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On the Theory of Representation of Finite Groups.

By Hirosi NAGAO

The ordinary representations of finite groups by linear transformations were first treated by G. Frobenius[1] and W. Burnside[2] in case that the coefficients are complex numbers, and the theory were extended by I. Schur[3] to the case where the field of coefficients is any algebraically closed field of characteristic 0. Later E. Noether gave a new foundation of the theory in her theory[4] of representation of algebras.

The modular representation of finite groups were first studied by L. E. Dickson[5] but the complete extension of the theory of ordinary representations to the modular case has been recently established by R. Brauer and C. Nesbitt in their remarkable joint paper.[6]

It seems to us that the main theorems in the general theory of group representations are the orthogonality relations for group characters and the theorems concerning the induced representations and the Kronecker product of two representations, and the existing constructions of the theory are all, even in the theory of R. Brauer and C. Nesbitt in the modular case, based on the orthogonality relations for ordinary characters.

In this paper we shall intend to construct the theory in the most general manner which involves the ordinary and the modular cases and further the case of collineations.

The representations of finite groups by collineations in the ordi-

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1) G. Frobenius; Sitzungsberichte der Preussischen Akademie der Wissenschaften, 1896, p. 1343, 1897, p. 994, 1899, p. 482, 1903, p. 401.

For the Frobenius' theory, see the accounts in L. E. Dickson, *Modern Algebraic Theories*, Chicago, 1926, Chapter XIV ; G. A. Millar, H. F. Blichfeldt, L. E. Dickson, *Theory and Application of Finite Groups*, Chicago, 1917, Chapter VI.

2) W. Burnside; Acta Mathematica, 28 (1904), p. 369, Procceedings of the London Mathematical Society (2), 1 (1904) p. 117.

3) I. Schur; Sitzungsber. der Preussischen Akad. der Wiss. (1905) p. 406.

4) E. Noether: Mathematische Zeitschrift, 30 (1907), p. 389.

5) L. E. Dickson; Transactions of the American Mathematical Society, 8 (1907), p. 389.

6) R. Brauer and C. Nesbitt; *On the modular representations of groups of finite ordcr* 1, University of Toronto Studies, Math. Series No. 4, (1937), refered to as B. N. M.

nary case were first treated by I. Schur [7], later by K. Shoda and K. Asano [8], and in the modular case by K. Asano, M. Osima and M. Takahasi [9]. From these theories we may reduce the problems of group representations by collineations to the linear case and the theorems concerning the induced representations and the Kronecker product in the linear case may be easily extended to that case (see Theorem 1 and 2).

In section 1 we state some preliminary lemmas on algebras which are essentially obtained by T. Nakayama [10]. In section 2 and 3, the theorems on the induced representations and Kronecker products are proved in the most general case. The theorems on group characters are stated in section 4, where we give another proof of the thorem [11] of R. Brauer and C. Nesbitt on the fundamental relation between Cartan invariants and decomposition numbers, which plays a principal rôle in their modular representation theory.

§ 1. Some Lemmas on Algebras.

Let A be a (finite dimensional) algebra with unit 1 over a field K. A is directly decomposed into directly indecomposable right (left) ideals as follows:

$$A=\sum_{\kappa=1}^{k}\sum_{i=1}^{f(\kappa)}E_{\kappa i}A\left(=\sum_{\kappa=1}^{k}\sum_{i=1}^{f(\kappa)}AE_{\kappa i}\right), \tag{1.1}$$

where $E_{\kappa i}$ ($\kappa=1, 2, \ldots\ldots, k$; $i=1, 2, \ldots, f(\kappa)$) are mutually orthogonal idempotents whose sum is equal to 1 and $E_{\kappa i}A$ ($AE_{\kappa i}$) is operator isomorphic to $E_{\lambda j}A$ ($AE_{\lambda j}$) if and only if $\kappa=\lambda$. We shall denote $E_{\kappa 1}$ by E_κ.

Denote the radical of A by $N(A)$ and a subset S of A mod $N(A)$ by $\bar{S}$. Then $\bar{A}=A/N(A)=\sum_{\kappa,i}\bar{E}_{\kappa i}\bar{A}(=\sum_{\kappa,i}\bar{A}\bar{E}_{\kappa i})$ is a direct decomposition of $\bar{A}$ into simple right (left) ideals $\bar{E}_{\kappa i}\bar{A}$ ($\bar{A}\bar{E}_{\kappa i}$) and $\bar{E}_{\kappa i}\bar{A}$ ($\bar{A}\bar{E}_{\kappa i}$) is operator isomorphic to $\bar{E}_{\lambda j}\bar{A}$ ($\bar{A}\bar{E}_{\lambda j}$) if and only if $\kappa=\lambda$.

The following lemma is a generalization of a theorem [12] obtained by T. Nakayama, and is proved in quite a similar way as it.

Lemma 1. *Let* $\mathfrak{m}$ *be a (finite dimensional) A-right module such*

7) I. Schur; Crelle 127 (1904), p. 20, 132 (1907), p. 85.

8) K. Asano and K. Shoda; Compositio Mathematica (1935).

9) K. Asano, M. Osima and M. Takahasi; Proceedings of the Physico-Mathematical Society of Japan, (3) 19 (1937) p. 197.

