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THE VANISHING OF COHOMOLOGY ASSOCIATED TO DISCRETE SUBGROUPS OF COMPLEX SIMPLE LIE GROUPS*

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1. Introduction

Let G denote a connected complex simple Lie group and K a maximal compact subgroup of G . The quotient $M=G/K$ is a riemannian symmetric space of non-compact type. Let Γ denote a discrete subgroup of G with compact quotient $\Gamma\backslash G$, and let ρ denote an irreducible non-trivial complex representation of G in a finite dimensional complex vector space F . In this paper we prove that for such representations a certain quadratic form defined by Matsushima and Murakami [3] is positive definite, and hence $H^*(\Gamma, M, \rho)$ vanishes.

The motivation for this paper is a result of Min-Oo and Ruh [4] on comparison theorems for non-compact symmetric spaces, where an estimate from below for the first eigenvalue of the Laplace operator on 2-forms with values in a bundle associated to the adjoint representation is essential. This estimate is an immediate consequence of the positivity of the above quadratic form. The vanishing of $H^*(\Gamma, M, \rho)$, without the information on the first eigenvalue, is a special case of [1, Ch. VII, Th. 6. 7].

2. The result

Let \mathfrak{g} denote the Lie algebra of left-invariant vector fields of the simple Lie group G , $\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(F)$ the representation induced by $\rho: G \rightarrow GL(F)$, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ a Cartan decomposition of \mathfrak{g} with \mathfrak{k} the Lie algebra of a maximal compact subgroup K . We identify the Lie algebra \mathfrak{g} with the corresponding vector fields on $\Gamma\backslash G$.

Let $A(\Gamma, M, \rho)$ ($A_0(\Gamma, M, \rho)$ in the notation of Matsushima and Murakami [3]) denote the vector space of F -valued differential forms on $\Gamma\backslash G$ which are horizontal and $\text{ad}K$ -equivariant, i.e., $\eta \in A(\Gamma, M, \rho)$ satisfies $i_X \eta = 0$ and $\theta_X \eta = -\rho(X)\eta$ for all $X \in \mathfrak{k}$, where i_X is interior multiplication and θ_X is the Lie

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derivative. A q -form $\eta \in A(\Gamma, M, \rho)$ is determined by its values $\eta_{i_1, \dots, i_q} = \eta(Y_{i_1}, \dots, Y_{i_q})$ on q -tuples of basis vectors of \mathfrak{p} . According to [3, (6.7)], the Laplace operator

$$\Delta: A(\Gamma, M, \rho) \rightarrow A(\Gamma, M, \rho)$$

is a sum of a differential operator Δ_D and an operator Δ_ρ associated to the representation ρ . Restricted to q -forms these operators have the following coordinate expressions.

$$(\Delta_D \eta)(Y_{i_1}, \dots, Y_{i_q}) = - \sum_{k=1}^n Y_k^2 \eta_{i_1, \dots, i_q} + \sum_{k=1}^n \sum_{u=1}^q (-1)^u [Y_{i_u}, Y_k] \eta_{ki_1, \dots, \hat{i}_u, \dots, i_q},$$

$$(\Delta_\rho \eta)(Y_{i_1}, \dots, Y_{i_q}) = \sum_{k=1}^n \rho(Y_k)^2 \eta_{i_1, \dots, i_q} - \sum_{k=1}^n \sum_{u=1}^q (-1)^u \rho([Y_{i_u}, Y_k]) \eta_{ki_1, \dots, \hat{i}_u, \dots, i_q},$$

where $\{Y_i; i=1, \dots, n=\dim M\}$ is an orthonormal basis of \mathfrak{p} with respect to the Killing form φ of \mathfrak{g} restricted to \mathfrak{p} . As in [3], the definition of Δ requires a choice of an admissible hermitean inner product on F . The inner product \langle, \rangle_F is called admissible if for all $u, v \in F$ the following conditions hold:

$$\begin{aligned} \langle \rho(X)u, v \rangle_F &= -\langle u, \rho(X)v \rangle_F & \text{for } X \in \mathfrak{k}, \\ \langle \rho(Y)u, v \rangle_F &= \langle u, \rho(Y)v \rangle_F & \text{for } Y \in \mathfrak{p}. \end{aligned}$$

Matsushima and Murakami [3] prove that admissible hermitean inner products always exist.

The following result is well known.

