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SUPER DIFFERENTIAL CALCULUS

KATSUMI YAGI

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The theory of super differential manifolds has been developed in recent years by many authors. While there are several approaches to the subject, we shall take the so-called geometric approach, developed by Boyer-Gitler [2], DeWitt [3] and Rogers [4]. On the other hand, Bernshtein-Rosenfel'd [1] have introduced a differential calculus on an infinite dimensional Euclidean space. Regarding the super Euclidean space as a projective limit of a family of finite dimensional Euclidean spaces as [1], in this note we shall propose a super differential calculus as a foundation for the theory of super manifolds. The underlying principle is to describe the concept of super differential calculus in terms of a differential calculus on an infinite dimensional Euclidean space. This gives a reduction of a "super" argument to an ordinary one and leads to an easier and clearer understanding of the theory of super manifolds. In section 4 we obtain the Cauchy-Riemann equations for a super smooth function, first obtained in [2], which is rather simplified. In the last section the inverse mapping theorem is also obtained following our principle.

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1. Super numbers and super Euclidean spaces

Let $\{\zeta^N : N \geqq 1\}$ be a set of countably infinite distinct letters. These are fixed once for all. We denote by Λ_N the Grassmann algebra of the vector space generated by $\{\zeta^1, \zeta^2, \cdots, \zeta^N\}$ over the real number field $\boldsymbol{R}$ where for $N=0$, Λ_0 denotes the real number field $\boldsymbol{R}$. The family $\{\Lambda_N : N \geqq 0\}$ and the natural projection of Λ_N onto Λ_{N-1} define the projective limit, denoted by Λ. In a natural way, Λ can be identified with the algebra of all formal series of the following form:

$$z = \sum_{K \in \Gamma} z_K \zeta^K$$

where Γ denotes the set of all h-tuples $K=(k_1, k_2, \cdots, k_h)$ of integers ($h \geqq 0$) with $1 \leqq k_1 < k_2 < \cdots < k_h$ and $z_k \in \boldsymbol{R}$ and $\zeta^K = \zeta^{k_1} \zeta^{k_2} \cdots \zeta^{k_h}$ ($\zeta^\phi = 1 \in \boldsymbol{R}$). The algebra

Λ is called the *super number algebra* and an element of Λ is called a *super number*. In super differential calculus, the super number algebra Λ is comparable to the real number field in ordinary differential calculus. The natural projection p_N of the projective limit Λ onto Λ_N maps the above $z\in\Lambda$ to the following $z_N\in\Lambda_N$:

$$z_N=\sum_{K\in\Gamma_N} z_K\,\zeta^K$$

where $\Gamma_N=\{K=(k_1, k_2, \cdots, k_h)\in\Gamma\colon 1\leqq k_1<k_2<\cdots<k_h\leqq N\}$. The algebra Λ_N is called the *N-th skeleton* of the super number algebra Λ. In the case where $N=0$, the 0-th skeleton $\Lambda_0=\boldsymbol{R}$ is called the *body* of the super number algebra Λ. The projection p_0 of Λ onto the body $\boldsymbol{R}$ of Λ will be denoted by p_B. For each super number $z\in\Lambda$, $p_B(z)\in\boldsymbol{R}$ is called the *body* of the super number z, denoted by z_B, and $z_S=z-z_B$ is called the *soul* of z. For each $K=(k_1, \cdots, k_h)\in\Gamma$, the *parity* $|K|$ of K is defined by $|K|=h$ mod $2\in\boldsymbol{Z}_2$. For $p\in\boldsymbol{Z}_2$, Γ_p and Λ_p are defined as follows:

$$\Gamma_p=\{K\in\Gamma\colon |K|=p\}$$
$$\Lambda_p=\{z\in\Lambda\colon z=\sum_{K\in\Gamma_p} z_K\,\zeta^K,\ z_K\in\boldsymbol{R}\}$$

Then $\Lambda=\Lambda_0+\Lambda_1$ and $\Lambda_p\cdot\Lambda_q\subset\Lambda_{p+q}$ $(p, q\in\boldsymbol{Z}_2)$. If $z\in\Lambda_p$, then $|z|=p\in\boldsymbol{Z}_p$ is called the *parity* of $z\in\Lambda$. If the parity of $z\in\Lambda$ is 0 (1), then the super number z is said to be *even* (*odd*) or *commutative* (*anti-commutative*), resp. Since Λ_N $(N\geqq 0)$ is a real vector space of 2^N dimension, Λ_N has a natural topology, and hence the projective limit Λ has the projective limit topology, with which Λ is a Fréchet space and the projection p_N of Λ onto Λ_N is continuous and open for $N\geqq 0$.

REMARK. The notation, Λ_0 and Λ_1, does have some ambiguity: that is, Λ_0 (Λ_1) denotes the 0-th (1-st) skeleton of Λ and also the set of all even (odd) super numbers, resp. In general, there will be no confusion. However, in those cases where there may be some doubt, the set of all even (odd) super numbers will be denoted by Λ_c (Λ_a).

REMARK. In a later argument the following simple fact will be often applied: If z is in Λ and $z\cdot w=0$ for each odd super number w there, then $z=0$.

