paired (see the proof of Theorem 1 in section 3). The author wishes to thank him for his helpful comments. The author also wishes to thank Professor H. Nagao for his advice and encouragement.

## 2. Notation, definitions and preliminary results

Let  $G$  be a permutation group on a finite set  $\Omega$ . For points  $a, b, c, \dots$  of  $\Omega$  we denote by  $G_{a,b,c,\dots}$  and  $G_{\{a,b,c,\dots\}}$  the pointwise and the global stabilizer in  $G$  of the set  $\{a, b, c, \dots\}$ , respectively. A subset  $\Delta$  of  $\Omega$  is a fixed block of  $G$  if  $G$  fixes  $\Delta$  as a set. If  $\Delta$  is a fixed block of  $G$  the restriction of  $G$  to  $\Delta$  and the kernel of the restriction of  $G$  to  $\Delta$  are denoted by  $G^\Delta$  and  $G_\Delta$ , respectively. For the remainder of this section  $G$  is assumed to be simply transitive on  $\Omega$ . Then an orbital of  $G$  is a mapping  $\Delta$  from  $\Omega$  into the subsets of  $\Omega$  such that

(i)  $\Delta(a)$  is an orbit of  $G_a$  for  $a \in \Omega$ , and

(ii)  $\Delta(a)^g = \Delta(a^g)$  for all  $a \in \Omega, g \in G$ .

An orbital of  $G$  is self-paired if  $b \in \Delta(a)$  implies  $a \in \Delta(b)$ . Now let  $G$  have a transitive extension  $T$  on  $\Omega \cup \{\infty\}$ . Then for  $c \in \Omega \cup \{\infty\}$  we denote by  $\Delta_c$  the orbital of  $T_c$  considered as a transitive group on  $\Omega \cup \{\infty\} - \{c\}$  such that

(i).  $\Delta_\infty = \Delta$ , and

(ii).  $\Delta_c(d) = \{\Delta_\infty(d^{g^{-1}})\}^g$  for all  $g = (\infty \dots \infty) \in T, d \in \Omega \cup \{\infty\} - \{c\}$ .

In this notation we have:

### Lemma 2.

(i).  $\{\Delta_a(b)\}^g = \Delta_{a^g}(b^g)$  for all  $a, b \in \Omega \cup \{\infty\}, g \in T$ .

- (ii). We have  $\Delta_a(b) = \Delta_b(a)$  for all  $a, b \in \Omega \cup \{\infty\}$  if there exists no orbital  $\Pi$  of  $G$  such that for  $c \in \Omega$   $(G_c, \Pi(c))$  is similar to  $(G_c, \Delta(c))$  on  $\Omega - \{c\}$ .
- (iii). If  $\Delta$  is self-paired then  $\Delta_a$  is self-paired for all  $a \in \Omega \cup \{\infty\}$ . If further  $\Delta_a(b) = \Delta_b(a)$ , then for  $c \in \Delta_a(b)$ ,  $T_{\{a, b, c\}}$  acts as  $S_3$  on  $\{a, b, c\}$ .

Proof.

- (i). Clear by the definition of  $\Delta_a$ .
- (ii). Let  $y$  be an element in  $T$  of the form  $(a, b) \dots$ . Then  $y$  normalizes  $T_{ab}$  and by the assumption on  $\Delta$ ,  $y$  fixes the orbit  $\Delta_a(b)$  of  $T_{ab}$  on  $\Omega \cup \{\infty\} - \{a, b\}$ . Hence by (i) we have  $\Delta_a(b) = \Delta_b(a)$ .
- (iii). Assume that  $\Delta = \Delta_\infty$  is self-paired, and let  $e \in \Delta(d)$ . Then  $T$  contains an element  $x$  of the form  $(\infty)(e, d) \dots$ . Then for an element  $y = (\infty \dots)$  in  $T$  we have that  $e^y \in \Delta_a(d^y)$ , and  $x^y = (a)(e^y, d^y) \dots$ . Hence  $\Delta_a$  is self-paired. Now let  $c \in \Delta_a(b) = \Delta_b(a)$ . Then  $T$  has elements of forms  $(a)(b, c) \dots$  and  $(b)(a, c) \dots$ . Thus  $T_{\{a, b, c\}}$  acts as  $S_3$  on  $\{a, b, c\}$ .

