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1. Introduction

In this paper, we shall give a proof of the following Theorem, which is a conjecture of B. Rickman [9]; in special case, \( C_G(\phi) \) has order 2, M.J. Collins and B. Rickman proved in [2].

Theorem. Let \( G \) be a finite group which admits an automorphism \( \phi \) of odd prime order \( r \) whose fixed-point-subgroup \( C_G(\phi) \) is a cyclic 2-group. Then \( G \) is solvable.

All groups considered in this paper are assumed finite. Our notation corresponds to that of Gorenstein [7].

An important tool that is brought to attack the problem is B. Baumann’s classification of finite simple groups whose Sylow 2-subgroups are maximal [1], and in analogy with Matsuyama [8] that used the results of [1], we shall prove that \( H_G(5;2) \), where \( S \) is a \( \phi \)-invariant Sylow 3-subgroup of \( G \).

C.A. Rowley has obtained a proof of the theorem under the additional hypothesis that \( G \) does not involve \( S_4 \), the symmetric group on 4 letters.

The Theorem is a contribution to the continuing problem of showing that finite groups which admit an automorphism \( \phi \) of odd prime order such that \( C_G(\phi) \) is a 2-group are solvable.

2. Preliminaries

We first quote some frequently used results.

2.1. (Thompson [12])

Let \( G \) be a group which admits a fixed-point-free automorphism of prime order. Then \( G \) is nilpotent.

2.2. (Rowley [10])

Let \( G \) be a solvable group admitting an automorphism of odd prime order \( p \) such that \( C_G(\phi) \), the fixed-point-subgroup of \( \phi \) in \( G \), is a cyclic \( q \)-group, \( q \neq p \). Then, for any prime \( r \), \( G \) is either \( r \)-nilpotent or \( r \)-closed.
2.3. (Glauberman [4])
Let $G$ be a group with a Sylow $p$-subgroup $P$, either $p$ odd or $p=2$ and $S_4$ is not involved in $G$, in which $C_2(Z(P))$ and $N_2(J(P))$ both have normal $p$-complements. Then $G$ possesses a normal $p$-complement.

2.4. (Gilman and Gorenstein [3])
If $G$ is a simple group with Sylow 2-subgroups of class 2, then $G=L_2(9)$, $q \equiv 7, 9 \pmod{16}$, $A_n$, $Sz(2^n)$, $n$ odd, $n>1$, $U_3(2^n)$, $n \geq 2$, $L_3(2^n)$, $n \geq 2$, or $Psp(4, 2^n)$, $n \geq 2$.

2.5. (Gorenstein [7])
Let $P$ be a Sylow $p$-subgroup of $G$, where $p$ is the smallest prime in $\pi(G)$. If $p>2$, assume $d_\phi(P)\leq 2$, while if $p=2$, assume $P$ is cyclic. Then $G$ has a normal $P$-complement.

2.6. (Matsuyama [8])
Let $Q$ be a 2-group admitting an automorphism $\phi$ of odd order $\neq 1$. If $d_\phi(Q)=1$, then $Q=E\ast R$, where $E$ is $\phi$-invariant, extra-special or 1, and $R$ is $\phi$-invariant, and $R$ is cyclic, $D_m, Q_m$, or $S_m, m \geq 4$.

2.7. (Collins-Rickman [2])
Let $T$ be an extra-special 2-group admitting an automorphism $\phi$ of odd prime order $r$ acting fixed-point-freely on $T/T'$. Let $S$ be the natural semidirect product $T\langle \phi \rangle$ and let $K$ be a field of nonzero characteristic different from 2 and $r$. Assume that there exists a $KS$-module $M$ for which $C_M(\phi)=C_M(T)=0$. Then

(i) $r=2^n+1$ is a Fermat prime,

(ii) $|T|=2^{2n+1}$, and

(iii) $T\cong Q\ast D$,

where $Q$ and $D$ denote the quaternion and dihedral groups of order 8, respectively, and $\ast$ denote the central product.