10) T. Nakayama: *Some studies on regular representations, induced representations.* Annals of Mathematics 39 (1938) p. 361.

11) See B. N. M. § 4.

12) See the paper in the footnote 10).

that $x \cdot 1 = x$ for all $x \in \mathfrak{m}$. Then the number of the factor groups isomorphic to $\bar{E}_\kappa \bar{A}$ which appear in a composition factor group series of A-right module $\mathfrak{m}$ is equal to the composition length of $E_\kappa A E_\kappa$-right module $\mathfrak{m}E_\kappa$. When K is algebraically closed it is equal to the dimension of $\mathfrak{m}E_\kappa$.

Proof. Let $\mathfrak{m} = \mathfrak{m}_0 \supset \mathfrak{m}_1 \supset \dots \supset \mathfrak{m}_s = 0$ be a composition series of an A-right module $\mathfrak{m}$. Then $\mathfrak{m}_i E_\kappa \neq \mathfrak{m}_{i+1} E_\kappa$ if and only if $\mathfrak{m}_i/\mathfrak{m}_{i+1} \simeq \bar{E}_\kappa \bar{A}$, and then $\mathfrak{m}_i E_\kappa / \mathfrak{m}_{i+1} E_\kappa$ is a simple $E_\kappa A E_\kappa$ -right module since $\mathfrak{m}_i E_\kappa / \mathfrak{m}_{i+1} E_\kappa \simeq (\mathfrak{m}_i/\mathfrak{m}_{i+1}) E_\kappa \simeq \bar{E}_\kappa \bar{A} \bar{E}_\kappa$. This proves the first part.

To prove the second part, suppose that K is algebraically closed. Then $\bar{E}_\kappa \bar{A} \bar{E}_\kappa$ is isomorphic to K. Let $\mathfrak{m}E_\kappa = \mathfrak{m}_{i_0} E_\kappa \supset \mathfrak{m}_{i_1} E_\kappa \supset \dots \supset \mathfrak{m}_{i_t} E_\kappa = 0$ is a composition series of $E_\kappa A E_\kappa$ -right module $\mathfrak{m}E_\kappa$, then $\mathfrak{m}_{i_\nu} E_\kappa / \mathfrak{m}_{i_{\nu+1}} E_\kappa$ is isomorphic to $\bar{E}_\kappa \bar{A} \bar{E}_\kappa$. Hence $\mathfrak{m}_{i_\nu} E_\kappa = (x_\nu K, \mathfrak{m}_{i_{\nu+1}} E_\kappa)$ for any element x_ν from $\mathfrak{m}_{i_\nu} E_\kappa$ not contained in $\mathfrak{m}_{i_{\nu+1}} E_\kappa$. Therefore $(x_1, x_2, \dots, x_t)$ is a basis of $\mathfrak{m}E_\kappa$ for K and t is equal to the dimension of $\mathfrak{m}E_\kappa$, q. e. d.

For left moduli, we have of course the similar assertion to this lemma.

Let B be a subalgebra of A containing 1 and $B = \sum_{q,i} e_{q,i}\ B (= \sum_{q,i} B e_{q,i})$ be a direct decomposition of B as (1.1). From Lemma 1 we have immediately the following lemma, where we shall suppose that K is algebraically closed for the sake of simplicity and because in the following sections we shall be concerned only with that case.

Lemma 2.

i) *If* $e_q A \leftrightarrow \sum_\kappa \alpha_{\kappa q} \bar{E}_\kappa \bar{A}$ *then* $A E_\kappa \leftrightarrow \sum_q \alpha_{\kappa q} \bar{B} \bar{e}_q$ *as B-left module.*

ii) *If* $e_q A \simeq \sum_\kappa \beta_{\kappa q} E_\kappa A$ *then* $\bar{A} \bar{E}_\kappa \leftrightarrow \sum_q \beta_{\kappa q} \bar{B} \bar{e}_q$ *as B-left module.*

Here, for an A (or B)-right module $\mathfrak{m}$ we indicate by

$$\mathfrak{m} \leftrightarrow \sum_\kappa \alpha_\kappa \bar{E}_\kappa \bar{A} \quad (\text{or } \sum_q \beta_q \bar{e}_q \bar{B})$$

that $\bar{E}_\kappa \bar{A}$ (or $\bar{e}_q \bar{B}$) will appear in a composition factor group series of $\mathfrak{m}$ with multiplicity α_κ (or β_q), and by

$$\mathfrak{m} \simeq \sum_\kappa \alpha_\kappa E_\kappa A$$

that $\mathfrak{m}$ will be directly decomposed into α_κ components isomorphic to $E_\kappa A$ ($\kappa = 1, 2, \dots, k$). For left moduli those notations have the similar significances.

Proof. i) From Lemma 1 $\alpha_{\kappa q}$ is equal to the dimension $e_q A E_\kappa$ and this is equal to the number of factor groups isomorphic to $\bar{B}\bar{e}_q$ which appear in a composition factor group series of B-left module AE_κ.

ii) If $e_q A \simeq \sum_{\kappa} \beta_{\kappa q} E_\kappa A$ then $\bar{e}_q \bar{A} \leftrightarrow \sum_{q} \beta_{\kappa q} \bar{E}_\kappa \bar{A}$, and hence we have ii) in a similar way as in i).

