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# ON THE PLANCHEREL FORMULAS FOR SOME TYPES OF SIMPLE LIE GROUPS

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The problem of finding the explicit Plancherel formulas for semisimple Lie groups has been solved completely in the case of complex semisimple Lie groups (see [3 (b)]). Moreover Harish-Chandra showed [3 (f)] that the problem is solved also for a real semisimple Lie group having only one conjugate class of Cartan subgroups. In the case of real semisimple Lie groups with several conjugate classes of Cartan subgroups, the problem is very difficult to attack. As far as the auther knows, the problem was taken up and solved for $\boldsymbol{SL}(2, \boldsymbol{R})$ by V. Bargman, [1], Harish-Chandra [3 (a)], R. Takahashi [9 (a)] and L. Pukánszky [7]; also for the universal covering group of $\boldsymbol{SL}(2, \boldsymbol{R})$ by L. Pukánszky. In the previous note [6], we gave a method of finding the Plancherel formula for the universal covering group of De Sitter group. The purpose of this paper is to generalize this method and to obtain the explicit Plancherel formulas for simple Lie groups $G$ which satisfy the following conditions (A. 1)$\sim$(A. 5).

(A. 1) There exists a simply connected complex simple analytic group $G^C$ containing $G$ as a real analytic subgroup corresponding to a real form of the Lie algebra of $G^C$.

(A. 2) $G$ has a compact Cartan subgroup.

(A. 3) $G$ has two conjugate classes of Cartan subgroups.

(A. 4) Every Cartan subgroup of $G$ is connected (c.f. Proposition 7).

(A. 5) Let $T_\Lambda$ be the invariant distribution defined by the formula (3. 8) in § 3. Then there exists a finite number of irreducible unitary representations $\omega_\Lambda^{(1)}, \cdots, \omega_\Lambda^{(s)}$ of $G$ such that the character of the representation $\omega_\Lambda^{(1)}\oplus\cdots\oplus\omega_\Lambda^{(s)}$ coincides with the distribution $T_\Lambda$ (c.f. Remark 1).

The Plancherel formula for such a group $G$, which is our main result, will be given in Theorem 2 in § 4 and Theorem 2′ in § 5.

We shall see that the assumptions (A. 1)$\sim$(A. 5) are satisfied by the universal covering group of De Sitter group. We shall thus obtain the explicit Plancherel formula for this group in Theorem 3 in §6. This formula was conjectured by R. Takahashi in [9 (b)].

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## 1. Preliminaries

Let $G$ be a simple Lie group which satisfies the conditions (A. 1), (A. 2) and (A. 3). We denote by $\mathfrak{g}$, $\mathfrak{g}^C$ the Lie algebras of $G$, $G^C$ respectively. Let $K$ be a connected maximal compact subgroup of $G$ and $\mathfrak{k}$ its Lie algebra. We put

$$\mathfrak{p} = \{X \in \mathfrak{g} : B(X, Y) = 0 \quad \text{for all} \quad Y \in \mathfrak{k}\},$$

where $B$ denotes the Killing form of the Lie algebra $\mathfrak{g}^C$. We have then

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}, \quad \mathfrak{k} \cap \mathfrak{p} = (0), \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}.$$

We take a maximal connected abelian subgroup $A_1$ of $K$ and fix it once for all and let $\mathfrak{h}_1$ denote its Lie algebra. Then from (A. 1) and (A. 2) $A_1$ is a Cartan subgroup of $G$ corresponding to $\mathfrak{h}_1$; i.e.

$$A_1 = \{g \in G : \mathrm{Ad}(g)H = H \quad \text{for all} \quad H \in \mathfrak{h}_1\},$$

where Ad denotes the adjoint representation of $G$. For any subspace $\mathfrak{l}$ of $\mathfrak{g}$, we denote its complexification by $\mathfrak{l}^C$. Let $\Sigma_1$ denote the set of all non zero roots of $\mathfrak{g}^C$ with respect to $\mathfrak{h}_1^C$. Let $\sigma$ be the conjugation of $\mathfrak{g}^C$ with respect to $\mathfrak{g}$ and let ${}^t\sigma$ be the linear transformation on the dual space $\hat{\mathfrak{h}}_1^C$ of $\mathfrak{h}_1^C$ defined by

$$({}^t\sigma\Lambda)(H) = \overline{\Lambda(\sigma H)} \qquad (H \in \mathfrak{h}_1^C)$$

for any $\Lambda \in \hat{\mathfrak{h}}_1^C$.

