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## A CHARACTERIZATION OF CONDITIONAL EXPECTATIONS FOR $L_\infty(X)$ -VALUED FUNGCTIONS

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**Introduction.** Let  $(\Omega, \mathcal{A}, \mu)$  be a probability space,  $(S, X, \lambda)$  a measure space and  $E$  a Banach space. We consider constant-preserving contractive projections of  $L_1(\Omega, \mathcal{A}, \mu; E)$  into itself. If  $E = \mathbb{R}$  or  $E$  is a strictly convex Banach space, then it is known (Ando [1], Douglas [2] and Landers and Rogge [4]) that such operators coincide precisely with the conditional expectation operators. If  $E = L_1(X, S, \lambda)$ , the author [5] proved that such operators which are translation invariant coincide with the conditional expectation operators. In this paper we deal with the case when  $E = L_\infty(X, S, \lambda)$ . If  $E = \mathbb{R}^2$  with the norm  $\|(x, y)\|_{\mathbb{R}^2} = |x| \vee |y|$ , then such operators can be expressed as a linear combination of two conditional expectation operators. On the other hand if  $E = L_\infty(X, S, \lambda)$  and  $E \neq \mathbb{R}^2$  with the norm  $\|(x, y)\|_{\mathbb{R}^2} = |x| \vee |y|$ , then such operators coincides with the conditional expectation given some  $\sigma$ -subalgebra.

**1. Definitions and lemmas.** Let  $(\Omega, \mathcal{A}, \mu)$  be a probability space and  $(X, S, \lambda)$  a measure space. Let  $S^+ = \{K: K \in S \text{ and } \lambda(K) > 0\}$  and  $L_\infty(X, S, \lambda)$  the class of essentially bounded measurable functions on  $(X, S, \lambda)$ . Let  $E = L_\infty(X, S, \lambda)$ , then  $E$  is a Banach space with the norm defined by  $\|a\|_E = \text{esssup}_{x \in X} |a(x)|$  for each  $a \in L_\infty(X, S, \lambda)$ . Let  $L_1(\Omega, \mathcal{A}, \mu, E)$  be the class of  $E$ -valued Bochner integrable functions on  $(\Omega, \mathcal{A}, \mu)$  with the norm defined by

$$\|f\|_L = \int \|f(\omega)\|_E d\mu(\omega) \quad \text{for each } f \in L_1(\Omega, \mathcal{A}, \mu, E).$$

For the definitions and properties of Bochner integral, see Hille and Phillips [3].

**DEFINITION 1.** For a  $\sigma$ -subalgebra  $\mathcal{B}$  of  $\mathcal{A}$ , a function  $g$  is called the conditional expectation of  $f$  given  $\mathcal{B}$  if  $g$  is weakly measurable with respect to  $\mathcal{B}$ , and

$$\int_B g d\mu = \int_B f d\mu \quad \text{for each } B \in \mathcal{B},$$

where the integral is the Bochner integral. We denote by  $f^{\mathcal{B}}$  the conditional

expectation of  $f$  given  $\mathcal{B}$ . We shall denote by  $\mathcal{R}$  the class of real numbers. For each  $\varphi \in L_1(\Omega, \mathcal{A}, \mu, R)$  we define  $(\varphi \cdot a) = \varphi(\omega) \cdot a$  for each  $\omega \in \Omega$ .

**DEFINITION 2.** Let  $P$  be a linear operator of  $L_1(\Omega, \mathcal{A}, \mu, E)$  into itself.  $P$  is said to be *contractive* if

$$\|P\| = \sup \{ \|P(f)\|_L : f \in L_1(\Omega, \mathcal{A}, \mu, E) \text{ and } \|f\|_L = 1 \} = 1,$$

$P$  is *constant-preserving* if  $P(l_\Omega \cdot a) = l_\Omega \cdot a$  for each  $a \in E$  and  $P$  is called a *projection* if  $P \circ P = P$ , where  $l_\Omega$  is the characteristic function of  $E$ .

**Lemma 1.1.** For each  $f \in L_1(\Omega, \mathcal{A}, \mu, E)$  the conditional expectation  $f^{\mathcal{B}}$  of  $f$  given  $\mathcal{B}$  exists uniquely up to almost every-where and the conditional expectation operator  $(\cdot)^{\mathcal{B}}$  is a constant-preserving contractive projection for each  $\sigma$ -subalgebra  $\mathcal{B}$  of  $\mathcal{A}$ .