### 3. Proof of Theorem 1

Let the orbitals  $\Delta$  and  $\Gamma$  of  $G$  satisfy the assumptions of Theorem 1. In a usual way we define a graph structure on  $\Omega$  as follows; a pair  $\{a, b\}$  of distinct points in  $\Omega$  is said to be an edge if  $b \in \Delta(a)$  or equivalently if  $a \in \Delta(b)$ . Assume that  $G$  has a transitive extension  $T$  on  $\Omega \cup \{\infty\}$ , and let  $a$  be a fixed point in  $\Omega$ . Then by making use of the orbital  $\Delta_a$  of  $T_a$  defined in section 2 we have as above a graph structure on  $\Omega \cup \{\infty\} - \{a\}$ . To distinguish the edges defined by  $\Delta_\infty$  and  $\Delta_a$  we say that

- (\*) a pair  $\{b, c\}$  of points on  $\Omega \cup \{\infty\}$  is a blue edge if  $b \in \Delta_\infty(c)$  and a red edge if  $b \in \Delta_a(c)$ .

Note that an element  $g = (\infty \dots)$  in  $T$  carries blue edges to red ones. Now consider the stabilizer  $T_{\infty a}$  of  $\infty$  and  $a$ , and let  $b$  be a point in  $\Delta_\infty(a)$  ( $= \Delta_a(\infty)$ ). Then the global stabilizer  $T_{\{\infty, a, b\}}$  in  $T$  of the set  $\{\infty, a, b\}$  acts as  $S_3$  on it by Lemma 2 (iii). Then an element in  $T_{\{\infty, a, b\}}$  of the form  $(\infty a)(b) \dots$  carries  $\Delta_\infty(a) \cap \Delta_\infty(b)$  to  $\Delta_a(\infty) \cap \Delta_a(b)$  (Lemma 2 (i)). Thus, if  $|\Delta_\infty(a) \cap \Delta_\infty(b)| \neq 0$ ,  $(T_{\infty ab}, \Delta_\infty(a) \cap \Delta_\infty(b))$  and  $(T_{\infty ab}, \Delta_a(\infty) \cap \Delta_a(b))$  are similar in our sense. Assume now that Case A of Theorem 1 does not occur. Then it follows that  $\Delta_\infty(a) \cap \Delta_\infty(b) = \Delta_a(\infty) \cap \Delta_a(b)$ . Then taking an element  $x$  in  $T_{\{\infty, a, b\}}$  of the form  $(\infty b)(a) \dots$  and considering the image of  $\Delta_\infty(a) \cap \Delta_\infty(b) = \Delta_a(\infty) \cap \Delta_a(b)$  under  $x$  we conclude that  $\Delta_\infty(a) \cap \Delta_\infty(b) = \Delta_a(\infty) \cap \Delta_a(b) = \Delta_\infty(b) \cap \Delta_a(b)$ . In particular  $\Delta_\infty(b) \cap \Delta_a(b)$  is contained in  $\Delta_\infty(a)$ . This implies that there is no pair  $\{b, d\}$  with  $d \in \Omega \cup \{\infty\} - \{\infty, a\} \cup \Delta_\infty(a)$  which is both a blue edge and a red edge. This is also true if  $|\Delta_\infty(a) \cap \Delta_\infty(b)| = 0$ . Then for a point  $c$  in  $\Gamma(a)$  with  $|\Delta(a) \cap \Delta(c)| \neq 0$ ,  $\Delta_\infty(a) \cap \Delta_\infty(c)$  and  $\Delta_a(\infty) \cap \Delta_a(c)$  are fixed

blocks of  $T_{\infty ac}$  which have no point in common. Furthermore since  $T_{\{\infty, a, c\}}$  acts as  $S_3$  on  $\{\infty, a, c\}$  it follows that  $(T_{\infty, a, c}, \Delta_a(\infty) \cap \Delta_a(c))$  and  $(T_{\infty, a, c}, \Delta_\infty(a) \cap \Delta_\infty(c))$  are similar on  $\Omega - \{a, c\}$ .

This completes the proof of Theorem 1.

#### 4. Some applications of Theorem 1

**Proposition 4.1.** *Let  $G$  be a 4-fold transitive group on a set  $\Sigma$ ,  $|\Sigma|=m$ . Assume that the rank 3 group  $G$  of degree  $\binom{m}{2}$  on 2-element subsets of  $\Sigma$  has a transitive extension  $T$ . Then one of the following holds:*

- (i).  $m=4$ ,  $G$  is  $S_4$ , and  $T$  is  $PSL(3, 2)$ ,
- (ii).  $m=6$ ,  $G$  is  $A_6$  or  $S_6$ , and  $T$  is  $A_6 \cdot E_{16}$  or  $S_6 \cdot E_{16}$ , the semi-direct product of elementary abelian 2-group  $E_{16}$  of order 16 by  $A_6$  or  $S_6$ ,
- (iii).  $m \geq 7$  and the stabilizer in  $G$  of four points in  $\Sigma$  has an orbit of length two on the remaining points.

