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<td>Author(s)</td>
<td>Okuyama, Takashi</td>
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<tr>
<td>Citation</td>
<td>Osaka Journal of Mathematics. 18(2) P.393-P.402</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1981</td>
</tr>
<tr>
<td>Text Version</td>
<td>publisher</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/3753">https://doi.org/10.18910/3753</a></td>
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<td>DOI</td>
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Okuyama, T.
Osaka J. Math.
18 (1981), 393–402

FINITE GROUPS ADMITTING AN AUTOMORPHISM OF PRIME ORDER FIXING A CYCLIC 2-GROUP

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(Received November 19, 1979)

1. Introduction

In this paper, we shall give a proof of the following Theorem, which is a conjecture of B. Rickman [9]; in special case, \( C_\phi(\phi) \) has order 2, M.J. Collins and B. Rickman proved in [2].

**Theorem.** Let \( G \) be a finite group which admits an automorphism \( \phi \) of odd prime order \( r \) whose fixed-point-subgroup \( C_\phi(\phi) \) is a cyclic 2-group. Then \( G \) is solvable.

All groups considered in this paper are assumed finite. Our notation corresponds to that of Gorenstein [7].

An important tool that is brought to attack the problem is B. Baumann's classification of finite simple groups whose Sylow 2-subgroups are maximal [1], and in analogy with Matsuyama [8] that used the results of [1], we shall prove that \( \mathbb{H}_S(S;2) \geq 1 \), where \( S \) is a \( \phi \)-invariant Sylow 3-subgroup of \( G \).

C.A. Rowley has obtained a proof of the theorem under the additional hypothesis that \( G \) does not involve \( \mathbb{S}_4 \), the symmetric group on 4 letters.

The Theorem is a contribution to the continuing problem of showing that finite groups which admit an automorphism \( \phi \) of odd prime order such that \( C_\phi(\phi) \) is a 2-group are solvable.

2. Preliminaries

We first quote some frequently used results.

2.1. (Thompson [12])

Let \( G \) be a group which admits a fixed-point-free automorphism of prime order. Then \( G \) is nilpotent.

2.2. (Rowley [10])

Let \( G \) be a solvable group admitting an automorphism of odd prime order \( p \) such that \( C_\phi(\phi) \), the fixed-point-subgroup of \( \phi \) in \( G \), is a cyclic \( q \)-group, \( q \neq p \). Then, for any prime \( r \), \( G \) is either \( r \)-nilpotent or \( r \)-closed.
2.3. (Glauberman [4])
Let $G$ be a group with a Sylow $p$-subgroup $P$, either $p$ odd or $p=2$ and $S_4$ is not involved in $G$, in which $C_0(Z(P))$ and $N_0(J(P))$ both have normal $p$-complements. Then $G$ possesses a normal $p$-complement.

2.4. (Gilman and Gorenstein [3])
If $G$ is a simple group with Sylow $2$-subgroups of class $2$, then $G=PSL(3,q)$, $q=7$, $9$ (mod $16$), $A_7$, $S_8(2^n)$, $n$ odd, $n>1$, $U_3(2^n)$, $n\geq 2$, $L_3(2^n)$, $n\geq 2$, or $Psp(4,2^n)$, $n\geq 2$.

2.5. (Gorenstein [7])
Let $P$ be a Sylow $p$-subgroup of $G$, where $p$ is the smallest prime in $\pi(G)$. If $p>2$, assume $d_n(P)\leq 2$, while if $p=2$, assume $P$ is cyclic. Then $G$ has a normal $P$-complement.

2.6. (Matsuyama [8])
Let $Q$ be a $2$-group admitting an automorphism $\phi$ of odd order $\neq 1$. If $d_n(Q)=1$, then $Q=E\ast R$, where $E$ is $\phi$-invariant, extra-special or $1$, and $R$ is $\phi$-invariant, and $R$ is cyclic, $D_m$, $Q_m$, or $S_m$, $m\geq 4$.

2.7. (Collins-Rickman [2])
Let $T$ be an extra-special $2$-group admitting an automorphism $\phi$ of odd prime order $r$ acting fixed-point-freely on $T/T'$. Let $S$ be the natural semi-direct product $T\langle \phi \rangle$ and let $K$ be a field of nonzero characteristic different from $2$ and $r$. Assume that there exists a $KS$-module $M$ for which $C_M(\phi)=C_M(T')=0$.