2.8. (Glauberman [5] [6])
Let $G$ be a solvable group with a Sylow 2-subgroup $Q$ with $G\cong C(Z(Q))N(J(Q))$, and $O(X)=1$. Put

$Z=\langle Z^* | G \triangleright Z^* : 2$-subgroup and $O_2(G/C(Z^*))=1 \rangle$

and $J=\langle x \in G | x : 2$-element, $|Z/C_2(x)|=2 \rangle$

and $H=\langle J, C(Z) \rangle$. Then the following hold;

(i) there exists a normal subgroup $G_i$ of $H$ containing $C(Z)$, $1 \leq i \leq m$, such that, for $i=1, \cdots, m, G_i/C(Z)\cong S_3$, and $H/C(Z)=G_1/C(Z)\times \cdots \times G_m/C(Z)$.

(ii) let $V_i=[G_i, Z]$, $1 \leq i \leq m$, and let $V=V_1\oplus \cdots \oplus V_m$, then $Z=V \oplus C_2(H)$ and $V_i\cong Z_2 \times Z_2$, $1 \leq i \leq m$. 

(iii) there is a 3-element $x_0$ of $H$ such that, for each $g \in H$, $H = \langle Q \cap H, x_0^g, C(Z) \rangle$ and $G/C(Z) = H/C(Z) C_{G/C(Z)}(x_0^g C(Z))$.

2.9. (Matsuyama [8])

Let $G$ be a group with a Hall $\tau$-subgroup $H$, and let $1 \neq P \in \text{Syl}_3(H)$, $Q$ 2-group. If $N_2(H) = HQ$, $d(Q) = 1$, $\Omega_1(Z(Q)) = \langle w \rangle$, $C_H(w) = 1$, and $\text{N}_G(P; \tau) = 1$, then, for each $P \neq P$, $g \in G$, $m(P \cap P^g) \leq 1$.

2.10. (Burnside's theorem [7])

If a Sylow $p$-subgroup of $G$ lies in the center of its normalizer in $G$, then $G$ has a normal $p$-complement.

2.11. (Burnside's theorem [7])

If $P$ is a Sylow $p$-subgroup of $G$, then two normal subsets of $P$ are conjugate in $G$ if and only if they are conjugate in $N_G(P)$. In particular, two elements of $Z(P)$ are conjugate in $G$ if and only if they are conjugate in $N_G(P)$.

2.12. (Smith-Tyrer [11])

Let $G$ be a group with an Abelian Sylow $p$-subgroup $P$ for some odd prime $p$. If $[N(P):C(P)] = 2$ and $P \cap N(P)^r$ is noncyclic, then $G$ is $p$-solvable.

2.13. (Thompson Transitivity theorem [7])

Let $G$ be a group in which the normalizer of every nonidentity $p$-subgroup is $p$-constrained. Then if $A \in SCN_3(P)$, $C_\sigma(A)$ permutes transitively under conjugation the set of all maximal $A$-invariant $q$-subgroups of $G$ for any prime $q \neq p$.

2.14. (Collins-Rickman [2])

Let $G$ be a group, and let $p$ and $q$ be distinct prime divisors of $G$. Assume that $G$ has an Abelian Sylow $p$-subgroup $P$ for which $m(P) \geq 3$ and that, whenever $P_0$ is a subgroup of $P$ with $m(P/P_0) \geq 2$, $N_G(P_0)$ is $p$-constrained. Then $C_\sigma(P)$ permutes the elements of $\text{N}_G(P; q)$ transitively under conjugation.

2.15. (Frobenius theorem [7])

$G$ is $p$-nilpotent if and only if $N_G(H)/C_\sigma(H)$ is a $p$-group for every nonidentity $p$-subgroup $H$ of $G$.

3. The proof of the Theorem

Let $G$ be a minimal counterexample to the Theorem, for the remainder of this paper.

Lemma 3.1. $G$ is simple.

Proof. By Lemma 5.1. of [2].
Lemma 3.2. Let \( p \) be a prime divisor of \( G \) and \( P \subseteq \text{Syl}_p(G) \). If \( N_0(P) \) has a normal \( p \)-complement, then \( p = 2 \) and the symmetric group \( S_4 \) is involved in \( G \).

Proof. By Lemma 5.2. of [2], (2.2) and (3.1).

For the remainder of this paper, \( Q \) denotes the \( \phi \)-invariant Sylow 2-subgroup of \( G \), and let \( C_\phi = \langle x \rangle \) and \( \Omega(C_\phi) = \langle \omega \rangle \).