*In particular if  $G$  is 5-fold transitive on  $\Sigma$  then  $m=6$ ,  $G$  is  $S_6$  and  $T$  is  $S_6 \cdot E_{16}$ .*

**Proof.** Let  $\Omega$  be the set of unordered pairs of points in  $\Sigma$ . For an element  $\{1, 2\}$  in  $\Omega$  set  $\Delta(\{1, 2\}) = \{\{i, j\} \mid |\{i, j\} \cap \{1, 2\}| = 1\}$  and  $\Gamma(\{1, 2\}) = \{\{i, j\} \mid |\{i, j\} \cap \{1, 2\}| = 0\}$ . Then  $\Delta$  and  $\Gamma$  are self-paired orbitals of  $G$  such that  $|\Delta(\{1, 2\})| = 2(m-2)$ ,  $|\Gamma(\{1, 2\})| = \binom{m-2}{2}$  and  $|\Delta(\{1, 2\}) \cap \Delta(\{i, j\})| = m-2$  or 4 according as  $\{i, j\} \in \Delta(\{1, 2\})$  or  $\Gamma(\{1, 2\})$ . Since  $G$  is 4-fold transitive on  $\Sigma$  the stabilizer in  $G$  of  $\{1, 2\}$  and  $\{1, 3\}$  in  $\Omega$  has three orbits on  $\Delta(\{1, 2\}) - \{1, 3\}$  of lengths 1,  $m-3$ , namely  $\{2, 3\}$ ,  $\{\{1, i\} \mid 4 \leq i \leq m\}$  and  $\{\{2, i\} \mid 4 \leq i \leq m\}$ . Now assume that  $G$  has a transitive extension  $T$  on  $\Omega \cup \{\infty\}$ . For  $\beta \in \Delta_\infty(\alpha)$  with  $\alpha = \{1, 2\}$ ,  $\beta = \{1, 3\} \in \Omega$  set  $\nu = |\Delta_\infty(\alpha) \cap \Delta_\alpha(\beta) \cap \Delta_\beta(\infty)|$ . Then since  $\Delta_\infty(\alpha) \cap \Delta_\alpha(\beta) \cap \Delta_\beta(\infty)$  is a fixed block of  $T_{\infty \alpha \beta}$  we have  $\nu = 1$  or  $m-2$ . Note that  $\nu \neq m-3$  if  $m > 4$ , because then both  $\Delta_\infty(\alpha) \cap \Delta_\infty(\beta)$  and  $\Delta_\alpha(\infty) \cap \Delta_\alpha(\beta)$  must contain  $\{2, 3\}$ . We first prove:

**Lemma 4.2.** *If  $\nu = 1$  then  $m = 4$  and  $G$  is  $S_4$ .*

**Proof.** We assume that  $m > 4$  and seek a contradiction. Let  $\{1, 2\}$  be a fixed element in  $\Omega$  and for simplicity we denote  $\{1, 2\}$  by  $\alpha$ ,  $\{1, i\}$  and  $\{2, i\}$  by  $\beta_i$  and  $\delta_i$  ( $i=3, \dots, m$ ), respectively. We say that a pair  $\{\varepsilon, \varepsilon'\}$  of points  $\varepsilon, \varepsilon'$  in  $\Delta_\infty(\alpha)$  is a blue edge if  $\varepsilon \in \Delta_\infty(\varepsilon')$  and a red edge if  $\varepsilon \in \Delta_\alpha(\varepsilon')$ . Since  $\nu = 1$  and  $m > 4$ ,  $\{\beta_i, \delta_i\}$ 's ( $i=3, \dots, m$ ) are the only edges in  $\Delta_\infty(\alpha)$  which are both red and blue. Now let  $x = (\infty)(\alpha)(\beta_3) \dots$  be an element of  $T$ . We first show that we can choose  $x$  to be an involution. Since  $G$  is 4-fold transitive on  $\Sigma$ ,  $G$  contains an involution  $y$  having the form  $(1)(23) \dots$  on  $\Sigma$ . Then the action of  $y$  on  $\Omega$  is of the form  $(\infty)(\alpha\beta_3) \dots$ . Since  $T_{\{\infty, \alpha, \beta_3\}}$  acts as  $S_3$  on  $\{\infty, \alpha, \beta_3\}$  we can take  $x$  to be conjugate to  $y$ . Now  $x$  carries red edges to blue ones and conversely.

