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FUNCTIONAL LIMIT THEOREM FOR OCCUPATION TIMES OF GAUSSIAN PROCESSES -NON-CRITICAL CASE

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1. Introduction

Let X^γ be a fractional Brownian motion with index γ $(0 < \gamma < 1)$. That is, X^γ is a real-valued centered Gaussian process such that

$$E[X^\gamma(t)X^\gamma(s)] = \frac{1}{2}\{t^{2\gamma} + s^{2\gamma} - |t-s|^{2\gamma}\}, \quad s,t \geq 0,$$

or equivalently,

$$X^\gamma(0) = 0, \quad \text{and} \quad E[(X^\gamma(t) - X^\gamma(s))^2] = |t-s|^{2\gamma}, \quad s,t \geq 0.$$

If $\gamma = 1/2$, then X^γ is the ordinary Brownian motion. From the definition of fractional Brownian motion, it is easy to see that it has self-similarity, i.e., $\{X^\gamma(ct)\} \stackrel{\mathcal{L}}{=} \{c^\gamma X^\gamma(t)\}$ for every $c > 0$.
A d-dimensional fractional Brownian motion is defined to be an $\mathbb{R}^d$-valued Gaussian process

$$X(t) := X^{\gamma,d}(t) = \left(X_1^\gamma(t), X_2^\gamma(t), \ldots, X_d^\gamma(t)\right)$$

where $X_1^\gamma, X_2^\gamma, \ldots$ are independent copies of X^γ. We shall consider only the case when fractional Brownian motion has jointly continuous local times which means the sample paths are point recurrent case, i.e., $0 < \gamma d < 1$.

Let $d \geq 2$. Suppose $f(x) \geq 0$ be a bounded integrable function on $\mathbb{R}^d$ such that $\bar{f} := \int_{\mathbb{R}^d} f(x)dx \neq 0$. We denote by $\ell_{\gamma,d}(t,x)$ the local time of fractional Brownian motion $X^{\gamma,d}$, then putting $\alpha = 1 - \gamma d$,

$$\begin{aligned}\frac{1}{\lambda^\alpha}\int_0^{\lambda t} f(X(s))ds &\stackrel{\mathcal{L}}{=} \frac{1}{\lambda^\alpha}\int_0^t \lambda f(\lambda^\gamma X(u))du \\ &= \frac{1}{\lambda^{\alpha-1}}\int_{\mathbb{R}^d} f(\lambda^\gamma x)\ell_{\gamma,d}(t,x)dx\end{aligned}$$

$$= \frac{1}{\lambda^{\alpha-1}} \int_{\mathbb{R}^d} f(y)\ell_{\gamma,d}\left(t, \frac{y}{\lambda^\gamma}\right) \frac{dy}{\lambda^{\gamma d}}$$

$$= \int_{\mathbb{R}^d} f(y)\ell_{\gamma,d}\left(t, \frac{y}{\lambda^\gamma}\right) dy$$

$$(1.1) \qquad \xrightarrow{\mathcal{L}} \int_{\mathbb{R}^d} f(y)dy\ell_{\gamma,d}(t,0) \qquad \text{as } \lambda \to \infty$$

over the function space $C([0,\infty))$.

The aim of the present paper is to extend (1.1) for a certain class of Gaussian process instead of fractional Brownian motion.

2. Main Theorem

Let $\xi(t)$ be a centered Gaussian process with stationary increments and $\xi(0) = 0$. Put $\varphi(h) = E[(\xi(t+h) - \xi(t))^2]$ for every t, $h \geq 0$. We shall assume that $\varphi(t)$ varies regularly at infinity, that is $\varphi(t) = t^{2\gamma}L(t)$, where $L(t)$ is a slowly varying function, i.e., $\lim_{t\to\infty} L(ct)/L(t) = 1$ for every $c > 0$. We also assume that $\varphi(t)$ is concave on $[0,\infty)$. Notice that a fractional Brownian motion X^γ with $\gamma < 1/2$ satisfies this condition with $L(t) = \mathbf{1}$.

As the same way as we defined a d-dimensional fractional Brownian motion, define $\mathbb{R}^d$-valued Gaussian process

$$Y(t) = (Y_1(t), Y_2(t), \ldots, Y_d(t)),$$

where $Y_1(t), \ldots, Y_d(t)$ are independent copies of $\xi(t)$. Throughout the paper we shall assume that all components are equally distributed, though it is not essential. In fact the independence of the coordinates is not crucial, either. However, for simplicity, we shall not go into details here. Our main result is the following.