The super Euclidean space $\boldsymbol{R}^{m|n}$ of dimension $(m|n)$ is, by definition, the product space $(\Lambda_0)^m\times(\Lambda_1)^n$ where are m copies of $\Lambda_0=\Lambda_c$ and n copies of $\Lambda_1=\Lambda_a$. The projection p_N of Λ onto Λ_N induces the projection of $\boldsymbol{R}^{m|n}$ onto $\boldsymbol{R}_N^{m|n}$ which is, by definition, the product space $((\Lambda_0)_N)^m\times((\Lambda_1)_N)^n$ where $(\Lambda_p)_N=p_N(\Lambda_p)$ $(p\in\boldsymbol{Z}_2)$. The space $\boldsymbol{R}_N^{m|n}$ is called the *N-th skeleton* of the super Euclidean space $\boldsymbol{R}^{m|n}$, which is a $2^{N-1}(m+n)$ dimensional real vector space for $N\geqq 1$ and the ordinary m-dimensional real Euclidean space $\boldsymbol{R}^m$ for $N=0$ (the 0-th skeleton of the super Euclidean space $\boldsymbol{R}^{m|n}$ is called the *body* of $\boldsymbol{R}^{m|n}$). Moreover $\boldsymbol{R}^{m|n}$ can be regarded as the projective limit of the family $\{\boldsymbol{R}_N^{m|n}: n\geqq 0\}$ of finite dimensional real vector spaces. Then the super Euclidean space $\boldsymbol{R}^{m|n}$ is a Fréchet space and the projection p_N of $\boldsymbol{R}^{m|n}$ onto $\boldsymbol{R}_N^{m|n}$ is continuous and open for each $N\geqq 0$. The image $p_N(z)$ $(p_N(U))$ of $z\in\boldsymbol{R}^{m|n}$ (a subset U of $\boldsymbol{R}^{m|n}$) by the projection p_N is denoted by $z_N\in\boldsymbol{R}^{m|n}$ ($U_N\subset\boldsymbol{R}_N^{m|n}$, called the N-th skeleton of U), resp. On the super Euclidean space $\boldsymbol{R}^{m|n}$ sometimes we consider another topology, the *coarse topology*, with respect to which an open set in $\boldsymbol{R}^{m|n}$ is an inverse image of an open set in the body $\boldsymbol{R}^m$ by the projection p_B. An open set in $\boldsymbol{R}^{m|n}$ with respect to the coarse topology will be called a *C-open set*. Unless otherwise stated, we consider the Fréchet topology on $\boldsymbol{R}^{m|n}$.

The projection of $\boldsymbol{R}^{m|n}=(\Lambda_0)^m\times(\Lambda_1)^n$ onto the i-th component Λ_p $(p=0\ (1)$ if $1\leqq i\leqq m$ $(m+1\leqq i\leqq m+n)$, resp.) will be denoted by z^i (for $1\leqq i\leqq m+n$). For $1\leqq i\leqq m$ $(m+1\leqq i\leqq m+n)$, sometimes z^i will be denoted by x^μ (θ^p), resp. where $1\leqq\mu\leqq m$ and $1\leqq p\leqq n$. Thus as usual, each $z\in\boldsymbol{R}^{m|n}$ can be written as follows:

$$\begin{aligned} z &= (z^1, \cdots, z^{m+n}) = (z^i) \\ &= (x^1, \cdots, x^m, \theta^1, \cdots, \theta^n) = (x^\mu, \theta^p)\,. \end{aligned}$$

The *parity* $|i|$ of the coordinate index i is defined as follows: $|i|=0\ (1)$ if $1\leqq i\leqq m$ $(m+1\leqq i\leqq m+n)$. On the N-th skeleton $\boldsymbol{R}_N^{m|n}$ of $\boldsymbol{R}^{m|n}(N\geqq 0)$, we consider the following natural coordinate system $\{z_K^i: 1\leqq i\leqq m+n, K\in\Gamma_N, |K|=|i|\}$. For each $z=(z^i)\in\boldsymbol{R}^{m|n}$, the component z^i can be written as follows:

$$z^i = \sum_{K\in\Gamma_p} z_K^i\,\zeta^K \quad \text{where} \quad p = |i|\,.$$

Thus $z_N=(z_N^i)\in\boldsymbol{R}_N^{m|n}$ has the coordinate $\{z_K^i: 1\leqq i\leqq m+n, K\in\Gamma_N, |K|=|i|\}$. Formally $\{z_K^i: 1\leqq i\leqq m+n, K\in\Gamma, |K|=|i|\}$ can be considered as a natural coordinate system of $\boldsymbol{R}^{m|n}$.

Let $\phi(t)$ be a Λ-valued function defined on an interval I in $\boldsymbol{R}$. Then $\phi(t)$ can written as follows:

$$\phi(t) = \sum_{K\in\Gamma} \phi_K(t)\,\zeta^K$$

where $\phi_K(t)$ is a real valued function on I for each $K\in\Gamma$. We can prove the following with respect to the Fréchet topology:

$$\lim_{t\to a}\phi(t) = \sum_{K\in\Gamma}(\lim_{t\to a}\phi_K(t))\,\zeta^K\,.$$

Thus $\phi(t)$ is continuous with respect to the Fréchet topology if and only if $\phi_K(t)$ is continuous in the usual sense for each $K\in\Gamma$. The differentiation and integral of such $\phi(t)$ are defined as follows:

$$\phi'(t) = \sum_{K\in\Gamma} \phi'_K(t)\, \zeta^K$$

$$\int_a^b \phi(t)\, dt = \sum_{K\in\Gamma} \left(\int_a^b \phi_K(t)\, dt\right) \zeta^K$$

where $\phi(t)$ is written in the above form. We have the following.

Lemma 1.1. *Let $\phi(t)$ and $\Phi(t)$ be continuous Λ-valued functions on an interval $[a, b]$ in $\boldsymbol{R}$. Then*

1) $\int_a^b \phi(t)\, dt$ *exists,*

2) *if* $\Phi'(t) = \phi(t)$ *on* $[a, b]$ *then*
$$\int_a^b \phi(t)\, dt = \Phi(b) - \Phi(a),$$

3) *if* $c\in\Lambda$ *is constant then* $\int_a^b \phi(t)\cdot c\, dt = \left(\int_a^b \phi(t)\, dt\right)\cdot c$.

2. C^∞-functions on $\boldsymbol{R}^{m|n}$

A real-valued function f defined on an open set U in $\boldsymbol{R}^{m|n}$ is said to be *admissible* on U if there exists some $N\geqq 0$ and a real-valued function ϕ defined on the N-skeleton U_N of U such that $f=\phi\circ p_N$ on U. Since U_N is an open set of a finite dimensional Euclidean space $\boldsymbol{R}_N^{m|n}$, the notion of C^r-function on U_N is well-defined as usual. In the above case, f is said to be $C^r (0\leqq r\leqq\infty)$ if and only if ϕ is so. When f is $C^r (r\geqq 1)$, then the *partial derivative* $\dfrac{\partial f}{\partial z_L^i}$ is canonically well-defined for each $1\leqq i\leqq m+n$, $L\in\Gamma$ and $|L|=|i|$.

