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CLASSIFICATION OF INVARIANT COMPLEX STRUCTURES ON IRREDUCIBLE COMPACT SIMPLY CONNECTED COSET SPACES

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Introduction

A compact simply connected homogeneous Kähler manifold is represented as a Kähler coset space G/U, where G is a compact connected semisimple Lie group and U is the centralizer of a toral subgroup S in G. Conversely, let G be a compact connected semisimple Lie group and U the centralizer of a toral subgroup in G. Then, G/U is a compact simply connected C^∞-manifold and carries a G-invariant complex structure. Moreover any G-invariant complex structure on G/U admits a G-invariant Kähler metric. In this paper, we shall consider the problem of classifying, up to equivalence, all G-invariant complex structures on the coset space G/U. Borel-Hirzebruch [2] showed that G-invariant complex structures on G/U are unique up to equivalence if U is a maximal torus of G or if U is a subgroup with one-dimensional center.

We shall consider exclusively the case where G is a simple compact Lie group and in this case we say that the coset space G/U is irreducible. We shall classify all G-invariant complex structures on an irreducible compact simply connected coset space G/U up to equivalence. An equivalence class of G-invariant complex structures on G/U gives rise to a pair of a simple root systems (π, π_0) such that π_0 is a subsystem of π and this pair is determined uniquely up to equivalence. Here two pairs (π, π_0) and (π', π'_0) are said to be equivalent if there is an isomorphism between the systems π and π' which maps π_0 to π'_0. Our classification will then be reduced to that of classifying, up to equivalence, all pairs (π, π_0) associated to G/U and in this way we shall count up the number of equivalence classes of G-invariant complex structures on G/U.

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1. *G*-invariant complex structures

Let G be a Lie group and U a closed subgroup of G. We denote by $\mathfrak{g}$

the Lie algebra of G and $\mathfrak{u}$ the Lie subalgebra corresponding to U in $\mathfrak{g}$, and we write $\mathfrak{g}^C$ and $\mathfrak{u}^C$ to denote their complexifications. Let M be the coset space G/U. Let T_0M denote the tangent vector space of M at the point $0=U$ in M and T_0M^C its complexification. Suppose I is a G-invariant complex structure on M. Then I defines a linear transformation I_0 on T_0M^C. Let T_0M^+ (resp. T_0M^-) be the eigenspace of I_0 with eigenvalue $\sqrt{-1}$ (resp. $-\sqrt{-1}$) of I_0. Then we have

$$T_0M^C = T_0M^+ + T_0M^- \qquad \text{(direct sum)}\,.$$

On the other hand, identifying $\mathfrak{g}$ with the tangent vector space of G at the unit element, the projection $\pi\colon G \to G/U$ induces a complex linear map $d\pi^C\colon \mathfrak{g}^C \to T_0M^C$. Let $\mathfrak{a}^+ = (d\pi_0^C)^{-1}(T_0M^+)$. Then, $\mathfrak{a}^+$ is Lie subalgebras of $\mathfrak{g}^C$ and we have

$$(1) \qquad \mathfrak{g}^C = \mathfrak{a}^+ + \overline{\mathfrak{a}^+}\,, \quad \mathfrak{u}^C = \mathfrak{a}^+ \cap \overline{\mathfrak{a}^+}$$

where — means the complex conjugation in $\mathfrak{g}^C$ with respect to $\mathfrak{g}$. Conversely any subalgebra $\mathfrak{a}^+$ satisfying (1) is obtained from a unique G-invariant complex structure on M in this way. Thus the classification of G-invariant complex structures on M reduces to that of subalgebras $\mathfrak{a}^+$ satisfying (1). (Fröhlicher [4]).

Now, let G be a compact connected semisimple Lie group, U the centralizer of a toral subgroup S of G. Then U contains the center of G. If G acts on G/U effectively, the center of G should be trivial. In the rest of this paper, we always assume that the center of G is trivial. Let T be a maximal torus containing S. Then it is a maximal torus of U. Let $\mathfrak{h}$ be the Lie algebra of T and $\mathfrak{h}^C$ its complexification. Then $\mathfrak{h}^C$ is a Cartan subalgebra of $\mathfrak{g}^C$. Let Δ be the root system of $\mathfrak{g}^C$ with respect to $\mathfrak{h}^C$, and

$$\mathfrak{g}^C = \mathfrak{h}^C + \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha$$

the decomposition of $\mathfrak{g}^C$ to the sum of eigenspaces of roots. Because $\mathfrak{u}^C$ contains $\mathfrak{h}^C$, there is a subset Δ_0 of Δ such that

$$\mathfrak{u}^C = \mathfrak{h}^C + \sum_{\alpha\in\Delta_0} \mathfrak{g}_\alpha\,.$$

Then, Δ_0 is a root system contained in Δ.

Now suppose I be a G-invariant complex structure on M and $\mathfrak{a}^+$ its defining Lie subalgebra of $\mathfrak{g}^C$ satisfying (1). Then $\mathfrak{a}^+ \supset \mathfrak{u}^C \supset \mathfrak{h}^C$, so there is a subset Δ^+ of Δ such that

$$\mathfrak{a}^+ = \mathfrak{u}^C + \sum_{\alpha\in\Delta^+} \mathfrak{g}_\alpha\,.$$

Then Δ^+ satisfies the following conditions.

(2) $\Delta = \Delta_0 \cup \Delta^+ \cup \Delta^-$ (disjoint union)

where Δ^- denotes $-\Delta^+ = \{-\alpha \mid \alpha \in \Delta^+\}$.

(3) If $\alpha \in \Delta_0 \cup \Delta^+$, $\beta \in \Delta^+$ and $\alpha+\beta \in \Delta$ then $\alpha+\beta \in \Delta^+$ (Koszul [8]).

Conversely if Δ^+ satisfies (2) and (3), then $\mathfrak{a}^+ = \mathfrak{u}^C + \sum_{\Delta \in \Delta^+} \mathfrak{g}_\alpha$ satisfies (1). Thus to count G-invariant complex structures on M, we may look for subsets Δ^+ of satisfying (2) and (3).

Lemma 1. *Let Δ be a root system in an Euclidean vector space $(E, (\ ,\ ))$, and Δ_0 a root system contained in Δ. Suppose that a subset Δ^+ of Δ satisfies* (2) *and* (3). *Then the element $s = \sum_{\alpha \in \Delta^+} \alpha$ satisfies $(s, \alpha) = 0$ if $\alpha \in \Delta_0$ and $(s, \alpha) > 0$ if $\alpha \ni \Delta^+$.*

Proof. See Koszul [8].

It is well known that a simple root system π of a root system Δ is given as the set of all simple roots in a certain positive root system (with respect to a given linear order), and we have a bijection between simple root systems and positive root systems in a root system. In general, for a subset π_0 of π, $[\pi_0]$ (resp. $[\pi_0]^+$) denotes the set of roots which are represented as a linear combination of elements of π_0 with integral (resp. non-negative integral) coefficients. The positive root system with respect to π coincides with $[\pi]^+$.

Theorem 1. *Let Δ be a root system in an Euclidean vector space $(E, (\ ,\ ))$ and Δ_0 a root system contained in Δ. Suppose that a subset Δ^+ of Δ satisfies* (2) *and* (3). *Then there exists a simple root system π such that $\pi_0 = \pi \cap \Delta_0$ is a simple root system of Δ_0 and $\Delta^+ = [\pi]^+ - [\pi_0]^+$.*

Conversely if π is a simple root system of Δ such that $\pi_0 = \pi \cap \Delta_0$ is a simple root system of Δ_0, then $\Delta^+ = [\pi]^+ - [\pi_0]^+$ satisfies (2) *and* (3).