Then

(i) $r=2^n+1$ is a Fermat prime,

(ii) $|T|=2^{n+1}$, and

(iii) $T\cong Q\ast D$,

where $Q$ and $D$ denote the quaternion and dihedral groups of order $8$, respectively, and $\ast$ denote the central product.

2.8. (Glauberman [5] [6])
Let $G$ be a solvable group with a Sylow $2$-subgroup $Q$ with $G=\langle Z(Q)\rangle N(J(Q))$, and $O(X)=1$. Put

$Z=\langle Z^* | G\supseteq Z^* : 2$-subgroup and $O_2(G/C(Z^*))=1 \rangle$

and $J=\langle x\in G | x: 2$-element, $|Z/C_2(x)|=2 \rangle$

and $H=\langle J, C(Z) \rangle$. Then the following hold;

(i) there exists a normal subgroup $G_i$ of $H$ containing $C(Z)$, $1\leq i\leq m$, such that, for $i=1, \cdots, m, G_i/C(Z)\cong S_3$, and $H/C(Z)=G_1/C(Z)\times \cdots \times G_m/C(Z)$.

(ii) let $V_i=[G_i, Z]$, $1\leq i\leq m$, and let $V=V_1\oplus \cdots \oplus V_m$, then $Z=V\oplus C_2(H)$ and $V_i\cong Z_2\times Z_2$, $1\leq i\leq m$. 

(iii) there is a 3-element $x_3$ of $H$ such that, for each $g \in H$, $H = \langle Q \cap H, x_3^g, C(Z) \rangle$ and $G/C(Z) = H/C(Z) C_{G/C(Z)}(x_3^g C(Z))$.

2.9. (Matsuyama [8])
Let $G$ be a group with a Hall $\pi$-subgroup $H$, and let $1 \neq P \in \text{Syl}_2(H)$, $Q$ 2-group. If $N_2(H) = HQ$, $d_2(Q) = 1$, $\Omega_2(Z(Q)) = \langle w \rangle$, $C_H(w) = 1$, and $N_2(P; \pi') = 1$, then, for each $P \neq P$, $g \in G$, $m(P \cap P^g) \leq 1$.

2.10. (Burnside's theorem [7])
If a Sylow $p$-subgroup of $G$ lies in the center of its normalizer in $G$, then $G$ has a normal $p$-complement.

2.11. (Burnside's theorem [7])
If $P$ is a Sylow $p$-subgroup of $G$, then two normal subsets of $P$ are conjugate in $G$ if and only if they are conjugate in $N_2(P)$. In particular, two elements of $Z(P)$ are conjugate in $G$ if and only if they are conjugate in $N_2(P)$.

2.12. (Smith-Tyrer [11])
Let $G$ be a group with an Abelian Sylow $p$-subgroup $P$ for some odd prime $p$. If $[N(P):C(P)] = 2$ and $P \cap N(P)'$ is noncyclic, then $G$ is $p$-solvable.

2.13. (Thompson Transitivity theorem [7])
Let $G$ be a group in which the normalizer of every nonidentity $p$-subgroup is $p$-constrained. Then if $A \in \text{SCN}_2(P)$, $C_2(A)$ permutes transitively under conjugation the set of all maximal $A$-invariant $q$-subgroups of $G$ for any prime $q \neq p$.

2.14. (Collins-Rickman [2])
Let $G$ be a group, and let $p$ and $q$ be distinct prime divisors of $G$. Assume that $G$ has an Abelian Sylow $p$-subgroup $P$ for which $m(P) \geq 3$ and that, whenever $P_0$ is a subgroup of $P$ with $m(P/P_0) \geq 2$, $N_2(P_0)$ is $p$-constrained. Then $C_2(P)$ permutes the elements of $N_2^*(P; q)$ transitively under conjugation.

2.15. (Frobenius theorem [7])
$G$ is $p$-nilpotent if and only if $N_2(H)/C_2(H)$ is a $p$-group for every nonidentity $p$-subgroup $H$ of $G$.