Let f be a Λ-valued function defined on an open set U in $\boldsymbol{R}^{m|n}$. Then f can be written in the following form:

$$f(z) = \sum_{K\in\Gamma} f_K(z)\, \zeta^K \quad (z\in U)$$

where f_K is a real-valued function on U. If each f_K $(K\in\Gamma)$ is admissible (C^r) on U, then f is said to be *admissible* (C^r) on U, respectively. When f is $C^r (r\geqq 1)$ on U, its partial derivatives are defined as follows:

$$\frac{\partial f}{\partial z_L^i} = \sum_{K\in\Gamma} \frac{\partial f_K}{\partial z_L^i}\, \zeta^K$$

where f is in the above form and $|i|=|L|$. A Λ-valued function f on U is said to be *projectable* if for each $N\geqq 0$ there exists a Λ_N-valued function f_N defined on U_N in $\boldsymbol{R}_N^{m|n}$ such that $p_N\circ f=f_N\circ p_N$ on U. f_N is called the *N-th projection* of f. Obviously a projectable function on U is admissible on U.

3. G^∞-functions on $\boldsymbol{R}^{m|n}$

Let f be a Λ-valued function on an open set U in $\boldsymbol{R}^{m|n}$. Then f is said to be G^1 on U if and only if there exist Λ-valued continuous functions F_i $(1\leqq i \leqq m+n)$ on U such that

$$\frac{d}{dt}f(z+th)\,|_{t=0}=\sum_{1\leqq i\leqq m+n}F_i(z)\cdot h^i$$

for each $z\in U$ and $h=(h^i)\in \boldsymbol{R}^{m|n}$ where $f(z+th)$ is considered as a Λ-valued function with one real variable $t\in\boldsymbol{R}$. If f is G^1 on U, then these F_i exist uniquely, and there also exist Λ-valued continuous functions G_i on U such that

$$\frac{d}{dt}f(z+th)\,|_{t=0}=\sum_{1\leqq i\leqq m+n}h^i\cdot G_i(z)$$

for each $z\in U$ and $h=(h^i)\in\boldsymbol{R}^{m|n}$. These F_i and G_i will be denoted as follows:

$$F_i(z)=f\frac{\overleftarrow{\partial}}{\partial z^i}(z)\,,\quad G_i(z)=\frac{\overrightarrow{\partial}}{\partial z^i}f(z)$$

These are called the *super partial derivatives* of f. For $r\geqq 2$, f is said to be G^r if all super partial derivatives are G^{r-1}. And f is siad to be G^∞ or *super smooth* if f is G^r for all $r\geqq 1$.

Theorem 1. *Let f be a Λ-valued function on a convex open set U in $\boldsymbol{R}^{m|n}$. If f is G^1 on U, then f is projectable and C^1 on U.*

Proof. Let z and $z+h$ be in U. Then $z+th$ is also in U for $0\leqq t\leqq 1$ since U is convex. We have the following for $0\leqq t\leqq 1$:

$$\frac{d}{dt}f(z+th)\,|_t=\sum_{1\leqq i\leqq m+n}F_i(z+th)\cdot h^i$$

Thus by Lemma 1.1, we have the following.

$$\begin{aligned}f(z+h)-f(z)&=\int_0^1\frac{d}{dt}f(z+th)\,|_t\,dt\\&=\sum_{1\leqq i\leqq m+n}\left(\int_0^1F_i(z+th)\,dt\right)\cdot h^i\end{aligned}$$

Therefore if z and w are in U and $z_N=w_N$, then we have $f(z)_N=f(w)_N$. This implies that f is projectable and in particular f is admissible on U. Let $\zeta^K_{(i)}$ be the element of $\boldsymbol{R}^{m|n}$ whose i-th component is ζ^K and whose other components are zero where $|i|=|K|$. Then we have

$$\frac{\partial}{\partial z^i_K}f(z)=\frac{d}{dt}f(z+t\,\zeta^K_{(i)})\,|_{t=0}=F_i(z)\cdot\zeta^K\,.$$

Thus the admissible function f is C^1 on U. This completes the proof.

4. Cauchy-Riemann equations

Let f be a Λ-valued C^∞ function on an open set U in $\boldsymbol{R}^{m|n}$ which is G^1 on U. Then as shown in the proof of Theorem 1, we obtain the following:

$$\frac{\partial}{\partial z_K^i} f(z) = f \frac{\overleftarrow{\partial}}{\partial z^i}(z) \cdot \zeta^K$$

where $1 \leqq i \leqq m+n$, $K \in \Gamma$ and $|i| = |K|$.
These equations imply the following:

$$\frac{\partial}{\partial x_K^\mu} f(z) = \frac{\partial}{\partial x_\phi^\mu} f(z) \cdot \zeta^K$$

$$\frac{\partial}{\partial \theta_L^p} f(z) \cdot \zeta^H + \frac{\partial}{\partial \theta_H^p} f(z) \cdot \zeta^L = 0$$

where $|K|=0$, $|L|=|H|=1$, $1 \leqq \mu \leqq m$ and $1 \leqq p \leqq n$. These equations are called the *Cauchy-Riemann equations.*

Theorem 2. *Let f be a Λ-valued C^∞ function on an open set U in $\boldsymbol{R}^{m|n}$. Then f is G^1 on U if and only if f satisfies the Cauchy-Riemann equations on U.*

Proof. We have already shown that a G^1 function satisfies the Cauchy-Riemann equations. Now suppose that f satisfies the Cauchy-Riemann equations on U. To show that f is G^1 on U, we have to construct continuous functions F_i $(1 \leqq i \leqq m+n)$ on U such that

$$\frac{d}{dt} f(z+th)\,|_{t=0} = \sum_{1 \leqq i \leqq m+n} F_i(z) \cdot h^i$$

for $z \in U$ and $h \in \boldsymbol{R}^{m|n}$. First for $1 \leqq \mu \leqq m$, let

$$F_\mu(z) = \frac{\partial}{\partial x_\phi^\mu} f(z) \quad (\text{for } z \in U)\,.$$

On the other hand $F_p(z)$ $(m+1 \leqq p \leqq m+n)$ can not be defined directly. Applying Lemma 4.4 below for $\{\frac{\partial}{\partial \theta_K^p} f(z) \colon K \in \Gamma_1\}$ while $m+1 \leqq p \leqq m+n$ and $z \in U$ are fixed, there exists $F_p(z)$ in Λ such that

$$\frac{\partial}{\partial \theta_K^p} f(z) = F_p(z) \cdot \zeta^K \quad \text{for each} \quad K \in \Gamma_1\,.$$