§ 2. Induced Representations.

In the following, we shall always suppose that K is an algebraically closed field. Let $G=\{1, a, b, \dots\}$ be a finite group. We shall call a set $\{\varepsilon_{a,b}\}$ of elements from K a *factor set* if it satisfies the conditions

i) $\varepsilon_{1,a}=\varepsilon_{a,1}=\varepsilon_{a,a^{-1}}=1$ for all $a \in G$.

ii) $\varepsilon_{a,b}\varepsilon_{ab,c}=\varepsilon_{a,bc}\varepsilon_{b,c}$ for all a, b and c in G.

If $\{\varepsilon_{a,b}\}$ is a factor set then

$$A(G, \varepsilon)=\sum_{a\in G} Ku_a\,;\ u_a u_b=\varepsilon_{a,b}u_{ab}$$

is a symmetric algebra [13], and u_1 is its unit element and $u_{a^{-1}}=u_a^{-1}$. We shall call it a group ring of G with a factor set $\{\varepsilon_{a,b}\}$. We denote by $A(G, 1)$ the group ring with such a factor set that $\varepsilon_{a,b}=1$ for all a and b in G.

By a (finite dimensional) $A(G, \varepsilon)$-right module $\mathfrak{m}$ we obtain a representation of $A(G, \varepsilon)$. Let $M(a)$ be the matrix corresponding to u_a in this representation. Then $M(a)M(b)=\varepsilon_{a,b}M(ab)$. We shall call the mapping $a \to M(a)$ the representation of G by $\mathfrak{m}$ with factor set $\{\varepsilon_{a,b}\}$. Every representation of G with factor set $\{\varepsilon_{a,b}\}$ may be obtained by an $A(G, \varepsilon)$-right module $\mathfrak{m}$.

Let H be a subgroup of G and $G=\sum_{i=1}^{t} Hs_i$ be the right coset decomposition of G mod H. Then the subalgebra $A(H, \varepsilon)=\sum_{c\in H} Ku_c$ of $A(G, \varepsilon)$ is the group ring with the factor set $\{\varepsilon_{c,d}\,;\ c, d \in H\}$ and $A(G, \varepsilon)$ is the direct sum of submoduli $A(H, \varepsilon)_{s_i}$ $(i=1, 2, \dots, t)$.

We shall define an $A(G, \varepsilon)$-right module induced by an $A(H, \varepsilon)$-right module $\mathfrak{m}$ as follows. Let $\tilde{\mathfrak{m}}=\sum_{i=1}^{t} \mathfrak{m} \circ vs_i$ be a direct sum of moduli $\mathfrak{m} \circ vs_i$ $(i=1, 2, \dots, t)$ which are isomorphic to $\mathfrak{m}$ by the correspondence $x \leftrightarrow x \circ vs_i$. When $us_i u_a = wus_j$ $(w \in A(H, \varepsilon))$ in $A(G, \varepsilon)$ we shall define the operation of u_a for $x \circ vs_i$ $(x \in \mathfrak{m})$ as $(x \circ vs_i)u_a=(x \cdot w) \circ vs_j$. Then $\tilde{\mathfrak{m}}$ becomes an $A(G, \varepsilon)$-right module and disregarding an operator isomorphism the definition dose not depend on the choice of representative system $\{s_i\}$. We shall call it an *$A(G, \varepsilon)$-right module*

13) See T. Nakayama: *On Frobeniusean algebras* I, Annals of Mathematics 40 (1939) p. 611.

induced by $\mathfrak{m}$.

Now let $A=A(G, \varepsilon)=\sum_{\kappa, i} E_{\kappa i}A(=\sum_{\kappa, i} AE_{\kappa i})$ and $B=A(H, \varepsilon)=\sum_{q, j} e_{qj}$ $B(=\sum Be_{qj})$ be decompositions as (1.1) of A and B respectively. Since A (B) is a symmetric algebras, $E_\kappa A$ ($e_q B$) and AE_κ (Be_q) induce the same representation of G (H). We shall denote by U_κ, V_q, F_κ, I_q, $\widetilde{V}_q$ and $\widetilde{I}_q$ representations by $E_\kappa A$, $e_q B$, $\bar{E}_\kappa \bar{A}$, $\bar{e}_q \bar{B}$, $\widetilde{e_q B}$ and $\widetilde{\bar{e}_q \bar{B}}$ respectively. And when M is a representation of G (or H) we indicate by

$$M \leftrightarrow \sum_{\kappa} \alpha_\kappa F_\kappa \text{ (or } M \leftrightarrow \sum_{q} \beta_q I_q)$$

that F_κ (or I_q) will appear in M as irreducible constituent with multiplicity α_κ (or β_q), and by

$$M \simeq \sum_{\kappa} \alpha_\kappa U_\kappa \text{ (or } M \simeq \sum_{q} \beta_q V_q)$$

that M will be directly decomposed into components U_κ (or V_q) with multiplicity α_κ (or β_q).