3. The proof of the Theorem

Let $G$ be a minimal counterexample to the Theorem, for the remainder of this paper.

Lemma 3.1. $G$ is simple.

Proof. By Lemma 5.1. of [2].
Lemma 3.2. Let $p$ be a prime divisor of $G$ and $P \in \text{Syl}_p(G)$. If $N_G(P)$ has a normal $p$-complement, then $p=2$ and the symmetric group $S_4$ is involved in $G$.

Proof. By Lemma 5.2. of [2], (2.2) and (3.1).

For the remainder of this paper, $Q$ denotes the $\phi$-invariant Sylow 2-subgroup of $G$, and let $C_G(\phi)=\langle x \rangle$ and $\Omega_4(C_G(\phi))=\langle \omega \rangle$.

Then $Q$ is a unique $\phi$-invariant Sylow 2-subgroup, and let $p$ be an odd prime in $\pi(G)$ and $P \in \text{Syl}_p(G)$, then, by (3.2), $N_G(P)\triangleright P$.

Lemma 3.3. $d_Q(\Omega_4)=1$.

Proof. If $d_Q(\Omega_4)=1$, by (2.6) and hypothesis $Q=E\ast R$ where $E$ is $\phi$-invariant, extra-special, and $R$ is $\phi$-invariant, cyclic. If $E=1$, by (2.5) $G$ is 2-nilpotent, contrary to (3.1). So $E \neq 1$. Since $cl(Q)=2$, by (2.4) this is a contradiction.

Lemma 3.4. Every $\phi$-invariant proper subgroup of $G$ is 2-nilpotent.

Proof. Assume otherwise. Let $M$ be a non-nilpotent maximal $\phi$-invariant subgroup of $G$ without a normal Sylow 2-subgroup. If $N(M)$ is 2-nilpotent, $M$ is nilpotent, a contradiction. By (2.2), $N(M)$ is 2-closed. Hence $M=N(M)$, $Q=M$, and $M=N_G(Q)$. Thus there is an odd prime $p$ dividing the index $[N_G(Q):C_G(Q)]$.

By (3.3), there is a characteristic subgroup $C$ of $Q$ such that $C=Z_2 \times \cdots \times Z_2$, $C$ contains $\Omega_4(Z(Q))$, and $[C, \phi]=1$. Let $P_0$ be a $\phi$-invariant Sylow $p$-subgroup of $N_G(Q)$ and let $P$ be a $\phi$-invariant Sylow $p$-subgroup containing $P_0$.

We now claim that $[P, P_0]=1$. We may assume that $w \in C$. $[w, P_0]=Q \cap P=1$. Since $P_0$ centralizes $C/C_G(P_0)$, $[P_0, C]=1$. Thus $C \subseteq N_G(P_0)$.

Let $M_0$ be a maximal $\phi$-invariant subgroup containing $N_G(P_0)$. If $M_0$ is 2-closed, $M_0=N_G(Q)$. Since $N_G(P_0)=P_0$, $P=P_0$. Let $Q_0$ be a $\phi$-invariant Sylow 2-subgroup of $N_G(P)$. Then $[P, Q_0]=P \cap Q=1$, so $N_G(P)$ is $P$-nilpotent, and by (3.2), $p=2$, a contradiction. Thus $M_0$ is 2-nilpotent. Hence $M_0=N_G(P)$. Since $C \subseteq N_G(P)$, $1=[C, \phi] \subseteq C_G(P)$.

Now put $Z_0=[\Omega_4(Z(Q)), \phi]$. If $Z_0=1$, $P, Q \subseteq C_G(Z_0)$. When $C_G(Z_0)$ is 2-closed, $P \subseteq N_G(Q)$, and $[Q_0, P_0] \subseteq Q \cap P=1$, a contradiction. Hence $C_G(Z_0)$ is 2-nilpotent. Therefore as $Q \subseteq N_G(P)$, $[Q, P_0] \subseteq Q \cap P=1$, a contradiction. Thus we may assume that $Z_0=1$, hence that $\Omega_4(Z(Q))=\langle \omega \rangle$.