Now we shall prove that

$$\frac{d}{dt} f(z+th)\,|_{t=0} = \sum_{1 \leqq i \leqq m+n} F_i(z) \cdot h^i$$

for $z\in U$ and $h\in \boldsymbol{R}^{m|n}$. Since f is admissible, for any $N\geqq 0$ there exist some $L\geqq N$ and a Λ_N-valued C^∞ function f_L on U_L in $\boldsymbol{R}_L^{m|n}$ such that $f(z)_N=f_L(z_L)$ for each $z\in U$. Then we can show that if $K\in\Gamma_L$, $\frac{\partial}{\partial z_K^i}f_L(z_L)=(\frac{\partial}{\partial z_K^i}f(z))_N$ and otherwise $\frac{\partial}{\partial z_K^i}f_L(z_L)=0$.

$$\begin{aligned}
\left(\frac{d}{dt}f(z+th)|_{t=0}\right)_N &= \frac{d}{dt}(f(z+th))_N|_{t=0} \\
&= \frac{d}{dt}f_L(z_L+th_L)|_{t=0} \\
&= \sum_i \sum_K \frac{\partial}{\partial z_K^i}f_L(z_L)\cdot(h_L)_K^i \\
&= \sum_i \sum_K \left(\frac{\partial}{\partial z_K^i}f(z)\right)_N\cdot(h_L)_K^i \\
&= \sum_i \sum_K (F_i(z)\cdot\zeta^K)_N\cdot(h_L)_K^i \\
&= \sum_i \sum_K (F_i(z))_N\cdot(\zeta^K)_N\cdot(h_L)_K^i \\
&= \sum_i (F_i(z))_N\cdot\sum_K (\zeta^K)_N\cdot(h_L)_K^i \\
&= \sum_i (F_i(z))_N\cdot\sum_K (\zeta^K\cdot(h_L)_K^i)_N \\
&= \sum_i (F_i(z))_N\cdot(h_L)_N^i \\
&= \sum_i (F_i(z))_N\cdot(h_N)^i \\
&= (\sum_i F_i(z)\cdot h^i)_N \qquad \text{for any} \quad N\geqq 0\,.
\end{aligned}$$

Thus we have

$$\frac{d}{dt}f(z+th)|_{t=0}=\sum_{1\leqq i\leqq m+n}F_i(z)\cdot h^i$$

The continuity of $F_i(z)$ can be seen easily.

Lemma 4.4. *Let $\{\xi_Q\in\Lambda\colon Q\in\Gamma_1\}$ be a set of super numbers such that $\xi_Q\,\zeta^R+\xi_R\,\zeta^Q=0$ for any R and $Q\in\Gamma_1$. Then there exists a unique super number $\xi\in\Lambda$ such that $\xi_Q=\xi\cdot\zeta^Q$ for each $Q\in\Gamma_1$.*

Proof. First we consider only the family $\{\xi_{(i)}\colon i=1, 2, \cdots\}$. Let $\xi_{(i)}=\sum_{S\in\Gamma}a_S^i\,\zeta^S$. Since $\xi_{(i)}\,\zeta^i=0$, we have $\sum_{i\notin S\in\Gamma}a_S^i\,\zeta^S=0$. Therefore each $\xi_{(i)}$ can be written uniquely as follows: $\xi_{(i)}=(\sum_{i\notin S\in\Gamma}b_S^i\,\zeta^S)\,\zeta^i$ for some $b_S^i\in\boldsymbol{R}$. Then the condition that $\xi_{(i)}\,\zeta^j=\xi_{(j)}\,\zeta^i$ holds for any i and j implies that $b_S^i=b_S^j$ for $i,j\notin S\in\Gamma$. Letting $b_S=b_S^i$ for $i\notin S\in\Gamma$, $\xi=\sum_{S\in\Gamma}b_S\,\zeta^S$ is well-defined and furthermore $\xi_{(i)}=\xi\,\zeta^i$ holds for each i. This ξ satisfies the above condition for each $Q\in\Gamma_1$.

Theorem 3. *Let f be a Λ-valued C^∞ function on an open set U in $\boldsymbol{R}^{m|n}$. If f is G^1 on U, then f is G^∞ on U.*

Proof. Let f be G^1 on U. Then f satisfies the Cauchy-Riemann equations on U. Since f is C^∞ on U, $\frac{\partial}{\partial x^v_\Phi}f(z)$ also satisfies the same equations and hence $f\frac{\partial}{\partial x^v}(z)$ is G^1 on U. Similarly $\frac{\partial}{\partial \theta^q_J}f(z)=f\frac{\overleftarrow{\partial}}{\partial \theta^q}\cdot\zeta^J$ is also G^1 on U. Therefore the following equations hold for any $J\in\Gamma_1$:

$$\frac{\partial}{\partial x^\mu_K}\left(f\frac{\overleftarrow{\partial}}{\partial\theta^q}\right)(z)\cdot\zeta^J=\frac{\partial}{\partial x^\mu_\Phi}\left(f\frac{\overleftarrow{\partial}}{\partial\theta^q}\right)(z)\cdot\zeta^K\cdot\zeta^J$$

$$\left\{\frac{\partial}{\partial\theta^p_L}\left(f\frac{\overleftarrow{\partial}}{\partial\theta^q}\right)(z)\cdot\zeta^H+\frac{\partial}{\partial\theta^p_H}\left(f\frac{\overleftarrow{\partial}}{\partial\theta^q}\right)(z)\cdot\zeta^L\right\}\cdot\zeta^J=0$$

Thus $f\frac{\partial}{\partial\theta^q}(z)$ satisfies the Cauchy-Riemann equations on U and hence it is G^1 on U. Therefore $f(z)$ is G^2 on U. By induction it can be seen that $f(z)$ is G^∞ on U.

REMARK. This assertion seems to contradict the theorem in [1]. In our argument we take $\boldsymbol{R}^{m|n}$ as base space while their space in [1] is the N-th skeleton $\boldsymbol{R}^{m|n}_N$ of $\boldsymbol{R}^{m|n}$. By considering the projective limit $\boldsymbol{R}^{m|n}$of the family $\{\boldsymbol{R}^{m|n}_N\}$ the argument is rather simplified.