Theorem 1.

i) *If* $\widetilde{V}_q \leftrightarrow \sum_{\kappa} \alpha_{\kappa q} F_\kappa$, *then* $U_\kappa \leftrightarrow \sum_{q} \alpha_{\kappa q} I_q$ *as a representation of* H.

ii) *If* $\widetilde{V}_q \simeq \sum_{\kappa} \beta_{\kappa q} U_\kappa$, *then* $F_\kappa \leftrightarrow \sum_{q} \beta_{\kappa q} I_q$ *as a representation of* H.

iii) *If* $\widetilde{I}_q \leftrightarrow \sum_{\kappa} \gamma_{\kappa q} F_\kappa$, *then* $U_\kappa \simeq \sum_{q} \gamma_{\kappa q} V_q$ *as a representation of* H.

Remark. Since $\widetilde{e_q B}$ is operator isomorphic to $(e_q B)A=e_q A$ and $e_q A$ is a component of a direct decomposition of A into right ideals, any directly indecomposable component of $\widetilde{V}_q$ is equivalent with some U_κ. And since A is, as a B-right module, a direct sum of submoduli $s_i' B (i=1, 2, \ldots, t)$ isomorphic to B (where $\{s_i'\}$ is the left representative system of G mod H), any directly indecomposable component of U_κ is equivalent with some V_q.

Proof. From i) and ii) in Lemma 2 we have immediately i) and ii) in the theorem. To prove iii), let $AE_\kappa \simeq \sum_{\kappa} \gamma_{\kappa q} Be_q$ (as B-left molule) and $\widetilde{\bar{e}_q \bar{B}} \leftrightarrow \sum \gamma'_{\kappa q} \bar{E}_\kappa \bar{A}$ (as A-right module). Since $AE_\kappa/N(B)AE_\kappa \simeq \sum_{q} \gamma_{\kappa q} \bar{B} \bar{e}_q$, $\gamma_{\kappa q}$ is equal to the dimension of $e_q(AE_\kappa/N(B)AE_\kappa)$. On the other hand, $\gamma'_{\kappa q}$ is equal to the dimension of $(e_q A/e_q N(B)A)E_\kappa$ since $\widetilde{\bar{e}_q \bar{B}} \simeq \widetilde{(e_q B/e_q N(B))} \simeq e_q A/e_q N(B)A$. Both $e_q(AE_\kappa/N(B)AE_\kappa)$ and $(e_q A/e_q N(B) A)E_\kappa$ are isomorphic to $e_q AE_\kappa/e_q N(B)AE_\kappa$ as K-module, hence $\gamma_{\kappa q}=\gamma'_{\kappa q}$.

§ 3, Kronecker Product.

Let $B^{(0)}$ be the groupring $A(G, 1)=\sum_{a} Ku_a^{(0)} (u_a^{(0)} u_b^{(0)}=u_{ab}^{(0)})$, and $B^{(1)}$

$=(AG, \varepsilon^{(1)})=\sum_a K u_a^{(1)}\ (u_a^{(1)} u_b^{(1)}=\varepsilon_{a,b}^{(1)} u_{ab}^{(1)})$ and $B^{(2)}=A(G, \varepsilon^{(2)})=\sum_a K u_a^{(2)}\ (u_a^{(2)} u_b^{(2)}=\varepsilon_{a,b}^{(2)} u_{ab}^{(2)})$. And let A be their direct product ($A=B^{(1)}\times B^{(2)}$). The subalgebra $B^{(3)}$ of A with the bases $u_a^{(1)}\times u_a^{(2)}$ is also a group ring of G with factor set $\{\varepsilon_{a,b}^{(1)}\varepsilon_{a,b}^{(2)}=\varepsilon_{a,b}^{(3)}\}$. We shall call $B^{(3)}$ the *Kronecker product of* $B^{(1)}$ and $B^{(2)}$, and denote it by $B^{(1)}\otimes B^{(2)}$. When $\mathfrak{m}^{(1)}$ and $\mathfrak{m}^{(2)}$ are respectively $B^{(1)}$ and $B^{(2)}$-right moduli, their direct product $\mathfrak{m}^{(1)}\times\mathfrak{m}^{(2)}$ is an A-right module. Of course this may be considered as $B^{(3)}$-right module, and then it is also called a *Kronecker product* of $\mathfrak{m}^{(1)}$ and $\mathfrak{m}^{(2)}$ and denoted by $\mathfrak{m}^{(1)}\otimes\mathfrak{m}^{(2)}$.

A and $B^{(3)}$ may be considered in a different way. Let $G^{(1)}$ and $G^{(2)}$ be both isomorphic to G by the correspondence $a\leftrightarrow a^{(i)}$ $(i=1, 2)$ and $G^{(3)}$ the subgroup consisting of elements $a^{(1)}\times a^{(2)}$ $(a\in G)$. Then $G^{(3)}$ is also isomorphic to G, A is a group ring of $G^{(1)}\times G^{(2)}$ and $B^{(3)}$ is the subalgebra corresponding to $G^{(3)}$.

We shall denote by $U_\kappa^{(i)}$ and $F_\kappa^{(i)}$ $(i=1, 2, 3)$ directly indecomposable constituents of the regular representation and irreducible representations of $B^{(i)}$ $(i=1, 2, 3)$ respectively, and by $\tilde{U}_\kappa^{(3)}(\tilde{F}_\kappa^{(3)})$ the representation of A induced by $U_\kappa^{(3)}(F_\kappa^{(3)})$ of $B^{(3)}$. Further when $M^{(1)}$ and $M^{(2)}$ are representations respectively by $B^{(1)}$-right module $\mathfrak{m}^{(1)}$ and $B^{(2)}$-right module $\mathfrak{m}^{(2)}$, we shall denote by $M^{(1)}\otimes M^{(2)}$ the representation of G by $\mathfrak{m}^{(1)}\otimes\mathfrak{m}^{(2)}$. It will be easily seen that $M^{(1)}\otimes M^{(2)}(a)=M^{(1)}(a)\times M^{(2)}(a)$ for all a in G.