For the proof see Schwartz [6].

By the definition of conditional expectation  $(\varphi \cdot a)^{\mathcal{B}} = \varphi^{\mathcal{B}} \cdot a$  for each  $\varphi \in L_1(\Omega, \mathcal{A}, \mu, R)$  and  $a \in E$ .

**Lemma 1.2.** If  $Q$  is a constant-preserving contractive projection of  $L_1(\Omega, \mathcal{A}, \mu, E)$  into itself, then for each  $\varphi \in L_1(\Omega, \mathcal{A}, \mu, R)$  with  $0 \leq \varphi \leq 1$  and  $a \in E$  there exists a  $\mu$ -null set  $N$  such that

$$\|a\|_E - \|Q(\varphi \cdot a)(\omega)\|_E = \|a - Q(\varphi \cdot a)(\omega)\|_E \quad \text{for each } \omega \in \Omega - N.$$

For the proof see Miyadera [5].

**Lemma 1.3.** Let  $Q$  be a constant-preserving contractive projection of  $L_1(\Omega, \mathcal{A}, \mu, E)$  into itself. If, for each  $\varphi \in L_1(\Omega, \mathcal{A}, \mu, R)$  and for each nonzero element  $a$  of  $E$ , there exists  $\varphi' \in L_1(\Omega, \mathcal{A}, \mu, R)$  such that  $Q(\varphi \cdot a) = \varphi' \cdot a$ , then there exists a  $\sigma$ -subalgebra  $\mathcal{B}$  of  $\mathcal{A}$  such that  $Q(f)$  is the conditional expectation of  $f$  given  $\mathcal{B}$  for each  $f \in L_1(\Omega, \mathcal{A}, \mu, E)$ . In particular each constant-preserving contractive projection of  $L_1(\Omega, \mathcal{A}, \mu, R)$  into itself is the conditional expectation given some  $\sigma$ -subalgebra.

For the proof see Miyadera [5].

**Lemma 1.4.** Let  $T \in S^+$  and  $A \in \mathcal{A}$ . Then there exists a  $\mu$ -null set  $N$  such that for each  $\omega \in \Omega - N$

$$|Q(l_A \cdot l_T)(\omega)| \leq 1 \quad (\text{a.e.x.})$$

and

$$0 \leq Q(l_A \cdot l_T)(\omega) \cdot l_T \leq 1 \quad (\text{a.e.x.})$$

**Proof.** By Lemma 1.2 there exists a  $\mu$ -null set  $N$  such that

$$\|l_T\|_E - \|Q(l_A \cdot l_T)(\omega)\|_E = \|l_T - Q(l_A \cdot l_T)(\omega)\|_E$$

for each  $\omega \in \Omega - N$ , and hence for each  $\omega \in \Omega - N$ ,

$$\|Q(l_A \cdot l_T)(\omega)\|_E \leq 1$$

and

$$\|l_T - Q(l_A \cdot l_T)(\omega)\|_E \leq 1.$$

Therefore by the definition of  $\|\cdot\|_E$  we have for each  $\omega \in \Omega - N$ ,

$$|Q(l_A \cdot l_T)(\omega)| \leq 1 \quad (\text{a.e.x})$$

and

$$0 \leq Q(l_A \cdot l_T)(\omega) \cdot l_T \leq 1 \quad (\text{a.e.x}).$$

**Lemma 1.5.** *Let  $A \in \mathcal{A}$  and  $T, T' \in S^+$  and  $T \cap T' = \phi$ . Then*

$$\int Q(l_A \cdot l_T) \cdot l_{T'} / d\mu = 0 \quad (\text{a.e.x}).$$

*Proof.* Since  $Q$  is constant-preserving and contractive,

$$\begin{aligned} 1 &= \int \|l_A \cdot l_T + l_\Omega \cdot l_{T'}\|_E d\mu \geq \int \|Q(l_A \cdot l_T + l_\Omega \cdot l_{T'})\|_E d\mu \\ &= \int \|Q(l_A \cdot l_T) + l_\Omega \cdot l_{T'}\|_E d\mu = \int \|Q(l_A \cdot l_T) \cdot l_{T'} + l_\Omega \cdot l_{T'}\|_E d\mu \\ &\geq \left\| \int (Q(l_A \cdot l_T) \cdot l_{T'} + l_\Omega \cdot l_{T'}) d\mu \right\|_E. \end{aligned}$$