Since ${}^t\sigma$ induces a substitution of roots, there exists a complex number $\kappa_\alpha$ for each $\alpha \in \Sigma_1$ such that

$$\sigma E_\alpha = \kappa_\alpha E_{{}^t\sigma\alpha}.$$

It is well known that we can find a basis $\{E_\alpha; \alpha \in \Sigma_1\}$ of $\mathfrak{p}^C \bmod \mathfrak{h}_1^C$ satisfying the following conditions;

$$[H, E_\alpha] = \alpha(H)E_\alpha \quad \text{for all} \quad H \in \mathfrak{h}_1^C, \tag{1.1}$$

$$B(E_\alpha, E_{-\alpha}) = -1, \tag{1.2}$$

(1.3) $$N_{\alpha,\beta} = N_{-\alpha,-\beta} \quad \text{(real number)},$$

(1.4) $$|\kappa_\alpha| = 1.$$

Since $[\mathfrak{h}_1^C, \mathfrak{k}^C] \subset \mathfrak{k}^C$ and $[\mathfrak{h}_1^C, \mathfrak{p}^C] \subset \mathfrak{p}^C$, it is clear that either $E_\alpha \in \mathfrak{k}^C$ or $E_\alpha \in \mathfrak{p}^C$. A root $\alpha \in \Sigma_1$ is called compact or non compact according to $E_\alpha \in \mathfrak{k}^C$ or $E_\alpha \in \mathfrak{p}^C$.

For any $\alpha \in \Sigma_1$, let $H_\alpha$ denote the unique element in $\mathfrak{h}_1^C$ such that

$$B(H_\alpha, H) = \alpha(H) \quad \text{for all} \quad H \in \mathfrak{h}_1^C.$$

Put

(1.5) $$U_\alpha = \sqrt{2}(E_\alpha + E_{-\alpha})/(\alpha(H_\alpha))^{1/2},$$

(1.6) $$V_\alpha = \sqrt{-1}\sqrt{2}(E_\alpha - E_{-\alpha})/(\alpha(H_\alpha))^{1/2}.$$

Fix a non compact root $\alpha_0$ in $\Sigma_1$ once for all. Then from (A.2) and (A.3), $\mathfrak{a} = \sqrt{-1}\boldsymbol{R}U_{\alpha_0}$ is a maximal abelian subalgebra in $\mathfrak{p}$ (see [8]).

We consider the automorphism $\nu = \exp\{(\pi/4)\,\mathrm{ad}\, V_{\alpha_0}\}$ of $\mathfrak{g}^C$ where ad denotes the adjoint representation of $\mathfrak{g}^C$. Then we have

(1.7) $$\nu(\sqrt{-1})U_{\alpha_0} = H_0, \quad \nu(H_0) = -\sqrt{-1}U_{\alpha_0},$$

where $H_0 = \dfrac{2}{\alpha_0(H_{\alpha_0})} H_{\alpha_0}$.

Put $\mathfrak{h}^- = \{H \in \mathfrak{h}_1 : [H, X] = 0 \text{ for all } X \in \mathfrak{a}\}$. Then $\mathfrak{h}_2 = \mathfrak{a} + \mathfrak{h}^-$ is a Cartan subalgebra of $\mathfrak{g}$ which is not conjugate to $\mathfrak{h}_1$ (see [8]). From the above assumption (A.3), every Cartan subalgebra of $\mathfrak{g}$ is conjugate to either $\mathfrak{h}_1$ or $\mathfrak{h}_2$. It is easy to see that

(1.8) $$\nu(\mathfrak{h}_2^C) = \mathfrak{h}_1^C \quad \text{and} \quad \nu(H) = H \quad \text{for all} \quad H \in \mathfrak{h}^-.$$

For any $\Lambda \in \hat{\mathfrak{h}}_1^C$, let $\tilde{\Lambda}$ denote the linear form on $\mathfrak{h}_2^C$ defined by