Proof. Let s be as in Lemma 1, and $\{v_1, \cdots, v_l\}$ ($l = \dim E$) a basis of E such that $v_1 = s$. Define $\lambda > \mu$ if $(\lambda - \mu, v_1) = \cdots = (\lambda - \mu, v_{i-1}) = 0$ and $(\lambda - \mu, v_i) > 0$ for some i $(1 \leqq i \leqq l)$. Then the simple roots with respect to this order in E form a simple root system π for which the positive root system contains Δ^+. Let $\pi_0 = \pi \cap \Delta_0$. We prove that π_0 is a simple root system of Δ_0. The simple roots in Δ_0 with respect to the above order form a simple root system π_0' of Δ_0. Because each element of π_0 is a simple root in Δ_0, we have $\pi_0' \supset \pi_0$. Suppose $\pi_0' \supsetneqq \pi_0$. Take $\alpha \in \pi_0' - \pi_0$. Thus we take $\alpha = \beta + \gamma$ where β and γ are positive roots in Δ. Then from Lemma 1 follows that $0 = (\alpha, s) = (\beta, s) + (\gamma, s)$ and $(\beta, s) \geqq 0$, $(\gamma, s) \geqq 0$. Thus we have $(\beta, s) = (\gamma, s) = 0$ and we conclude $\beta, \gamma \in \Delta_0 \cap [\pi]^+$, which contradicts our assumption. Therefore $\pi_0 = \pi_0'$ and π_0 is a simple root system of Δ_0. Combining Lemma 1 and the definition of order, we see

$$\Delta^+ = [\pi]^+ - [\pi]^+ \cap \Delta_0 .$$

Hence to get $\Delta^+=[\pi]^+-[\pi_0]^+$, it suffices to prove $[\pi]^+\cap\Delta_0=[\pi_0]^+$. Put $\pi=\{\alpha_1, \cdots, \alpha_l\}$ and assume $\pi_0=\{\alpha_1, \cdots, \alpha_k\}$. If $\alpha\in[\pi]^+\cap\Delta_0$, then $\alpha=n_1\alpha_1+\cdots+n_l\alpha_l$ for some $n_1\geqq 0, \cdots, n_l\geqq 0$. Since $0=(\alpha, s)=n_1(\alpha_1\ s)+\cdots+n_l(\alpha_l, s)$ and $(\alpha_1, s)=\cdots=(\alpha_k, s)=0$, $(\alpha_{k+1}, s)>0, \cdots, (\alpha_l, s)>0$, we have $n_{k+1}=\cdots=n_l=0$. Thus we have $\alpha=n_1\alpha_1+\cdots+n_k\alpha_k\in[\pi_0]^+$. If $\alpha\in[\pi_0]^+$, then $\alpha=n_1\alpha_1+\cdots+n_k\alpha_k$ with $n_1\geqq 0, \cdots, n_k\geqq 0$. Since $(\alpha, s)=n_1(\alpha_1, s)+\cdots+n_k(\alpha_k, s)=0$, it follows that $\alpha\in\Delta_0\cap[\pi]^+$. Thus we have $[\pi_0]^+=\Delta_0\cap[\pi]^+$.

Conversely, let π be a simple root system of Δ such that $\pi_0=\pi\cap\Delta_0$ is a simple root system of Δ_0. Let $\Delta^+=[\pi]^+-[\pi_0]^+$. We prove first that Δ^+ satisfies (2). By the definition of Δ^+, $\Delta=[\pi_0]\cup\Delta^+\cup\Delta^-$ (disjoint union) where Δ^- denotes $-\Delta^+$. It is sufficient to prove $\Delta_0=[\pi_0]$. Let $\pi=\{\alpha_1, \cdots, \alpha_1\}$ and $\pi_0=\{\alpha_1, \cdots, \alpha_k\}$. Suppose $\alpha\in[\pi_0]^+$. Then α is represented as $\alpha=n_1\alpha_1+\cdots+n_k\alpha_k$ with $n_1\geqq 0, \cdots, n_k\geqq 0$. The property of the root system yields that α is represented as $\alpha=\alpha_{i_1}+\cdots+\alpha_{i_p}$ with $\alpha_{i_1}, \cdots, \alpha_{i_p}\in\pi_0$ where $\alpha_{i_1}+\cdots+\alpha_{i_j}\in\Delta$ for any $j=1, \cdots, p$. Because Δ_0 is a root subsystem of Δ, if $\alpha, \beta\in\Delta_0$, $\alpha+\beta\in\Delta$ then $\alpha+\beta\in\Delta_0$. Hence we have $\alpha=\alpha_{i_1}+\cdots+\alpha_{i_p}\in\Delta_0$. Therefore $[\pi_0]^+\subset\Delta_0$. Clearly $\Delta_0\subset[\pi_0]$. So we have $\Delta_0=[\pi_0]$. The property (3) of Δ^+ follows from the fact: A root $\alpha=n_1\alpha_1+\cdots+n_l\alpha_l$ is in Δ^+ if and only if $n_i>0$ for some $i>k$. This proves Theorem 1. ■

Now, let $M=G/U$, Δ and Δ_0 be as before. We denote by $\mathcal{J}_0$ the set of all G-invariant complex structures on M. Also we write $\mathcal{S}_1$ for the set of all simple root systems π of Δ such that $\pi\cap\Delta_0$ is a simple root system of Δ_0. Then we get a surjection from $\mathcal{S}_1$ onto $\mathcal{J}_0$. Namely, for a given $\pi\in\mathcal{S}_1$, we define Δ^+ as in Theorem 1 and, putting $\mathfrak{a}^+=\mathfrak{u}^c+\sum_{\alpha\in\Delta^+}\mathfrak{g}_\alpha$, we make correspond to π the G-invariant complex structure on M defined by $\mathfrak{a}^+$. We denote $\mathcal{W}(\Delta)$ and $\mathcal{W}(\Delta_0)$ the Weyl groups of Δ and Δ_0 respectively. We may consider $\mathcal{W}(\Delta_0)\subset\mathcal{W}(\Delta)$.

Theorem 2. *Let π_0 be a simple root system of Δ_0. We denote by $\mathcal{S}_0$ the set of all simple root systems π of Δ such that $\pi\cap\Delta_0=\pi_0$. Then the mapping $\mathcal{S}_1\rightarrow\mathcal{J}_0$ defined above induces a bijection $\mathcal{S}_0\rightarrow\mathcal{J}_0$.*

Proof. First we see that the mapping is surjective. For a given $I\in\mathcal{J}_0$, we get a unique Δ^+ satisfying (2) and (3). By Theorem 1, there corresponds to Δ^+ an element $\pi'\in\mathcal{S}_1$. Let $\pi_0'=\pi'\cap\Delta_0$. Because π_0' is a simple root system of Δ_0, there exists $\sigma\in\mathcal{W}(\Delta_0)$ such that $\sigma\pi_0'=\pi_0$. Let $\pi=\sigma\pi'$. Then $\pi\in\mathcal{S}_0$. Now we claim $\sigma\Delta^+=\Delta^+$. Let σ_α be the reflection defined by $\alpha\in\Delta$. For $\alpha\in\pi_0$, we have $\sigma_\alpha\Delta^+\subset[\pi']^+$ because $\sigma_\alpha([\pi']^+-\{\alpha\})=[\pi']^+-\{\alpha\}$ and $\alpha\notin\Delta^+$.

Furthermore since $\sigma_\alpha\Delta_0=\Delta_0$, we get $\sigma_\alpha\Delta^+\cap\Delta_0=\phi$. Hence we have $\sigma_\alpha\Delta^+=\Delta^+$. Since $\sigma_\alpha(\alpha\in\pi_0)$ generate $\mathcal{W}(\Delta_0)$, we have $\sigma\Delta^+=\Delta^+$. Since $[\pi]^+-[\pi_0]^+=[\sigma\pi']^+-[\sigma\pi_0']^+=\sigma([\pi']^+-[\pi_0']^+)=\sigma\Delta^+$, we have $\Delta^+=[\pi]^+-[\pi_0]^+$. Therefore the mapping is surjective.

Next we see that the mapping is injective. Since $\Delta^+=[\pi]^+-[\pi_0]^+$, $[\pi]^+=\Delta^+\cup[\pi_0]^+$. Therefore π is the simple root system with respect to the positive root system $[\pi_0]^+\cup\Delta^+$. Thus Δ^+ defines π uniquely. This proves that the mapping is injective, and we get Theorem 2. ■

We note that by a theorem of H.C. Wang [1], $\mathcal{J}_0$ is not an empty set, and so $\mathcal{S}_0$ is not empty.

REMARK. We may choose and fix π belonging to $\mathcal{S}_0$, and put $\pi_0=\pi\cap\Delta_0$. Let

$$\mathcal{W}_0=\{\sigma\in\mathcal{W}(\Delta_0)\,|\,\sigma\pi\supset\pi_0\}\ .$$

Then we have a natural bijection from $\mathcal{S}_0$ to $\mathcal{W}_0$. Thus we can count the number of the elements in $\mathcal{J}_0$ by counting of the cardinality of $\mathcal{W}_0$. Hou-Tze-sin [6] counted it when G is a simple Lie group of classical type.