Similarly

$$\begin{aligned} 1 &= \int \|l_A \cdot l_T - l_\Omega \cdot l_{T'}\|_E d\mu \\ &\geq \int \|Q(l_A \cdot l_T - l_\Omega \cdot l_{T'})\|_E d\mu = \int \|Q(l_A \cdot l_T) - l_\Omega \cdot l_{T'}\|_E d\mu \\ &\geq \left\| \int (Q(l_A \cdot l_T) \cdot l_{T'} - l_\Omega \cdot l_{T'}) d\mu \right\|_E. \end{aligned}$$

Therefore we have proved that

$$1 \geq \left\| \int (Q(l_A \cdot l_T) \cdot l_{T'} + l_\Omega \cdot l_{T'}) d\mu \right\|_E$$

and

$$1 \geq \left\| \int (Q(l_A \cdot l_T) \cdot l_{T'} - l_\Omega \cdot l_{T'}) d\mu \right\|_E.$$

Since

$$\int Q(l_A \cdot l_T) l_{T'} d\mu = 0 \quad \text{on } (T')^c$$

and

$$\int (l_A \cdot l_{T'}) d\mu = l_{T'},$$

we have

$$\int Q(l_A \cdot l_T) \cdot l_{T'} d\mu = 0 \quad (\text{a.e.x}).$$

**Lemma 1.6.** *Let  $A \in \mathcal{A}$  and  $T \in S^+$ . Suppose that there exists  $T', T'' \in S^+$  such that  $T' \cap T'' = \emptyset$  and  $T' \cup T'' = T^c$ , then there exists a  $\mu$ -nullset  $N$  such that  $Q(l_A \cdot l_T)(\omega) \cdot l_{T^c} = 0$  (a.e.x) for each  $\omega \in \Omega - N$ .*

*Proof.* Since  $Q$  is constant-preserving and contractive,

$$\begin{aligned} 1 &= \int \|l_A \cdot l_T + l_{\Omega} \cdot l_{T''} + (-1)^k l_{\Omega} \cdot l_{T'}\|_E d\mu \\ &\geq \int \|Q(l_A \cdot l_T + l_{\Omega} \cdot l_{T''} + (-1)^k l_{\Omega} \cdot l_{T'})\|_E d\mu \\ &= \int \|Q(l_A \cdot l_T) + l_{\Omega} \cdot l_{T''} + (-1)^k l_{\Omega} \cdot l_{T'}\|_E d\mu \\ &\geq \int \| (Q(l_A \cdot l_T) \cdot l_{T''} + l_{\Omega} \cdot l_{T''}) \vee (Q(l_A \cdot l_T) \cdot l_{T'} + (-1)^k l_{\Omega} \cdot l_{T'}) \|_E d\mu \\ &\geq \int \|Q(l_A \cdot l_T) \cdot l_{T''} + l_{\Omega} \cdot l_{T''}\|_E d\mu \wedge \int \|Q(l_A \cdot l_T) \cdot l_{T'} + (-1)^k l_{\Omega} \cdot l_{T'}\|_E d\mu \\ &\geq \int \| (Q(l_A \cdot l_T) \cdot l_{T''} + l_{\Omega} \cdot l_{T''}) d\mu \|_E \wedge \int \| (Q(l_A \cdot l_T) \cdot l_{T'} + (-1)^k l_{\Omega} \cdot l_{T'}) d\mu \|_E \\ &= 1. \end{aligned}$$

Here we also used Lemma 1.5 that

$$\int Q(l_A \cdot l_T) \cdot l_{T'} d\mu = 0$$

and

$$\int Q(l_A \cdot l_T) \cdot l_{T''} d\mu = 0.$$

We have proved that

$$1 = \int (\|Q(l_A \cdot l_T) \cdot l_{T''} + l_{\Omega} \cdot l_{T''}\|_E \vee \|Q(l_A \cdot l_T) \cdot l_{T'} + (-1)^k l_{\Omega} \cdot l_{T'}\|_E) d\mu$$

and

$$\|Q(l_A \cdot l_T) \cdot l_{T''} + l_{\Omega} \cdot l_{T''}\|_E = \|Q(l_A \cdot l_T) \cdot l_{T'} + (-1)^k l_{\Omega} \cdot l_{T'}\|_E \quad (\text{a.e.x}).$$