As K is an algebraically closed field, any indecomoposable constituent of the regular representation of A is equivalent with some $U_\rho^{(1)}\times U_\kappa^{(2)}$ and any irreducible represntation with some $F_\rho^{(1)}\times F_\kappa^{(2)}$.. As the immediate consequence of Theorem 1, we have

Lemma 3.

i) *If* $U_\rho^{(1)}\otimes U_\kappa^{(2)}\leftrightarrow\sum_\sigma \alpha_{\kappa\rho\sigma}F_\sigma^{(3)}$ *then* $\tilde{U}_\sigma^{(3)}\leftrightarrow\sum_{\kappa,\rho}\alpha_{\kappa\rho\sigma}F_\rho^{(1)}\times F_\kappa^{(2)}$.

ii) *If* $F_\rho^{(1)}\otimes F_\kappa^{(2)}\leftrightarrow\sum_\sigma \beta_{\kappa\rho\sigma}F_\rho^{(3)}$ *then* $\tilde{U}_\sigma^{(3)}\simeq\sum_{\kappa,\rho}\beta_{\kappa\rho\sigma}U_\rho^{(1)}\times U_\kappa^{(2)}$.

iii) *If* $U_\rho^{(1)}\otimes U_\kappa^{(2)}\simeq\sum_\sigma \gamma_{\kappa\rho\sigma}U_\sigma^{(3)}$ *then* $\tilde{F}_\sigma^{(3)}\leftrightarrow\sum_{\kappa,\rho}\gamma_{\kappa\rho\sigma}F_\rho^{(1)}\times F_\kappa^{(2)}$.

Now, let $\mathfrak{m}$ be a $B^{(3)}$-right module and $\tilde{\mathfrak{m}}$ be an A-right module induced by $\mathfrak{m}$. $A(G, \varepsilon^{(2)-1})=\sum_a K u_a^{(2')}(u_a^{(2')} u_b^{(2')}=\frac{1}{\varepsilon_{a,b}^{(2)}} u_{ab}^{(2')})$ is reciprocally isomorphic to $B^{(2)}$ by the correspondence $u_a^{(2)}\leftrightarrow u_a^{(2')-1}$, $B^{(1)}\simeq B^{(2)}\otimes B^{(2')}$. $B^{(2)}$ may be considered as $B^{(2')}$-right module by defining the operation as $x u_a^{(2')}=u_a^{(2)-1}x$ $(x\in B^{(2)})$, and hence $\mathfrak{m}\times B^{(2)}$ may be considered as $B^{(2)}\otimes B^{(2')}$-right module, namely $B^{(1)}$-right module. On the other hand, $\mathfrak{m}$ may be considered as $B^{(0)}$-right module by defining the operation as $xu_a^{(0)}=x\,(x\in\mathfrak{m})$ and $B^{(2)}$ as $B^{(2)}$-right module in a natural manner. Then $\mathfrak{m}\times B^{(2)}$ may be considered as $B^{(0)}\otimes B^{(2)}$-right module, namely $B^{(2)}$-right module. As is easily seen, these operations of elements of $B^{(1)}$ and $B^{(2)}$ for $\mathfrak{m}\times B^{(2)}$ are mutually commutative, and hence $\mathfrak{m}\times B^{(2)}$ may be considered as $A\,(=B^{(1)}\times B^{(2)})$-right module.

Lemma 4. *The above defined A-right module $\mathfrak{m}\times B^{(2)}$ is isomorphic to $\tilde{\mathfrak{m}}$.*

Proof. As a represetative system of $G^{(1)}\times G^{(2)}$ mod $G^{(3)}$ we choose $\{1\times c^{(2)}\ ;\ c\in G\}$. Then $\tilde{\mathfrak{m}}$ is isomorphic to the direct sum of moduli $\mathfrak{m}(1\times u_c^{(2)})$ isomorphic to $\mathfrak{m}$. If $x\in\mathfrak{m}$, then

$$x(1\times u_c^{(2)})(u_a^{(1)}\times u_b^{(2)})=x(u_a^{(1)}\times u_a^{(2)})(1\times u_a^{(2)-1}u_c^{(2)}u_b^{(2)})$$
$$=(xu_a^{(3)})(1\times u_a^{(2)-1}u_c^{(2)}u_b^{(2)}).$$

On the other hand, in $\mathfrak{m}\times B^{(2)}$

$$(x\times u_c^{(2)})(u_a^{(1)}\times u_b^{(2)})=xu_a^{(3)}\times u_a^{(2)-1}u_c^{(3)}u_b^{(2)}$$

Hence $\tilde{\mathfrak{m}}$ is isomorphic to $\mathfrak{m}\times B^{(2)}$ by the correspondence $x(1\times u_c^{(2)})\leftrightarrow x\times u_c^{(2)}$.