Proposition 1. *The vector space $H^*(\Gamma, M, \rho)$ is canonically isomorphic to the vector space $\{\eta \in A(\Gamma, M, \rho); \Delta\eta=0\}$ of harmonic forms.*

The restriction of the Killing form φ to \mathfrak{p} together with the scalar product \langle, \rangle_F on F induce a hermitean scalar product $(,)$ on $A(\Gamma, M, \rho)$, obtained by integrating the pointwise defined scalar product

$$\langle \eta, \omega \rangle = \sum_{i_1 < \dots < i_q} \langle \eta_{i_1, \dots, i_q}, \omega_{i_1, \dots, i_q} \rangle_F.$$

Here η_{i_1, \dots, i_q} and ω_{i_1, \dots, i_q} are the coordinates of q -forms with respect to an orthonormal basis in \mathfrak{p} , and $\langle \eta, \omega \rangle$ is defined to be zero if η and ω are of different degrees.

The following result is proved in [3].

Proposition 2. *The quadratic forms $\eta \mapsto (\Delta_D \eta, \eta)$ and $\eta \mapsto (\Delta_\rho \eta, \eta)$ are positive semi-definite.*

A differential form $\eta \in A(\Gamma, M, \rho)$ is a section of the trivial vector bundle on

$\Gamma \backslash G$ with fibre $\text{Hom}(\Lambda\mathfrak{p}, F)$, the homomorphisms from the exterior algebra over \mathfrak{p} to F . The operator Δ_ρ does not involve derivatives and thus can be viewed as a linear map

$$\Delta_\rho: \text{Hom}(\Lambda\mathfrak{p}, F) \rightarrow \text{Hom}(\Lambda\mathfrak{p}, F).$$

Our main result concerns the positivity of the quadratic form $\eta \mapsto \langle \Delta_\rho \eta, \eta \rangle$ on $\text{Hom}(\Lambda\mathfrak{p}, F)$, which by Proposition 2 implies the vanishing of the cohomology vector space $H^*(\Gamma, M, \rho)$.

Theorem. *Let ρ denote an irreducible non-trivial complex representation of a complex simple Lie algebra \mathfrak{g} on a finite dimensional complex vector space F . Then the quadratic form $\eta \mapsto \langle \Delta_\rho \eta, \eta \rangle$ on $\text{Hom}(\Lambda\mathfrak{p}, F)$ is positive definite, and therefore $H^*(\Gamma, M, \rho) = (0)$.*

The basic ideas of the proof are similar to those of Raghunathan [6]. Our restriction to complex Lie groups allows us to prove the optimal result. In addition, Assertions III and IV of [6], which lead to difficulties, can be avoided.

3. The proof

The restriction to complex Lie algebras \mathfrak{g} allows us to identify $\text{Hom}_{\mathbf{R}}(\Lambda\mathfrak{p}, F)$ with $\text{Hom}_{\mathbf{C}}(\Lambda\mathfrak{g}, F)$. In the following we suppress the subscripts \mathbf{R} and \mathbf{C} . Since $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and $\mathfrak{p} = i\mathfrak{k}$, multiplication with i is a \mathbf{R} -vector space isomorphism $J: \mathfrak{k} \rightarrow \mathfrak{p}$. Let $\Lambda J: \Lambda\mathfrak{k} \rightarrow \Lambda\mathfrak{p}$ denote the induced isomorphism and define

$$\text{Hom}(\Lambda\mathfrak{p}, F) \xrightarrow{\cong} \text{Hom}(\Lambda\mathfrak{k}, F) \xrightarrow{\cong} \text{Hom}(\Lambda\mathfrak{g}, F),$$

where the first isomorphism is composition with ΛJ , and the image ξ of $\xi' \in \text{Hom}(\Lambda\mathfrak{k}, F)$ under the second isomorphism is defined by $\xi(X \otimes \lambda) = \lambda \xi'(X)$, for $X \in \Lambda\mathfrak{k}$ and $\lambda \in \mathbf{C}$.

From now on we identify $\text{Hom}(\Lambda\mathfrak{g}, F)$ with $F \otimes \Lambda\mathfrak{g}^*$ and view Δ_ρ as an element in the endomorphism ring of $F \otimes \Lambda\mathfrak{g}^*$. Let c denote the Casimir element with respect to the Killing form φ in the universal enveloping algebra $U(\mathfrak{g})$ of \mathfrak{g} . The representation ρ extends to $U(\mathfrak{g})$. In the following lemma σ denotes the dual of the representation Λad induced by the adjoint representation of \mathfrak{g} .