Put $Q=Q/\langle \omega \rangle$ and let $C_1$ be the inverse image of $Z(Q) \cap C$ in $Q$. As $[C_1, x] \subseteq \langle \omega \rangle$, $C_1 \subseteq N_G(Q)$. On the other hand, let $y \in C_1$. Then $[y, \phi] \subseteq C_1\langle \langle x \rangle \rangle$, since $(y^{-1}x)^{y^{-1}}=y^{-1}xy$. Put $C_0=[C_1, \phi]$, so that $1=C_0 \subseteq N_G(P)$, hence $C_0(P_0)$ contains $P_0$ and $x$.

Now let $M_1$ be a maximal $\phi$-invariant subgroup of $G$ containing $C_G(Z_0)$. If $M_1$ is 2-closed, $M_1=N_G(Q)$, and $[Q_0, P_0]=1$, contradiction. Thus $M_1$ is 2-nilpotent,
i.e. $M_1 = N_G(P)$.

Put $\tilde{Q} = Q/\Phi(Q)$. Since $P_0$ centralizes $\tilde{Q}/C_{\tilde{Q}}(P_0)$, $P_0$ centralizes $Q$. Hence $[P_0, Q] = 1$, a contradiction. Hence the lemma is proved.

For the remainder of this paper, in analogy with Matsuyama [8], we shall prove the following result;

(i) $3/|G|$;
(ii) $|C_G(S)|$ is odd, where $S$ is a $\phi$-invariant Sylow 3-subgroup of $G$;
(iii) $U_G(S; 2) = 1$; and
(iv) $m(S) \geq 4$.

On the other hand, in analogy with Collins-Rickman [2], we shall prove that $U_G(S; 2) = 1$. Hence this contradicts above.

For the remainder of this paper, we shall write down the results which can be similarly proved as [8].

(3.5) $C_G(w) \subseteq Q$.

(3.6) If $p$ is an odd prime in $\pi(G)$ and $P \subseteq \text{Syl}_p(G)$, then $P$ is Abelian.

(3.7) If $p$ is an odd prime in $\pi(G)$ and $A$ is any $p$-subgroup of $G$, then $\text{Aut}_G(A) = N_G(A)/C_G(A)$ is a 2-group.

(3.8) If $\Omega_3(Z(Q)) \neq \{w\}$, then $N_G(T)$ is a 2-group for any nontrivial $\phi$-invariant 2-subgroup $T$ of $G$.

Now put $P$ be a $\phi$-invariant Sylow $p$-subgroup of $G$ for any odd prime $p$ in $\pi(G)$. Let $K_p$ be a normal 2-complement of $N_G(P)$ and $Q_p = Q \cap N_G(P)$. Then $N_G(P) = Q_p K_p$, $Q_p \subseteq Q$. Furthermore let $Q_p^* = C_{Q_p}(K_p)$, and then $Q_p^* = [Q_p^*, \phi]$, since $w \subseteq Q_p^*$.

Hence, for any $s \in \pi(K_s)$, $K_s = K_p$, $Q_p = Q_s$, and $Q_p^* = Q_s^*$. In particular, $K_p$ is a nilpotent Hall subgroup of $G$.

(3.9) $C_{Q_p}(P) = Q_p^*$.

(3.10) $d(Q_p/Q_p^*) = 1$.

Furthermore let $M_p = N_G(P)$ and $\bar{M}_p = M_p/Q_p^*K_p$. Then by (2.6) and hypothesis, $\bar{M}_p = \bar{E}_p^* E_p^*$, where either $\bar{E}_p = 1$ or $\bar{E}_p$ is $\phi$-invariant, extra-special and $\bar{R}_p$ is $\phi$-invariant, cyclic.

On the other hand, by (3.4), $N_G(Q)$ is nilpotent, and then $N_G(Q) = Q$ by (3.5). Hence by (3.2), $S_4$ is involved in $G$, yields $3/|G|$. Furthermore let $S \subseteq \text{Syl}_3(G)$, and then $m(S) \geq 3$.

**Lemma 3.11.** Let $p$ be an odd prime in $\pi(G)$. We can write $\bar{M}_p = E_p^* \bar{R}_p$, where either $E_p = 1$ or $E_p$ is $\phi$-invariant, extra-special, and $\bar{R}_p$ is $\phi$-invariant,
If $E_p \neq 1$, then $r=2^n+1$ is a Fermat prime.