5. *Z*-expansion of G^∞ functions

First we consider G^∞ functions on $\boldsymbol{R}^{0|n}$: i.e., G^∞ functions with only odd variables.

Lemma 5.1. *Let f be a G^∞ function on an open set U in $\boldsymbol{R}^{0|n}$. Then f can be written as follows:*

$$f(\theta^1,\cdots,\theta^n)=\sum_P C_P\,\theta^P$$

where $P=(p_1,\cdots,p_k)$, $1\leq p_1<\cdots<p_k\leq n$, $\theta^P=\theta^{p_1}\cdots\theta^{p_k}$ and $C_P\in\Lambda$ is constant. In particular any G^∞ function on a connected open set in $\boldsymbol{R}^{0|n}$ can be extended to the whole space $\boldsymbol{R}^{0|n}$ uniquely.

Proof. First we consider the case where f has only one odd variable. Let f be a G^∞ function on an open set U in $\boldsymbol{R}^{0|1}$. Thus we have

$$\frac{\partial}{\partial\theta_L}f(\theta)=f\frac{\overleftarrow{d}}{d\theta}(\theta)\cdot\zeta^L\,.$$

Hence

$$\frac{\partial}{\partial\,\theta_H}\frac{\partial}{\partial\,\theta_L}f(\theta)=\Big(f\frac{\overleftarrow{d}}{d\theta}\frac{\overleftarrow{d}}{d\theta}\Big)(\theta)\cdot\zeta^L\cdot\zeta^H\,.$$

Since $|H|=|L|=1$, we have

$$\frac{\partial}{\partial\,\theta_H}\frac{\partial}{\partial\,\theta_L}f(\theta)=-\frac{\partial}{\partial\,\theta_L}\frac{\partial}{\partial\,\theta_H}f(\theta)\,.$$

Thus we have

$$\frac{\partial}{\partial\,\theta_H}\frac{\partial}{\partial\,\theta_L}f(\theta)=0\quad\text{for any}\quad H,L\in\Gamma_1\,.$$

This implies that each component $f_K(\theta)$ of $f(\theta)=\sum_K f_K(\theta)\cdot\zeta^K$ is a polynomial of degree 1 with variables $\{\theta_L\colon L\in\Gamma_K\}$. Then $f\frac{\overleftarrow{d}}{d\theta}(\theta)\cdot\zeta^K=\frac{\partial}{\partial\,\theta_K}f(\theta)$ is constant for any $K\in\Gamma_1$. Thus $f\frac{\overleftarrow{d}}{d\theta}(\theta)$ is constant, denoted by $a\in\Lambda$. Then $(f(\theta)-a\theta)\frac{\overleftarrow{d}}{d\theta}=0$. This shows that $f(\theta)=a\theta+b$ for some $b\in\Lambda$. Now let f be a G^∞ function on an open set U in $\boldsymbol{R}^{0|n}$. While $\theta^1,\cdots,\theta^{n-1}$ are fixed, $f(\theta^1,\cdots,\theta^{n-1},\theta^n)$ is a G^∞ function with one odd variable θ^n. Thus we have the following:

$$\begin{aligned}&f(\theta^1,\cdots,\theta^n)=g(\theta^1,\cdots,\theta^{n-1})\cdot\theta^n+h(\theta^1,\cdots,\theta^{n-1})\\&f\frac{\overleftarrow{\partial}}{\partial\,\theta^n}(\theta^1,\cdots,\theta^n)=g(\theta^1,\cdots,\theta^{n-1})\,.\end{aligned}$$

Therefore g is G^∞ on $\theta^1,\cdots,\theta^{n-1}$ and hence h is also G^∞ on $\theta^1,\cdots,\theta^{n-1}$. By induction we can show that $f(\theta^1,\cdots,\theta^n)$ can be written in the above form.

Now we consider G^∞ functions on $\boldsymbol{R}^{m|0}$: i.e., G^∞ functions with only even variables.

Lemma 5.2. *Let f be a G^∞ function on a C-open set U in $\boldsymbol{R}^{m|0}$ which vanishes identically on the body U_B of U. Then f vanishes identically on U.*

Proof. Clearly it is sufficient to consider the case of only one even variable. Let f be a G^∞ function on a C-open set U in $\boldsymbol{R}^{1|0}$. Let t be an arbitrary point in U_B. We shall consider the behavior of f on $p_B^{-1}(t)$. Let $z\in p_B^{-1}(t)$ and $z_N=p_N(z)$. Then $\{x_K\colon \phi\neq K\in\Gamma_N,\ |K|=0\}$ is a coordinate for $(p_B^{-1}(t))_N$ as the ordinary Euclidean space. Let f_N be the N-th projection of f. Then $\frac{\partial}{\partial\,x_K}f_N(z_N)=\frac{\partial}{\partial\,x_\phi}f_N(z_N)\cdot\zeta^K$ for $K\in\Gamma_N$ and $|K|=0$. If $K_1,\cdots,K_h\in\Gamma_N$, $|K_j|=0$ and none of them is ϕ and $2h>N$, then $\zeta^{K_1}\cdots\zeta^{K_h}=0$ and $\frac{\partial}{\partial\,x_{K_1}}\cdots\frac{\partial}{\partial\,x_{K_h}}f_N(z_N)=0$.

This implies that f_N is a polynomial on $(p_B^{-1}(t))_N$. Moreover for any $h\geqq 0$,

$$\frac{\partial}{\partial x_{K_1}}\cdots\frac{\partial}{\partial x_{K_h}}f_N(t)=\left(\frac{\partial}{\partial x_\phi}\right)^h f_N(t)\cdot\zeta^{K_1}\cdots\zeta^{K_h}\,.$$

Since f vanishes identically on the body U_B, we have

$$\left(\frac{\partial}{\partial x_\phi}\right)^h f=0 \quad \text{on} \quad U_B$$

and hence

$$\frac{\partial}{\partial x_{K_1}}\cdots\frac{\partial}{\partial x_{K_h}}f_N(t)=0 \qquad \text{for any} \quad h\geqq 0 \quad \text{and}$$
$$\phi\neq K_1,\cdots,K_h\in\Gamma_N,\ |K_j|=0\,.$$

Thus the polynomial $f_N|(p_B^{-1}(t))_N$ must vanish identically and hence $f_N\equiv 0$ on U_N. This holds for any $N\geqq 0$. Thus $f\equiv 0$ on U.