Then \( Q \) is a unique \( \phi \)-invariant Sylow 2-subgroup, and let \( p \) be an odd prime in \( \pi(G) \) and \( P \subseteq \text{Syl}_p(G) \), then, by (3.2), \( N_0(P) \equiv w \).

Lemma 3.3. \( d(Q) \geq 2 \).

Proof. If \( d(Q) = 1 \), by (2.6) and hypothesis \( Q = E \ast R \) where \( E \) is \( \phi \)-invariant, extra-special, and \( R \) is \( \phi \)-invariant, cyclic. If \( E = 1 \), by (2.5) \( G \) is 2-nilpotent, contrary to (3.1). So \( E \neq 1 \). Since \( c(c) = 2 \), by (2.4) this is a contradiction.

Lemma 3.4. Every \( \phi \)-invariant proper subgroup of \( G \) is 2-nilpotent.

Proof. Assume otherwise. Let \( M \) be a non-nilpotent maximal \( \phi \)-invariant subgroup of \( G \) without a normal Sylow 2-subgroup. If \( N(O_2(M)) \) is 2-nilpotent, \( M \) is nilpotent, a contradiction. By (2.2), \( N(O_2(M)) \) is 2-closed. Hence \( M = N(O_2(M)) \), \( O_2(M) = Q \), and \( M = N_0(Q) \). Thus there is an odd prime \( p \) dividing the index \( [N_0(Q) : C_0(Q)] \).

By (3.3), there is a characteristic subgroup \( C \) of \( Q \) such that \( C \cong Z_2 \times \cdots \times Z_2 \), \( C \) contains \( \Omega(Z(Q)) \), and \( [C, \phi] = 1 \). Let \( P_0 \) be a \( \phi \)-invariant Sylow \( p \)-subgroup of \( N_0(Q) \) and \( P \) be a \( \phi \)-invariant Sylow \( p \)-subgroup containing \( P_0 \).

We now claim that \([C, P_0] = 1\). We may assume that \( w \in C \). \([w, P_0] \subseteq Q \cap P = 1\). Since \( P_0 \) centralizes \( C/C_0(P_0) \), \([P_0, C] = 1\). Thus \( C \subseteq N_0(P_0) \).

Let \( M_0 \) be a maximal \( \phi \)-invariant subgroup containing \( N_0(P_0) \). If \( M_0 \) is 2-closed, \( M_0 = N_0(Q) \). Since \( N_0(P_0) = P_0 = P \). Let \( Q_0 \) be a \( \phi \)-invariant Sylow 2-subgroup of \( N_0(P) \). Then \([P, Q_0] \subseteq P \cap Q = 1\), so \( N_0(P) \) is \( P \)-nilpotent, and by (3.2), \( p = 2 \), a contradiction. Thus \( M_0 \) is 2-nilpotent. Hence \( M_0 = N_0(P) \). Since \( C \subseteq N_0(P) \), \( 1 = [C, \phi] \subseteq C_0(P) \).

Now put \( Z_0 = [\Omega(Z(Q)), \phi] \). If \( Z_0 = 1 \), \( P, Q \subseteq C_0(Z_0) \). When \( C_0(Z_0) \) is 2-closed, \( P \subseteq N_0(Q) \), and \([Q_0, P_0] \subseteq Q \cap P = 1 \), a contradiction. Hence \( C_0(Z_0) \) is 2-nilpotent. Therefore as \( Q \subseteq N_0(P) \), \([Q, P_0] \subseteq Q \cap P = 1 \), a contradiction. Thus we may assume that \( Z_0 = 1 \), hence that \( \Omega(Z(Q)) = \langle \omega \rangle \).

Put \( \bar{Q} = Q/\langle \omega \rangle \) and let \( C_1 \) be the inverse image of \( Z(\bar{Q}) \cap \bar{C} \) in \( Q \). As \([C_1, x] \subseteq \langle \omega \rangle \), \( C_1 \subseteq N_0(Q) \). On the other hand, let \( y \in C_1 \). Then \([y, \phi] \in C_0(\langle x \rangle) \), since \( (y^{-1}xy)^* = y^{-1}xy \). Put \( C_0 = [C_1, \phi] \), so that \( 1 + C_0 \subseteq N_0(P) \), hence \( C_0(P) \) contains \( P_0 \) and \( x \).