2. Equivalent complex structures

Let $M=G/U$ be as in section 1. For a given G-invariant complex structures I on M, let (M, I) denote the complex manifold defined by I. Let A be the complex Lie group of biholomorphic automorphisms on (M, I). (See Bochner and Montgomery [1].) Let $H(M, I)$ be the maximal connected subgroup of A. Because G is supposed to be semisimple and have a trivial center, we have $G=G_1\times\cdots\times G_m$ (direct sum), where $G_1, \cdots, G_m$ are compact simple Lie subgroups of G. Let S be a center of U. Then U coincides with the centralizer of S in G. Let T be a maximal torus in G containing S. Let $S_i=G_i\cap S$, $U_i=G_i\cap U$ and $T_i=G_i\cap T$ $(i=1, \cdots, m)$. Then S_i is a torus in G_i, U_i is a centralizer of S_i in G_i, T_i is a torus which is maximal in both U_i and G_i and contains S_i. Let $M_i=G_i/U_i$. We have $M=M_1\times\cdots\times M_m$ (direct product). Moreover the complex structure I on M defines G_i-invariant complex structure I_i on M_i for each i. Then we have $(M, I)=(M_1, I_1)\times\cdots\times(M_m, I_m)$ (direct product). The following theorem is due to Oniščik [10].

Theorem 3. *In the above situation, we have* $H(M, I)=H(M_1, I_1)\times\cdots\times H(M_m, I_m)$. *Furthermore if the group G is simple, then except the three cases indicated in Table* 1, *the Lie algebra* $\mathfrak{g}$ *of G is a compact real form of* $\tilde{\mathfrak{g}}$, *where* $\tilde{\mathfrak{g}}$ *denotes the complex Lie algebra of* $H(M, I)$.

Table 1

Case	$\mathfrak{g}$	$\mathfrak{u}$	$\tilde{\mathfrak{g}}$
1	$C_l(l>1)$	$C_{l-1}+\mathfrak{t}$	A^{C}_{2l-1}
2	G_2	$A_1+\mathfrak{t}$	B^{C}_3
3	$B_l(l>2)$	$A_{l-1}+\mathfrak{t}$	D^{C}_{l+1}

Here $\mathfrak{t}$ denotes the real one dimensional abelian Lie algebra, and the Lie algebra $\mathfrak{u}$ of U is unique up to inner automorphisms of $\mathfrak{g}$.

From now on, we assume always that G is simple.

DEFINITION. Two elements I and I' in $\mathcal{J}_0$ are said to be *equivalent*, noted $I\sim I'$, if the complex manifolds (M, I') and (M, I) are biholomorphic.

Denoting by (π, π_0) a pair of simple root systems with $\pi\supset\pi_0$, two pairs (π, π_0) and (π', π_0') are said to be *equivalent*, if there exists a simple root system somorphism ψ from π onto π' such that $\psi\pi_0=\pi_0'$. We write $(\pi, \pi_0)\sim(\pi', \pi_0')$ in this case. Let $[\pi, \pi_0]$ denote the equivalence class containing a pair (π, π_0).

For $M=G/U$, let Δ and Δ_0 be as in section 1, and $\tau(g)$ denotes the action of $g\in G$ on M. Fix a root system π_0 of Δ_0, and define $\mathcal{S}_0$ as in section 1.

Theorem 4. *For two complex structures I and I' belonging to $\mathcal{J}_0$, let π and π' be the elements of $\mathcal{S}_0$ corresponding to I and I' respectively (Theorem 2). Then $I\sim I'$ if and only if $(\pi, \pi_0)\sim(\pi', \pi_0)$.*

Proof. Suppose $I\sim I'$. We show $(\pi, \pi_0)\sim(\pi', \pi_0)$ first when $\tau(G)$ is a compact real form of $H(M, I)$. Let f be a biholomorphic mapping from (M, I) onto (M, I'). Then we have $df\circ I=I'\circ df$ and $df^{-1}\circ I'=I\circ df^{-1}$. We may assume $f(0)=0$ since f can be replaced by $\tau(g^{-1})\cdot f$ for $g\in G$ such that $\tau(g)0=f(0)$. For $g\in G$, let $\eta(g)$ be the automorphism of M defined by $\eta(g)x=f^{-1}\cdot\tau(g)\cdot f(x)$ for $x\in M$. Then $\eta(G)$ acts on M. By the definition of η, we have $d\eta(g)\circ I=I\circ d\eta(g)$. Thus it follows that $\eta(G)\subset H(M, I)$. Since $\tau(G)$ is a compact real form of $H(M, I)$, so is $\eta(G)$. Since all compact real forms of $H(M, I)$ are conjugate, there exists $a\in H(M, I)$ such that $a^{-1}\eta(G)a=\tau(G)$. We may assume $a0=0$ since a can be replaced by $\eta(g^{-1})\cdot a$ for $g\in G$ such that $\eta(g)0=a0$. Then we have $\tau(U)=a^{-1}\eta(U)a$. Thus $a^{-1}\eta(T)a$ is a maximal torus of $\tau(U)$. Since all maximal tori in $\tau(U)$ are conjugate, there exists $b\in\tau(U)$ such that $b^{-1}(a^{-1}\eta(T)a)b=\tau(T)$. Since $\tau(G)=a^{-1}\eta(G)a$, there exists an automorphism ϕ of G such that $\tau(\phi(g))=a^{-1}\eta(g)a$ for all $g\in G$. Then we have $\phi(U)=U$. Thus ϕ induces an automorphism $\tilde{\phi}$ on $M=G/U$. By the property of $\tilde{\phi}$, $\tilde{\phi}=a^{-1}\circ f^{-1}$, and hence

$$(4) \qquad d\tilde{\phi}\circ I' = I\circ d\tilde{\phi}\,.$$

Moreover we have $\phi(T)=T$. Thus ϕ induces an automorphism ψ' of Δ such that $d\phi^C(\mathfrak{g}_\alpha)=\mathfrak{g}_{\psi'(\alpha)}$ for all $\alpha\in\Delta$. Since $d\phi^C(\mathfrak{u}^C)=\mathfrak{u}^C$, we have $\psi'(\Delta_0)=\Delta_0$. Let $\Delta^+=[\pi]^+-[\pi_0]^+$ and $\Delta'^+=[\pi']^+-[\pi_0]^+$. Let $\alpha\in\Delta'^+$. For any $X\in\mathfrak{g}_\alpha$ with $X\neq 0$, we have $d\phi^C(X)\in\mathfrak{g}_{\psi'(\alpha)}$ and

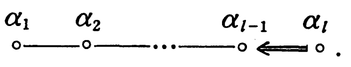
$$(5)\qquad I'(d\pi^C(X))=\sqrt{-1}\,d\pi^C(X)\,.$$

Combining (4) and (5) we have $I(d\pi^C(d\phi^C(X)))=\sqrt{-1}\,d\pi^C(d\phi^C(X))$. Thus $\psi'(\alpha)\in\Delta^+$. Therefore we see that $\psi'\Delta'^+=\Delta^+$. Since $\psi'\Delta_0=\Delta_0$, $\psi'\pi_0$ and π_0 are simple root systems of Δ_0, and hence there exists $\mu\in\mathcal{W}(\Delta_0)$ such that $\mu\psi'\pi_0=\pi_0$. By the same argument as in the proof of Theorem 2 we see that $\mu\Delta^+=\Delta^+$. Let $\psi=(\mu\psi')^{-1}$. Then ψ is an automorphism of Δ such that $\psi\pi_0=\pi_0$ and $\psi\Delta^+=\Delta'^+$. Thus we have $\psi\pi=\pi'$ and $(\pi,\pi_0)\sim(\pi',\pi_0)$.

We show $(\pi,\pi_0)\sim(\pi',\pi_0)$ when $\tau(G)$ is not a compact real form of $H(M,I)$. By Theorem 3, it suffices to prove this in three cases in Table 1. We denote by $D(\pi)$ the Dynkin diagram of a simple root system π.