Proof. By (2.7), it is immediate that $C_{Q_p}(\phi) = C_{Q_p}(E_p) = 0$. By (2.7), it suffices to prove that $\phi$ acts on $E_p/E_p'$ fixed-point-freely. First we may assume that $|E_p| = 2$. Then, since we can suppose that $\phi$ centralizes an element of $E_p$ of order 4, it is not necessarily trivial.

Now suppose that there exists an element $y$ of $E_p$ of order 4 such that $[y, \phi] = 1$. As $E_p$ is extra-special, the conjugate class of $y$ is $\{y, yw\}$. Hence $[E_p: C_{E_p}(y)] = 2$. Then $\phi$ acts on the set, $E_p - C_{E_p}(y)$, fixed-point-freely. It is impossible.

**Lemma 3.12.** Let $S$ be a $\phi$-invariant Sylow 3-subgroup of $G$. If $[Q_3/Q_3^*, \phi] = 1$, then $S$ is a $T.I.$-set.

Proof. If not, there exists an element $g$ of $G$ such that $S^g \neq S$ and $S^g \cap S = 1$. First we shall show that $C_{Q_3}(x) = Q_3^*$ for any $x \in S^g$. It is immediate that $C_{Q_3}(x) \supseteq Q_3^*$. If $C_{Q_3}(x) \supseteq Q_3^*$ for some $x \in S^g$, $w \in C_{Q_3}(x)$, by hypothesis. But this is impossible. Next we will prove that, for any $x \in S^g$, $C_{Q_3}(x)$ is 3-nilpotent.

Now put $C_G(x) = C$ and let $S_1$ be a nontrivial subgroup of $S$. By (3.7), $Aut_C(S_1)$ is a 2-group. Put $Aut_C(S_1) \supseteq t \neq 1$. Then $t$ is a 2-element. Furthermore there exists an element $y$ of $S_1$ such that $y' \neq y$, i.e. $y$ and $y'$ are conjugate in $C_G(x)$. By (2.11), $y$ and $y'$ are conjugate in $N_G(S)$. Thus we may assume that $t \in N_G(S)$, and $t \in Q_3$. Then $t \in C_{Q_3}(x) = Q_3^* = C_{Q_3}(S)$, a contradiction. Hence $C_{Q_3}(x)$ is 3-nilpotent by (2.15), especially $C_{Q_3}(x)$ is 3-constrained.

Furthermore put $3 \neq p \in \pi(K_3)$, and let $P$ be a $\phi$-invariant Sylow $p$-subgroup of $G$. $N_G(S) = N_G(P)$. Thus $C_G(x)$ is $\pi(K_3)$-nilpotent.

Next put $1 \neq y \in S^g \cap S$, and let $M$ be a $\pi(K_3)$-complement of $C_G(y)$, and then we will prove that $M$ is a 2-group.

$S$ normalizes $M$ and $(|S|, |M|) = 1$. Now suppose that $M$ is not a 2-group. There exists an odd prime $q$ in $\pi(M)$ such that $q \in \pi(K_3)$. Furthermore there exists a Sylow $q$-subgroup $Q_i$ of $M$ normalized by $S$. Since $Aut_G(Q_i)$ is a 2-group, $S \subseteq C_G(Q_i)$, and hence $Q_i \subseteq K_3$. It is impossible. Thus $M$ is a 2-group.

On the other hand, it is easy to show that $M \supseteq Q_3^*$. Now suppose that $M = Q_3^*$. Then $C_G(y) = Q_3^* K_3$, and since $S, S^g \subseteq C_G(y)$, $S = S^g$, a contradiction. Hence $M \supseteq Q_3^*$, $C_G(S) \subseteq N_G(M)$.

Let $\bar{M}$ be the intersection of all elements of $U_G(S; 2)$. By (2.14), $\bar{M} \subseteq M$. On the other hand, as $\bar{M}$ is $\phi$-invariant, $S \bar{M}$ is $\phi$-invariant. By (3.4), $S \bar{M}$ is 2-nilpotent. Thus $[\bar{M}, S] \subseteq \bar{M} \cap S = 1$. Hence $\bar{M} \subseteq C_G(S) = Q_3^*$, a contradiction. This completes the proof of Lemma 3.12.