Theorem. *Let $d \geq 2$, and $0 < \gamma d < 1$. Suppose $f(x) \geq 0$ be a bounded integrable function on $\mathbb{R}^d$ such that $\bar{f} := \int_{\mathbb{R}^d} f(x)dx \neq 0$. Then*

$$(2.1) \quad A_\lambda(t) := \frac{(\varphi(\lambda))^{d/2}}{\lambda} \int_0^{\lambda t} f(Y(u))du \xrightarrow{\mathcal{L}} \bar{f}\ell_{\gamma,d}(t,0), \qquad \text{as} \quad \lambda \to \infty,$$

where "$\mathcal{L}$" denotes the convergence in law on the function space $C([0,\infty))$, and $\ell_{\gamma,d}(t,0)$ is the local time of a fractional Brownian motion $X^{\gamma,d}$ as before.

We remark that $\ell_{\gamma,d}(t,0)$ in Theorem is not the local time of Y, but that of fractional Brownian motion $X^{\gamma,d}$.

3. Proof of Theorem

DEFINITION 3.1. Throughout the paper, we set $t_0 = 0$ and let $0 < t_1 < t_2 < \ldots < t_n$, and $\Delta t_j = t_j - t_{j-1}, j = 1, \ldots, n$.
We denote by $C(t_1, t_2, \ldots, t_n)$ the covariance matrix of

$$(\xi(t_1) - \xi(t_0), \xi(t_2) - \xi(t_1), \ldots, \xi(t_n) - \xi(t_{n-1})),$$

and for every $\lambda > 0$, we put

$$C_\lambda(t_1, t_2, \ldots, t_n) = C(\lambda t_1, \lambda t_2, \ldots, \lambda t_n).$$

We also denote by $\mathbb{C}_\lambda(t_1, t_2, \ldots, t_n)$ the covariance matrix of

$$(Y(\lambda t_1) - Y(\lambda t_0), Y(\lambda t_2) - Y(\lambda t_1), \ldots, Y(\lambda t_n) - Y(\lambda t_{n-1})).$$

Since $Y_j (1 \le j \le d)$ are equally distributed, $\mathbb{C}_\lambda(t_1, \ldots, t_n)$ is the $nd \times nd$ matrix such that

$$\mathbb{C}_\lambda = \begin{pmatrix} C_\lambda & & & \mathbf{0} \\ & C_\lambda & & \\ & & \ddots & \\ \mathbf{0} & & & C_\lambda \end{pmatrix}$$

Let $x_j = (x_j^{(1)}, \ldots, x_j^{(d)}) \in \mathbb{R}^d, j = 1, \ldots, n$, then

$$\mathbf{x} = \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \\ \vdots \\ x_n^{(1)} \\ x_1^{(2)} \\ \vdots \\ \vdots \\ x_n^{(d)} \end{pmatrix} \in \mathbb{R}^{nd},$$

So,

$$(\mathbb{C}_\lambda(t_1, \ldots, t_n)^{-1}\mathbf{x}, \mathbf{x}) = \sum_{k=1}^{d} \sum_{i,j=1}^{n} (C_\lambda(t_1, \ldots, t_n)^{-1})_{ij} x_i^{(k)} x_j^{(k)}.$$

Without loss of generality, we may and do assume that $f(x)$ vanishes outside a compact set (see Kôno [5]). By the assumption that $f(x) \ge 0$, the continuity of the limiting process leads to the tightness. Therefore, it is enough to show the weak convergence of all finite-dimensional distributions. To this end, we shall use Bingham's method(see [1]), which is to show the convergence of the Laplace transforms.

$$(3.1)\qquad \begin{aligned} &\lim_{\lambda\to\infty} s_1\dots s_n \int_0^\infty \dots \int_0^\infty \mathbf{e}^{-\sum s_j t_j} E[A_\lambda(t_1)\dots A_\lambda(t_n)]dt_1\dots dt_n \\ &= s_1\dots s_n \int_0^\infty \dots \int_0^\infty \mathbf{e}^{-\sum s_j t_j} \bar{f}^n E[\ell_{\gamma,d}(t_1,0),\dots,\ell_{\gamma,d}(t_n,0)]dt_1\dots dt_n. \end{aligned}$$