(1.9) $$\tilde{\Lambda}(H) = \Lambda(\nu H) \qquad (H \in \mathfrak{h}_2^C).$$

Put $\Sigma_2 = \{\tilde{\alpha} : \alpha \in \Sigma_1\}$. Since $\nu$ is an automorphism of $\mathfrak{g}^C$ it follows that $\Sigma_2$ is exactly the set of all non zero roots of $\mathfrak{g}^C$ with respect to $\mathfrak{h}_2^C$. Select compatible orderings in the dual spaces of $\mathfrak{a}$ and $\mathfrak{a} + \sqrt{-1}\mathfrak{h}^-$ and let $\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_l$ be all the simple roots in $\Sigma_2$ under this order. Since $\{\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_l\}$ is a fundamental root system of $\Sigma_2$, $\{\alpha_1, \alpha_2, \cdots, \alpha_l\}$ is a fundamental root system of $\Sigma_1$. Hence we can define an order in $\Sigma_1$ such that $\{\alpha_1, \alpha_2, \cdots, \alpha_l\}$ is exactly the set of all simple roots in this order. Moreover we may assume $\alpha_0 > 0$. We put $H_i = \dfrac{2}{\alpha_i(H_{\alpha_i})} H_{\alpha_i}$ $(i = 1, \cdots, l)$. Let $P_1$ (resp. $P_2$) be the set of all positive roots of $\Sigma_1$ (resp. $\Sigma_2$). We put

$$P_2^+ = \{\tilde{\alpha} \in P_2 : \tilde{\alpha}(\mathfrak{a}) \neq (0)\},$$
$$P_2^- = P_2 - P_2^+.$$

Then $P_2^-$ is the set of all compact positive roots in $\Sigma_2$. Let $\mathfrak{F}$ (resp. $\mathfrak{F}_0$) be the set of all integral (resp. dominant integral) forms on $\mathfrak{h}_1^C$. Then we have

$$\mathfrak{F} = \{\Lambda = \sum_{i=1}^{l} m_i\Lambda_i : m_i \in \boldsymbol{Z} \ (i=1, \cdots, l)\} ,$$

$$\mathfrak{F}_0 = \{\Lambda = \sum_{i=1}^{l} m_i\Lambda_i : m_i \geqslant 0, \ m_i \in \boldsymbol{Z} \ (i=1, \cdots, l)\} ,$$

where $\{\Lambda_1, \Lambda_2, \cdots, \Lambda_l\}$ is the dual basis of $\{H_1, H_2, \cdots, H_l\}$.

Since $A_1$ is a connected abelian Lie group, the mapping $H \rightarrow \exp H (H \in \mathfrak{h}_1)$ is a homomorphism of $\mathfrak{h}_1$ onto $A_1$. Let $\Lambda$ be a linear form on $\mathfrak{h}_1$ such that $e^{\Lambda(H)}=1$ for all $H \in \Gamma(\mathfrak{h}_1)$, where

$$\Gamma(\mathfrak{h}_1) = \{H \in \mathfrak{h}_1 : \exp H = e\} .$$

Then we can define a function $\xi_\Lambda$ on $A_1$ by

$$\xi_\Lambda(\exp H) = e^{\Lambda(H)} \qquad (H \in \mathfrak{h}_1) . \tag{1.10}$$

Moreover $\xi_\Lambda$ is uniquely extended to a holomorphic function on $A_1^C = \exp \mathfrak{h}_1^C$.

Although the following proposition is well known, it is fundamental in the present paper so that we shall give a proof of it.

**Proposition 1.** *Let $\hat{A}_1$ be the character group of $A_1$. Then $\hat{A}_1 = \{\xi_\Lambda : \Lambda \in \mathfrak{F}\}$.*

Proof. Put

$$\Gamma = \{2\pi\sqrt{-1} \sum_{i=1}^{l} m_i H_i : m_i \in \boldsymbol{Z} \ (i=1, 2, \cdots, l)\} .$$

Then, since $\mathfrak{h}_1/\Gamma(\mathfrak{h}_1) \cong A_1$, the proposition follows immediately if we prove $\Gamma = \Gamma(\mathfrak{h}_1)$.

Let $H \in \Gamma(\mathfrak{h}_1)$. Then $\exp H = e$, where $e$ is the identity of $G$. Since $\Lambda_i \, (i=1, \cdots, l)$ is a dominant integral form, there exists an irreducible finite dimentional representation $\tau_i$ of $\mathfrak{g}^C$ with the highest weight $\Lambda_i$. Since $G^C$ is simply connected, there exists a representation $\tilde{\tau}_i$ of $G^C$ such that $d\tilde{\tau}_i = \tau_i$. Let $u_i$ be a weight vector corresponding to $\Lambda_i$. Then we get

$$e^{\Lambda_i(H)} u_i = \exp(\tau_i H) u_i = \tilde{\tau}_i(\exp H) u_i = u_i \neq 0 .$$

Hence $\Lambda_i(H) \in 2\pi\sqrt{-1}\boldsymbol{Z}$ $(i=1, 2, \cdots, l)$. This means $H \in \Gamma$.