Now if $E_p \neq 1$, $r=2^n+1$ is a Fermat prime by (3.11), where $r = |\phi|$. 

On the other hand, when $E_3 = 1$, by (3.12), $S$ is a T.I.-set, where $S$ is a $\phi$-invariant Sylow 3-subgroup of $G$.

By B. Baumann [1], $Q$ is not a maximal subgroup of $G$, and thus there exists a proper subgroup $X$ of $G$ containing $Q$ such that $Q$ is a maximal subgroup of $X$.

In analogy with Matsuyama [8], we can say the following.

$X$ is a solvable $\{2, 3\}$-subgroup with $O(X) = 1$, and $X$ satisfies the hypothesis of (2.8). Thus the structure of $X$ is one of the following two type.

(Type I)

$X/O_2(X)$ is isomorphic to $S_3$, the symmetric group on 4 letters.

$Z(O_2(X))$ contains $Z(Q)$ and $Z(O_2(X)) = [Z(O_2(X)), X] \oplus C_{Z(O_2(X))}(X)$, where $[Z(O_2(X)), X]$ is isomorphic to $Z_2 \times Z_2$.

(Type II)

$X$ has a subgroup $H$ containing $O_2(X)$ such that $[X: H] = 2$. $H/O_2(X)$ contains $Z(Q)$ and $Z(O_2(X)) = [Z(O_2(X)), X] \oplus [Z(O_2(X)), X] \oplus C_{Z(O_2(X))}(H)$, where $[Z(O_2(X)), X]$ is isomorphic to $Z_2 \times Z_2$, $i = 1, 2$.

On the other hand, considering the structure of $X$, $Z(Q)$ is noncyclic, by (3.8), $Q_3 \neq 1$.

Now we will show that $H_G(S; \pi(K_3)) \neq 1$. For the remainder of this paper, let $S$ be a $\phi$-invariant Sylow 3-subgroup of $G$.

**Lemma 3.13.** $H_G(S; \pi(K_3)) = H_G(S; 2)$.

**Proof.** It is easy that $H_G(S; \pi(K_3)) \supseteq H_G(S; 2)$. If there exists an element $A$ of $H_G(S; \pi(K_3))$ that is not a 2-group, by [7; 6.2.2], $S$ normalizes some Sylow $p$-subgroup $S^*$ of $A$. As $\text{Aut}_G(S^*)$ is a 2-group, $[S, S^*] = 1$. But it contradicts $C_G(S) = K_3$.

By (3.13), it suffices to prove that $H_G(S; \pi(K_3)) \neq 1$. Now we suppose that $H_G(S; \pi(K_3)) = 1$. By Matsuyama [8], we can say the following.

(3.14) If $S^g \neq S, g \in G$, then $m(S \cap S^g) \leq 1$.

(3.15) There exists a nontrivial proper subgroup $Z_1$ of $Z(Q)$ such that $3/|C_G(Z_1)|$ and $[Z(Q): Z_1] = 2$.

Furthermore, in analogy with Matsuyama [8], we can show the next lemma.

**Lemma 3.16.** There exists a nontrivial element $a$ of $\Omega_1(Z(Q))$ such that $|a^{(b)} \cap Z_1| > \frac{1}{2} |a^{(b)}|$ or $\Omega_1(Z(Q))^* = \{a^{(b)}\}$. 

Proof. Put $a_1 \in \Omega_1(Z(Q))^\sharp$, $a_1 \neq w$. Let $A_1 = \{a_1^{(\phi)}\}$. If there exists an element of $\Omega_1(Z(Q))^\sharp - A_1$ that does not equal $w$, let $a_2$ denote this element. So let $A_2 = \{a_2^{(\phi)}\}$, and then $A_1 \cap A_2 = \emptyset$. Inductively, if there exists an element of $\Omega_1(Z(Q))^\sharp - \bigcup_{i=1}^n A_i$ that does not equal $w$, we let $a_i$ denote this element. Then we can write the following,

$$\Omega_1(Z(Q))^\sharp - (w) = \bigcup_{i=1}^n A_i,$$

where $A_i \cap A_j = \emptyset$ if $i \neq j$, $1 \leq i, j \leq m$.