Hence  $x$  fixes  $\delta_3$  and carries  $\beta_i (i \geq 4)$  to some  $\delta_j (j \geq 4)$ . Furthermore if  $x$  carries  $\beta_i$  to  $\delta_j$  (hence  $\delta_j$  to  $\beta_i$ ),  $\{\beta_i, \delta_j\}$  is an edge which are both blue and red. Hence we conclude that  $x$  is of the form  $(\infty \alpha)(\beta_3)(\delta_3)(\beta_4 \delta_4)(\beta_5 \delta_5) \cdots (\beta_m \delta_m)$ . Now let  $z$  be an involution of the form  $(\infty \alpha)(\beta_4) \cdots$ . Then, similarly to the above,  $z$  has the form  $(\infty \alpha)(\beta_3 \delta_3)(\beta_4)(\delta_4)(\beta_5 \delta_5) \cdots (\beta_m \delta_m)$ . Hence it follows that  $xz = (\infty)(\alpha)(\beta_3 \delta_3)(\beta_4 \delta_4)(\beta_5)(\delta_5) \cdots (\beta_m)(\delta_m)$ . Then  $xz$  is an element of  $G$  and the action of  $xz$  on  $\Sigma$  must be of the form  $(12)(3)(4) \cdots$ . But then  $xz$  can not fix  $\beta_5$ , a contradiction. This complete the proof of Lemma 4.2.

We now complete the proof of proposition 4.1. we may assume that  $v = m - 2$ . This implies that blue edges and red edges on  $\Delta_\infty(\{1, 2\})$  coincide and that there is no pair  $\{\delta, \gamma\}$  with  $\delta \in \Delta_\infty(\{1, 2\})$  and  $\gamma \in \Gamma_\infty(\{1, 2\})$  which is both a blue edge and a red edge. Now set  $\alpha = \{1, 2\}$ ,  $\gamma = \{3, 4\}$  and let  $g = (\infty \alpha)(\gamma) \cdots$  be an element of  $T$ . Then  $(\Delta_\infty(\alpha) \cap \Delta_\infty(\gamma))^g = \Delta_\alpha(\infty) \cap \Delta_\alpha(\gamma)$  is a fixed block of  $T_{\infty \alpha \gamma}$  on  $\Delta_\infty(\alpha)$  which is disjoint from  $\Delta_\infty(\alpha) \cap \Delta_\infty(\gamma)$ . Furthermore since edges in  $\Delta_\infty(\alpha) \cap \Delta_\infty(\gamma)$  are carried by  $g$  to edges in  $\Delta_\alpha(\infty) \cap \Delta_\alpha(\gamma)$  we see that  $\Delta_\alpha(\infty) \cap \Delta_\alpha(\gamma) = \{\{1, i\}, \{1, j\}, \{2, i\}, \{2, j\}\}$  for some  $i, j$  in  $\Sigma - \{1, 2, 3, 4\}$ . This implies that  $\{i, j\}$  is a fixed block of  $G_{\{1, 2\}\{3, 4\}}$  on  $\Sigma - \{1, 2, 3, 4\}$ . Then  $G_{1234}$  fixes  $\{i, j\}$  pointwise or as a set. If the former case occurs  $G$  is  $A_6$  or  $M_{11}$  by a result of H. Nagao [3]. But  $M_{11}$  considered as a rank 3 group of degree 55 has no transitive extension on 56 points. This is seen as follows. Let  $T$  be a transitive extension of  $M_{11}$ . Then since  $M_{11}$  is simple,  $T$  is also simple and has order equal to  $|M_{22}|$ , whence  $T$  is  $M_{22}$  by [4], contradicting a well known fact that  $M_{11}$  is not a subgroup of  $M_{22}$  (see [1]). This completes the proof of Proposition 4.1.

We now prove the following

**Theorem 3.4.** *The symmetric group  $S_m$  or the alternating group  $A_m$  on a set  $\Sigma$ ,  $|\Sigma| = m$  considered as permutation group of degree  $\binom{m}{r}$  on  $r$ -element subsets of  $\Sigma$  with  $1 < r < m - 1$  has no transitive extension except the cases  $(r, m) = (2, 4), (2, 5)$  and  $(2, 6)$ .*

**Proof.** We may assume without any loss of generality that  $2r \leq m$ . Let  $\Omega$  be the set of  $r$ -element subsets of  $\Sigma$ . For an element  $\alpha = \{1, 2, \dots, r\}$  of  $\Omega$  set

$$\Delta(\alpha) = \{\{i_1, i_2, \dots, i_r\} \mid |\{i_1, i_2, \dots, i_r\} \cap \{1, 2, \dots, r\}| = r - 1\}, \text{ and}$$

$$\Gamma(\alpha) = \{\{i_1, i_2, \dots, i_r\} \mid |\{i_1, i_2, \dots, i_r\} \cap \{1, 2, \dots, r\}| = r - 2\}.$$

