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| Title | A note on Azumaya's theorem |
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| Citation | Osaka Journal of Mathematics. 1967, 4(1), p. 157-160 |
| Version Type | VoR |
| URL | https://doi.org/10.18910/3729 |
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A NOTE ON AZUMAYA'S THEOREM

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(Received May 4, 1967)

We say that a left Λ -module M is a cogenerator of the category of left Λ -modules if for every submodule N_1 of a left Λ -module N there exists a Λ -homomorphism f from N to M such that $f(N_1) \neq 0$. Let $\{I_\alpha\}$ be the full set of non isomorphic irreducible Λ -modules, and $\{E_\alpha\}$ be the set of their injective hulls. Then a left Λ -module M is a cogenerator if and only if M contains every E_α . In this case the sum of all E_α 's is a direct sum by Zorn's Lemma (see Lemma 1[3]). A cogenerator is a faithful module (Lemma 2). The aim of this paper is to compare the ring for which every faithful module is a cogenerator with the ring for which every faithful module is generator (see G. Azumaya [1]). We assume every ring has units and every module is unitary.

Lemma 1. *Let M be a left Λ -module and A be an arbitrary set of index. Then the followings are equivalent.*

- (1) M is a cogenerator
- (2) $\sum_{\nu \in A}^\oplus M_\nu$ ($M_\nu \cong M$) is a cogenerator
- (3) $\prod_{\nu \in A} M_\nu$ ($M_\nu \cong M$) is a generator

Proof. (1) \Rightarrow (2) \Rightarrow (3) is clear by Lemma 1 [3]. So we shall prove (3) \Rightarrow (1).

Choose any E_α , then we have Λ -maps $E_\alpha \xrightarrow{\tau} \prod M_\nu \xrightarrow{\pi_\nu} M_\nu$, where τ is a monomorphism and π_ν are the canonical maps. Let $f_\nu = \pi_\nu \cdot \tau$ then we see $\bigcap \ker f_\nu = 0$. If $\ker f_\nu \neq 0$ for every $\nu \in A$, then $I_\alpha \subseteq \bigcap \ker f_\nu \neq 0$ since I_α is irreducible and E_α is an essential extension of I_α . Hence M_ν has an isomorphic image of E_α and M is a cogenerator.

Corollary 1. *A ring Λ is a self-cogenerator ring if and only if every E_α , i.e., $\sum^\oplus E_\alpha$ is projective.*

Proof. If $\sum^\oplus E_\alpha$ is projective, $\sum^\oplus E_\alpha < \bigoplus \sum^\oplus \Lambda$, and $\sum^\oplus \Lambda$ is a cogenerator. Hence Λ is a cogenerator by Lemma 1. Conversely if Λ is a cogenerator, $\Lambda \oplus > E_\alpha$, and E_α is projective.

Lemma 2. *If M is a cogenerator and \mathfrak{I} is a left ideal of Λ , then $\mathfrak{I} = l_\Lambda(r_M(\mathfrak{I}))$. Hence every cogenerator is faithful. Conversely assume that Λ is a left self-cogenerator ring. Then every faithful module is a cogenerator.*

Proof. In general we have $l_\Lambda(r_M(I)) \supseteq I$. If $l_\Lambda(r_M(I)) \neq I$, we have a Λ -homomorphism f from Λ/I to M such that $f(l_\Lambda(r_M(I)/I)) \neq 0$. This means that there exists $m \in M$ such that $Im = 0$ and $l_\Lambda(r_M(I)) m \neq 0$, which is a contradiction since $r_M(l_\Lambda(r_M(I))) = r_M(I)$. Thus $I = l_\Lambda(r_M(I))$. Since $0 = l_\Lambda(r_M(0)) = l(M)$, M is faithful. If Λ is a left self cogenerator ring and M is a faithful left module, we have a monomorphism f from Λ to ΠM such that $f(\lambda) = (\lambda m)_{m \in M}$. Hence ΠM is a cogenerator, consequently M is a cogenerator by Lemma 1.