Now let \( M_1 \) be a maximal \( \phi \)-invariant subgroup of \( G \) containing \( C_0(C_0) \). If \( M_1 \) is 2-closed, \( M_1 = N_0(Q) \), and \([Q_0, P] = 1 \), contradiction. Thus \( M_1 \) is 2-nilpotent,
i.e. $M_1 = N_G(P)$.

Put $\bar{Q} = Q/\Phi(Q)$. \([x, P] \subseteq P \cap Q = 1\). Since $P_0$ centralizes $\bar{Q}/C_{\bar{Q}}(P_0)$, $P_0$ centralizes $Q$. Hence $[P_0, Q] = 1$, a contradiction. Hence the lemma is proved.

For the remainder of this paper, in analogy with Matsuyama [8], we shall prove the following result:

(i) $3 \mid |G|$;
(ii) $|C_G(S)|$ is odd, where $S$ is a $\phi$-invariant Sylow 3-subgroup of $G$;
(iii) $U_0(S; 2) \neq 1$; and
(iv) $m(S) \geq 4$.

On the other hand, in analogy with Collins-Rickman [2], we shall prove that $U_0(S; 2) = 1$. Hence this contradicts above.

For the remainder of this paper, we shall write down the results which can be similarly proved as [8].

(3.5) $C_G(w) \subseteq Q$.

(3.6) If $p$ is an odd prime in $\pi(G)$ and $P \subseteq Syl_p(G)$, then $P$ is Abelian.

(3.7) If $p$ is an odd prime in $\pi(G)$ and $A$ is any $p$-subgroup of $G$, then $\text{Aut}_G(A) = N_G(A)/C_G(A)$ is a 2-group.

(3.8) If $\Omega_3(Z(Q)) \neq \langle w \rangle$, then $N_G(T)$ is a 2-group for any nontrivial $\phi$-invariant 2-subgroup $T$ of $G$.

Now put $P$ be a $\phi$-invariant Sylow $p$-subgroup of $G$ for any odd prime $p$ in $\pi(G)$. Let $K_p$ be a normal 2-complement of $N_G(P)$ and $Q_p = Q \cap N_G(P)$. Then $N_G(P) = Q_p K_p$, $Q_p \subseteq Q$. Furthermore let $Q_p^* = C_{Q_p}(K_p)$, and then $Q_p^* = [Q_p^*, \phi]$, since $w \in Q_p^*$.

Hence, for any $s \in \pi(K_s)$, $K_s = K_s$, $Q_s = Q_s$, and $Q_s^* = Q_s^*$. In particular, $K_p$ is a nilpotent Hall subgroup of $G$.

(3.9) $C_{Q_p}(P) = Q_p^*$.

(3.10) $d(Q_p/Q_p^*) = 1$.

Furthermore let $M_p = N_G(P)$ and $\bar{M}_p = M_p/Q_p^* K_p$. Then by (2.6) and hypothesis, $\bar{M}_p = \bar{E}_p$ $\Phi$ $\bar{E}_p$, where either $\bar{E}_p = 1$ or $\bar{E}_p$ is $\phi$-invariant, extra-special and $\bar{K}_p$ is $\phi$-invariant, cyclic.

On the other hand, by (3.4), $N_G(Q)$ is nilpotent, and then $N_G(Q) = Q$ by (3.5). Hence by (3.2), $S_s$ is involved in $G$, yields $3 \mid |G|$. Furthermore let $S \subseteq Syl_3(G)$, and then $m(S) \geq 3$.

Lemma 3.11. Let $p$ be an odd prime in $\pi(G)$. We can write $\bar{M}_p = E_p \Phi \bar{R}_p$, where either $E_p = 1$ or $E_p$ is $\phi$-invariant, extra-special, and $\bar{R}_p$ is $\phi$-invariant,
cyclic.
If \(E_p = 1\), then \(r = 2^s + 1\) is a Fermat prime.

Proof. By (2.7), it is immediate that \(C_{q(p)}(\phi) = C_{q(p)}(E_p) = 0\). By (2.7), it suffices to prove that \(\phi\) acts on \(E_p/E'_p\) fixed-point-freely. First we may assume that \(|R_p| = 2\). Then, since we can suppose that \(\phi\) centralizes an element of \(E_p\) of order 4, it is not necessarily trivial.