Lemma 1. $2\Delta_\rho = 3(\rho \otimes 1)(c) + (1 \otimes \sigma)(c) - (\rho \otimes \sigma)(c)$

This lemma proves in particular that Δ_ρ is a selfadjoint endomorphism with respect to the scalar product \langle, \rangle introduced earlier.

Proof. Let $\{X_k; k=1, \dots, n\}$ be an orthonormal basis of \mathfrak{k} with respect to $-\varphi$ restricted to \mathfrak{k} , and $\{Y_k\} = \{iX_k\}$ the corresponding basis in \mathfrak{p} . The image

$\xi \in \text{Hom}(\Lambda\mathfrak{g}, F)$ of $\eta \in \text{Hom}(\Lambda\mathfrak{p}, F)$ under the isomorphism defined above evaluated on $(X_{i_1}, \dots, X_{i_q})$ is

$$\xi(X_{i_1}, \dots, X_{i_q}) = \eta(iX_{i_1}, \dots, iX_{i_q}) = \eta(Y_{i_1}, \dots, Y_{i_q}).$$

With this identification of $\text{Hom}_{\mathbb{C}}(\Lambda\mathfrak{g}, F)$ and $\text{Hom}_{\mathbb{R}}(\Lambda\mathfrak{p}, F)$, Δ_ρ operates on ξ as follows:

$$\begin{aligned} (\Delta_\rho \xi)(X_{i_1}, \dots, X_{i_q}) &= \sum_{k=1}^n \rho(iX_k)^2 \xi(X_{i_1}, \dots, X_{i_q}) \\ &\quad - \sum_{k=1}^n \sum_{u=1}^q (-1)^u \rho([iX_{i_u}, iX_k]) \xi(X_k, X_{i_1}, \dots, \hat{X}_{i_u}, \dots, X_{i_q}) \\ &= (S\xi)(X_{i_1}, \dots, X_{i_q}) + (T\xi)(X_{i_1}, \dots, X_{i_q}). \end{aligned}$$

In view of the identification $\text{Hom}(\Lambda\mathfrak{g}, F) = F \otimes \Lambda\mathfrak{g}^*$, the first summand is given in terms of the Casimir element c as

$$S = (\rho \otimes 1)(c),$$

since $\{X_k\}$ and $\{-X_k\}$ are dual bases with respect to φ and therefore $c = -\sum X_k^2$. To deal with the second summand, we abbreviate $E = \Lambda\mathfrak{g}^*$ and specialize to $\xi = f \otimes e$ with $f \in F$ and $e \in E$. The immediate goal is to prove that in this case

$$T(f \otimes e) = \sum_{k=1}^n \rho(X_k) f \otimes \sigma(X_k) e.$$

Let c_{ij}^k denote the structure constants of \mathfrak{k} (and \mathfrak{g}) with respect to the basis $\{X_k\}$;

thus $\sum_{k=1}^n c_{ij}^k X_k = [X_i, X_j]$, and

$$\rho([iX_{i_u}, iX_k]) = -\rho([X_{i_u}, X_k]) = -\sum_{s=1}^n c_{i_u k}^s \rho(X_s) = -\sum_{s=1}^n c_{s i_u}^k \rho(X_s),$$

where the last equality holds because c_{ij}^k , in terms of an orthonormal basis with respect to $-\varphi$, is skew symmetric in each pair of indices. We have

$$(T\xi)(X_{i_1}, \dots, X_{i_q}) = \sum_{s=1}^n \sum_{u=1}^q (-1)^u \rho(X_s) \xi([X_s, X_{i_u}], X_{i_1}, \dots, \hat{X}_{i_u}, \dots, X_{i_q}).$$

Abbreviating $X = X_{i_1} \wedge \dots \wedge X_{i_q}$ we obtain $(T\xi)(X) = \sum_{k=1}^n \rho(X_k) \xi(-\Lambda \text{ad}(X_k)X)$, and for $\xi = f \otimes e$ and σ the dual representation of Λad we obtain

$$T(f \otimes e) = \sum_{k=1}^n \rho(X_k) f \otimes \sigma(X_k) e.$$

To conclude the proof we compute as in [6]

$$2\rho(X_k) \otimes \sigma(X_k) = (\rho \otimes \sigma)(X_k)^2 - \rho(X_k)^2 \otimes id_E - id_F \otimes \sigma(X_k),$$

and obtain

$$2T = (\rho \otimes 1)(c) + (1 \otimes \sigma)(c) - (\rho \otimes \sigma)(c).$$

To prove the Theorem we will show that all the eigenvalues of Δ_ρ are positive. The basic observation (see Lemma 2 below) is that for any irreducible representation ρ , the endomorphism $\rho(c)$ is a scalar operator whose eigenvalue is given in terms of the highest weight of ρ . This fact will be applied individually to the irreducible components of $\rho \otimes 1$, $1 \otimes \sigma$, and $\rho \otimes \sigma$.