Therefore we have

$$1 = \int (\|Q(l_A \cdot l_T) \cdot l_{T'} + l_{\Omega} \cdot l_{T'}\|_E \vee \|Q(l_A \cdot l_T) \cdot l_{T'} - l_{\Omega} \cdot l_{T'}\|_E) d\mu$$

Since

$$\begin{aligned} & \|Q(l_A \cdot l_T) \cdot l_{T'} + l_\Omega \cdot l_{T'}\|_E \vee \|Q(l_A \cdot l_T) \cdot l_{T'} - l_\Omega \cdot l_{T'}\|_E \geq 1, \\ & \|Q(l_A \cdot l_T)(\omega) \cdot l_{T'} + l_{T'}\|_E = \|Q(l_A \cdot l_T)(\omega) \cdot l_{T'} - l_{T'}\|_E = 1 \quad (\text{a.e.x}). \end{aligned}$$

Similarly we can prove that

$$\|Q(l_A \cdot l_T)(\omega) \cdot l_{T''} + l_{T''}\|_E = \|Q(l_A \cdot l_T)(\omega) \cdot l_{T''} - l_{T''}\|_E = 1 \quad (\text{a.e.x}).$$

Therefore there exists a  $\mu$ -nullset  $N$  such that

$$Q(l_A \cdot l_T)(\omega) \cdot l_{T' \cup T''} = 0 \quad (\text{a.e.x}).$$

**2. A characterization of conditional expectation for  $L_\infty(X)$ -valued function**

**Theorem 1.** *If there exist pairwise disjoint elements  $X_1, X_2$  and  $X_3$  of  $S^+$  such that  $X_1 \cup X_2 \cup X_3 = X$ , then a constant-preserving contractive projection  $Q$  of  $L_1(\Omega, \mathcal{A}, \mu, E)$  into itself is a conditional expectation operator given some  $\sigma$ -sub-algebra.*

*Proof.* Let  $A \in \mathcal{A}$  and  $T \in S^+$  and  $T \subset X_i$ . Then by Lemma 1.6 there exists a  $\mu$ -nullset  $N$  such that for each  $\omega \in \Omega - N$

$$(1) \quad Q(l_A \cdot l_T)(\omega) \cdot l_{T^c} = 0 \quad (\text{a.e.x})$$

and

$$(2) \quad Q(l_{A^c} \cdot l_T)(\omega) \cdot l_{T^c} = 0 \quad (\text{a.e.x}).$$

Since  $Q$  is constant-preserving

$$(3) \quad Q(l_A \cdot l_T) + Q(l_{A^c} \cdot l_T) = Q(l_\Omega \cdot l_T) = l_\Omega \cdot l_T.$$

Since  $Q$  is constant-preserving and contractive

$$\begin{aligned} 1 &= \mu(A) + \mu(A^c) = \int (\|l_A \cdot l_T\|_E + \|l_{A^c} \cdot l_T\|_E) d\mu \\ &\geq \int (\|Q(l_A \cdot l_T)\|_E + \|Q(l_{A^c} \cdot l_T)\|_E) d\mu \\ &\geq \int \|Q(l_A \cdot l_T) + Q(l_{A^c} \cdot l_T)\|_E d\mu = \int \|l_\Omega \cdot l_T\|_E d\mu = 1. \end{aligned}$$

Therefore there exists a  $\mu$ -nullset  $N'$  such that for each  $\omega \in \Omega - N'$

$$\|Q(l_A \cdot l_T)(\omega)\|_E + \|Q(l_{A^c} \cdot l_T)(\omega)\|_E = 1.$$

This together with (1), (2) and (3), implies that for each  $\omega \in \Omega - (N \cup N')$  there exists a real number  $k(\omega)$  such that  $Q(l_A \cdot l_T)(\omega) = k(\omega) \cdot l_T$ . Obviously  $k(\cdot) \in L_1(\Omega, \mathcal{A}, \mu, R)$ . Since  $Q$  is linear,  $k(\cdot)$  is independent of the choice of  $T$ . Let