Case 1. Let $\alpha_1,\cdots,\alpha_l$ be the elements of π such that

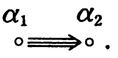
$D(\pi)$:



In this case we have $\pi_0=\{\alpha_2,\cdots,\alpha_l\}$. For any simple root system $\pi'\in\mathcal{S}_0$, there exists $\sigma\in\mathcal{W}(\Delta)$ such that $\sigma\pi=\pi'$. Since the longer root α_l in π is in π_0, we have $\sigma\alpha_l=\alpha_l$. Thus $\sigma\pi_0=\pi_0$ and $(\pi,\pi_0)\sim(\pi',\pi_0)$.

Case 2. Let α_1, α_2 be the elements of π such that

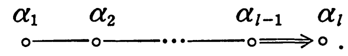
$D(\pi)$:



Also in this case we have $\pi_0=\{\alpha_2\}$. By the same argument as for Case 1, it follows that $(\pi,\pi_0)\sim(\pi',\pi_0)$.

Case 3. Let $\alpha_1,\cdots,\alpha_l$ be the elements of π such that

$D(\pi)$:



In this case we have $\pi_0=\{\alpha_1,\cdots,\alpha_{l-1}\}$. For any π' in $\mathcal{S}_0$, the set of longer roots in π' coincides with π_0. Thus for $\sigma\in\mathcal{W}(\Delta)$ with $\sigma\pi=\pi'$, it follows that $\sigma\pi_0=\pi_0$. Therefore we have $(\pi,\pi_0)\sim(\pi',\pi_0)$. Thus we have proved for all cases that $I\sim I'$ yields $(\pi,\pi_0)\sim(\pi',\pi_0)$.

Conversely suppose $(\pi,\pi_0)\sim(\pi',\pi_0)$. Then there exists an isomorphism ψ from π onto π' such that $\psi\pi_0=\pi_0$. We may extend ψ as an automorphism of Δ naturally. Then ψ induces an automorphism ϕ of $\mathfrak{g}^C$ such that $\phi(\mathfrak{h})=\mathfrak{h}$,

$\phi(\mathfrak{g}_\alpha)=\mathfrak{g}_{\psi(\alpha)}$ and $\Phi(\mathfrak{g})=\mathfrak{g}$. And thus we have $\phi(\mathfrak{u})=\mathfrak{u}$ and $\phi(\mathfrak{a}^+)=\mathfrak{a}'^+$, where $\mathfrak{a}^+$ and $\mathfrak{a}'^+$ are the subalgebras of $\mathfrak{g}^C$ corresponding to I and I' respectively. Since G is connected, $\phi|\mathfrak{g}$ induces an automorphism f of G. Let $\tilde{f}$ and $\tilde{\phi}$ denote the automorphisms on M and T_0M respectively induced from f and ϕ. Then $d\tilde{f}_0=\tilde{\phi}$ and $d\tilde{f}_0(d\pi^C(\mathfrak{a}^+))=d\pi^C(\mathfrak{a}'^+)$. Thus we have $d\tilde{f}\circ I'=I\circ d\tilde{f}$. It follows that $I\sim I'$, which completes the proof. ■

3. The number of the elements in $\mathcal{I}_0/\sim$

For a given $M=G/U$, we shall count the number of elements in $\mathcal{I}_0/\sim$. We shall denote this number by n. Let

$$\mathcal{D}_0=\ \{[\pi,\ \pi\cap\Delta_0]\,|\,\pi\in\mathcal{S}_1\}\ .$$

If we choose a simple root system π_0 of Δ_0, then

$$\mathcal{D}_0=\ \{[\pi,\ \pi_0]\,|\,\pi\in\mathcal{S}_0\}\ .$$

By Theorem 4, we get a bijection between $\mathcal{D}_0$ and $\mathcal{I}_0/\sim$. Thus the number n is equal to the number of elements in $\mathcal{D}_0$. Let l denote the rank of Δ and k the rank of Δ_0. Let $(E, (\ ,\ ))$ denote the Euclidean vector space in which Δ is defined. Note that the inner product $(\ ,\ )$ in E is defined uniquely up to scalar multiplication, since Δ is assumed to be irreducible root system. We shall regard E as a subspace of the Euclidean space R^m of an appropriate dimension m. Let $\{\varepsilon_1, \cdots, \varepsilon_m\}$ be the canonical basis of R^m with the usual inner product.

Fix $\pi\in\mathcal{S}_1$, and let $\pi_0=\pi\subset\Delta_0$. Let $\mathcal{D}_1$ denote the set of $[\pi,\ \phi\pi_0]$ wnere ϕ is ahy mapping from π_0 into π with the following condition:

(*) ϕ is injective and $(\phi\alpha, \phi\beta)=(\alpha, \beta)$ for all $\alpha, \beta\in\pi_0$.

Then $\mathcal{D}_1$ does not depend on the choice of $\pi\in\mathcal{S}_1$. Obviously we have $\mathcal{D}_0\subset\mathcal{D}_1$.

Lemma 2. *Suppose* Δ *is of type* A_l, B_l *or* C_l. *Then we have* $\mathcal{D}_1=\mathcal{D}_0$.

Proof. If $\Delta_0=\phi$, there is nothing to prove. Suppose $\Delta_0\neq\phi$. Fix $\pi\in\mathcal{S}_1$ and let $\pi_0=\pi\cap\Delta_0$ $(\neq\phi)$. It suffices to show that $[\pi, \phi\pi_0]\in\mathcal{D}_0$ for any ϕ with (*). Let first Δ be of type A_l. Then π may be assumed to consist of $\varepsilon_1-\varepsilon_2, \varepsilon_2-\varepsilon_3, \cdots, \varepsilon_l-\varepsilon_{l+1}$. For any irreducible component π_0' of π_0, there are i and p with $0\leqq p\leqq l-i\leqq l-1$ such that $\pi_0'=\{\varepsilon_i-\varepsilon_{i+1}, \cdots, \varepsilon_{i+p}-\varepsilon_{i+p+1}\}$. Let ϕ be a mapping from π_0 into π with (*). Since we have $\phi\pi_0'\subset\pi$ and $\phi\pi_0'$ is an irreducible component of $\phi\pi_0'$, there is j with $j+p\leqq l$ such that $\phi\pi_0'=\{\varepsilon_j-\varepsilon_{j+1}, \cdots, \varepsilon_{j+p}-\varepsilon_{j+p+1}\}$. Thus ϕ may be assumed to satisfy $\phi(\varepsilon_{i+q}-\varepsilon_{i+q+1})=\varepsilon_{j+q}-\varepsilon_{j+q+1}$ for $q=0, \cdots, p$. Then it is easily seen that there exists $\sigma\in\mathfrak{S}_{l+1}$ (the symmetric group of $l+1$ letters which is identified with $\mathcal{W}(\Delta)$) such that $\sigma(j)=i$

whenever $\phi(\varepsilon_i-\varepsilon_{i+1})=\varepsilon_j-\varepsilon_{j+1}$. Also we obtain $\sigma\pi\supset\pi_0$, and hence $\sigma\pi\in\mathcal{S}_0$. Therefore we have $[\pi, \phi\pi_0]=[\sigma\pi, \pi_0]\in\mathcal{D}_0$.

