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# AN APPLICATION OF INNER-EXTENSION OF HIGHER DERIVATIONS TO p-ALGEBRAS

Dedicated to Professor Keizo Asano on his 60th birthday

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Let A be a central separable algebra over a field K, and assume that it contains a normal extension L of K as a maximal commutative subalgebra. When L/K is moreover separable, one obtains a description of A as a crossed product by extending automorphisms of L to inner automorphisms of A. When L/K is purely inseparable of exponent 1, one obtains similarly a concrete description of A by extending derivations of L to inner derivations of A (e.g. Jacobson [5]). Here we apply a similar procedure using higher derivations in case L is purely inseparable of exponent 2, and derive the normal form  $(\alpha | \beta_0, \beta_1]$  for A given by Schmid [7] and Witt [9]. For this purpose we prove in §1 some facts about inner-extension of higher derivations.

# 1. Inner-extension of higher derivations

Let A be an algebra over a commutative ring R. Let  $A[T]_q = A[T]/(T^{q+1})$ , where T is an indeterminate. We denote  $T \pmod{T^{q+1}}$  by t, so that any element of  $A[T]_q$  is written uniquely as  $a_0 + a_1 t + \cdots + a_q t^q$  ( $a_i \in A$ ). Let B be an R-algebra containing A. A higher derivation D of A into B of rank q is a sequence  $\{D_1, \dots, D_q\}$  of R-linear maps  $D_i: A \to B$  such that

$$D_t: A \to B[T]_q; \quad D_t(a) = a + D_1(a)t + \dots + D_q(a)t^q \qquad (a \in A)$$

is an algebra homomorphism, or what is the same thing,  $D_t$  defines an algebra homomorphism  $A[T]_q \rightarrow B[T]_q$  over  $R[T]_q$ . If  $D = \{D_1, \dots, D_q\}$  is a higher derivation of rank q,  $\{D_1, \dots, D_k\}$   $(k \leq q)$  is a higher derivation of rank k, which will be called the *k*-section of D.

For any  $d_1, \dots, d_q \in A$ , the polynomial

$$d_t = 1 + d_1 t + \dots + d_q t^q$$

is invertible in  $A[T]_q$ . It yields a higher derivation  $\tilde{d}$  of A, via inner automorphism of  $A[T]_q$ , i.e.

A. HATTORI

$$\widetilde{d}_t(a) = d_t a d_t^{-1} \qquad (a \in A)$$

We call such  $\tilde{d}$  an *inner higher derivation* of A determined by  $\{d_1, \dots, d_q\}$ . In other words, a higher derivation D is inner if there exist  $d_1, \dots, d_q \in A$  such that for any  $a \in A$ 

$$d_{k}a = D_{k}(a) + D_{k-1}(a)d_{1} + \dots + D_{1}(a)d_{k-1} + ad_{k} \qquad (k = 1, \dots, q)$$

EXAMPLE 1. Embed A into End (A) as the set of left multiplications. Any higher derivation  $D: A \to A$  is then extended to the inner higher derivation of End (A) defined by  $D_1, \dots, D_q \in \text{End} (A)$ .

EXAMPLE 2. Let 1, ..., q be invertible in R. A derivation  $\delta: A \to A$  gives rise to a higher derivation  $e_q(\delta)$  defined by

$$e_q(\delta)_t(a) = a + \delta(a)t + \frac{1}{2!}\delta^2(a)t^2 + \dots + \frac{1}{q!}\delta^q(a)t^q \qquad (a \in A)$$

If  $\delta$  is an inner derivation defined by  $d \in A$ , then  $e_q(\delta)$  is an inner higher derivation defined by

$$e_q(d)_t = 1 + dt + \frac{1}{2!} d^2 t^2 + \dots + \frac{1}{q!} d^q t^q$$

Let B be a subalgebra of A, and D:  $B \rightarrow A$  a higher derivation. If there exists an inner higher derivation  $\tilde{d}$  of A which coincides on B with D, we say that  $\tilde{d}$  is an *inner-extension* of D.

**Theorem 1.** If B is a separable algebra [2] over R, any higher derivation  $D: B \rightarrow A$  has an inner-extension  $A \rightarrow A$ .

Proof. There exist  $u_i, v_i \in B$   $(i=1, \dots, n)$  such that

i)  $\sum u_i v_i = 1$ , and

ii)  $\sum bu_i \otimes v_i = \sum u_i \otimes v_i b$  (in  $B \otimes B$ ).

Set

$$d_t = \sum_i D_t(u_i)v_i = 1 + (\sum_i D_1(u_i)v_i)t + \cdots$$

For  $b \in B$ , we have

$$D_t(b)d_t = \sum D_t(b)D_t(u_i)v_i = \sum D_t(bu_i)v_i$$
$$= \sum D_t(u_i)(v_ib) = d_tb$$

This shows that  $\tilde{d}$  is an inner-extension of D.