Lemma 3. *If Λ is a self cogenerator ring, $l(x)$ is 0 or a minimal left ideal for every maximal right ideal of Λ and $r(I)$ is an essential extension of a minimal right ideal for every maximal left ideal. Consequently every right ideal (resp. if Λ is right lower distinguished, every left ideal) contains a minimal right (resp. left) ideal of Λ .*

Proof. If $l(x) \neq 0$, $r = r(l(x)) = r(I')$, and $l(x) = l(r(I')) = I'$, where I' is an arbitrary left subideal of $l(x)$. Thus $l(x)$ is a minimal left ideal. Let I be a maximal left ideal. Then there exists a minimal left ideal Λx such that $\Lambda/I \cong \Lambda x$ and $E(\Lambda x) \subseteq \Lambda$. Let $y \in r(I)$, then $\Lambda y \cong \Lambda/I \cong \Lambda x$. Hence there exists $z \in E(\Lambda x)$ such that $yz = x$, since $E(\Lambda x)$ is injective. Therefore $y\Lambda \supseteq x\Lambda$ for any $y \in r(I)$, and we see $x\Lambda$ is minimal and $r(I)$ is an essential extension of $x\Lambda$. Let y be an arbitrary element of Λ and I a maximal left ideal such that $I \supseteq l(y)$. Then there is a Λ -homomorphism $\Lambda y \cong \Lambda/I(y) \rightarrow \Lambda/I \cong \Lambda x$, where $E(\Lambda x) \subseteq \Lambda$. Then $yz = x$ for some $z \in E(\Lambda x)$, and $x\Lambda \subseteq y\Lambda$. Since $x \in r(I)$, $x\Lambda$, consequently, $y\Lambda$ contains a minimal right ideal. Thus every right ideal contains a minimal right ideal. If Λ is right lower distinguished, $l(x) \neq 0$ for every maximal right ideal. Let I be an arbitrary left ideal. Then $I = l(r(I)) \supseteq l(x)$, where r is a maximal right ideal such that $r \supseteq r(I)$. Therefore I contains a minimal left ideal.

Proposition 1. *Let Λ be a left self cogenerator ring and $\{E_\alpha\}, \{I_\alpha\}$ be as before. If $E_\alpha = \Lambda e_\alpha$, then $e_\alpha \Lambda$ is an essential extension of a minimal right ideal $x_\alpha \Lambda$. $x_\alpha \Lambda$ and $x_\beta \Lambda$ are not isomorphic for any $e_\alpha \neq e_\beta$.*

Proof. $l(e_\alpha) = \Lambda(1 - e_\alpha)$. By Lemma 3 [3] $\Lambda(1 - e_\alpha) \oplus Ne_\alpha$ is a maximal ideal. Hence $\Lambda/\Lambda(1 - e_\alpha) \oplus Ne_\alpha \cong \Lambda e_\alpha/Ne_\alpha \cong \Lambda x_\alpha$ where $E(\Lambda x_\alpha) \subseteq \Lambda$. Then the same argument as the proof of Lemma 3 shows that $x_\alpha \Lambda$ is a minimal right ideal and $e_\alpha \Lambda$ is an essential extension of $x_\alpha \Lambda$. If $x_\alpha \Lambda \cong x_\beta \Lambda$, then $r(x_\alpha) = r(x_\beta z)$ for some $z \in \Lambda$. Then $\Lambda x_\alpha = l(r(\Lambda x_\alpha)) = l(r(\Lambda x_\beta z)) = \Lambda x_\beta z \cong \Lambda x_\beta$, since $\Lambda x_\beta \cong \Lambda e_\beta/Ne_\beta$ is minimal. This is a contradiction since $\Lambda e_\alpha/Ne_\alpha \not\cong \Lambda e_\beta/Ne_\beta$. Hence $e_\alpha \Lambda \not\cong e_\beta \Lambda$.

The following lemma is well known.

Lemma 4. *Let P_1, P_2 be finitely generated projective module. Then $P_1 \cong P_2$ if and only if $P_1/NP_1 \cong P_2/NP_2$.*

Proposition 2. *If Λ/N is a semi-simple ring where N is the radical of Λ ,*

and Λ is a left self cogenerator ring, then Λ is a finite direct sum of injective hulls of irreducible left ideals. Every commutative self cogenerator ring is self-injective and a direct sum of a finite number of injective hulls of irreducible ideals such that any two of which are not isomorphic.