Let $B^{(\nu)}=\sum\limits_{\kappa,i}e_{\kappa i}^{(\nu)}B^{(\nu)}\,(=\sum\limits_{\kappa,i}B^{(\nu)}e_{\kappa,i}^{(\nu)})\,(\nu=1,2,2',3)$ be a decomposition of $B^{(\nu)}$ as (1,1), and suppose that $B^{(2)}e_{\kappa i}^{(2)}$ corresponds to $e_{\kappa i}^{(2')}B^{(2')}$ in the reciprocally isomorphic correspondence between $B^{(2)}$ and $B^{(2')}$. Then $B^{(2)}e_\kappa^{(2)}$ is, considered as $B^{(2')}$-right module, isomorphic to $e_\kappa^{(2')}B^{(2')}$

Since $(\mathfrak{m}\times B^{(2)})(e_\rho^{(1)}\times e_\kappa^{(2)})=(\mathfrak{m}\times B^{(2)}e_\kappa^{(2)})(e_\rho^{(1)}\times 1)$, we have immediately

Lemma 5. *Let $\mathfrak{m}$ be a $B^{(3)}$-right module. Then the number of the factor groups isomorphic to $\bar{e}_\rho^{(1)}\bar{B}^{(1)}\times\bar{e}_\kappa^{(2)}\bar{B}^{(2)}$ which appear in a composition factor group siries of A-right module $\tilde{\mathfrak{m}}$ is equal to the number of the factor groups isomorphic to $\bar{e}_\rho^{(1)}\bar{B}^{(1)}$ which appear in a composition factor group siries of $B^{(1)}$-right module $\mathfrak{m}\times B^{(2)}e_\kappa^{(2)}$.*

Now we have the main theorem concerning the Kronecker product of representations.

Theorem 2.

i) *If* $U_\rho^{(1)}\otimes U_\kappa^{(2)}\leftrightarrow\sum\limits_\sigma\alpha_{\kappa\rho\sigma}F_\sigma^{(3)}$ *then* $U_\sigma^{(3)}\otimes U_\kappa^{(2')}\leftrightarrow\sum\limits_\rho\alpha_{\kappa\rho\sigma}F_\rho^{(1)}$.

ii) *If* $F_\rho^{(1)}\otimes F_\kappa^{(2)}\leftrightarrow\sum\limits_\sigma\beta_{\kappa\rho\sigma}F_\sigma^{(3)}$ *then* $U_\sigma^{(3)}\otimes F_\kappa^{(2')}\simeq\sum\limits_\rho\beta_{\kappa\rho\sigma}U_\rho^{(1)}$.

iii) *If* $U_\rho^{(1)}\otimes U_\kappa^{(2)}\simeq\sum\limits_\sigma\gamma_{\kappa\rho\sigma}U_\sigma^{(3)}$ *then* $F_\sigma^{(3)}\otimes U_\kappa^{(2')}\leftrightarrow\sum\limits_\rho\gamma_{\kappa\rho\sigma}F_\rho^{(1)}$.

Proof. i) From lemma 3 $\widetilde{U}_{\rho}^{(3)}\leftrightarrow\sum\alpha_{\kappa\rho\sigma}F_{\sigma}^{(1)}\otimes F_{\kappa}^{(2)}$, hence from lemma 5 $\alpha_{\kappa\rho\sigma}$ is equal to the number of the factor groups isomorphic to $\bar{e}_{\rho}^{(1)}\bar{B}^{(1)}$ which appear in a composition factor group series of $B^{(1)}$-right module $e_{\sigma}^{(3)}B^{(3)}\times B^{(2)}e_{\kappa}^{(2)}$, and as is easily seen we have the representation $U_{\sigma}^{(3)}\times U_{\kappa}^{(2')}$ of G by $B^{(1)}$-right module $e_{\sigma}^{(3)}B^{(3)}\times B^{(2)}e_{\kappa}^{(2)}$. This proves i).

ii) From Lemma 3, $\widetilde{U}_{\sigma}^{(3)}\simeq\sum_{\kappa,\rho}\beta_{\kappa\rho\sigma}U_{\rho}^{(1)}\times U_{\kappa}^{(2)}$, hence $\widetilde{e_{\sigma}^{(3)}B^{(3)}}\simeq\sum_{\kappa,\rho}\beta_{\kappa\rho\sigma}e_{\rho}^{(1)}B^{(1)}\times e_{\kappa}^{(2)}B^{(2)}$. From lemma 4 $\widetilde{e_{\sigma}^{(3)}B^{(3)}}$ is isomorphic to $e_{\sigma}^{(3)}B^{(3)}\times B^{(2)}$ considered as A-right module. Since

$$e_{\sigma}^{(3)}B^{(3)}\times B^{(2)}/(e_{\sigma}^{(3)}B^{(3)}\times B^{(2)})(1\times N(B^{(2)}))\simeq e_{\sigma}^{(3)}B^{(3)}\times\bar{B}^{(2)}$$
$$\simeq\sum_{\kappa,\rho}\beta_{\kappa\rho\sigma}e_{\rho}^{(1)}B^{(1)}\times\bar{e}_{\kappa}^{(2)}\bar{B}^{(2)},$$
$$(e_{\sigma}^{(3)}B^{(3)}\times\bar{B}^{(2)})(1\times e_{\kappa}^{(2)})=e_{\sigma}^{(3)}B^{(3)}\times\bar{B}^{(2)}\bar{e}_{\kappa}^{(2)}\simeq\sum_{\kappa,\rho}\beta_{\kappa\rho\sigma}e_{\rho}^{(1)}B^{(1)}$$
$$\times\bar{e}_{\kappa}^{(2)}\bar{B}\bar{e}_{\kappa}^{(2)}.$$