Let U be a C-open set in $\boldsymbol{R}^{m|o}$ and ϕ a Λ-valued C^∞ function on the body U_B of U. Then the so-called Z-expansion $\tilde{\phi}$ of ϕ is defined as follows: For any $x=(x^\mu)\in U$,

$$\tilde{\phi}(x)=\sum_I\frac{1}{I!}(D_t^I\phi)(x_B)\cdot x_S^I$$

where $x_B=(x_B^\mu)$ is the body of x and $x_S=x-x_B$ and $I=(i_1,\cdots,i_m)$ and $x_S^I=(x_S^1)^{i_1}\cdots(x_S^m)^{i_m}$ and $I!=i_1!\cdots i_m!$ and $D_t^I=\left(\frac{\partial}{\partial t^1}\right)^{i_1}\cdots\left(\frac{\partial}{\partial t^m}\right)^{i_m}$ and $t=(t^\mu)$ denotes the canonical coordinate on the body $\boldsymbol{R}^m$ of $\boldsymbol{R}^{m|n}$.

Lemma 5.3. *Let U be a C-open set of $\boldsymbol{R}^{m|o}$ and $\phi(t)$ a Λ-valued C^∞ function on the body U_B of U. Then the Z-expansion $\tilde{\phi}(x)$ is G^∞ on U. Moreover for $1\leqq\mu\leqq m$*

$$\tilde{\phi}\overleftarrow{\frac{\partial}{\partial x^\mu}}(x)=\overrightarrow{\frac{\partial}{\partial x^\mu}}\tilde{\phi}(x)=\widetilde{\left(\frac{\partial}{\partial t^\mu}\phi\right)}(x)\,.$$

Proof.

$$\begin{aligned}\frac{d}{dt}\tilde{\phi}(x+th)|_{t=0}&=\frac{d}{dt}\sum_I\frac{1}{I!}(D_t^I\phi)(x_B+th_B)\cdot(x_S+th_S)^I|_{t=0}\\&=\sum_{I,\mu}\frac{1}{I!}\left(D_t^I\left(\frac{\partial}{\partial t^\mu}\right)\phi\right)(x_B)\cdot h_B^\mu\cdot x_S^I\\&\quad+\sum_{I,\mu}\frac{1}{I!}(D_t^I\phi)(x_B)\cdot i_\mu\cdot h_S^\mu\cdot x_S^{I-(\mu)}\\&=\sum_{I,\mu}\frac{1}{I!}\left(D_t^I\left(\frac{\partial}{\partial t^\mu}\right)\phi\right)(x_B)\cdot x_S^I\cdot(h_B^\mu+h_S^\mu)\end{aligned}$$

$$= \sum_{\mu} \sum_{I} \frac{1}{I!} (D_t^I \left(\frac{\partial}{\partial t^\mu} \right) \phi) (x_B) \cdot x_S^I \cdot h^\mu .$$

Therefore we have

$$\tilde{\phi} \frac{\overleftarrow{\partial}}{\partial x^\mu}(x) = \sum_{I} \frac{1}{I!} (D_t^I \left(\frac{\partial}{\partial t^\mu} \right) \phi) (x_B) \cdot x_S^I .$$

Hence $\tilde{\phi}(x)$ is G^∞ on U.

Combining the above lemmas, we obtain the following theorem.

Theorem 4. *Let f be a G^∞ function on a C-open set U in $\boldsymbol{R}^{m|n}$. Then there exist Λ-valued C^∞ functions $\phi_P(P=(p_1, \cdots, p_k),\ 1 \leqq p_1 < \cdots < p_k \leqq n)$ on U_B such that*

$$f(z) = f(x, \theta) = \sum_P \tilde{\phi}_P(x) \cdot \theta^P$$

where $z=(x, \theta) \in U$. Moreover this expression is unique.

6. The linear groups

Regarding $\boldsymbol{R}^{m|n}$ as a real vector space, a linear mapping Φ of $\boldsymbol{R}^{m|n}$ into $\boldsymbol{R}^{\bar{m}|\bar{n}}$ is said to be *projectable* if for each $N \geqq 0$, there exists a linear mapping Φ_N of $\boldsymbol{R}_N^{m|n}$ into $\boldsymbol{R}_N^{\bar{m}|\bar{n}}$ such that $\Phi_N \circ p_N = p_N \circ \Phi$ on $\boldsymbol{R}^{m|n}$. In this case Φ_N is called the *N-th projection* of Φ. Let $\mathcal{M}(\bar{m}|\bar{n}, m|n)$ denote the set of all projectable linear mappings of $\boldsymbol{R}^{m|n}$ into $\boldsymbol{R}^{\bar{m}|\bar{n}}$ and $\mathcal{M}_N(\bar{m}|\bar{n}, m|n)$ the set of all N-th projections Φ_N of $\Phi \in \mathcal{M}(\bar{m}|\bar{n}, m|n)$. Then $\mathcal{M}(\bar{m}|\bar{n}, m|n)$ can be identified with the projective limit of the family $\{\mathcal{M}_N(\bar{m}|\bar{n}, m|n)\}$ in a canonical way. In case of $m=\bar{m}$ and $n=\bar{n}$, these sets will be denoted by $\mathcal{M}(m|n)$ and $\mathcal{M}_N(m|n)$ respectively, which form algebras over $\boldsymbol{R}$. The set of all invertible elements of $\mathcal{M}(m|n)$ (resp. $\mathcal{M}_N(m|n)$) is denoted by $\mathcal{Gl}(m|n)$ (resp. $\mathcal{Gl}_N(m|n)$). Then $\mathcal{Gl}(m|n)$ and $\mathcal{Gl}_N(m|n)$ are groups, and moreover $\mathcal{Gl}_N(m|n)$ is a closed subgroup of the general linear group $GL(\boldsymbol{R}_N^{m|n})$ over the vector space $\boldsymbol{R}_N^{m|n}$. Furthermore $\mathcal{Gl}(m|n)$ can be identified with the projective limit of the family $\{\mathcal{Gl}_N(m|n)\}$, which is called the *general linear group* over the vector space $\boldsymbol{R}^{m|n}$.