$E_i = \{a: a \in E, a(x) = 0 \text{ for each } x \in X_i^c\}$  ( $i = 1, 2$  and  $3$ ).  $Q$  is linear and continuous, and hence for each  $\varphi \in L_1(\Omega, \mathcal{A}, \mu, R)$  and  $a \in E_i$ , there exists  $\varphi' \in L_1(\Omega, \mathcal{A}, \mu, E_i)$  such that  $Q(\varphi \cdot a) = \varphi' \cdot a$ . Therefore by Lemma 1.3 there exists a  $\sigma$ -subalgebra  $\mathcal{B}_i$  of  $\mathcal{A}$  such that  $Q(f) = f^{\mathcal{B}_i}$  for each  $f \in L_1(\Omega, \mathcal{A}, \mu, E_i)$ . We shall prove that  $\mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B}_3$ . Let  $B \in \mathcal{B}_1$ . Then

$$\begin{aligned} \int l_B d\mu &= \int l_B \cdot \|l_{X_1} + l_{X_2}\|_E d\mu = \int \|l_B \cdot l_{X_1} + l_B \cdot l_{X_2}\|_E d\mu \\ &\geq \int \|Q(l_B \cdot l_{X_1} + l_B \cdot l_{X_2})\|_E d\mu = \int \|Q(l_B \cdot l_{X_1}) + Q(l_B \cdot l_{X_2})\|_E d\mu \\ &= \int \|(l_B)^{\mathcal{B}_1} \cdot l_{X_1} + (l_B)^{\mathcal{B}_2} \cdot l_{X_2}\|_E d\mu = \int \|l_B \cdot l_{X_1} + (l_B)^{\mathcal{B}_2} \cdot l_{X_2}\|_E d\mu \\ &= \int (l_B \vee (l_B)^{\mathcal{B}_2}) d\mu. \end{aligned}$$

Hence  $l_B(\omega) = l_B(\omega) \vee (l_B)^{\mathcal{B}_2}(\omega)$ , which implies that  $l_B(\omega) = (l_B)^{\mathcal{B}_2}(\omega)$ . Since  $\|(l_B)^{\mathcal{B}_2}\|_L = \|l_B\|_L$ ,  $l_B = (l_B)^{\mathcal{B}_2}$ . Since  $B$  is an arbitrary element of  $\mathcal{B}_1$ , we have proved that  $\mathcal{B}_1 \subset \mathcal{B}_2$ . Similarly we can prove that  $\mathcal{B}_2 \subset \mathcal{B}_3$  and  $\mathcal{B}_3 \subset \mathcal{B}_1$ , which imply that  $\mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B}_3$ . Write  $\mathcal{B} = \mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B}_3$ , then  $Q(f) = f^{\mathcal{B}}$  for each  $f \in L_1(\Omega, \mathcal{A}, \mu, E)$ .

**3. A characterization of a constant-preserving contractive projection of for  $R^2$ -valued functions.** If there do not exist pairwise disjoint element  $X_1, X_2$  and  $X_3$  such that  $X_1 \cup X_2 \cup X_3 = X$  and  $X_1, X_2$  and  $X_3 \in S^+$ , then  $E \cong R^2$  with the norm  $\|(x, y)\|_E = |x| \vee |y|$  or  $E \cong R$  with the norm  $\|x\|_E = |x|$ . Douglas [2] showed that a constant-preserving contractive projection of  $L_1(\Omega, \mathcal{A}, \mu, R)$  into itself is a conditional expectation given some  $\sigma$ -subalgebra. Therefore our next aim is to consider the case that  $E \cong R^2$  with the norm  $\|(x, y)\|_E = |x| \vee |y|$ . Note that for each  $f \in L_1(\Omega, \mathcal{A}, \mu, R^2)$  there exist  $f_1, f_2 \in L_1(\Omega, \mathcal{A}, \mu, R)$  such that  $f(\omega) = (f_1(\omega), f_2(\omega))$ .