Then  $\Delta$  and  $\Gamma$  are self-paired orbitals of  $G$  such that  $|\Delta(\alpha)| = r(m - r)$ ,  $|\Gamma(\alpha)| = \binom{r}{2} \binom{m - r}{2}$ , and

$$|\Delta(\alpha) \cap \Delta(\beta)| = \begin{cases} m-2 & \text{if } \beta \in \Delta(\alpha), \\ 4 & \text{if } \beta \in \Gamma(\alpha). \end{cases}$$

$\Delta$  and  $\Gamma$  satisfy the assumptions of Theorem 1 except the case  $m=2r^*$ . Let  $\beta = \{1, 2, \dots, r-1, r+1\}$  be an element of  $\Delta(\alpha)$ . Then  $G_{\alpha\beta} = G_{\{1,2,\dots,r-1\}\{r\}\{r+1\}}$  has three orbits on  $\Delta(\alpha) - \{\beta\}$ , namely

$$\begin{aligned} \Phi_1 &= \{\{i_1, i_2, \dots, i_{r-1}, r+1\} \mid \{1, 2, \dots, r-1\} \neq \{i_1, i_2, \dots, i_{r-1}\} \subset \{1, 2, \dots, r\}\}, \\ \Phi_2 &= \{\{1, 2, \dots, r-1, i\} \mid r+2 \leq i \leq m\}, \text{ and} \\ \Phi_3 &= \{\{i_1, i_2, \dots, i_{r-2}, r, i\} \mid \{i_1, i_2, \dots, i_{r-2}\} \subset \{1, 2, \dots, r-1\}, r+2 \leq i \leq m\}. \end{aligned}$$

Here we have that  $|\Phi_1| = r-1$ ,  $|\Phi_2| = m-r-1$  and  $|\Phi_3| = (r-1)(m-r-1)$  and  $\Delta(\alpha) \cap \Delta(\beta) = \Phi_1 \cup \Phi_2$ . Therefore if  $G_{\alpha\beta}$  has a fixed block  $\Lambda$  on  $\Delta(\alpha) - \{\beta\}$  such that  $\Lambda \neq \Delta(\alpha) \cap \Delta(\beta)$  and  $(G_{\alpha\beta}, \Lambda)$  is similar to  $(G_{\alpha\beta}, \Delta(\alpha) \cap \Delta(\beta))$  it follows that  $(r-1)(m-r-1) = r-1$  or  $m-r-1$ , hence  $m=r+1$  or  $r=2$ . The former case is out of our consideration and the latter was treated in Prop 4.1. Thus we may assume that Case A of Theorem 1 does not occur.

Now let  $\delta = \{1, 2, \dots, r-1, r+2\}$  be an element of  $\Gamma(\alpha)$ . Then  $G_{\alpha\delta} = G_{\{1,2,\dots,r-2\}\{r-1,r\}\{r+1,r+2\}}$  has four orbits on  $\Delta(\alpha)$ , namely

$$\begin{aligned} \Psi_1 &= \{\{1, 2, \dots, r-2, i, j\} \mid r-1 \leq i \leq r, r+1 \leq j \leq r+2\}, \\ \Psi_2 &= \{\{i_1, i_2, \dots, i_{r-1}, j\} \mid \{1, 2, \dots, r-2\} \not\subset \{i_1, i_2, \dots, i_{r-1}\} \subset \{1, 2, \dots, r\}, r+1 \leq j \leq r+2\}, \\ \Psi_3 &= \{\{1, 2, \dots, r-2, i, j\} \mid r-1 \leq i \leq r, r+3 \leq j \leq m\} \text{ and} \\ \Psi_4 &= \{\{i_1, i_2, \dots, i_{r-1}, j\} \mid \{1, 2, \dots, r-2\} \not\subset \{i_1, i_2, \dots, i_{r-1}\} \subset \{1, 2, \dots, r\}, r+3 \leq j \leq m\}. \end{aligned}$$

Here we have that  $|\Psi_1| = 4$ ,  $|\Psi_2| = 2(r-2)$ ,  $|\Psi_3| = 2(m-r-2)$ ,  $|\Psi_4| = (m-r-2)(r-2)$  and  $\Psi_1 = \Delta(\alpha) \cap \Delta(\delta)$ . Hence Case B of Theorem 1 may possibly hold only if  $|\Psi_1| = |\Psi_2|$ ,  $|\Psi_3|$  or  $|\Psi_4|$ , namely  $r=4$ ,  $r=m-4$  or  $(r, m) = (3, 9)$ . If  $r=m-4$  then  $(r, m) = (3, 7)$  or  $(4, 8)$  because  $2r \leq m$ . We first eliminate the cases  $(r, m) = (3, 7)$  and  $(3, 9)$ . Assume that  $A_7$  or  $S_7$  of degree  $\binom{7}{3}$  has a transitive extension  $T$ , and let  $N$  denote a minimal normal subgroup of  $T$ . Then  $N$  is simple and since  $N \cap S_7$  is a normal subgroup of  $S_7$  it follows that either  $N = T$  or  $N$  has index two in  $T$ , contradicting a result of M. Hall [2]. Now assume that  $|\Psi_1| = |\Psi_4|$  and hence  $(r, m) = (3, 9)$ . In this case the kernels of the restrictions of  $G_{\alpha\beta}$  to  $\Psi_1$  and  $\Psi_4$  are  $G_{12345\{6789\}}$  and  $G_{1\{2,3\}\{45\}6789}$ , respectively and hence are not isomorphic as abstract groups. Hence  $(G_{\alpha\beta}, \Psi_1)$  and