Now suppose that $m \geq 2$. Let $|\Omega_1(Z(Q))| = 2^n$, and as $[\Omega_1(Z(Q)):\Omega_1(Z_i)] = 2$, $|\Omega_i(Z_i)| = 2^{n-1}$. If, any $i$, $1 \leq i \leq m$, $|a_i^{(\phi)} \cap Z_1| \leq \frac{1}{2}|a_i^{(\phi)}|$, then since $|a_i^{(\phi)}| = r$ is odd, $|\bigcup_{i=1}^n (a_i^{(\phi)} \cap Z_i)| \leq |\Omega_i(Z(Q)) - \Omega_i(Z_i)| - 2$.

But, on the other hand, $|\Omega_i(Z(Q)) - \Omega_i(Z_i)| = 2^{n-1} - 1$, and $|\Omega_i(Z_i)| = 2^{n-1} - 1$. It is impossible. Hence $m = 1$. $\Omega_i(Z(Q))^\sharp = \{a_i^{(\phi)}\}$. This lemma is proved.

(3.17) $a^{\phi}$ normalizes some Sylow $3$-subgroup of $G$, $0 \leq i \leq r - 1$.

Now put $\Delta_i = (a_i^{(\phi)})^G \cap Q_3$, and then $\Delta_i = \emptyset$, and $\Delta_i = \Delta_{i+1}$, $0 \leq i \leq r - 1$. Furthermore, as $Q_3^* = 1$, $Q_3 = E_3 \ast R_3$.

If $E_3 = 1$, then $S$ is a T.I.-set, by (3.13). In analogy with the above argument, we can show that $\Delta_i \neq \emptyset$, $0 < i < r - 1$.

But, in this time, $w$ is an only involution in $Q_3$. This is a contradiction.

Hence, for the remainder of this paper, we may assume that $E_3 \neq 1$, i.e. $r = 2^n + 1$ is a Fermat prime. Then (3.16) is reduced that there exists a nontrivial element $a$ of $\Omega_i(Z(Q))$ such that $|a^{(\phi)} \cap Z_i| > \frac{1}{2}|a^{(\phi)}|$.

On the other hand, $m(S) \geq 4$.

(3.18) There exists an element $b_i, b_j$ of $\Delta_i, \Delta_j$, respectively, $0 \leq i, j \leq r - 1$, $i \neq j$, $[b_i, b_j] = 1$.

Next $\Delta_i$ is determined as the following.

**Lemma 3.19.** $\Delta_i = \{b_i, \phi w\}$, $0 \leq i \leq r - 1$, $b_i \neq w$.

Proof. If $w \in \Delta_i$, then $w$ centralizes some element of order $3$, a contradiction. Thus $w \notin \Delta_i$.

For the remainder, we set $b = b_i$.

Suppose that $b, b^e \in \Delta_i, g \in G, b \neq b^e$. Then $b, b^e \in Q_3$. Since $S = C_3(w) \oplus C_3(bw)$, $\frac{1}{2}m(S) = m(C_3(b)) = m(C_3(bw)) \geq 2$.

Let $S^*$ be a Sylow $3$-subgroup of $C_3(b^e)$ containing $C_3(b^e)$. There exists an element $h$ of $C_3(b^e)$ such that $(C_3(b))^h \subseteq S^*$. On the other hand, let $S_0$ be a
Sylow 3-subgroup of $G$ containing $S^*$, and then $S=S_0$ as $C_S(b^g) \subseteq S \cap S_0$. Since $(C_S(b))^t \subseteq S \cap S^t$, $gh \in N_G(S)$. Since $b^g=b^t$, $b$ and $b^t$ are conjugate in $N_G(S)$. As $N_G(S)=Q_3K_3$, $b$ and $b^t$ are conjugate in $Q_3$. Hence $\Delta_i=\{b, b^t\}$.

Now put $\Delta=\langle \Delta_i | 0 \leq i \leq r-1 \rangle$, and then, by (3.19), $\Delta$ is $\phi$-invariant Abelian. Furthermore, as $[\Delta, \phi]=1$, $[\Delta, \phi]K_3$ is nilpotent, and

$$1+[\Delta, \phi] \subseteq C_{G_0}(K_3) = Q_3^* = 1,$$

this is a contradiction. Hence $N_G(S;2) \neq 1$.