Proof. If Λ/N is semisimple, Λ has only a finite number of non isomorphic irreducible modules, say I_1, I_2, \dots, I_n . Let $E_i (i=1, \dots, n)$ be their injective hulls. Then E_i/NE_i 's are all the non isomorphic irreducible modules by Lemma 3 [3] and Lemma 4. Since $\Lambda/N \cong I_1^{\alpha_1} + I_2^{\alpha_2} + \dots + I_n^{\alpha_n} \cong (E_1/NE_1)^{\alpha_1} + (E_2/NE_2)^{\alpha_2} + \dots + (E_n/NE_n)^{\alpha_n}$, $\Lambda \cong E_1^{\alpha_1} + E_2^{\alpha_2} + \dots + E_n^{\alpha_n}$ by Lemma 4. Let R be a commutative self cogenerator ring. Then $R \supseteq \sum^{\oplus} E_{\alpha}$. Since $\sum^{\oplus} E_{\alpha}$ is faithful $\sum^{\oplus} E_{\alpha} = l(r(\sum^{\oplus} E_{\alpha})) = l(0) = R$. Then the number of E_{α} 's is finite and any two of which are not isomorphic.

Now we consider following two properties of a ring Λ .

- (*) Every faithful left module is a generator
- (**) Every faithful left module is a cogenerator.

Then we have

Theorem 1. Λ has Property (*) if and only if Λ/N is semisimple and Λ has Property (**). In case Λ is commutative Property (*) and Property (**) are equivalent.

Proof. By Theorem 6 G. Azumaya [1] we see that a ring Λ has Property (*) if and only if Λ is left self injective and a finite direct sum of indecomposable left ideals each of which is an essential extension of a minimal left ideal. In this case Λ/N is a semisimple ring (see Theorem 7 [7].) Hence the proof is straightforward by Proposition 2 and Lemma 2.

REMARK 1. Proposition 1 means that a left self cogenerator ring contains at least the same number of the isomorphism types of irreducible right ideals as that of irreducible left ideals. Hence if in addition Λ/N is a semisimple ring Λ is right lower distinguished, since Λ/N , consequently Λ has the same number of isomorphism types of irreducible right ideals as that of irreducible left ideals (see Theorem 7 [1]).

REMARK 2. Lemma 3 [3] can be stated more generally as follows.

Lemma 5. If M is a projective, injective and indecomposable Λ -module, then M/NM is an irreducible Λ -module, and M is isomorphic to a principal left ideal of Λ .

Proof. We need only to prove that $M \cong \Lambda e, e^2 = e$, and $\Omega = \text{Hom}(\Lambda e, \Lambda e)$ is a local ring. Since M is Λ -projective, M is contained in a free Λ -module, $F = \sum \Lambda u_{\alpha}$. Let m be a non zero element of M and $m = \sum_{i=1}^m \lambda_i u_{\alpha_i}, \lambda_i \in \Lambda$.

If $\lambda \lambda_i = 0$ implies $\lambda m = 0$ for any $\lambda \in \Lambda$ and some i , then $\Lambda m \cong \Lambda \lambda_i \subseteq \Lambda$. If we consider the projection π_{α_i} of F to Λ , then $M \cong \pi_{\alpha_i}(M)$ since M is an essential extension of Λm . If $\lambda m \neq 0$ and $\lambda \lambda_i = 0$ for some $\lambda \in \Lambda$, then replacing m by λm we may assume $m = \sum_{i=2}^n \lambda_i u_{\alpha_i}$. Repeating this argument we can show as above that $M \cong \Lambda e$, $e^2 = e$. Since Λe is an essential extension of every subideal, we can easily show that $\text{Hom}(\Lambda e, \Lambda e) \cong e \Lambda e$ is a local ring. Then the same proof as Lemma 3 [3] shows that M/NM is irreducible.

The author rewrote this paper after Osofsky's paper appeared, and he gives thanks to M. Harada for helpful suggestions.

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References

- [1] G. Azumaya: *Completely faithful modules and self-injective rings*, Nagoya Math. J. **27** (1966), 697–708.
- [2] E. Matlis: *Injective modules over Noetherian rings*, Pacific J. Math. **8** (1958), 511–528.
- [3] B.L. Osofsky: *A generalization of quasi-Frobenius rings*, J. Algebra **4** (1966), 373–387.