\* Even in this case the conclusion of Theorem 1 holds since all orbitals of  $G$  are self-paired (see §1).

$(G_{\omega\beta}, \Psi_4)$  are not similar in our sense. Finally we consider the case  $r=4$ . Let  $\theta$  and  $\pi$  be the orbitals of  $G$  defined as follows:

$$\begin{aligned}\theta(\{1, 2, 3, 4\}) &= \{\{i_1, i_2, i_3, i_4\} \mid \{i_1, i_2, i_3, i_4\} \cap \{1, 2, 3, 4\} = \{1\}\}, \text{ and} \\ \pi(\{1, 2, 3, 4\}) &= \{\{i_1, i_2, i_3, i_4\} \mid \{i_1, i_2, i_3, i_4\} \cap \{1, 2, 3, 4\} = \emptyset\}.\end{aligned}$$

Assume first that  $m \geq 10$ . Then  $\theta$  and  $\pi$  satisfy the assumptions of Theorem 1 for  $\Delta$  and  $\Gamma$ , respectively. We have that  $|\theta(\alpha)| = 4 \binom{m-4}{3}$ ,  $|\pi(\alpha)| = \binom{m-4}{4}$  and

$$|\theta(\alpha) \cap \theta(\varepsilon)| = \begin{cases} \binom{m-7}{3} + 9 \binom{m-7}{2} & \text{if } \varepsilon \in \theta(\alpha), \\ 16 \binom{m-8}{2} & \text{if } \varepsilon \in \pi(\alpha). \end{cases}$$

For  $\alpha = \{1, 2, 3, 4\}$  and  $\varepsilon = \{1, 5, 6, 7\}$  of  $\theta(\alpha)$ ,  $G_{\omega\varepsilon} = G_{\{1\}\{2,3,4\}\{5,6,7\}}$  has seven orbits on  $\theta(\alpha) - \{\varepsilon\}$ , namely

$$\begin{aligned}P_1 &= \{\{1, i, j, k\} \mid \{i, j, k\} \subset \{8, 9, \dots, m\}\}, \\ P_2 &= \{\{i, j, k, l\} \mid 5 \leq i \leq 7, 2 \leq j \leq 4, 8 \leq k, l \leq m\}, \\ P_3 &= \{\{1, i, j, k\} \mid 5 \leq i \leq 7, 8 \leq j, k \leq m\}, \\ P_4 &= \{\{i, j, k, l\} \mid 5 \leq i, j \leq 7, 2 \leq k \leq 4, 8 \leq l \leq m\}, \\ P_5 &= \{\{5, 6, 7, i\} \mid 2 \leq i \leq 4\}, \\ P_6 &= \{\{1, i, j, k\} \mid 5 \leq i, j \leq 7, 8 \leq k\}, \text{ and} \\ P_7 &= \{\{i, j, k, l\} \mid 2 \leq i \leq 4, 8 \leq j, k \leq m\}.\end{aligned}$$

Here  $|P_1| = \binom{m-7}{3}$ ,  $|P_2| = 9 \binom{m-7}{2}$ ,  $|P_3| = 3 \binom{m-7}{2}$ ,  $|P_4| = 9(m-7)$ ,  $|P_5| = 3$ ,  $|P_6| = 3(m-7)$ ,  $|P_7| = 3 \binom{m-7}{3}$ , and  $\theta(\alpha) \cap \theta(\varepsilon) = P_1 \cup P_2$ .