First we introduce some notation. As above we fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where \mathfrak{k} is a compact real form of \mathfrak{g} and $\mathfrak{p} = i\mathfrak{k}$. Let \mathfrak{h} be a Cartan subalgebra of \mathfrak{g} compatible with the given Cartan decomposition. Then $\mathfrak{h} = \mathfrak{h}_\mathfrak{k} \oplus \mathfrak{h}_\mathfrak{p}$, where $\mathfrak{h}_\mathfrak{k} = \mathfrak{h} \cap \mathfrak{k}$ and $\mathfrak{h}_\mathfrak{p} = \mathfrak{h} \cap \mathfrak{p} = i\mathfrak{h}_\mathfrak{k}$. Let Δ denote the root system of the pair $(\mathfrak{g}, \mathfrak{h})$. To each $\alpha \in \Delta$ we associate $H_\alpha \in \mathfrak{h}$ such that $\alpha(H) = \langle H_\alpha, H \rangle$ for all $H \in \mathfrak{h}$, where the Killing form is denoted by \langle, \rangle from now on. Then $\mathfrak{h}_\mathfrak{p}$ coincides with the real vector space spanned by $\{H_\alpha; \alpha \in \Delta\}$, so Δ may be viewed as a subset of $\mathfrak{h}_\mathfrak{p}^*$, the real dual of $\mathfrak{h}_\mathfrak{p}$. The Killing form \langle, \rangle is real and positive definite on $\mathfrak{h}_\mathfrak{p}$, hence it induces a scalar product \langle, \rangle on $\mathfrak{h}_\mathfrak{p}^*$. By fixing a basis of Δ we once and for all determine a set Δ^+ of positive roots. We define $\delta = \sum_{\alpha \in \Delta^+} \alpha$.

Lemma 2. *Let $\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(F)$ be any irreducible representation of \mathfrak{g} with highest weight λ , then $\rho(c) = \langle \lambda, \lambda + \delta \rangle \cdot id_F$.*

For a proof see Raghunathan [5, Lemma 4], or Bourbaki [2, Ch. 8, §6, n° 4].

Lemma 2 immediately applies to our given representation ρ and thus enables us to compute the contribution of $3(\rho \otimes 1)(c)$ to the eigenvalues of $2\Delta_\rho$. The second term $(1 \otimes \sigma)(c)$ involves the representation $\sigma = \Lambda ad^*$ of \mathfrak{g} on $E = \Lambda \mathfrak{g}^*$. This representation is no longer irreducible, so Lemma 2 applies to each component of σ separately. Thus the knowledge of the highest weights of the irreducible components of σ is required.

Lemma 3. *Let μ be the highest weight of an irreducible component of σ . Then μ is of the form $\mu = \sum_{\alpha \in \Delta} m_\alpha \alpha$, with $m_\alpha \in \{0, 1\}$.*

Proof. The weight space decomposition of $ad: \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$ with respect to the Cartan subalgebra \mathfrak{h} equals the root space decomposition of the pair $(\mathfrak{g}, \mathfrak{h})$, i.e.,

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha.$$

The dual representation $ad^*: \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g}^*)$ leads to the analogous decomposition

$$\mathfrak{g}^* = \mathfrak{h}^* \oplus \sum_{\alpha \in \Delta} (\mathfrak{g}^*)_\alpha, \quad \text{with} \quad (\mathfrak{g}^*)_\alpha = (\mathfrak{g}_{-\alpha})^*.$$

Now let $n = \dim \mathfrak{g}^*$, $r = \dim \mathfrak{h}^*$, and observe $\dim(\mathfrak{g}^*)_\alpha = 1$. Then $E = \Lambda \mathfrak{g}^* = \Lambda(\mathfrak{h}^* \oplus \sum_{\alpha \in \Delta} (\mathfrak{g}^*)_\alpha)$ is isomorphic to a sum of subspaces of the form

$$\Lambda^h(\mathfrak{h}^*) \otimes (\mathfrak{g}^*)_{\alpha_1} \otimes \cdots \otimes (\mathfrak{g}^*)_{\alpha_k},$$

where $0 \leq h \leq r$, $0 \leq k \leq n-1$, $\alpha_i \in \Delta$, and $\alpha_i \neq \alpha_j$ for $i \neq j$.