Lemma 6.1. *Let Φ be in $\mathcal{M}(m|n)$. Then Φ is in $\mathcal{gl}(m|n)$ if and only if Φ_N is in $\mathcal{gl}_N(m|n)$ for all $N \geqq 0$.*

Proof. Suppose that Φ_N is in $\mathcal{gl}_N(m|n)$ for any $N \geqq 0$. Let $z \in \boldsymbol{R}^{m|n}$ be in the kernel of Φ. Then $\Phi_N(z_N) = (\Phi(z))_N = 0$ for any $N \geqq 0$. Thus $z_N = 0$ for any $N \geqq 0$ and hence $z=0$. This shows that Φ is injective. Let w be an arbitrary element in $\boldsymbol{R}^{m|n}$. Then for each $N \geqq 0$, there exists a unique element $z_N \in \boldsymbol{R}_N^{m|n}$ such that $\Phi_N(z_N) = w_N$. Then $\{z_N\}$ defines an element $z \in \boldsymbol{R}^{m|n}$ such that $z_N = p_N(z)$ for each $N \geqq 0$ and moreover $\Phi(z) = w$. Thus Φ is surjective. The converse is obvious.

Now we regard $\boldsymbol{R}^{m|n}$ as the set of all $(m+n)$-column vectors whose first m components are even super numbers and last n components are odd super numbers. We denote by $M(\bar{m}|\bar{n}, m|n)$ the set of all $(\bar{m}+\bar{n}, m+n)$ matrices of the following form.

$$\phi=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A, B, C and D are $(\bar{m}, m)$, $(\bar{m}, n)$, $(\bar{n}, m)$ and $(\bar{n}, n)$ matrices, respectively and components of A and D are even and components of B and C are odd. For each $\phi\in M(\bar{m}|\bar{n}, m|n)$ and $N\geqq 0$, we define a matrix $\phi_{(N)}$ by

$$\phi_{(N)}=\begin{pmatrix} A_N & B_N \\ C_N & D_N \end{pmatrix}$$

where A_N, B_N, C_N and D_N denote the matiices whose components are the N-th projections of those of A, B, C and D. When $N=0$, $\phi_{(0)}$ will be also denoted by ϕ_B, called the body of ϕ. For $p\in\boldsymbol{Z}_2$ and $N\geqq 0$, we define $(\Lambda_p)'_N$ as follows;

$$(\Lambda_p)'_N=\begin{cases}(\Lambda_p)_N & \text{if } p\not\equiv N \bmod 2 \\ (\Lambda_p)_N/\boldsymbol{R}\cdot\chi_N & \text{if } p\equiv N \bmod 2\end{cases}$$

where $\chi_N=\zeta^1\cdots\zeta^N\in\Lambda_N$.
Now for each $\phi\in M(\bar{m}|\bar{n}, m|n)$ and $N\geqq 0$, we define a matrix ϕ_N by

$$\phi_N=\begin{pmatrix} A_N & (B_N)' \\ C_N & (D_N)' \end{pmatrix}$$

where $(B)'$ and $(D_N)'$ denote the matrices whose components are the projections of those of B_N and D_N into $(\Lambda_1)'_N$ and $(\Lambda_0)'_N$, resp. We define $M_{(N)}(\bar{m}|\bar{n}, m|n)$ and $M_N(\bar{m}|\bar{n}, m|n)$ by

$$M_{(N)}(\bar{m}|\bar{n}, m|n)=\{\phi_{(N)}\colon \phi\in M(\bar{m}|\bar{n}, m|n)\}$$
$$M_N(\bar{m}|\bar{n}, m|n)=\{\phi_N\colon \phi\in M(\bar{m}|\bar{n}, m|n)\}\ .$$

Then in a natural way $M_N(\bar{m}|\bar{n}, m|n)$ acts on $\boldsymbol{R}_N^{m|n}$ on the left hand side effectively. Moreover we have the following commutative diagram:

$$\begin{array}{ccccc} & M(\bar{m}|\bar{n}, m|n) & \xrightarrow{i} & \mathscr{M}(\bar{m}|\bar{n}, m|n) & \\ p_{(N)}\swarrow & & \searrow p_N & \downarrow p_N & \\ M_{(N)}(\bar{m}|\bar{n}, m|n) & \xrightarrow{\pi} & M_N(m|\bar{n}, \underline{u\!u}|n) & \xrightarrow{i} & \mathscr{M}_N(\bar{m}|\bar{n}, m|n) \end{array}$$

where both i are natuarl imbeddings. In particular when $N=1$ or 0 and ϕ

is a matrix in the above form,

$$\phi_1 = \begin{pmatrix} A_B & 0 \\ C_1 & D_B \end{pmatrix} \quad \text{and} \quad \phi_B = \begin{pmatrix} A_B & 0 \\ 0 & D_B \end{pmatrix}$$

where A_B and D_B are the bodies of matrices A and D.
By Theorem 2, the characterization of $M(\bar{m}|\bar{n}, m|n)$ in $\mathcal{M}(\bar{m}|\bar{n}, m|n)$ is given as follows.

Lemma 6.2. *Let Φ be in $\mathcal{M}(\bar{m}|\bar{n}, m|n)$. Then Φ is in $M(\bar{m}|\bar{n}, m|n)$ if and only if Φ satisfies the Cauchy-Riemann equations.*

When $\bar{m}=m$ and $\bar{n}=n$, $M(m|n, m|n)$ will be denote by $M(m|n)$. Then $M(m|n)$ forms a subalgebra of $\mathcal{M}(m|n)$. The set of all invertible elements of $M(m|n)$ is denoted by $GL(m|n)$, which forms a subgroup of $\mathcal{gl}(m|n)$ and is called the *super general linear group* over $\boldsymbol{R}^{m|n}$.

Lemma 6.3. *Let ϕ be in $M(m|n)$. Then ϕ is in $GL(m|n)$ if and only if $det(A_B) \neq 0$ and $det(D_B) \neq 0$ where*

$$\phi = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Moreover we have $GL(m|n) = M(m|n) \cap \mathcal{gl}(m|n)$.