Now suppose that there exists an element \(y\) of \(E_p\) of order 4 such that \([\overline{y}, \overline{\phi}] = 1\). As \(E_p\) is extra-special, the conjugate class of \(y\) is \(\{\overline{y}, \overline{yw}\}\). Hence \([E_p, C_{\overline{y}}(\overline{y})] = 2\). Then \(\phi\) acts on the set, \(E_p - C_{\overline{y}}(\overline{y})\), fixed-point-freely. It is impossible.

Lemma 3.12. Let \(S\) be a \(\phi\)-invariant Sylow 3-subgroup of \(G\). If \([Q_3/Q_3^*, \phi] = 1\), then \(S\) is a T.I.-set.

Proof. If not, there exists an element \(g\) of \(G\) such that \(S^g \neq S\) and \(S^g \cap S = \pm 1\). First we shall show that \(C_{Q_3}(z) = Q_3^*\) for any \(z \in S^g\). It is immediate that \(C_{Q_3}(z) \supseteq Q_3^*\). If \(C_{Q_3}(z) = Q_3^*\) for some \(z \in S^g\), \(w \in C_{Q_3}(z)\), by hypothesis. But this is impossible. Next we will prove that, for any \(z \in S^g\), \(C_{Q_3}(z) = Q_3^*\) is 3-nilpotent.

Now put \(C_{Q_3}(z) = C\) and let \(S_1\) be a nontrivial subgroup of \(S\). By (3.7), \(\text{Aut}_C(S_1)\) is a 2-group. Put \(\text{Aut}_C(S_1) \ni t \neq 1\). Then \(t\) is a 2-element. Furthermore there exists an element \(y\) of \(S_1\) such that \(y^t = y\), i.e. \(y\) and \(y^t\) are conjugate in \(C_{Q_3}(z)\). By (2.11), \(y\) and \(y^t\) are conjugate in \(N_C(S)\). Thus we may assume that \(t \in N_C(S)\), and \(t \in Q_3^*\). Then \(t \in C_{Q_3}(z) = Q_3^* = C_{Q_3}(S)\), a contradiction. Hence \(C_{Q_3}(z) = Q_3^*\) is 3-nilpotent by (2.15), especially \(C_{Q_3}(z) = 3\)-constrained.

Furthermore put \(3 \equiv p \in \pi(K_3)\), and let \(P\) be a \(\phi\)-invariant Sylow \(p\)-subgroup of \(G\). \(N_{Q_3}(S) = N_{Q_3}(P)\). Thus \(C_{Q_3}(z) = \pi(K_3)\)-nilpotent.

Next put \(1 \neq y \in S^g \cap S\), and let \(M\) be a \(\pi(K_3)\)-complement of \(C_{Q_3}(y)\), and then we will prove that \(M\) is a 2-group.

\(S\) normalizes \(M\) and \(|S|, |M| = 1\). Now suppose that \(M\) is not a 2-group. There exists an odd prime \(q\) in \(\pi(M)\) such that \(q \in \pi(K_3)\). Furthermore there exists a Sylow \(q\)-subgroup \(Q_1\) of \(M\) normalized by \(S\). Since \(\text{Aut}_C(Q_1)\) is a 2-group, \(S \subseteq C_{Q_3}(Q_1)\), and hence \(Q_1 \subseteq K_3\). It is impossible. Thus \(M\) is a 2-group.

On the other hand, it is easy to show that \(M \supseteq Q_3^*\). Now suppose that \(M = Q_3^*\). Then \(C_{Q_3}(y) = Q_3^*K_3\), and since \(S, S^g \subseteq C_{Q_3}(y)\), \(S = S^g\), a contradiction. Hence \(M \supseteq Q_3^*\), \(C_{Q_3}(S) \subseteq N_{Q_3}(M)\).

Let \(\overline{M}\) be the intersection of all elements of \(U_{C_3}(S; 2)\). By (2.14), \(\overline{M} \supseteq M\). On the other hand, as \(\overline{M}\) is \(\phi\)-invariant, \(S\overline{M}\) is \(\phi\)-invariant. By (3.4), \(S\overline{M}\) is 2-nilpotent. Thus \([\overline{M}, S] \subseteq \overline{M} \cap S = 1\). Hence \(\overline{M} \subseteq C_{Q_3}(S) = Q_3^*\), a contradiction. This completes the proof of Lemma 3.12.