Let us consider the special case of fractional Brownian motion, in which case we have, by (1.1), that $A_\lambda(t) \longrightarrow \bar{f}\ell_{\gamma,d}(t,0)$. Therefore, we shall evaluate the left side of (3.1) compared with the case of fractional Brownian motion $X^{\gamma,d}$ and show that (3.1) holds for a certain class of Gaussian process Y.
Integrating by parts, the left side of (3.1) may be rewritten as

$$(3.2)\qquad \lim_{\lambda\to\infty}\int_0^\infty \dots \int_0^\infty \varphi(\lambda)^{nd/2}\mathbf{e}^{-\sum s_j t_j} E\Big[\prod_{j=1}^n f(Y(\lambda t_j))\Big]dt_1\dots dt_n.$$

Here, we define

$$(3.3)\qquad \phi_n^{(\lambda)}(s_1,\dots,s_n) = \int\cdots\int_{0<t_1<\dots<t_n} \varphi(\lambda)^{nd/2}\mathbf{e}^{-\sum s_j t_j} E\Big[\prod_{j=1}^n f(Y(\lambda t_j))\Big]dt_1\dots dt_n.$$

By the symmetry of domain of integration in (3.2), it is enough to calculate the value of $\lim_{\lambda\to\infty}\phi_n^{(\lambda)}(s_1,\dots,s_n)$. Considering that $Y(t)$ is a Gaussian process, we have only to evaluate the following:

$$(3.4)\qquad \begin{aligned} &\lim_{\lambda\to\infty}\int\cdots\int_{0<t_1<\dots<t_n} \varphi(\lambda)^{nd/2}\mathbf{e}^{-\sum s_j t_j} \\ &\times\int\cdots\int_{\mathbb{R}^d\times\dots\times\mathbb{R}^d} \frac{\exp\{-1/2(\mathbb{C}_\lambda(t_1,\dots,t_n)^{-1}\mathbf{x},\mathbf{x})\}}{(2\pi)^{nd/2}(\det C_\lambda(t_1,\dots,t_n))^{d/2}} \\ &\qquad\times f(x_1)\dots f(x_1+\dots+x_n)dx_1\dots dx_n dt_1\dots dt_n. \end{aligned}$$

Lemma 3.1. *For every $M\geq 1$, let*

$$G(n;M) = \{(t_1,\dots,t_n)\in\mathbb{R}^n \mid \triangle t_j \geq M, \forall j=1,\dots,n\},$$

then for every $K\geq 1$,

$$\lim_{M\to\infty}\sup_{G(n;M)}\sup\{(C(t_1,\dots,t_n)^{-1}x,x) \mid x\in\mathbb{R}^n, |x|\leq K\} = 0.$$

Proof. As in the proof of Lemma 3.7 in [3], we have

$$(C(t_1,\dots,t_n)^{-1}x,x) \le 2^n n^{n-1}\sum_{k=1}^{n}\frac{x_k^2}{\varphi(\Delta t_k)} \le \frac{2^n n^{n-1}}{\varphi(M)}|x|^2,$$

which proves the assertion. □

Lemma 3.2. *For $1 \le i,\ j \le n$, and for every $0 < t_1 < \dots < t_n$,*

$$\lim_{\lambda\to\infty}\frac{C_\lambda(t_1,\dots,t_n)_{ij}}{\varphi(\lambda)} = \frac{1}{2}\{|t_{j-1}-t_i|^{2\gamma}+|t_j-t_{i-1}|^{2\gamma}-|t_j-t_i|^{2\gamma}-|t_{j-1}-t_{i-1}|^{2\gamma}\}. \tag{3.5}$$

Proof. By the definition of $C_\lambda(t_1,\dots,t_n)$ and the property of regularly varying function, we have

$$\begin{aligned}
&\lim_{\lambda\to\infty}\frac{C_\lambda(t_1,\dots,t_n)_{ij}}{\varphi(\lambda)}\\
&= \lim_{\lambda\to\infty}\frac{1}{2\varphi(\lambda)}\{\varphi(\lambda t_{j-1}-\lambda t_i)+\varphi(\lambda t_j-\lambda t_{i-1})\\
&\qquad\qquad -\varphi(\lambda t_j-\lambda t_i)-\varphi(\lambda t_{j-1}-\lambda t_{i-1})\}\\
&= \frac{1}{2}\{|t_{j-1}-t_i|^{2\gamma}+|t_j-t_{i-1}|^{2\gamma}\\
&\qquad\qquad -|t_j-t_i|^{2\gamma}-|t_{j-1}-t_{i-1}|^{2\gamma}\}.
\end{aligned}$$

□

Notice that the right side of (3.5) is exactly the same element of covariance matrix of $\{X^\gamma(t_j)-X^\gamma(t_{j-1})\}_{j=1}^n$, where X^γ is the fractional Brownian motion as before.