**Theorem 2.** Let A be a central separable algebra over R, and B be a left (or right) semisimple subalgebra [3] of A. Then any higher derivation D of B into

308

### A has an inner-extension $A \rightarrow A$ .

REMARK. A special case is proved in Jacobson [5], and the theorem itself is essentially a special case of Sweedler [8, Th. 9. 5].

Proof. We proceed by induction on q. The case q=1 is proved in the following manner (after Hochschild [4]). Let  $A^{\circ}$  be an anti-isomorphic copy of A. The direct sum  $(A, A)=A\oplus A$  is considered as a  $B\otimes A^{\circ}$ -module by setting

$$b(a_1, a_2)a = (ba_1a, D_1(b)a_1a + ba_2a).$$

The map  $(a_1, a_2) \mapsto a_1$  defines an *R*-split  $(B \otimes A^0)$ -epimorphism  $A \oplus A \to A$ , where *A* is considered naturally as a  $B \otimes A^0$ -module. Since  $B \otimes A^0$  is left semisimple [3, Prop. 2. 4], there exists a  $B \otimes A^0$ -monomorphism  $\alpha : A \to A \oplus A$ such that  $(A, A) = (0, A) \oplus im(\alpha)$ . If  $\alpha(1) = (u, v)$ , *u* is invertible, and we have  $D(b) = (vu^{-1})b - b(vu^{-1})$  (cf. [4]). Let q > 1 and assume that  $d_1, \dots, d_{q-1} \in A$  give an inner-extension of the (q-1)-section  $D = \{D_1, \dots, D_{q-1}\}$  of *D*. Set

$$d'_t = 1 + d_1 t + \dots + d_{q-1} t^{q-1} \in A[T]_q$$

For every  $b \in B$ , the terms of degree  $\langle q \text{ in } D_t(b)d'_t - d'_t b$  all vanish. So there exists  $f(b) \in A$  such that

$$f(b)t^q = D_t(b)d'_t - d'_t b$$

We have

$$\begin{aligned} f(b_1b_2)t^q &= D_t(b_1)D_t(b_2)d_t' - d_t'b_1b_2 \\ &= (D_t(b_1)f(b_2) + f(b_1)b_2)t^q \,, \end{aligned}$$

whence

$$f(b_1b_2) = b_1f(b_2) + f(b_1)b_2$$
.

Hence there exists  $d_q \in A$  such that  $f(b) = d_q b - b d_q ({}^{\nu}b \in B)$ . Setting

$$d_t = d'_t + d_q t^q ,$$

we have

$$d_t b = D_t(b) d_t$$
 ( ${}^{\vee}b \in B$ ). q. e. d.

If both  $\tilde{d}$  and  $\tilde{d}'$  are inner-extensions of  $D: B \to A$ , then it is clear that  $d_t^{-1}d_t' \in V_A(B)[T]_q$ , where  $V_A(B)$  denotes the commuter of B in A.

**Proposition 3.** Let D be a higher derivation  $B \to A$  which admits an innerextension  $A \to A$ . If the k-section of D  $(k \le q)$  has an inner-extension  $A \to A$ determined by  $d_1, \dots, d_k$ , then we can find  $d_{k+1}, \dots, d_q \in A$  so that D is extended to the inner higher derivation defined by  $\{d_1, \dots, d_k, d_{k+1}, \dots, d_q\}$ . A. HATTORI

Proof. Let the inner higher derivation by  $\{d'_1, \dots, d'_q\}$  yields D when restricted to B. Then there exist  $c_1, \dots, c_k \in V_A(B)$  such that

 $1+d_1t+\cdots+d_kt^k \equiv d'_t(1+c_1t+\cdots+c_kt^k) \pmod{t^{k+1}}$ 

Determine  $d_{k+1}, \dots, d_q \in A$  by the identity (in  $A[T]_q$ )

$$1 + d_1 t + \dots + d_k t^k + d_{k+1} t^{k+1} + \dots + d_q t^q = d'_t (1 + c_1 t + \dots + c_k t^k).$$

It is clear that  $\{d_1, \dots, d_k, d_{k+1}, \dots, d_q\}$  induces the higher derivation D of B.

#### 2. p-algebras of exponent 2

Let A be a central separable algebra over a field K of characteristic  $p \neq 0$ , and assume that there exists a maximal commutative subalgebra L which is a purely inseparable extension of K such that

$$(1) L = K(u), u^{p^2} = \alpha \in K$$

Since  $L \simeq K[X]/(X^{p^2} - \alpha)$ , a higher derivation  $D: L \to A$  of rank q is determined by assigning to u a polynomial  $D_t(u) \in A[T]_q$  such that  $D_t(u)^{p^2} = \alpha$ . It follows that for  $q < p^2$ , there exists a (unique) higher derivation  $D = \{D_1, \dots, D_q\}: L \to L$ such that  $D_i(u) = a_i, i = 1, \dots, q$ , for any preassigned values  $a_1, \dots, a_q \in L$ .