Also for an element  $\rho = \{5, 6, 7, 8\}$  of  $\pi(\alpha)$   $G_{\omega\rho} = G_{\{1,2,3,4\}\{5,6,7,8\}}$  has four orbits on  $\theta(\alpha)$ , namely

$$\begin{aligned}O_1 &= \{\{i, j, k, l\} \mid 5 \leq i \leq 8, 1 \leq j \leq 4, 9 \leq k, l \leq m\}, \\ O_2 &= \{\{i, j, k, l\} \mid 5 \leq i, j \leq 8, 1 \leq k \leq 4, 9 \leq l \leq m\}, \\ O_3 &= \{\{i, j, k, l\} \mid 5 \leq i, j, k \leq 8, 1 \leq l \leq 4\}, \text{ and} \\ O_4 &= \{\{i, j, k, l\} \mid 1 \leq i \leq 4, 9 \leq j, k, l \leq m\}.\end{aligned}$$

Here  $|O_1| = 16 \binom{m-8}{2}$ ,  $|O_2| = 24(m-8)$ ,  $|O_3| = 16$ ,  $|O_4| = 4 \binom{m-8}{3}$  and  $O_1 = \theta(\alpha) \cap \theta(\rho)$ . Hence we see that the conclusion of Theorem 1 may possibly hold only in the following cases.

Case 1.  $|P_1| + |P_2| = |P_1| + |P_7|$ .

Case 2.  $|P_1| + |P_2| = |P_3| + |P_2|$ .

Case 3.  $|O_1| = |O_4|$ .

Case 4.  $|O_1| = |O_2|$ .

We treat these cases separately.

Case 1. In this case  $m=18$ . We see that  $G_{\alpha\beta}$  is faithful on  $P_2$ , but not on  $P_7$ . Hence Case A of theorem 1 does not hold in this case.

Case 2. In this case  $m=18$ , and the kernels of the restrictions of  $G_{\alpha\beta}$  to  $P_1$  and  $P_3$  have distinct orders.

Case 3. In this case  $m=28$ , and  $G_{\alpha\beta}$  is faithful on  $O_1$ , but not on  $O_4$ .

Case 4. In this case  $m=12$ . Assume that  $G$  has a transitive extension  $T$  on  $\Omega \cup \{\infty\}$ , where  $\Omega$  denotes the set of four element subsets of  $\Sigma$ ,  $|\Sigma|=12$ . Let  $x$  be an element of order three in  $G$  having the form  $(i, j, k)$  on  $\Sigma$ . Then  $x$  has 135 fixed points on  $\Omega$ , hence 136 fixed points on  $\Omega \cup \{\infty\}$ . In particular, if  $x^t \in G$  for  $t \in T$ , then  $x^t = x^g$  for some  $g \in G$ . Then since  $G$  contains 440 conjugates of  $x$  it follows that the number of conjugates of  $x$  in  $T$  is equal to  $\frac{136}{496} \times 440$ , which is not an integer, a contradiction.

Finally the cases  $(r, m)=(4, 8)$  and  $(4, 9)$  are eliminated by a similar argument to Case 4.

**Theorem 4.4** *A subgroup of  $P\Gamma L(m, q)$  containing  $PSL(m, q)$  with  $q > 2$  and  $m \geq 4$  considered as a permutation group of degree  $\frac{(q^m-1)(q^m-q)\cdots(q^m-q^r)}{(q^{r+1}-1)(q^{r+1}-q)\cdots(q^{r+1}-q^r)}$  on  $r$ -dimensional subspaces of  $PG(m-1, q)$  with  $1 \leq r \leq m-3$  has no transitive extension.*

**Proof.** Let  $\Omega$  denote the set of  $r$ -dimensional subspaces of  $PG(m-1, q)$ . For an element  $\alpha$  of  $\Omega$  set

$$\Delta(\alpha) = \{\beta \in \Omega \mid \dim. \alpha \cap \beta = r-1\}, \text{ and} \\ \Gamma(\alpha) = \{\beta \in \Omega \mid \dim. \alpha \cap \beta = r-2\}.$$

Then  $\Delta$  and  $\Gamma$  are self-paired orbitals of  $G$  such that

$$|\Delta(\alpha)| = \frac{q(q^{r+1}-1)(q^{m-r-1}-1)}{(q-1)^2}, \\ |\Gamma(\alpha)| = \frac{q^4(q^{r+1}-1)(q^r-1)(q^{m-r-1}-1)(q^{m-r-2}-1)}{(q-1)^2(q^2-1)^2}, \text{ and} \\ |\Delta(\alpha) \cap \Delta(\beta)| = \begin{cases} \frac{q(q^{m-r-1}-1)}{q-1} - 1 + \frac{q^2(q^r-1)}{q-1} & \text{if } \beta \in \Delta(\alpha), \\ (q+1)^2 & \text{if } \beta \in \Gamma(\alpha). \end{cases}$$

It is easy to see that  $\Delta$  and  $\Gamma$  satisfy the assumptions of Theorem 1. Let  $\beta$  be an element of  $\Delta(\alpha)$ . We may assume that

$$\alpha = \left\{ \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{r+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \mid \alpha_i \in GF(q) \right\} \text{ and } \beta = \left\{ \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_r \\ 0 \\ \beta_{r+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \mid \beta_i \in GF(q) \right\}$$