Now let Δ be of type B_l. Then π may be assumed to consist of $\varepsilon_1-\varepsilon_2$, $\varepsilon_2-\varepsilon_3, \cdots, \varepsilon_{l-1}-\varepsilon_l, \varepsilon_l$. If $\pi_0\not\ni\varepsilon_l$, then $\phi\pi_0\not\ni\varepsilon_l$. Thus we have $\pi_0\subset\{\varepsilon_1-\varepsilon_2, \cdots, \varepsilon_{l-1}-\varepsilon_l\}$ and the image of ϕ is contained in $\{\varepsilon_1-\varepsilon_2, \cdots, \varepsilon_{l-1}-\varepsilon_l\}$. By the same argument as for the previous case, it follows that $[\pi, \phi\pi_0]$ is an element of $\mathcal{D}_0$. Now suppose $\pi_0\ni\varepsilon_l$. Then we have $\phi\varepsilon_l=\varepsilon_l$. Let π_0' be the irreducible component of π_0 containing ε_l. Then we have $\phi\pi_0'=\pi_0'$. We denote by $\varepsilon_1-\varepsilon_2, \cdots, \varepsilon_p-\varepsilon_{p+1}$ the elements of $\pi-\pi_0'$. Let π_0'' denote $\pi_0-\pi_0'$. Then we have $\pi''\subset\{\varepsilon_1-\varepsilon_2, \cdots, \varepsilon_{p-1}-\varepsilon_p\}$ and the image of the restriction of ϕ to π_0'' is contained in $\{\varepsilon_1-\varepsilon_2 \cdots, \varepsilon_{p-1}-\varepsilon_p\}$. Let $\mathfrak{S}_p$ be considered as the subgroup of $\mathcal{W}(\Delta)$ which is generated by the reflections of $\{\varepsilon_1-\varepsilon_2, \cdots, \varepsilon_{p-1}-\varepsilon_p\}$. By the same argument as for the case of A_l, we see there exists $\sigma\in\mathfrak{S}_p$ with $\sigma\phi\pi_0''=\pi_0''$. Since π_0' is contained in $\{\varepsilon_{p+1}-\varepsilon_{p+2}, \cdots, \varepsilon_l\}$, we have $\sigma\pi_0'=\pi_0'$, and hence we obtain $\sigma\phi\pi_0=\pi_0$. Thus we have $[\pi, \phi\pi_0]=[\sigma\pi, \pi_0]\in\mathcal{D}_0$. The same argument as in the case of B_l works for the case of C_l. Thus we have $\mathcal{D}_0=\mathcal{D}_1$ for all cases. ■

By counting the number of the elements in $\mathcal{D}_1$, we get the following theorem. To state the theorem, we need some notations. If $k_1, \cdots, k_p$ are positive integers, we write $\alpha(k_1, \cdots, k_p)$ for the number of the permutations of $\{k_1, \cdots, k_p\}$. And we write $\beta(k_1, \cdots, k_p)$ for the number of the permutations σ of $\{k_1, \cdots, k_p\}$ such that $k_{\sigma(q)}=k_{\sigma(p-q)}$ for $q=1, \cdots, [p/2]$.

Theorem 5. (i) *Suppose Δ is of type A_l and Δ_0 is of type $A_{k_1}+\cdots+A_{k_p}$. (Note that $0\leqq p\leqq k_1+\cdots+k_p=k\leqq k+p\leqq l+1$). Then the number n of elements in $\mathcal{J}_0/\sim$ is given by the following formula.*

If both $(l-k)$ and p are odd number, then

$$n=\frac{1}{2}\binom{l-k+1}{p}\cdot\alpha(k_1, \cdots, k_p)\,.$$

In other cases, if $p\neq 0$

$$n=\frac{1}{2}\left\{\binom{l-k+1}{p}\cdot\alpha(k_1, \cdots, k_p)+\binom{\left[\frac{l+p-k-1}{2}\right]}{\left[\frac{p}{2}\right]}\cdot\beta(k_1, \cdots, k_p)\right\}.$$

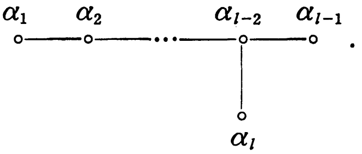
If $p=0$, then $n=1$.

(ii) *Suppose Δ is of type B_l (resp. C_l) and Δ_0 is of type $B_t+A_{k_1}+\cdots+A_{k_p}$ (resp. $C_t+A_{k_1}+\cdots+A_{k_p}$). Here B_t (resp. C_t) denotes the type of the irreducible component of Δ_0 containing shorter roots (resp. longer roots). Note that $B_0=C_0=\phi$, $B_1\cong C_1\cong A_1$, $B_2\cong C_2$, and $0\leqq p\leqq k_1+\cdots+k_p+t=k\leqq k+p\leqq l+1$. Then we get*

If $p \neq 0$, *then* $n = \binom{l-k}{p} \cdot \alpha(k_1, \cdots, k_p)$.

If $p=0$, *then* $n = 1$.

Before to give a theorem for the case of type D_l, we need some notations. Suppose Δ is of type D_l. Fix $\pi \in \mathcal{S}_1$ and let $\pi_0 = \pi \cap \Delta_0$. Let $\alpha_1, \cdots, \alpha_l$ denote the elements of π such that

$D(\pi)$:


We may assume that $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i=1, \cdots, l-1$, and $\alpha_l = \varepsilon_{l-1} + \varepsilon_l$. Then $\mathcal{W}(\Delta)$ consists of such elements as $\sigma = (\tau, a_1, \cdots, a_l)$ where $\tau \in \mathfrak{S}_l$, $a_i = 1$ or -1, and the number of -1 in $\{a_1, \cdots, a_l\}$ is even, whose action is given by $\sigma(\varepsilon_i \pm \varepsilon_j) = a_i \varepsilon_{\sigma(i)} \pm a_j \varepsilon_{\sigma(j)}$. Put

$$\pi_0' = \begin{cases} \phi, \quad \text{if} \quad \pi_0 \not\supset \{\alpha_{l-1}, \alpha_l\} \\ \{\alpha_{l-1}, \alpha_l\}, \quad \text{if} \quad \pi_0 \supset \{\alpha_{l-1}, \alpha_l\} \quad \text{and} \quad \pi_0 \not\ni \alpha_{l-2} \\ \text{the irreducible component of } \pi_0 \text{ containing} \\ \{\alpha_{l-2}, \alpha_{l-1}, \alpha_l\}, \quad \text{if} \quad \pi_0 \supset \{\alpha_{l-2}, \alpha_{l-1}, \alpha_l\}, \end{cases}$$

and

$\mathcal{D}_2 = \{[\pi, \phi\pi] \mid \phi$ is any mapping from π_0 into π with $(*)$ such that $\phi\pi_0' = \pi_0'\}$,

if $\pi_0' \neq \phi$.

$\mathcal{D}_3 = \{[\pi, \phi\pi_0] \mid \phi$ is any mapping from π_0 into π with such that $\phi\pi_0 \not\supset \{\alpha_{l-1}, \alpha_l\}\}$,

if $\pi_0' = \phi$.

Lemma 3. *Suppose* $\Delta_0 \neq \phi$. *If* $\pi_0' \neq \phi$, *we have* $\mathcal{D}_0 = \mathcal{D}_2$. *If* $\pi' = \phi$, *we have* $\mathcal{D}_0 = \mathcal{D}_3$.

Proof. First we consider the case where $\pi_0' \neq \phi$. For any $[\pi', \pi_0] \in \mathcal{D}_0$, there exists $\sigma \in \mathcal{W}(\Delta)$ with $\sigma\pi = \pi'$. Let $\sigma = (\tau, a_1, \cdots, a_l)$. Since $\{\varepsilon_{l-1} \pm \varepsilon_l\}$ is contained in π_0, it is also contained in $\sigma\pi = \{a_1\varepsilon_{\tau(1)} - a_2\varepsilon_{\tau(2)}, \cdots, a_{l-1}\varepsilon_{\tau(l-1)} - a_l\varepsilon_{\tau(l)}, a_{l-1}\varepsilon_{\tau(l-1)} + a_l\varepsilon_{\tau(l)}\}$. We can show easily that $\{a_{l-1}\varepsilon_{\tau(l-1)} \pm a_l\varepsilon_{\tau(l)}\} = \{\varepsilon_{l-1} \pm \varepsilon_l\}$. Thus we obtain $\sigma\{\alpha_{l-1}, \alpha_l\} = \{\alpha_{l-1}, \alpha_l\}$, and hence we have $\sigma\pi_0' = \pi_0'$. Therefore $[\pi', \pi_0] = [\pi, \sigma^{-1}\pi_0] \in \mathcal{D}_2$. Conversely, let ϕ satisfy the condition as in $\mathcal{D}_2$. We denote by $\varepsilon_1 - \varepsilon_2, \cdots, \varepsilon_p - \varepsilon_{p+1}$ the elements of $\pi - \pi_0'$. Put $\pi_0'' = \pi_0 - \pi_0'$. Then we have $\pi_0'' \subset \{\varepsilon_1 - \varepsilon_2, \cdots, \varepsilon_{p-1} - \varepsilon_p\}$ and the image of the restriction of ϕ to π_0'' is contained in $\{\varepsilon_1 - \varepsilon_2, \cdots, \varepsilon_{p-1} - \varepsilon_p\}$. Then by the same argument as in the Case B_l, we see that there exists an element $\sigma \in \mathcal{W}(\Delta)$ with