Conversely let $H \in \Gamma$. Then $H = 2\pi\sqrt{-1} \sum_{i=1}^{l} m_i H_i$ for some $m_i \in \boldsymbol{Z}$ $(i=1, 2, \cdots, l)$. Let $\tau$ be a faithful representation of $G^C$ on a finite dimensional vector space $V$. It is known that every weight $\Lambda$ of $\tau$ is an integral form on $\mathfrak{h}_1^C$ and that $V$ is the direct sum of eigenspaces $V_\Lambda$, $\Lambda$ being a weight of $\tau$. For any $u \in V_\Lambda$, we have

$$\begin{aligned}\tau(\exp H)u &= \exp\left(2\pi\sqrt{-1}\sum_{i=1}^{l} m_i d\tau(H_i)\right)u \\ &= e^{2\pi\sqrt{-1}\sum_{i=1}^{l} m_i \Lambda(H_i)}u\,.\end{aligned}$$

This is equal to $u$, since $\Lambda$ is an integral form. Hence $\tau(\exp H)$ is an identity transformation. Since $\tau$ is a faithful representation, it follows that $\exp H = e$. This implies $H \in \Gamma(\mathfrak{h}_1)$. Thus the proposition is proved.

## 2. Some results of Harish-Chandra

In this section we gather some results of Harish-Chandra which will be used in this paper.

In this section we assume that $G$ satisfies the conditions (A.1)$\sim$(A.4).

For any submanifold $U$ of $G$, let $C_c^\infty(U)$ denote the set of all complex valued $C^\infty$-functions on $U$ with the compact supports. Then for any $f \in C_c^\infty(G)$ and a fixed $g \in G$, the function $f^g: x \to f(gxg^{-1})$ $(x \in G)$ is again in $C_c^\infty(G)$, and if $T$ is a distribution on $G$, the mapping $T^g: f \to T(f^{g^{-1}})$ $(f \in C_c^\infty(G))$ is also a distribution. We say $T$ is invariant if $T^g = T$ for all $g \in G$. Let $\mathfrak{Z}$ be the algebra of all differential operators on $G$ which are invariant under both left and right translations. We denote by $D(x)$ the coefficitent of $t^l$ in $\det(t+1-\mathrm{Ad}(x))$ $(x \in G)$. Then $D$ is an analytic function on $G$ and an element $x \in G$ is called regular if $D(x) \neq 0$. Let $G'$ be the set of all regular elements in $G$. Then $G'$ is an open and dense subset of $G$ whose complement is of measure zero with respect to the Haar measure of $G$. For any subset $B$ of $G$, we define $B' = B \cap G'$. A distribution $T$ on an open submanifold $U$ of $G$ is called an eigendistribution of $\mathfrak{Z}$ on $U$ if it satisfies the equation $\Delta T = \chi(\Delta)T$ for any $\Delta \in \mathfrak{Z}$, where $\chi$ denotes a homomorphism of $\mathfrak{Z}$ into $\boldsymbol{C}$.

**Lemma 1.** *Let $T$ be an invariant eigendistribution of $\mathfrak{Z}$ on $G$. Then $T$ is a locally summable function (as distribution) which is analytic on $G'$.* (see [3 (g)]).

Since $A_k$ is connected by virtue of (A.4), we can define a continous function $\Delta_k$ on $A_k$ by

$$(2.1) \qquad \Delta_k(h) = \left|\prod_{\alpha \in P_k^+} \left(e^{\alpha(H)/2} - e^{-\alpha(H)/2}\right)\right| \prod_{\alpha \in P_k^0 \cup P_k^-} \left(e^{\alpha(H)/2} - e^{-\alpha(H)/2}\right),$$

for all $h = \exp H \in A_k$.

Where $P_1^0$ (resp. $P_1^-$) is the set of all non compact (resp. compact) roots in $\Sigma_2$ and $P_1^+ = P_2^0 = \phi$ (empty set). Let $x \to x^{(k)}$ $(x \in G)$ be the canonical projection of $G$ onto $G/A_k$ $(k=1, 2)$. For any $f\in C_c^\infty(G)$, we put

$$(2.2) \qquad F_f^{(k)}(h) = \overline{\Delta_k(h)} \int_{G/A_k} f(h^{x(k)})\,dx^{(k)} \qquad (h\in A_k'),$$

where $dx^{(k)}$ is the invariant measure on $G/A_k$ and $h^{x(k)}=xhx^{-1}(h\in A_k)$.