**Lemma 3.1.** *Suppose that  $E = R^2$  and  $Q$  is a constant-preserving contractive projection of  $L_1(\Omega, \mathcal{A}, \mu, E)$  into itself. If  $f \in L_1(\Omega, \mathcal{A}, \mu, R)$  with  $0 \leq f(\omega) \leq 1$  (a.e.  $\omega$ ) and  $Q((f, f)) = (f_1, f_2)$  and  $Q((f, -f)) = (g_1, g_1)$ , then  $f_2 = f_2$  and  $g_1 = -g_2$ .*

*Proof.* By Lemma 1.2 there exists  $\mu$ -nullsets  $N_1$  and  $N_2$  such that for each  $\omega \in \Omega - N_1$

$$\|(1, 1)\|_E - \|Q((f, f))(\omega)\|_E = \|(1, 1) - Q((f, f))(\omega)\|_E$$

and for each  $\omega \in \Omega - N_2$

$$\|(1, -1)\|_E - \|Q((f, -f))(\omega)\|_E = \|(1, -1) - Q((f, -f))(\omega)\|_E.$$

Therefore we have for each  $\omega \in \Omega - N_1$

$$1 - (|f_1(\omega)| \vee |f_2(\omega)|) = |1 - f_1(\omega)| \vee |1 - f_2(\omega)|$$

and for each  $\omega \in \Omega - N_2$

$$1 - (|g_1(\omega)| \vee |g_2(\omega)|) = |1 - g_1(\omega)| \vee |1 - g_2(\omega)|$$

and hence  $f_1(\omega) = f_2(\omega)$  for each  $\omega \in \Omega - N_1$  and  $g_1(\omega) = -g_2(\omega)$  for each  $\omega \in \Omega - N_2$ .

**Theorem 2.** *Let  $E = R^2$  with the norm  $\|(x, y)\|_E = |x| \vee |y|$ . Then  $Q$  is a constant-preserving contractive projection if and only if there exist  $\sigma$ -subalgebras  $\mathcal{B}$  and  $\mathcal{C}$  of  $\mathcal{A}$  such that*

$$Q((f, g)) = (1/2(f^{\mathcal{B}} + g^{\mathcal{B}} + f^{\mathcal{C}} - g^{\mathcal{C}}), 1/2(f^{\mathcal{B}} + g^{\mathcal{B}} + g^{\mathcal{C}} - f^{\mathcal{C}})).$$

Proof. Suppose that  $Q$  is a constant-preserving contractive projection. Then by Lemma 3.1 we can get two operators  $Q_1$  and  $Q_2$  of  $L_1(\Omega, \mathcal{A}, \mu, R)$  into itself such that for each  $f \in L_1(\Omega, \mathcal{A}, \mu, R)$ ,

$$Q((f, f)) = (Q_1(f), Q_1(f)) \quad \text{and} \quad Q((f, -f)) = (Q_2(f), -Q_2(f)).$$

Since  $Q$  is a constant-preserving contractive projection,  $Q_1$  and  $Q_2$  are constant-preserving contractive projections. Therefore by Lemma 1.3 there exist  $\sigma$ -subalgebra  $\mathcal{B}$  and  $\mathcal{C}$  such that

$$Q_1 = (\cdot)^{\mathcal{B}} \quad \text{and} \quad Q_2 = (\cdot)^{\mathcal{C}}.$$

Then

$$\begin{aligned} Q((f, g)) &= Q(1/2(f+g) + 1/2(f-g), 1/2(f+g) - 1/2(f-g)) \\ &= (Q_1(1/2(f+g)) + Q_2(1/2(f-g)), Q_1(1/2(f+g)) - Q_2(1/2(f-g))) \\ &= ((1/2(f^{\mathcal{B}} + g^{\mathcal{B}})) + (1/2(f^{\mathcal{C}} - g^{\mathcal{C}})), (1/2(f^{\mathcal{B}} + g^{\mathcal{B}})) - (1/2(f^{\mathcal{C}} - g^{\mathcal{C}}))) \\ &= (1/2(f^{\mathcal{B}} + g^{\mathcal{B}} + f^{\mathcal{C}} - g^{\mathcal{C}}), 1/2(f^{\mathcal{B}} + g^{\mathcal{B}} + g^{\mathcal{C}} - f^{\mathcal{C}})). \end{aligned}$$