$\sigma\phi\pi_0=\pi_0$. Thus we obtain $[\pi, \phi\pi_0]=[\sigma\pi, \pi_0]\in\mathscr{D}_0$. And hence we have $\mathscr{D}_0=\mathscr{D}_2$. Next we consider the case where $\pi_0'=\phi$. For any $[\pi', \pi_0]\in\mathscr{D}_0$ there exists $\sigma\in\mathcal{W}(\Delta)$ with $\sigma\pi=\pi'$. Since $\{\varepsilon_{l-1}\pm\varepsilon_l\}$ is not contained in π_0, $\sigma^{-1}\pi_0$ does not contain $\{\varepsilon_{l-1}\pm\varepsilon_l\}$. Therefore $[\pi', \pi_0]=[\pi, \sigma^{-1}\pi_0]\in\mathscr{D}_3$. Conversely let ϕ satisfy the condition as in $\mathscr{D}_3$. Let f denote the following automorphism of π.

$$f(\alpha)=\begin{cases}\alpha_l & \text{if } \alpha=\alpha_{l-1}\\ \alpha_{l-1} & \text{if } \alpha=\alpha_l\\ \alpha & \text{otherwise.}\end{cases}$$

Since $[\pi, f\pi_0]=[\pi, \pi_0]$, it is sufficient to prove the case where $\alpha_l\notin\pi_0$. Suppose $\phi\pi_0\not\ni\alpha_l$. Then we have $\pi_0\subset\{\alpha_1, \cdots, \alpha_{l-1}\}$ and the image of ϕ is contained in $\{\alpha_1, \cdots, \alpha_{l-1}\}$. Thus by the same argument as in the case where Δ is A_l, we have $[\pi, \phi\pi_0]\in\mathscr{D}_0$. Suppose $\phi\pi_0\ni\alpha_l$. Then we have $\phi\pi_0\not\ni\alpha_{l-1}$. Since $[\pi, f\circ\phi\pi_0]=[\pi, \pi_0]$ and $f\circ\phi\pi_0\not\ni\alpha_l$, we obtain $[\pi, f\circ\phi\pi_0]\in\mathscr{D}_0$. Thus we have $\mathscr{D}_0=\mathscr{D}_3$ and we have proved the lemma. ■

From Lemma 3, by counting the number of elements in $\mathscr{D}_2$ or $\mathscr{D}_3$, we get

Theorem 6. *Suppose that Δ is of type D_l and Δ_0 is of type $D_t+A_{k_1}+\cdots+A_{k_p}$. Here D_t denotes the type of π_0'. Note that $D_0=\phi$, $D_1\cong A_1$, $D_3\cong A_1+A_1$, $D_3\cong A_3$ and $0\leqq p\leqq k_1+\cdots+k_p+t=k\leqq k+p\leqq l+1$. Then we have following formula for the number n of elements in $\mathscr{J}_0/\sim$.*

$$\textit{If}\quad p\neq 0,\quad \textit{then}\quad n=\binom{l-k}{p}\cdot\alpha(k_1, \cdots, k_p)$$

$$\textit{If}\quad p=0,\quad \textit{then}\quad n=1\,.$$

Before giving our theorems for the cases where Δ are of types E, F or G, we need a lemma. Fix an irreducible root system Δ. For a subset π_0 of Δ, put

$$\mathscr{D}(\pi_0)=\{[\pi', \pi_0]\,|\,\pi' \text{ is any simple root system containing } \pi_0\}.$$

Lemma 4. *In above notation, let π_0' be another subset of Δ. If $\mathscr{D}(\pi_0)\cap\mathscr{D}(\pi_0')\neq\phi$ then we have $\mathscr{D}(\pi_0)=\mathscr{D}(\pi_0')$.*

Proof. Suppose $[\pi, \pi_0'']\in\mathscr{D}(\pi_0)\cap\mathscr{D}(\pi_0')$. Then there exist simple root systems π' and π'' of Δ such that $(\pi', \pi_0)\sim(\pi, \pi_0'')$ and $(\pi'', \pi_0')\sim(\pi, \pi_0'')$. Thus we have $(\pi', \pi_0)\sim(\pi'', \pi_0')$, and hence there exists $\sigma\in\mathrm{Aut}(\Delta)$ with $\sigma\pi_0=\pi_0'$. Therefore we obtain $\mathscr{D}(\pi_0)=\mathscr{D}(\pi_0')$. ■

REMARK. For a given Δ and Δ_0, let $\mathscr{D}_0$ and $\mathscr{D}_1$ denote the sets defined before. Fix $[\pi, \pi_0]\in\mathscr{D}_1$. If we show $\mathscr{D}_1=\mathscr{D}(\pi_0)$, then we obtain $\mathscr{D}_0=\mathscr{D}_1$. In fact, we have $\mathscr{D}_0\cap\mathscr{D}(\pi_0)\neq\phi$. On the other hand, for $\pi'\in\mathcal{S}_1$, let $\pi_0'=\pi'\cap\Delta_0$. Then we have $\mathscr{D}_0=\mathscr{D}(\pi_0')$. Since $\mathscr{D}_0\cap\mathscr{D}_1\neq\phi$, by Lemma 4, $\mathscr{D}_0=\mathscr{D}(\pi_0)$. Thus we obtain $\mathscr{D}_0=\mathscr{D}_1$.

In the case where Δ is of type E, F or G, this argument yields $\mathcal{D}_0=\mathcal{D}_1$.

Theorem 7. *Suppose that Δ is of type F_4. Then we have $\mathcal{D}_0=\mathcal{D}_1$ and we get the following table for the number n of elements in $\mathcal{J}_0/\sim$.*

Table 2

type of Δ_0	n	type of Δ_0	n
ϕ	1	A_1+A_1	3
A_1	2	B_3	1
A_2	1	C_3	1
B_2	1	A_1+A_2	1

Proof. We may assume that π consists of $\varepsilon_2-\varepsilon_3$, $\varepsilon_3-\varepsilon_4$, ε_4, $\frac{1}{2}(\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4)$. For each element $[\pi, \pi_0']$ in $\mathcal{D}_1$, $D[\pi, \pi_0']$ denotes the Dynkin diagram of π whose vertices not belonging to π_0' are marked by X. Fix $[\pi, \pi_0]\in\mathcal{D}_1$ and for any $[\pi, \pi_0']\in\mathcal{D}_1$, we can find a simple root system π' such that $[\pi', \pi_0]=[\pi, \pi_0]$ as in the following table. Thus we have $\mathcal{D}_1=\mathcal{D}(\pi_0)$ and, by above remark, $\mathcal{D}_0=\mathcal{D}_1$. ■

type of π_0	$D[\pi, \pi_0']$ and π'	n
A_1	π $\quad \circ - X \Rightarrow X - X$ $\varepsilon_2-\varepsilon_3$	2
	π' $\quad X - \circ \Rightarrow X - X$ $\varepsilon_4-\varepsilon_2 \quad \varepsilon_2-\varepsilon_3 \quad \frac{1}{2}(\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4)$	
A_1	π $\quad X - X \Rightarrow \circ - X$ ε_4	2
	π' $\quad X - X \Rightarrow X - \circ$ $\varepsilon_1-\varepsilon_2 \quad \varepsilon_2-\varepsilon_3 \quad \frac{1}{2}(-\varepsilon_1-\varepsilon_2+\varepsilon_3-\varepsilon_4)$	
A_2	π $\quad \circ - \circ \Rightarrow X - X$	1
A_2	π $\quad X - X \Rightarrow \circ - \circ$	1
B_2	π $\quad X - \circ \Rightarrow \circ - X$	1
A_1+A_1	π $\quad \circ - X \Rightarrow \circ - X$ $\varepsilon_2-\varepsilon_3 \qquad \varepsilon_4$	3
	π' $\quad \circ - X \Rightarrow X - \circ \ \varepsilon_4$ $\varepsilon_2-\varepsilon_3 \quad \varepsilon_1-\varepsilon_2 \quad \frac{1}{2}(\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4)$	
	π' $\quad X - \circ \Rightarrow X - \circ \ \varepsilon_4$ $\varepsilon_1-\varepsilon_2 \quad \varepsilon_2-\varepsilon_3 \quad \frac{1}{2}(-\varepsilon_1-\varepsilon_2+\varepsilon_3-\varepsilon_4)$	