Proof. Suppose that $\det(A_B) \neq 0$ and $\det(D_B) \neq 0$. Then the inverse of ϕ is gievn by

$$\phi^{-1} = \phi_B^{-1} \sum_k (-(\phi - \phi_B)\, \phi_B^{-1})^k .$$

Conversely if ϕ is in $GL(m|n)$ then ϕ_B is in $GL(m+n, \boldsymbol{R})$ and $\det(A_B) \neq 0$ and $\det(D_B) \neq 0$. Now let $\phi \in M(m|n) \subset \mathcal{gl}(m|n)$. Then regarding ϕ as a projectable $\boldsymbol{R}$ linear endomorphism of $\boldsymbol{R}^{m|n}$, the 1st projecton projection ϕ_1 of ϕ is in $\mathcal{Gl}_1(m|n)$. On the other hand, ϕ_1 is given by

$$\phi_1 = \begin{pmatrix} A_B & 0 \\ C_1 & D_B \end{pmatrix}.$$

Therefore $\det(A_B) \neq 0$ and $\det(D_B) \neq 0$ and hence ϕ is in $GL(m|n)$.

7. Jacobi matrixes

Let f be a projectable C^∞ mapping of an open set U in $\boldsymbol{R}^{m|n}$ into $\boldsymbol{R}^{\bar{m}|\bar{n}}$. Then the *Jacobi matrix* $\mathcal{J}f(z) \in \mathcal{M}(\bar{m}|\bar{n}, m|n)$ of f at $z \in U$ is defined as follows. For any $h \in \boldsymbol{R}^{m|n}$,

$$\mathscr{J}f(z)(h)=\frac{d}{dt}f(z+th)|_{t=0}.$$

With respect to the natural coordinate systems of $\boldsymbol{R}^{m|n}$ and $\boldsymbol{R}^{\bar{m}|\bar{n}}$, the Jacobi matrix $\mathscr{J}f(z)$ can expressed by

$$(\mathscr{J}f(z)(h))^i_K=\sum_{j,L}\left(\frac{\partial}{\partial z^j_L}f^i_K\right)(z)\cdot h^j_L\,.$$

Thus formally $\mathscr{J}f(z)$ can be regarded as the matrix $\left(\frac{\partial}{\partial z^i_K}f^j_L\right)(z)$.

Let f be a G^∞ mapping of an open set U in $\boldsymbol{R}^{m|n}$ into $\boldsymbol{R}^{\bar{m}|\bar{n}}$. Then the *super Jacobi matrix* $Jf(z)\in M(\bar{m}|\bar{n}, m|n)$ of f at $z\in U$ is defined as follows.

$$Jf(z)=\left(f^i\frac{\overleftarrow{\partial}}{\partial z^j}(z)\right).$$

With the canonical inclusion of $M(\bar{m}|\bar{n}, m|n)$ into $\mathscr{M}(\bar{m}|\bar{n}, m|n)$, the super Jacobi matrix $Jf(z)$ and the Jacobi matrix $\mathscr{J}f(z)$ coincides for a G^∞ mapping f of U into $\boldsymbol{R}^{m|n}$. Applying Theorem 2 and Theorem 3, we can show that if f is a projectable C^∞ mapping of an open set U in $\boldsymbol{R}^{m|n}$ into $\boldsymbol{R}^{m|n}$ whose Jacobi matrix $\mathscr{J}f(z)$ is in $M(\bar{m}|\bar{n}, m|n)$ for each $z\in U$, then f is G^∞ on U.

Theorem 5. *Let f be a G^∞ mapping of a C-open set U in $\boldsymbol{R}^{m|n}$ into $\boldsymbol{R}^{m|n}$ whose super Jacobi matrix $Jf(z)$ at a point z in U is invertible. Then we can find a C-open neighborhood V of z and a C-open neighborhood W of $f(z)$ such that the restriction of f to V is a one-to-one mapping of V onto W and its inverse mapping is also G^∞ on W.*

Proof. Let f_N be the N-th projection of f. Considering $\boldsymbol{R}^{m|n}_N$ as a vector bundle over $\boldsymbol{R}^{m|n}_{N-1}$, f_N is a fibre-preserving mapping over a base-mapping f_{N-1} and moreover the restriction of f_N to each fibre is an affine mapping of a vector space. In the case of $N=0$, the Jacobi matrix of the 0-th projection f_0 of f is the 0-th projection of the super Jacobi matrix $\mathscr{J}f(z)$ of f, which is the body A_B of the matrix A where $\mathscr{J}f(z)=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in GL(m|n)$. Therefore there exists an open neighborhood $V_B\subset\boldsymbol{R}^m$ of z_B and an open neighborhood $W_B\subset\boldsymbol{R}^m$ of $(f(z))_B$ such that the 0-th projection f_0 of f is a bijective mapping of V_B onto W_B and the inverse mapping $(f_0)^{-1}$ of f is C^∞ on W_B. Let V and W be the inverse images of V_B and W_B by the projection p_B of $\boldsymbol{R}^{m|n}$ onto $\boldsymbol{R}^m$. We shall show by induction that the N-th projection f_N of f is a bijection of the N-th skeleton V_N onto the N-th skeleton W_N and its inverse mapping is C^∞ on W_N. In the case of $N=0$, this has been shown already. Now suppose that the assertion holds in the case of $N-1$. The Jacobi matrix of f_N at z_N is invertible and the restriction of

f_N to each fibre is an affine mapping when $\boldsymbol{R}_N^{m|n}$ is regarded as a vector bundle over $\boldsymbol{R}_{N-1}^{m|n}$. On the other hand f_{N-1} is a diffeomorphism of V_{N-1} onto W_{N-1}. Thus f_N is a diffeomorphism of V_N onto W_N. Therefore there exists a C^∞ inverse mapping f^{-1} of f which maps W onto V. Moreover $\mathscr{J}f\cdot\mathscr{J}(f^{-1})$=identity of $\boldsymbol{R}^{m|n}$ and hence $\mathscr{J}(f^{-1})(w)$ is $GL(m|n)$. Therefore f^{-1} is also G^∞. This completes the proof.

REMARK. The inverse mapping theorem does not hold for a projectable mapping nor an admissible mapping of $\boldsymbol{R}^{m|n}$.

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Kyoto Pref. University of Medicine
Kyoto 603, Japan