Lemma 3.3. *For sufficiently large λ, and every $\epsilon > 0$ such that $\epsilon < (2/d - 2\gamma)\wedge(2\gamma)$,*

$$\frac{\varphi(\lambda)^{nd/2}}{(2\pi)^{nd/2}(\det C_\lambda(t_1,\dots,t_n))^{d/2}} \le C_1\frac{1}{\prod_{j=1}^n \Delta t_j^{\gamma d}(\Delta t_j^{\epsilon}\wedge\Delta t_j^{-\epsilon})^{d/2}},$$

where $C_1 > 0$ is some constant which does not depend on λ.

Proof. Applying Lemma 3.3 in [2], we have

$$\det C_\lambda(t_1,\dots,t_n) \ge 2^{-n}\Bigl(\prod_{j=1}^n \varphi(\lambda t_j-\lambda t_{j-1})\Bigr).$$

Therefore,

$$\begin{aligned}
&\frac{\varphi(\lambda)^{nd/2}}{(2\pi)^{nd/2}(\det C_\lambda(t_1,\dots,t_n))^{d/2}}\\
&= \frac{\prod_{j=1}^n\{\varphi(\lambda t_j-\lambda t_{j-1})\}^{d/2}}{(2\pi)^{nd/2}(\det C_\lambda(t_1,\dots,t_n))^{d/2}}\frac{\varphi(\lambda)^{nd/2}}{\prod_{j=1}^n\{\varphi(\lambda t_j-\lambda t_{j-1})\}^{d/2}}\\
&\le \frac{2^n}{(2\pi)^{nd/2}}\frac{L(\lambda)^{nd/2}}{\prod_{j=1}^n\{(t_j-t_{j-1})^{2\gamma}L(\lambda(t_j-t_{j-1}))\}^{d/2}}.
\end{aligned}$$

Notice that $L(\lambda)/L(\lambda\Delta t_j)$ is also a slowly varying function. When $\lambda\Delta t_j$ is small, the following inequation is shown in [6]: Consider $y(<1)$ an arbitrary positive small number. Then for every η $(0<\eta<1)$,

$$\limsup_{x\to\infty}\left|\int_0^y \frac{L(x)}{L(xt)}dt\right| \le M\int_0^y t^{-\eta}dt, \tag{3.6}$$

where M does not depend on y. According to (3.6), we have only to consider the case when $\lambda\Delta t_j$ is large enough. For some $C_2>0$, every $\epsilon>0$, and sufficiently large $\lambda\Delta t_j$,

$$\left|\frac{L(\lambda)}{L(\lambda\Delta t_j)}\right| < C_2\left\{\left(\frac{1}{\Delta t_j}\right)^{-\epsilon}\vee\left(\frac{1}{\Delta t_j}\right)^{\epsilon}\right\}.$$

Therefore,

$$\begin{aligned}
\frac{L(\lambda)^{nd/2}}{\prod_{j=1}^n\{\Delta t_j^{2\gamma}L(\lambda\Delta t_j)\}^{d/2}} &\le \prod_{j=1}^n\left[\frac{C_2}{\Delta t_j^{2\gamma}}\left\{\left(\frac{1}{\Delta t_j}\right)^{-\epsilon}\vee\left(\frac{1}{\Delta t_j}\right)^{\epsilon}\right\}\right]^{d/2}\\
&= \frac{C_2^{nd/2}}{\prod_{j=1}^n\Delta t_j^{\gamma d}(\Delta t_j^{-\epsilon}\wedge\Delta t_j^{\epsilon})^{d/2}},
\end{aligned}$$

and this proves the assertion. □

Now we are ready to prove Theorem. From Lemma 3.3, we can apply dominated convergence theorem to (3.4). From Lemma 3.2, we obtain that (3.4) has the same limiting law as the case of fractional Brownian motion, i.e., $Y(t)=X^{\gamma,d}(t)$, for $\gamma<1/2$; and thus the assertion follows.

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