On the other hand, we will prove the next lemma, and then, in analogy with Collins-Rickman [2], the proof of the main theorem is complete.

**Lemma 3.20.** Let $S_0$ be a proper subgroup of $S$ such that $m(S/S_0) \leq 2$. Then $N_G(S_0)$ is 3-solvable.

**Proof.** First we shall consider the case $m(S_0)>2$. In this case, we will show that $C_G(S_0)$ is 3-nilpotent. Put $C=C_G(S_0)$, and let $S_1$ be a nontrivial subgroup of $S$. If there exists a nontrivial element $t$ of $\text{Aut}_G(S_1)$, $t$ is a 2-element as $\text{Aut}_G(S_1)$ is a 2-group. Then there exists an element $y$ of $S_1$ such that $y^t \neq y$.

Thus $y$ and $y^t$ are conjugate in $C_G(S_0)$. By (2.11), $y$ and $y^t$ are conjugate in $N_G(S)$. Hence we may assume that $t \in Q_3 \cap C \cong Z_2 \times \cdots \times Z_2$. As $t \neq w$, $S=C(t) \oplus C_S(tw)$. Hence

$$\frac{1}{2} m(S) = m(C(t)) = m(C_S(tw)).$$

This is a contradiction. By (2.15), $C_G(S_0)$ is 3-nilpotent. $C_G(S_0)/S_0$ is 3-solvable. Hence $N_G(S_0)$ is 3-solvable.

Now we may assume that $m(S)=4$ and $m(S_0)=2$. In this case, similarly, if $C_{M_S}(S_0)=C(S)$, $M_3=N_G(S)$, then by (2.10), $C_G(S_0)$ is 3-nilpotent. Hence, furthermore, we may assume that $C_{M_S}(S_0) \cong C(S)$.

If there exists an element $x_0$ of $C_{M_S}(S_0)$ such that $|x_0|=4$, then $x_0^2=tw \in C_{M_S}(S_0)$, a contradiction.

If there exists a four-group $\langle x_1 \rangle \times \langle x_2 \rangle$ in $C_{M_S}(S_0)$, then $S=\langle C_S(x_1), C_S(x_2), C_S(x_1x_2) \rangle$. On the other hand, $S_0$ is contained in $C_S(x_1), C_S(x_2)$, and $C_S(x_1x_2)$, and since

$$m(C_S(x_1)) = m(C_S(x_2)) = m(C_S(x_1x_2)) = 2,$$

$C_S(x_1)=C_S(x_2)=C_S(x_1x_2)=S_0$, a contradiction. Hence we can write the following;

$$C_{M_S}(S_0) = C(S)\langle t \rangle,$$

where $t \in C(S)$ and $S=S_0 \oplus [S, t]$. 


Put $\bar{C}_0(S_0) = C_0(S_0)/S_0$. Then $\bar{S} = S \cap N_{\bar{C}_0(S_0)}(\bar{S})'$. By (2.12), $\bar{C}_0(S_0)$ is 3-solvable. Hence, in this case, $N_0(S_0)$ is 3-solvable. This lemma is complete.

Now we already proved that $\mathcal{U}_0^*(S;2) \neq 1$. Next we will show that there exists a $\phi$-invariant element $Q_1$ of $\mathcal{U}_0^*(S;2)$. Suppose false. Since $\mathcal{U}_0^*(S;2)$ is $\phi$-invariant, $r$ divides $|\mathcal{U}_0^*(S;2)|$. On the other hand, by (2.14), the element of $\mathcal{U}_0^*(S;2)$ permuted by $C(S)$ transitively. This is a contradiction.

Let $N = SQ_1$. By (3.4), $N$ is nilpotent. Hence

$$Q_1 \subseteq C_0(S) = C_0(K_3).$$

On the other hand, as $Q_3^* = 1$, $|C_0(S)|$ is odd. This is a contradiction. The main theorem is proved.

Acknowledgement.
The author thanks to Mr. Fukushima and Mr. Matsuyama for this theme and for their helpful advices.

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