Now if \(E_p = 1\), \(r = 2^s + 1\) is a Fermat prime by (3.11), where \(r = |\phi|\).
On the other hand, when $E_3 = 1$, by (3.12), $S$ is a T.I.-set, where $S$ is a $\phi$-invariant Sylow 3-subgroup of $G$.

By B. Baumann [1], $Q$ is not a maximal subgroup of $G$, and thus there exists a proper subgroup $X$ of $G$ containing $Q$ such that $Q$ is a maximal subgroup of $X$.

In analogy with Matsuyama [8], we can say the following.

$X$ is a solvable $\{2, 3\}$-subgroup with $O(X) = 1$, and $X$ satisfies the hypothesis of (2.8). Thus the structure of $X$ is one of the following two type.

\begin{itemize}
  \item \textbf{Type I}
    \begin{align*}
    X/O_2(X) \text{ is isomorphic to } S_3, \text{ the symmetric group on 4 letters.}
    \\
    Z(O_2(X)) \text{ contains } Z(Q) \text{ and } Z(O_2(X)) = [Z(O_2(X)), X] \oplus C_{Z(O_2(X))}(X),
    \\
    \text{where } [Z(O_2(X)), X] \text{ is isomorphic to } Z_2 \times Z_2.
    
    \end{align*}

  \item \textbf{Type II}
    \begin{align*}
    X \text{ has a subgroup } H \text{ containing } O_2(X) \text{ such that } [X:H] = 2.
    \\
    H/O_2(X) = X_i/O_2(X) \times X_i/O_2(X), \text{ for } i = 1, 2.
    \\
    Z(O_2(X)) \text{ contains } Z(Q) \text{ and } Z(O_2(X)) = [Z(O_2(X)), X] \oplus [Z(O_2(X)), X] \oplus C_{Z(O_2(X))}(H),
    \\
    \text{where } [Z(O_2(X)), X] \text{ is isomorphic to } Z_2 \times Z_2, \text{ for } i = 1, 2.
    
    \end{align*}
\end{itemize}

On the other hand, considering the structure of $X$, $Z(Q)$ is noncyclic, by (3.8), $Q_3^* = 1$.

Now we will show that $U_C(K_3; 2) \neq 1$. For the remainder of this paper, let $S$ be a $\phi$-invariant Sylow 3-subgroup of $G$.

\textbf{Lemma 3.13.} \quad $U_C(S; \pi(K_3)) = U_C(S; 2)$.

Proof. It is easy that $U_C(S; \pi(K_3)) \supseteq U_C(S; 2)$. If there exists an element $A$ of $U_C(S; \pi(K_3))$ that is not a 2-group, by [7; 6.2.2], $S$ normalizes some Sylow $\pi$-subgroup $S^*$ of $A$. As $\text{Aut}_C(S^*)$ is a 2-group, $[S, S^*] = 1$. But it contradicts $C_C(S) = K_3$.

By (3.13), it suffices to prove that $U_C(S; \pi(K_3)) \neq 1$.

Now we suppose that $U_C(S; \pi(K_3)) = 1$. By Matsuyama [8], we can say the following.

\begin{itemize}
  \item (3.14) \quad If $S^g = S$, $g \in G$, then $m(S \cap S^g) \leq 1$.
  \item (3.15) \quad There exists a nontrivial proper subgroup $Z_1$ of $Z(Q)$ such that
      \[ 3/|C_C(Z_1)| \text{ and } [Z(Q); Z_1] = 2. \]
\end{itemize}

Furthermore, in analogy with Matsuyama [8], we can show the next lemma.

\textbf{Lemma 3.16.} \quad \textit{There exists a nontrivial element } $a$ \textit{of } $\Omega_1(Z(Q))$ \textit{such that}
\[ |a^{(h)} \cap Z_1| > \frac{1}{2} |a^{(h)}| \text{ or } \Omega_1(Z(Q))^* = \{a^{(h)}\}. \]
Proof. Put $a_i \in \Omega_i(Z(Q))^\delta$, $a_i \neq w$. Let $A_i = \{a_i^{(y)}\}$. If there exists an element of $\Omega_i(Z(Q))^\delta - A_i$ that does not equal $w$, let $a_2$ denote this element. So let $A_2 = \{a_2^{(y)}\}$, and then $A_1 \cap A_2 = \phi$. Inductively, if there exists an element of $\Omega_i(Z(Q))^\delta - \bigcup_{i=1}^r A_i$ that does not equal $w$, we let $a_i$ denote this element. Then we can write the following:

$$\Omega_i(Z(Q))^\delta - \langle w \rangle = \bigcup_{i=1}^r A_i,$$

where $A_i \cap A_j = \phi$ if $i \neq j$, $1 \leq i, j \leq m$.