On the other hand let  $\mathcal{B}$  and  $\mathcal{C}$  be  $\sigma$ -subalgebras of  $\mathcal{A}$  and

$$Q((f, g)) = (1/2(f^{\mathcal{B}} + g^{\mathcal{B}} + f^{\mathcal{C}} - g^{\mathcal{C}}), 1/2(f^{\mathcal{B}} + g^{\mathcal{B}} + g^{\mathcal{C}} - f^{\mathcal{C}})).$$

Since  $(\cdot)^{\mathcal{B}}$  and  $(\cdot)^{\mathcal{C}}$  are constant-preserving projections,  $Q$  is a constant-preserving projection. In the following we denote

$$\{\omega: f^{\mathcal{B}} + g^{\mathcal{B}} \geq 0\} \quad \text{by} \quad \{f^{\mathcal{B}} + g^{\mathcal{B}} \geq 0\}$$

and

$$\{\omega: f^{\mathcal{C}} - g^{\mathcal{C}} < 0\} \quad \text{by} \quad \{f^{\mathcal{C}} - g^{\mathcal{C}} < 0\}, \quad \text{etc.}$$

It holds that



$$\begin{aligned}
\|Q((f, g))\|_L &= \int \|Q((f, g))\|_E d\mu \\
&= \int |(1/2(f^{\mathcal{B}}+g^{\mathcal{B}}+f^{\mathcal{C}}-g^{\mathcal{C}})) \vee |1/2(f^{\mathcal{B}}+g^{\mathcal{B}}+g^{\mathcal{C}}-f^{\mathcal{C}})| d\mu \\
&= \int_{\{f^{\mathcal{B}}+g^{\mathcal{B}}<0\} \cap \{f^{\mathcal{C}}-g^{\mathcal{C}} \geq 0\}} 1/2(f^{\mathcal{B}}+g^{\mathcal{B}}+f^{\mathcal{C}}-g^{\mathcal{C}}) d\mu + \int_{\{f^{\mathcal{B}}+g^{\mathcal{B}}=0\} \cap \{f^{\mathcal{C}}-g^{\mathcal{C}}<0\}} 1/2(f^{\mathcal{B}}+g^{\mathcal{B}}+g^{\mathcal{C}}-f^{\mathcal{C}}) d\mu \\
&\quad + \int_{\{f^{\mathcal{B}}+g^{\mathcal{B}}<0\} \cap \{f^{\mathcal{C}}-g^{\mathcal{C}} \geq 0\}} 1/2(-f^{\mathcal{B}}-g^{\mathcal{B}}+f^{\mathcal{C}}-g^{\mathcal{C}}) d\mu + \int_{\{f^{\mathcal{B}}+g^{\mathcal{B}}<0\} \cap \{f^{\mathcal{C}}-g^{\mathcal{C}}<0\}} 1/2(-f^{\mathcal{B}}-g^{\mathcal{B}}+g^{\mathcal{C}}-f^{\mathcal{C}}) d\mu \\
&= \int_{\{f^{\mathcal{B}}+g^{\mathcal{B}} \geq 0\}} 1/2(f^{\mathcal{B}}+g^{\mathcal{B}}) d\mu + \int_{\{f^{\mathcal{B}}+g^{\mathcal{B}}<0\}} 1/2(-f^{\mathcal{B}}-g^{\mathcal{B}}) d\mu \\
&= \int_{\{f^{\mathcal{C}}-g^{\mathcal{C}} \geq 0\}} 1/2(f^{\mathcal{C}}-g^{\mathcal{C}}) d\mu + \int_{\{g^{\mathcal{C}}-f^{\mathcal{C}} \geq 0\}} 1/2(g^{\mathcal{C}}-f^{\mathcal{C}}) d\mu \\
&= \int |1/2(f^{\mathcal{B}}+g^{\mathcal{B}})| d\mu + \int |1/2(f^{\mathcal{C}}-g^{\mathcal{C}})| d\mu \\
&= \int |1/2(f+g)| d\mu + \int |1/2(f-g)| d\mu \\
&\leq \int (|f| \vee |g|) d\mu = \int \|(f, g)\|_E d\mu.
\end{aligned}$$

Therefore  $Q$  is contractive.

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