Let $S(\mathfrak{h}_k^C)$ be the universal enveloping algebra of $\mathfrak{h}_k^C$. Let $B$ be an open subset of $A_k$. We regard it as an open submanifold of $A_k$ and consider the space $\mathfrak{D}(B)$ of all complex valued functions $F$ on $B$ of class $C^\infty$ satisfying the following two conditions.

(1) The closure in $A_k$ of the support of $F$ is compact.

(2) For every $u\in S(\mathfrak{h}_k^C)$,

$$\tau_u(F) = \sup_{h\in B} |F(h\,;u)| < \infty \quad \text{where} \quad F(h\,;u) = (uF)(h).$$

Define a topology in $\mathfrak{D}(B)$ by means of the collection of seminorms $\tau_u\,(u\in S(\mathfrak{h}_k^C))$. Then $\mathfrak{D}(B)$ is a locally convex space and the same holds for $C_c^\infty(G)$ under its usual topology (introduced by Schwartz).

**Lemma 2.** *The mapping $f\to F_f^{(k)}$ is a continuous mapping of $C_c^\infty(G)$ into $\mathfrak{D}(A_k')$. Moreover, for any relatively compact open subset $U$ of $G$, there exists an open subset $B$ of $A_k'$ such that $\bar{B}$ is compact and that $F_f^{(k)}$ is zero outside $B$ for every $f\in C_c^\infty(U)$.*

For the proof, see [3 (f)].

Let $A_1''$ be the set of all points $h=\exp H\in A_1$ such that

$$\prod_{\alpha\in P_1^0} (e^{\alpha(H)/2}-e^{-\alpha(H)/2}) \neq 0.$$

**Lemma 3.** (1) *Let $B$ be any connected component of $A_1'$. Then $uF_f^{(1)}$ $(u\in S(\mathfrak{h}_1^C))$ can be extended to a continuous function on the closure of $B$ in $A_1$ with the compact support which is of class $C^\infty$ on $A_1''$.*

(2) *$F_f^{(2)}$ can be extended to a function of class $C^\infty$ on $A_2$ with the compact support.*

For the proof, see [3 (f)].

Let $\mathfrak{U}$ be the universal enveloping algebra of $\mathfrak{g}^C$. Let $W_k$ be the Weyl group of $\mathfrak{g}^C$ with respect to $\mathfrak{h}_k^C$ $(k=1, 2)$. For any $s\in W_k$, let $u\to su$ $(u\in S(\mathfrak{h}_k^C))$ denote the automorphism of $S(\mathfrak{h}_k^C)$ which coincides with $s$ on $\mathfrak{h}_k^C$ $(k=1, 2)$. Let $\mathfrak{J}_k$ be the subalgebra of all elements $u\in S(\mathfrak{h}_k^C)$ such that $su=u$ for all $s\in W_k$.

**Lemma 4.** *There exists an algebraic isomorphism $\gamma_k:\Delta\to\gamma_k(\Delta)$ $(\Delta\in\mathfrak{Z})$ of $\mathfrak{Z}$ onto $\mathfrak{J}_k$ which satisfies the following conditions* $(k=1, 2)$;

(1) *Let $u\to u^*$ $(u\in\mathfrak{U})$ denote the anti-automorphism of $\mathfrak{U}$ which maps $X$ on $-X$ $(X\in\mathfrak{g}^C)$. Then*

$$\gamma_k(\Delta^*) = (\gamma_k(\Delta))^* \qquad (\Delta \in \mathfrak{Z}).$$

(2) $$F_{\Delta f}^{(k)} = \gamma_k(\Delta) F_f^{(k)} \quad \textit{for all} \quad f \in C_c^\infty(G) \quad \textit{and} \quad \Delta \in \mathfrak{Z}.$$

For the proof of this lemma, see [3 (d), (f)]. One should notice that our definition of $F_f^{(k)}$ is a little different from the one given in [3 (f)] and consequetly (2) in Lemma 4 is a slight modification of [3 (f)].

**Lemma 5.** *For arbitrarily normalized Haar measures $dg$ and $dh$, the invariant measures $dx^{(k)}$ can be normalized so that we have*

$$\int_G f(g) dg = \sum_{k=1}^{2} \int_{A_k} \Delta_k(h) F_f^{(k)}(h) dh \quad \textit{for all} \quad f \in C_c^\infty(G).$$

This lemma is proved in the same way as [3 (b)].