Now suppose that $m \geq 2$. Let $|\Omega_i(Z(Q))| = 2^m$, and as $[\Omega_i(Z(Q)) : \Omega_i(Z_i)] = 2$, $|\Omega_i(Z_i)| = 2^{m-1}$. If, any $i$, $1 \leq i \leq m$, $|a_i^{(y)} \cap Z_i| \leq \frac{1}{2} |a_i^{(y)}|$, then since $|a_i^{(y)}| = r$ is odd. $|\bigcup_{i=1}^r (a_i^{(y)} \cap Z_i)| \leq |\Omega_i(Z(Q)) - \Omega_i(Z_i)| = 2^m - 2$. 

But, on the other hand, $|\Omega_i(Z(Q)) - \Omega_i(Z_i)| = 2^{m-1}$, and $|\Omega_i(Z_i)| = 2^{m-1} - 1$. It is impossible. Hence $m = 1$. $\Omega_i(Z(Q))^\delta = \{a_i^{(y)}\}$. This lemma is proved.

(3.17) $a^\delta$ normalizes some Sylow 3-subgroup of $G$, $0 \leq i \leq r-1$.

Now put $\Delta_i = \langle a^\delta \rangle^G \cap Q_3$, and then $\Delta_i \triangleleft \Phi$, and $\Delta_i^\Phi = \Delta_i^{i+1}$, $0 \leq i \leq r-1$. Furthermore, as $Q_3^* = 1$, $Q_3 = E_3^* R_3$.

If $E_3 = 1$, then $S$ is a T.I.-set, by (3.13). In analogy with the above argument, we can show that $\Delta_i \triangleleft \Phi$, $0 < i < r-1$.

But, in this time, $w$ is an only involution in $Q_3$. This is a contradiction.

Hence, for the remainder of this paper, we may assume that $E_3 \neq 1$, i.e. $r = 2^s + 1$ is a Fermat prime. Then (3.16) is reduced that there exists a nontrivial element $a$ of $\Omega_1(Z(Q))$ such that $|a^{(y)} \cap Z_1| > \frac{1}{2} |a^{(y)}|$.

On the other hand, $m(S) \geq 4$.

(3.18) There exists an element $b_i, b_j$ of $\Delta_i$, respectively, $0 \leq i, j \leq r-1$, $i \neq j$, $[b_i, b_j] = 1$.

Next $\Delta$ is determined as the following.

**Lemma 3.19.** $\Delta_i = \{b_i, b_i^w\}, 0 \leq i \leq r-1, b_i \neq w$.

Proof. If $w \in \Delta_i$, then $w$ centralizes some element of order 3, a contradiction. Thus $w \notin \Delta_i$.

For the remainder, we set $b = b_i$.

Suppose that $b, b^\varepsilon \in \Delta_i, g \in G, b \neq b^\varepsilon$. Then $b, b^\varepsilon \in Q_3$. Since $S = C_3(w) \oplus C_3(bw), \frac{1}{2} m(S) = m(C_3(b)) = m(C_3(bw)) \geq 2$.

Let $S^*$ be a Sylow 3-subgroup of $C_3(b^\varepsilon)$ containing $C_3(b^\varepsilon)$. There exists an element $h$ of $C_3(b^\varepsilon)$ such that $(C_3(b))^{b^\varepsilon} \subseteq S^*$. On the other hand, let $S^*$ be a
Sylow 3-subgroup of $G$ containing $S^*$, and then $S=S_0$ as $C_3(b^x) \subseteq S \cap S_0$. Since $(C_3(b))^x \subseteq S \cap S^x$, $gh \in N_3(S)$. Since $b^x=b^t$, $b$ and $b^x$ are conjugate in $N_3(S)$. As $N_3(S)=Q_3K_3$, $b$ and $b^x$ are conjugate in $Q_3$. Hence $\Delta_i=\{b, b^x\}$.