And since $\bar{e}_{\kappa}^{(2)}\bar{B}^{(2)}\bar{e}_{\kappa}^{(2)}$ is isomorphic to K, $\beta_{\kappa\rho\sigma}$ is the number of direct factors of $e_{\rho}^{(3)}B^{(3)}\times\bar{B}^{(2)}\bar{e}_{\kappa}^{(2)}$ which are isomorphic to $e_{\rho}^{(1)}B^{(1)}$. Further, as is easily seen, we have the representation $U_{\rho}^{(3)}\otimes F_{\kappa}^{(2')}$ by $B^{(1)}$-right module $e_{\rho}^{(3)}B^{(3)}\times\bar{B}^{(2)}\bar{e}_{\kappa}^{(2)}$. This proves ii).

iii) Considering $\bar{e}_{\sigma}^{(3)}\bar{B}^{(3)}$ for $e_{\rho}^{(3)}B^{(3)}$ in the proof of i), we have iii) in a similar way as in i).

§ 4. Group Characters.

When the field of coefficients K is of characteristic $\neq 0$, we understand the group characters as in the sense of B. N. M. § 6. We shall denote by $\eta^{(\kappa)}$ and $\varphi^{(\kappa)}$ ($\kappa=1, 2, \dots, k$) directly indecomposable characters and irreducible characters of G respectively, and by $\zeta^{(i)}$ ($i=1, 2, \dots, k^*$) ordinary irreducible characters of G, and further by $c_{\kappa\lambda}$ ($\kappa, \lambda=1, 2, \dots, k$) Cartan invariants[14], by $d_{i\kappa}$ ($i=1, 2, \dots, k^*$; $\kappa=1, 2, \dots, k$) decomposition numbers[15].

In section 3, if $B^{(2)}=B^{(0)}$ then $U_{\kappa}^{(0')}$ and $F_{\kappa}^{(0')}$ are respectively contragredient to $U_{\kappa}^{(0)}$ and $F_{\kappa}^{(0)}$. We shall denote them by $U_{\kappa'}^{(0)}$ and $F_{\kappa'}^{(0)}$.

As a special case of Lemma 3 iii), we have that if $U_{\rho}^{(0)}\otimes U_{\kappa}^{(0)}\simeq\sum_{\sigma}\gamma_{\kappa\rho\sigma}U_{\sigma}^{(0)}$ then $\widetilde{F}_{\sigma}^{(0)}\leftrightarrow\sum_{\kappa,\rho}\gamma_{\kappa\rho\sigma}F_{\rho}^{(0)}\times F_{\kappa}^{(0)}$ and then, from Theorem 2 iii),

14) See B. N. M.
15) See B. N. M.

$F_\sigma^{(0)} \otimes U_{\kappa'}^{(0)} \leftrightarrow \sum_\rho \gamma_{\kappa\rho\sigma} F_\rho^{(0)}$. If we denote the unit representation by $F_1^{(0)}$ then $U_{\kappa'}^{(0)} \leftrightarrow \sum_\rho \gamma_{\kappa\rho 1} F_\rho^{(0)}$ and hence $c_{\kappa\rho} = \gamma_{\kappa'\rho 1}$. Therefore

$$\tilde{F}_1^{(0)} \leftrightarrow \sum_{\kappa, \rho} c_{\kappa\rho} F_\rho^{(0)} \times F_{\kappa'}^{(0)} \tag{4.1}$$

Since $\tilde{F}_1^{(0)}(a \times b) = (\delta_{x, a^{-1}xb})$ (x; row index, $a^{-1}xb$; column index), $\tilde{F}_1^{(0)}(a \times b)$ may be considered as a matrix with coefficients of complex numbers and its trace coincides with the character in the sense of B. N. M. for all p-regular elements a and b in G. Further

$$tr\ \tilde{F}_1^{(0)}(a \times b) = 0 \text{ if } a \text{ is not conjugate with } b \text{ in } G. \tag{4.2}$$
$$= \frac{g}{g(a)} \text{ if } a \text{ and } b \text{ are mutually coinjugate}$$

in G, where $g(a)$ is the number of elements of the conjugate class containing a.

From (4.1) and (4.2) we have the following orthogonality relation for group characters.

$$\sum_\kappa \eta^{(\kappa)}(a) \varphi^{(\kappa')}(b) = 0 \text{ if } a \text{ is not conjugate with } b \text{ in } G. \tag{4.3}$$
$$= \frac{g}{g(a)} \text{ if } a \text{ and } b \text{ are mutually conjugate in } G.$$

Let $K_1, K_2, \ldots, K_{h^*}$ be the conjugate classes of G and $K_1, K_2, \ldots, K_h$ be p-regular classes among them. We denote $\eta^{(\kappa)}(a)$, $\varphi^{(\kappa)}(a)$ by $\eta_\mu^{(\kappa)}$, $\varphi_\mu^{(\kappa)}$ when $a \in K_\mu$. Further, suppose that $U_{\kappa'}$ and $F_{\kappa'}$ are respectively contragredient to U_κ and F_κ, and $K_{\mu'}$ is the conjugate class consisting of inverses of elements in K_μ. Then (4.3) may be expressed as follows.