Such a subspace is invariant under the action of \mathfrak{h} , it has weight $\alpha_1 + \cdots + \alpha_k$. This implies in particular that the highest weights of the irreducible components occurring in σ are of the form $\alpha_1 + \cdots + \alpha_k$, $\alpha_i \in \Delta$, $\alpha_i \neq \alpha_j$ for $i \neq j$.

Let now $E = \sum E_\mu$ be the decomposition of E into its irreducible components E_μ indexed by their respective highest weights. Lemma 2 enables us to compute the eigenvalues of $(1 \otimes \sigma)(c)$ on $F \otimes E_\mu$. The third term, $(\rho \otimes \sigma)(c)$, in Lemma 1 involves the representation $\rho \otimes \sigma$ of \mathfrak{g} on $F \otimes E$. This space certainly decomposes into the sum $\sum F \otimes E_\mu$, but each of the components $F \otimes E_\mu$ may further decompose into a sum $F \otimes E_\mu = \sum V_\mu^\nu$, where the subspaces V_μ^ν , irreducible under $\rho \otimes \sigma$, are indexed by their respective highest weights ν .

For each μ there is exactly one component $V_\mu^{\lambda+\mu}$ of $F \otimes E_\mu$ with highest weight $\lambda + \mu$. All other components V_μ^ν have highest weights $\nu < \lambda + \mu$. The following lemma allows us to restrict our attention to the spaces $V_\mu^{\lambda+\mu}$.

Lemma 4. *Let ρ_1, ρ_2 be two irreducible representations of \mathfrak{g} with respective highest weights λ_1, λ_2 . Then $\lambda_1 > \lambda_2$ implies*

$$\langle \lambda_1, \lambda_1 + \delta \rangle > \langle \lambda_2, \lambda_2 + \delta \rangle$$

Proof. Let $\beta = \lambda_1 - \lambda_2$ and assume $\beta > 0$. Then

$$\begin{aligned} \langle \lambda_1, \lambda_1 + \delta \rangle - \langle \lambda_2, \lambda_2 + \delta \rangle &= \\ 2\langle \lambda_2, \beta \rangle + \langle \beta, \beta \rangle + \langle \beta, \delta \rangle &> 0, \end{aligned}$$

since λ_2 and δ are dominant.

According to Lemma 4, the maximal eigenvalue of $(\rho \otimes \sigma)(c)$ restricted to $F \otimes E_\mu$ is attained on the space $V_\mu^{\lambda+\mu}$. Since $(\rho \otimes 1)(c)$ and $(1 \otimes \sigma)(c)$ are positive scalar operators on the whole space $F \otimes E_\mu$, and $(\rho \otimes \sigma)(c)$ occurs with a minus sign in $2\Delta_\rho$, the *minimal* eigenvalue of $2\Delta_\rho$ restricted to $F \otimes E_\mu$ is attained on the space $V_\mu^{\lambda+\mu}$. This minimal eigenvalue involves only λ and μ , according to Lemma 2. Our claim is now reduced to the

Assertion. *Let μ be any of the highest weights occurring in the decomposition $E = \sum E_\mu$. Then the eigenvalue of $2\Delta_\rho$ is positive on $V_\mu^{\lambda+\mu}$.*

Proof. On $V_\mu^{\lambda+\mu}$ we have

$$2\Delta_\rho = \{3\langle \lambda, \lambda + \delta \rangle + \langle \mu, \mu + \delta \rangle - \langle \lambda + \mu, \lambda + \mu + \delta \rangle\} \cdot \text{id}.$$

By a straightforward computation this reduces to

$$\Delta_p = \{\langle \lambda, \lambda \rangle + \langle \lambda, \delta - \mu \rangle\} \cdot \text{id}.$$

The term $\langle \lambda, \lambda \rangle$ is obviously positive, since λ is the highest weight of a non-trivial representation. Now $\delta = \sum_{\alpha \in \Delta^+} \alpha$, and according to Lemma 3, $\mu = \sum_{\alpha \in \Delta} m_\alpha \alpha$ with $m_\alpha \in \{0, 1\}$, hence $\delta - \mu = \sum_{\alpha \in \Delta^+} n_\alpha \alpha$ with $n_\alpha \in \{0, 1, 2\}$. Therefore $\langle \lambda, \delta - \mu \rangle = \sum_{\alpha \in \Delta^+} n_\alpha \langle \lambda, \alpha \rangle \geq 0$, since λ is dominant.

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