Now put $\Delta=\langle \Delta_i | 0 \leq i \leq r-1 \rangle$, and then, by (3.19), $\Delta$ is $\phi$-invariant Abelian. Furthermore, as $[\Delta, \phi]=1$, $[\Delta, \phi]K_3$ is nilpotent, and

$$1+\phi[\Delta, \phi]=C_{\phi}(K_3)=Q_3^{*}=1,$$

this is a contradiction. Hence $\mu_3(S;2)=1$.

On the other hand, we will prove the next lemma, and then, in analogy with Collins-Rickman [2], the proof of the main theorem is complete.

**Lemma 3.20.** Let $S_0$ be a proper subgroup of $S$ such that $m(S/S_0) \leq 2$. Then $N_3(S_0)$ is 3-solvable.

Proof. First we shall consider the case $m(S_0) \geq 2$. In this case, we will show that $C_3(S_0)$ is 3-nilpotent. Put $C=C_3(S_0)$, and let $S_1$ be a nontrivial subgroup of $S$. If there exists a nontrivial element $t$ of $\text{Aut}_C(S_1)$, $t$ is a 2-element as $\text{Aut}_C(S_1)$ is a 2-group. Then there exists an element $y$ of $S_1$ such that $y^t=y$. Thus $y$ and $y^t$ are conjugate in $C_3(S_0)$. By (2.11), $y$ and $y^t$ are conjugate in $N_3(C)$. Hence we may assume that $t \in Q_3 \cap C=Z_2 \times \cdots \times Z_2$. As $t \neq w$, $S=C(t) \oplus C_3(tw)$. Hence

$$\frac{1}{2} m(S)=m(C_3(t))=m(C_3(tw)).$$

This is a contradiction. By (2.15), $C_3(S_0)$ is 3-nilpotent. $C_3(S_0)/S_0$ is 3-solvable. Hence $N_3(S_0)$ is 3-solvable.

Now we may assume that $m(S)=4$ and $m(S_0)=2$. In this case, similarly, if $C_M(S_0)=C_3(S)$, $M_3=N_3(S_0)$, then by (2.10), $C_3(S_0)$ is 3-nilpotent. Hence, furthermore, we may assume that $C_3(M_3(S_0))=C(S)$.

If there exists an element $x_0$ of $C_3(M_3(S_0))$ such that $|x_0|=4$, then $x_0^2=w \in C_3(M_3(S_0))$, a contradiction.

If there exists a four-group $\langle x_1 \rangle \times \langle x_2 \rangle$ in $C_3(M_3(S_0))$, then $S=\langle C_3(x_1), C_3(x_2), C_3(x_1x_2) \rangle$. On the other hand, $S_0$ is contained in $C_3(x_1), C_3(x_2),$ and $C_3(x_1x_2)$, and since

$$m(C_3(x_1))=m(C_3(x_2))=m(C_3(x_1x_2))=2,$$

$C_3(x_1)=C_3(x_2)=C_3(x_1x_2)=S_0$, a contradiction. Hence we can write the following;

$$C_3(M_3(S_0)=C(S) \langle t \rangle,$$

where $t^2 \in C(S)$ and $S=S_0 \oplus \{S, t\}$.  


Put \( C_0(S_0) = C_0(S_0)/S_0 \). Then \( \overline{S} = S \cap N_{C_0(S_0)}(S)' \). By (2.12), \( C_0(S_0) \) is 3-solvable. Hence, in this case, \( N_0(S_0) \) is 3-solvable. This lemma is complete.

Now we already proved that \( \mu_0^*(S; 2) \neq 1 \). Next we will show that there exists a \( \phi \)-invariant element \( Q_1 \) of \( \mu_0^*(S; 2) \). Suppose false. Since \( \mu_0^*(S; 2) \) is \( \phi \)-invariant, \( r \) divides \( |\mu_0^*(S; 2)| \). On the other hand, by (2.14), the element of \( \mu_0^*(S; 2) \) permuted by \( C(S) \) transitively. This is a contradiction.

Let \( N = SQ_1 \). By (3.4), \( N \) is nilpotent. Hence

\[ Q_1 \subseteq C_0(S) = C_0(K_3). \]

On the other hand, as \( Q_3^* = 1 \), \( |C_0(S)| \) is odd. This is a contradiction. The main theorem is proved.

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Reference