Then  $G_{\alpha\beta} \cap PGL(m, q)$  has the following form:

$$\begin{matrix} & & r & & 2 & & \\ & & \boxed{\begin{matrix} * & & * & & * \end{matrix}} & & & & \\ & r & & & & & \\ & & & & \boxed{\begin{matrix} * & 0 \\ 0 & * \end{matrix}} & & \\ 2 & & 0 & & & & \\ & & & & & & \boxed{\begin{matrix} * \end{matrix}} \\ & & & 0 & & & \end{matrix}$$

It is then easy to see that  $G_{\alpha\beta}$  has the following orbits on  $\Delta(\alpha) - \{\beta\}$ .

$$\Phi_1 = \{\varepsilon \in \Delta(\alpha) \mid \dim. \varepsilon \cap \beta = r-1, \varepsilon \cap \beta \neq \varepsilon \cap \alpha\},$$

$$\Phi_2 = \{\varepsilon \in \Delta(\alpha) \mid \dim. \varepsilon \cap \beta = r-1, \varepsilon \cap \beta = \varepsilon \cap \alpha, \varepsilon \subset \alpha \cap \beta\},$$

$$\Phi_3 = \{\varepsilon \in \Delta(\alpha) \mid \dim. \varepsilon \cap \beta = r-1, \varepsilon \cap \beta = \varepsilon \cap \alpha, \varepsilon \not\subset \alpha \cup \beta\}, \text{ and}$$

$$\Phi_4 = \{\varepsilon \in \Delta(\alpha) \mid \dim. \varepsilon \cap \beta = r-2\}.$$

Here we have  $\Delta(\alpha) \cap \Delta(\beta) = \Phi_1 \cup \Phi_2 \cup \Phi_3$ ,  $|\Phi_1| = \frac{q^2(q^r-1)}{q-1}$ ,  $|\Phi_2| = q-1$ ,  $|\Phi_3| = \frac{q^2(q^{m-r-2}-1)}{q-1}$  and  $|\Phi_4| = \frac{q^3(q^r-1)(q^{m-r-2}-1)}{(q-1)^2}$ . Therefore Case A of Theorem 1 does not hold.

$$\text{Now let } \delta = \left\{ \begin{pmatrix} \delta_1 \\ \vdots \\ \delta_{r-1} \\ 0 \\ 0 \\ \delta_{r+2} \\ \delta_{r+3} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \mid \delta_i \in GF(q) \right\} \text{ be an element of } \Gamma(\alpha).$$

Then  $G_{\alpha\delta} \cap PGL(m, q)$  is of the form;

$$r-1 \quad 4$$

$$\begin{pmatrix} r-1 & & & & & \\ & * & & & & * \\ & & * & 0 & & \\ & & 0 & * & & \\ & & & & \square & \\ 4 & & & & & 0 \\ & & & & & & * \end{pmatrix},$$

and  $G_{\alpha\delta}$  has the following orbits on  $\Delta(\alpha)$ .

$$\Psi_1 = \{\varepsilon \in \Delta(\alpha) \mid \dim. \varepsilon \cap \delta = r-1\},$$

$$\Psi_2 = \{\varepsilon \in \Delta(\alpha) \mid \dim. \varepsilon \cap \delta = r-2, \varepsilon \cap \delta \neq \alpha \cap \delta\},$$

$$\Psi_3 = \{\varepsilon \in \Delta(\alpha) \mid \dim. \varepsilon \cap \delta = r-2, \varepsilon \cap \delta = \alpha \cap \delta, \varepsilon \subset \alpha \cup \delta\},$$

$$\Psi_4 = \{\varepsilon \in \Delta(\alpha) \mid \dim. \varepsilon \cap \delta = r-2, \varepsilon \cap \delta = \alpha \cap \delta, \varepsilon \not\subset \alpha \cup \delta\} \text{ and}$$

$$\Psi_5 = \{\varepsilon \in \Delta(\alpha) \mid \dim. \varepsilon \cap \delta = r-3\}.$$

$$\text{Here we have } \Psi_1 = \Delta(\alpha) \cap \Delta(\delta), |\Psi_1| = (q+1)^2, |\Psi_2| = \frac{q^3(q+1)(q^{r-1}-1)}{(q-1)},$$

$$|\Psi_3| = (q+1)(q^2-1), |\Psi_4| = \frac{q^3(q+1)(q^{m-r-3}-1)}{(q-1)} \text{ and}$$

$$|\Psi_5| = \frac{q^5(q^{r-1}-1)(q^{m-r-3}-1)}{(q-1)^2}. \text{ Then since } q \neq 2, \text{ Case B of Theorem 1 does not}$$

hold, and the proof of Theorem 4.4 is completed.

OSAKA UNIVERSITY

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