Table continued

Type of π_0	$D[\pi_0, \pi'_0]$ and π'	n
B_3	π ○——○⟹○——X	1
C_3	π X——○⟹○——○	1
A_1+A_2	π ○——X⟹○——○	1
A_1+A_2	π ○——○⟹X——○	1

Theorem 8. *Suppose that Δ is of type G_2. Then we have $\mathscr{D}_0=\mathscr{D}_2$ and the following table holds.*

Table 3

type of Δ_0	n
ϕ	1
A_1	1

Proof. Obviously $\mathscr{D}_1$ contains only one element in any case. Since $\mathscr{D}_0\subset\mathscr{D}_1$, we obtain the theorem. ■

Theorem 9. *Suppose that Δ is of type E. Then we have $\mathscr{D}_0=\mathscr{D}_1$ and get the following table for the number n of elements in $\mathscr{J}_0^{\sim}$.*

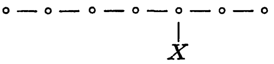
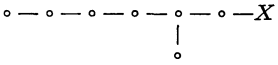
Table 4

type of Δ_0	n			type of Δ_0	n		
	E_6	E_7	E_8		E_6	E_7	E_8
ϕ	1	1	1	$A_1+A_1+A_1+A_1$	—	2	7
A_1	4	7	8	A_5	1	3	4
A_2	3	6	7	D_5	1	2	2
A_1+A_1	6	15	21	A_4+A_1	1	5	12
A_3	3	6	7	$A_2+A_2+A_1$	1	3	8
A_2+A_1	5	18	28	D_4+A_1	—	1	2
$A_1+A_1+A_1$	4	11	21	A_3+A_2	—	3	10
A_4	2	5	6	$A_3+A_1+A_1$	—	3	10
D_4	1	1	1	$A_2+A_1+A_1+A_1$	—	1	8
A_3+A_1	2	11	20	A_6	—	1	3
A_2+A_2	1	4	8	D_6	—	1	1
$A_2+A_1+A_1$	3	12	28	E_6	—	1	1

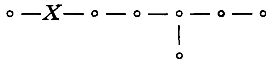
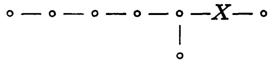
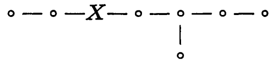
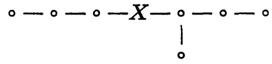
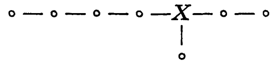
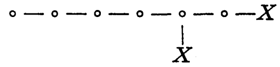
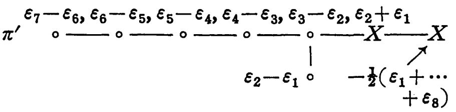
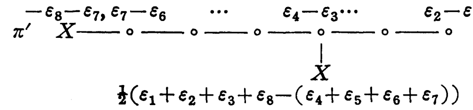
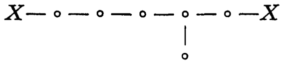
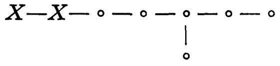
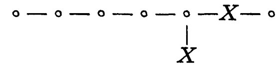
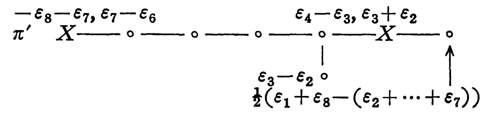
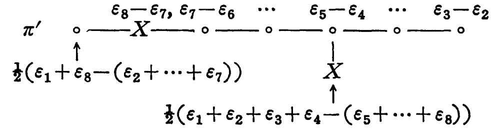
Table 4 continued

type of Δ_0	n			type of Δ_0	n		
	E_6	E_7	E_8		E_6	E_7	E_8
A_5+A_1	—	—	3	A_7	—	—	1
D_5+A_1	—	1	3	D_7	—	—	1
A_4+A_2	—	—	4	E_7	—	—	1
$A_4+A_1+A_1$	—	1	4	E_6+A_1	—	—	1
D_4+A_2	—	1	1	D_5+A_2	—	—	1
A_3+A_3	—	1	2	$D_5+A_1+A_1$	—	—	1
$A_3+A_2+A_1$	—	—	4	A_4+A_3	—	—	1
$A_2+A_2+A_1+A_1$	—	—	2	$A_4+A_2+A_1$	—	—	1

Proof. Since root systems of type E_6 and E_7 are canonically root subsystems of that of type E_8, it is sufficient to show our assertion for the case of E_8. The system π may be assumed to consists of $\varepsilon_7-\varepsilon_8$, $\varepsilon_6-\varepsilon_5$, $\varepsilon_5-\varepsilon_4$, $\varepsilon_4-\varepsilon_3$, $\varepsilon_3-\varepsilon_2$, $\varepsilon_2-\varepsilon_1$, $\varepsilon_2+\varepsilon_1$, $\frac{1}{2}(\varepsilon_1+\varepsilon_8-(\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5+\varepsilon_6+\varepsilon_7))$. The following table is as in the case of F_4. In the table, each equivalence class $[\pi, \pi_0']$ is numbered. Suppose that $[\pi, \pi_a]$, $[\pi, \pi_b]$, $[\pi, \pi_c]$ and $[\pi, \pi_d]$ are numbered by a, b, c and d. Then "$a \rightarrow b$" has the following meaning: "$[\pi, \pi_a] \in \mathscr{D}_1$ has already been proved. Suppose π_a do not contain the element $\varepsilon_1+\varepsilon_2$. Let π_a' be all irreducible components contained in $\{\varepsilon_7-\varepsilon_6, \cdots, \varepsilon_2-\varepsilon_1\}$ and put $\pi_a''=\pi-\pi_a'$. Moreover suppose there exist a mapping ϕ from π_a onto π_b with (*) such that $\phi\pi_a''=\pi_a''$ and $\phi\pi_a' \subset \{\varepsilon_7-\varepsilon_6, \cdots, \varepsilon_2-\varepsilon_1\}$. Then we can show $[\pi, \pi_b] \in \mathscr{D}_1$ by the same argument as in the case of A_l." "$a \rightarrow b$ $(c \rightarrow d)$" has the following meaning: "$[\pi, \pi_a] \in \mathscr{D}_1$ has already been proved. And the existence of π' such that $[\pi, \pi_d]=[\pi', \pi_c]$ has already been shown. Suppose π_a and π_b are subsets of π_c and π_d respectively. Moreover suppose for $\sigma \in \mathscr{W}(\Delta)$ with $\sigma\pi=\pi'$ (note that then $\sigma\pi_d=\pi_c$), we have $\sigma\pi_a=\pi_b$. Then we can show $[\pi, \pi_b] \in \mathscr{D}_1$."

number	type of π_0	$D[\pi, \pi_0']$ and π' such that $D[\pi, \pi_0']=D[\pi', \pi_0]$	n
1	A_7	π 	1
2	D_7	π 	1
3	E_7	π 	1

continued

number	type of π_0	$D[\pi, \pi_0']$ and π' such that $D[\pi, \pi_0']=D[\pi', \pi_0]$	n
4	E_6+A_1	π 	1
5	A_6+A_1	π 	1
6	D_5+A_2	π 	1
7	A_4+A_3	π 	1
8	$A_4+A_2+A_1$	π 	1
9	A_6	π 	3
10		π' 	
11		π' 	
12	D_6	π 	1
13	E_6	π 	1
14	A_5+A_1	π 	3
15		π' 	
16		π' 	