In particular, there exists a higher derivation  $D: L \rightarrow L$  of rank p such that

$$D_i(u) = \frac{1}{i!} D_1^i(u) = \frac{1}{i!} u, \quad i = 1, \dots, p-1,$$
  
 $D_p(u) = 0.$ 

By Theorem 2, D has an inner-extension  $A \to A$ . If  $D_1$  is given by the inner derivation by  $d_1 \in A$ ,  $\{D_1, \dots, D_{p-1}\}$  is given by  $\{d_1, \dots, d_{p-1}\}$  where  $d_i = (1/i!) d_1^i$   $(i=1, \dots, p-1)$ . Hence, by proposition 3, D is extended to an inner higher derivation defined by

$$d_1, \dots, d_p;$$
 where  $d_i = \frac{1}{i!} d_1^i, \quad i = 1, \dots, p-1.$ 

In particular we have

$$(2) u^{-1}d_1u = d_1 + 1,$$

(3) 
$$u^{-1}d_{p}u = d_{p} + \sum_{i=1}^{p-1} \frac{1}{i!(p-i)!} d_{1}^{i}.$$

By (2) we have

(2') 
$$u^{-1}d_1^p u = d_1^p + 1$$
.

Hence  $d_1^n - d_1$  commutes with u, and  $d_1^n - d_1 \in L$ . It commutes moreover with  $d_1$ . Hence  $d_1^n - d_1 \in K(u^p)$ . It follows that

310

**INNER-EXTENSION OF HIGHER DERIVATIONS** 

$$(d_1^p)^p - d_1^v = (d_1^p - d_1)^p \in K$$

By (2') we may start with  $d_1^{\ell}$  instead of  $d_1$ . So we may assume

$$(4) d_1^p - d_1 = \beta_0 \in K.$$

Set  $v = [d_1, d_p] = d_1 d_p - d_p d_1$ . We have

$$u^{-1}vu = (d_1+1)(d_p+S)-(d_p+S)(d_1+1) = v$$
,

since  $S = u^{-1}d_p u - d_p$  commutes with  $d_1$  (cf. (3)). Hence  $v \in L$ . We have

$$d_1^p d_p - d_p d_1^p = [\widetilde{d_1, \cdots, [d_1, d_p]} \cdots] = D_1^{p-1}(v)$$

This together with (4) shows

$$d_1d_p - d_pd_1 = D_1^{p-1}(v) = d_1D_1^{p-2}(v) - D_1^{p-2}(v)d_1$$

This means that  $d'_p = d_p - D_1^{p-2}(v)$  commutes with  $d_1$ . Since  $d'_p$  satisfies (3), we can use this  $d'_p$  in place of  $d_p$ .

Hence we may assume

$$(5) d_1 d_p = d_p d_1$$

 $d_1$  and  $d_p$  generate a commutative subalgebra P. Let  $W_2(P)$  be the group of Witt vectors of length 2 in P. By definition, we have

$$(b_0, b_1) + \underline{1} = \left(b_0 + 1, b_1 + \sum_{i=1}^{p-1} \frac{1}{i!(p-i)!} b_0^i\right)$$

where 1 = (1, 0). (Notice  $(p-1)! \equiv -1 \pmod{p}$ .) Hence (2) and (3) mean

$$(6) u^{-1}(d_1, d_p)u = (d_1, d_p) + \underline{1}$$

Similarly, (2') and the identity

(3') 
$$u^{-1}d_{p}^{p}u = d_{p}^{p} + \sum_{i=1}^{p-1} \frac{1}{i!(p-i)!} d_{1}^{p_{i}}$$

which is derived from (3), mean

(6') 
$$u^{-1}(d_1^p, d_p^p)u = (d_1^p, d_p^p) + \underline{1}$$

Putting

(7) 
$$\mathscr{O}(d_1, d_p) = (d_1^p, d_p^p) - (d_1, d_p) = (\beta_0, \beta_1)$$

we have (by (6) and (6'))

311

$$u^{-1}(\beta_0, \beta_1)u = (\beta_0, \beta_1)$$

Since  $\beta_1$  (as well as  $\beta_0$ ) commutes with  $d_1$ ,  $d_p$  and u, it must lie in K. Finally it is clear that  $d_1$ ,  $d_p$  and u generate the whole algebra. A. The structure of A is thus completely determined by (1), (5), (6) and (7), and we have arrived to the normal form  $(\alpha | \beta_0, \beta_1]$  given by Schmid [7] and Witt [9].

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