Theorem 3.

$$\sum_{i=1}^{k} \eta_\mu^{(i)} \varphi_\nu^{(i)} = \frac{g}{g_\mu} \delta_{\mu, \nu'} \text{ for } \mu, \nu = 1, 2, \ldots, h. \tag{4.4}$$

As an immediate consequence of the theorem, we have for ordinary characters

Corollary.

$$\sum_{i=1}^{k^*} \zeta_\mu^{(i)} \zeta_\nu^{(i)} = \frac{g}{g_\mu} \delta_{\mu, \nu'} \text{ for } \mu, \nu = 1, 2, \ldots, h^*. \tag{4.5}$$

Further we can easily show that $h=k$.

We arrange $\varphi_\lambda^{(\kappa)}$, $\eta_\lambda^{(\kappa)}$, $\zeta_\lambda^{(i)}$ in matrix form

$$\Phi = (\varphi_\lambda^{(\kappa)}),\ H = (\eta_\lambda^{(\kappa)}),\ Z = (\zeta_\lambda^{(i)})$$

($\kappa=1, \ldots, k$; row indices, $\lambda=1, \ldots, k$; column indices in Φ, H; $i=1, \ldots, k^*$; row indices, $\lambda=1, \ldots, k$; column indices in Z). Then $C=C'$, $H=C\Phi$, $Z=D\Phi$, and (4.4) and (4.5) show the equation

$$Z'Z = H'\Phi = \left(\delta_{\mu,\nu'}\frac{g}{g_\mu}\right). \tag{4.6}$$

Let $\Phi^*=(\varphi_{\mu'}^{(\kappa)})$ (κ; row index, μ; column index) then from (4.6) we have

$$\Phi^* H \begin{pmatrix} g_1 & & 0 \\ & \ddots & \\ 0 & & g_k \end{pmatrix} = gE \ (E\text{; unit matrix}) \tag{4.7}$$

This proves

Theorem 4.

$$\sum_\mu g_\mu \varphi_{\mu'}^{(\kappa)} \eta_\mu^{(\lambda)} = g\delta_{\kappa,\lambda}. \tag{4.8}$$

From (4.6) $\Phi' C\Phi = \Phi' D' D\Phi$ and $|\Phi| \neq 0$, hence we have

Theorem 5.[16]. $C = D'D$.

From Theorem 1 and 2 we have the following relations on induced characters and multiplication of characters.

Let H be a subgroup of G. We denote by $\eta^{(q)*}$, $\varphi^{(q)*}$ a directly indecomposable character of H, and by $\tilde{\eta}^{(q)*}$, $\tilde{\varphi}^{(q)*}$ characters of G induced by them.

Theorem 6.

i) *If* $\tilde{\eta}^{(q)*} = \sum_\kappa \alpha_{\kappa q}\varphi^{(\kappa)}$ *for p-regular elements of G, then* $\eta^{(\kappa)} = \sum_q \alpha_{\kappa q}\varphi^{(q)*}$ *for p-regular elemetts of H.*

ii) *If* $\tilde{\eta}^{(q)*} = \sum_\kappa \beta_{\kappa q}\eta^{(\kappa)}$ *for p-regular elements of G, then* $\varphi^{(\kappa)} = \sum_q \beta_{\kappa q}\varphi^{(q)*}$ *for p-regular elements of H.*

iii) *If* $\tilde{\varphi}^{(q)*} = \sum_\kappa \gamma_{\kappa q}\varphi^{(\kappa)}$ *for p-regular elements of G, then* $\eta^{(\kappa)} = \sum_q \gamma_{\kappa q}\eta^{(q)*}$ *for p-regular elements of H.*

Theorem 7.

i) *If* $\eta^{(\rho)}\eta^{(\kappa)} = \sum_\sigma \alpha_{\kappa\rho\sigma}\varphi^{(\sigma)}$ *then* $\eta^{(\sigma)}\eta^{(\kappa')} = \sum_\rho \alpha_{\kappa\rho\sigma}\varphi^{(\rho)}$.

ii) *If* $\varphi^{(\rho)}\varphi^{(\kappa)} = \sum_\sigma \beta_{\kappa\rho\sigma}\varphi^{(\sigma)}$ *then* $\eta^{(\sigma)}\varphi^{(\kappa')} = \sum_\rho \beta_{\kappa\rho\sigma}\eta^{(\rho)}$.

iii) *If* $\eta^{(\rho)}\eta^{(\kappa)} = \sum_\sigma \gamma_{\kappa\rho\sigma}\eta^{(\sigma)}$ *then* $\varphi^{(\sigma)}\eta^{(\kappa')} = \sum_\rho \gamma_{\kappa\rho\sigma}\varphi^{(\rho)}$.

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16) The same proof of the theorem has been independently obtained by M. Osima. For the existing proofs, see B. N. M., the paper in the foot note 10) and R. Brauer: Proceedings of the National Academy of Science, 25 (1939), p. 252.