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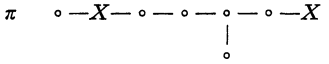
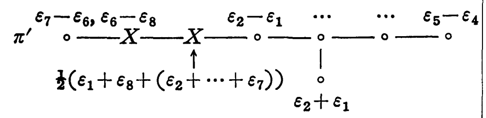
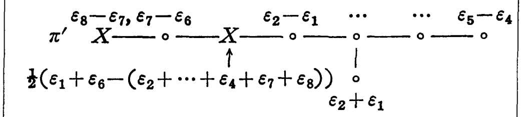
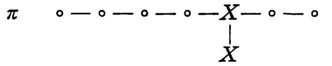
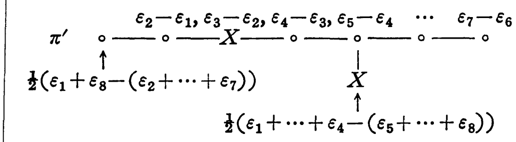
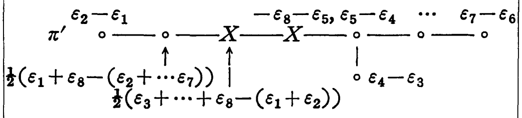
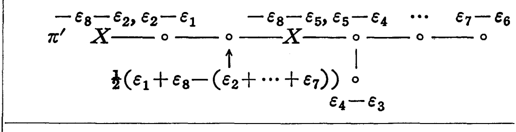
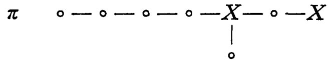
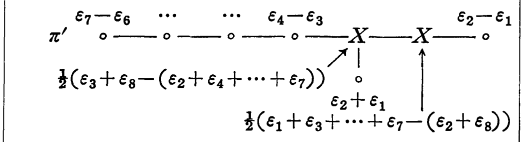
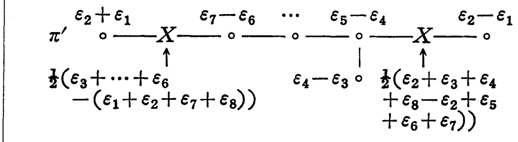
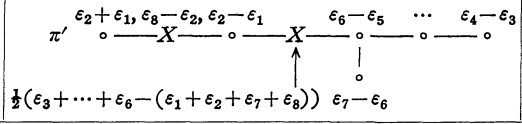
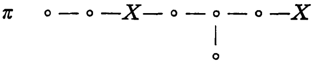
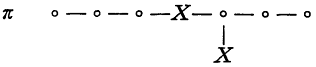
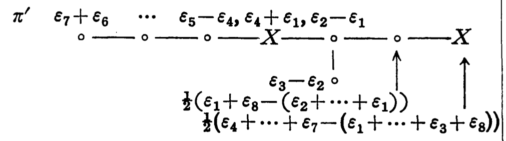
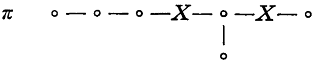
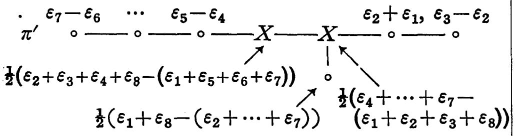
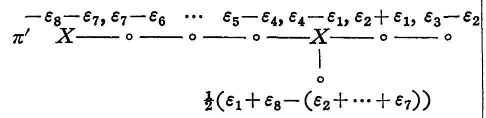
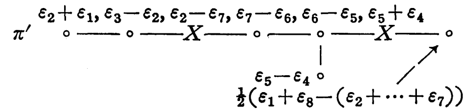
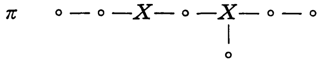
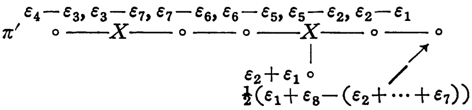
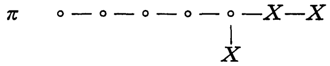
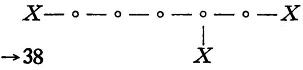
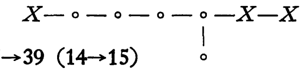
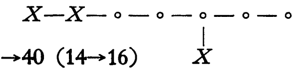
number	type of π_0	$D[\pi, \pi_0']$ and π' such that $D[\pi, \pi_0']=D[\pi', \pi_0]$	n
17	D_5+A_1		3
18			
19			
20	A_4+A_2		4
21			
22			
23			
24	$A_4+A_1+A_1$		4
25			
26			
27			

continued

number	type of π_0	$D[\pi, \pi_0']$ and π' such that $D[\pi, \pi_0']=D[\pi', \pi_0]$	n
28	D_4+A_2		1
29	A_3+A_3		2
30			
31	$A_1+A_2+A_1$		4
32			
33			
34			
35	$A_2+A_2+A_1+A_1$		2
36			
37	A_5		4
38		 37→38	
39		 37→39 (14→15)	
40		 37→40 (14→16)	

continued

number	type of π_0	$D[\pi, \pi'_0]$ and π' such that $D[\pi, \pi'_0]=D[\pi', \pi_0]$	n
41	D_5	π X—X— ∘ — ∘ — ∘ — ∘ —X | ∘	2
42		X—X—X— ∘ — ∘ — ∘ — ∘ | 41→42 (17→18) ∘	
43	A_4+A_1	π ∘ — ∘ — ∘ — ∘ —X— ∘ —X | X	12
44		∘ — ∘ — ∘ — ∘ —X—X— ∘ | 43→44 (24→25) X	
45		∘ — ∘ — ∘ — ∘ —X—X—X | 43→45 (9→10) ∘	
46		X— ∘ — ∘ — ∘ — ∘ —X— ∘ | 43→46 (9→11) X	
47		∘ —X— ∘ — ∘ — ∘ — ∘ —X | 43→47 X	
48		$\varepsilon_2-\varepsilon_1, \varepsilon_1+\varepsilon_8$ $\varepsilon_7-\varepsilon_6$ ⋯ ⋯ $\varepsilon_4-\varepsilon_3$ π' ∘ ——X——X—— ∘ —— ∘ —— ∘ —— ∘ ↑ | $\frac{1}{2}(\varepsilon_3+\cdots+\varepsilon_6$ | X $-(\varepsilon_1+\varepsilon_2+\varepsilon_7+\varepsilon_8))$ $\varepsilon_7+\varepsilon_6$	
49		X— ∘ —X— ∘ — ∘ — ∘ — ∘ | 48→49 X	
50		X—X— ∘ — ∘ — ∘ —X— ∘ | 46→50 (14→15) ∘	
51		∘ —X— ∘ — ∘ — ∘ —X—X | 47→51 (9→10) ∘	
52		∘ —X—X—X— ∘ — ∘ — ∘ | 48→52 (21→22) ∘	
53		X— ∘ —X—X— ∘ — ∘ — ∘ | 52→53 (18→19) ∘	
54		$\varepsilon_8-\varepsilon_2, \varepsilon_2-\varepsilon_1$ $\varepsilon_6-\varepsilon_5, \varepsilon_5-\varepsilon_4$ π' X——X—— ∘ ——X—— ∘ —— ∘ —— ∘ ↑ ↑ | $\varepsilon_4-\varepsilon_3$ $\frac{1}{2}(\varepsilon_1+\varepsilon_8-(\varepsilon_2+\cdots+\varepsilon_7))$ | ∘ $\varepsilon_7-\varepsilon_6$ $\frac{1}{2}(\varepsilon_1+\varepsilon_3+\varepsilon_4+\varepsilon_5-(\varepsilon_2+\varepsilon_6+\varepsilon_7+\varepsilon_8))$	

continued

number	type of π_0	$D[\pi, \pi_0']$ and π' such that $D[\pi, \pi_0']=D[\pi', \pi_0]$	n
55	D_4+A_1	π o—X—X—o—o—o—X | o	2
56		X—o—X—o—o—o—X | 55→56 (18→19) o	
57	A_3+A_2	π o—o—o—X—o—o—X | X	10
58		X—o—o—o—X—o—o | 57→58 (9→11) X	
59		o—o—o—X—X—o—o | 58→59 X	
60		o—o—o—X—o—X—X | 57→60 (9→10) o	
61		o—o—X—o—o—o—X | 57→61 X	
62		o—o—X—o—o—X—X | 61→62 (9→10) o	
63		o—o—X—X—o—o—X | 61→63 (21→22) o	
64		X—o—o—X—o—o—X | 63→64 (22→23) o	
65		o—o—X—X—o—o—o | 63→65 (30→29) X	
66		X—o—o—X—o—o—o | 65→66 X	

We omit the rest of this table because we may write it in the same way. ■

From Theorems 5, 6, 7, 8 and 9, we get the next corollary which has been shown by Borel-Hirzebruch [2] in a different way.

Corollary. *If U is a maximal torus of G or if U has one-dimensional center, then G-invariant complex structures on G/U are unique up to biholomorphism.*

Proof. Suppose that U is a maximal torus of G. Then we have $\Delta_0=\phi$.

Thus we obtain $n=1$. Let S be the center of U. Then we have rank $[U, U]=$ rank $U-\dim S$. Suppose $\dim S=1$. Then we have rank $\Delta_0=$rank $[U, U]=l-1$. From above theorems we obtain $n=1$. ■

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