Now we fix the normalizations of the Haar measures of $G$ and $A_2$ arbitrarily. As for the Haar measure of $A_1$, we normalize the measure $dh$ such that

(2. 3) $$\int_{A_1} dh = 1.$$

After this, we normalize the invariant measure $dx^{(k)}$ so that the equality in Lemma 5 holds.

Now let $S(\hat{\mathfrak{h}}_k^C)$ be the symmetric algebra over $\hat{\mathfrak{h}}_k^C$, where $\hat{\mathfrak{h}}_k^C$ is the vector space of all linear forms on $\mathfrak{h}_k^C$ $(k=1, 2)$. For any $\lambda \in \hat{\mathfrak{h}}_k^C$, we denote by $H_\lambda$ the unique element of $\mathfrak{h}_k^C$ such that $B(H_\lambda, H) = \lambda(H)$ for all $H \in \mathfrak{h}_k^C$. Then the mapping $\lambda \to H_\lambda$ $(\lambda \in \hat{\mathfrak{h}}_k^C)$ can be uniquely extended to an isomorphism of $S(\hat{\mathfrak{h}}_k^C)$ onto $S(\mathfrak{h}_k^C)$ which we denote by $\partial$. Put $\pi_k = \prod_{\alpha \in P_k} \alpha$ and $\rho_k = \frac{1}{2} \sum_{\alpha \in P_k} \alpha$. We also put $\pi = \pi_1$ and $\rho = \rho_1$. Then $\partial(\pi) \in S(\mathfrak{h}_1^C)$.

**Lemma 6.** *For any $f \in C_c^\infty(G)$, $\partial(\pi) F_f^{(1)}$ can be extended to a continuous function on $A_1$. Moreover, there exists a positive number $c$ independent of $f$ such that*

$$\lim_{h \to e} F_f^{(1)}(h; \partial(\pi)) = c(-1)^{n+q} f(e) \qquad (h \in A_1'),$$

*for all $f \in C_c^\infty(G)$, where $n$ (resp. $q$) is the number of the elements of* $\mathrm{P}_1$ (*resp.* $\mathrm{P}_1^0$).

For the proof, see [3 (f), (h)], and notice that $\overline{\Delta_1(h)} = (-1)^n \Delta_1(h)$.

### 3. Definition and Properties of $T_\Lambda$

In this section we shall define a distribution $T_\Lambda$ for $\Lambda \in \mathfrak{F}$ and get some formulas on $T_\Lambda$ under the additional assumption (A. 5). Let $G$

be a simple Lie group which satisfies the conditions (A. 1)~(A. 4).

$S(\hat{\mathfrak{h}}_k^C)$ can be regarded as the algebra of all polynomial functions on $\mathfrak{h}_k^C$. For any $\lambda\in\hat{\mathfrak{h}}_k^C$ and $p\in S(\hat{\mathfrak{h}}_k^C)$, let $\langle\lambda, p\rangle=\langle\lambda, \partial(p)\rangle$ denote the value of $p$ at $H_\lambda$. Let $\mathfrak{F}'$ be the set of all elements $\Lambda\in\mathfrak{F}$ such that $\langle\Lambda+\rho, \pi\rangle\neq 0$. We put $W=W_1$ and for any $s\in W$ define $\tilde{s}$ by

$$\tilde{s}H = s(\nu H) \quad \text{for all} \quad H\in\mathfrak{h}_2^C. \tag{3.1}$$

For any $\Lambda\in\mathfrak{F}'$ we put

$$_+A_\Lambda = \{h = \exp H\in A_2:\ |e^{(\tilde{\Lambda}+\tilde{\rho})(H)}|\leqq 1\}, \tag{3.2}$$

$$_-A_\Lambda = \{h = \exp H\in A_2:\ |e^{(\tilde{\Lambda}+\tilde{\rho})(H)}|> 1\}. \tag{3.3}$$

Now we define a bounded continuous function $\xi_\Lambda^{(2)}$ on $A_2$ as follows;

$$\xi_\Lambda^{(2)}(h) = \begin{cases} e^{(\tilde{\Lambda}+\tilde{\rho})(H)} & \text{if} \quad h = \exp H\in {}_+A_\Lambda, \\ e^{\tilde{s}_0(\tilde{\Lambda}+\tilde{\rho})(H)} & \text{if} \quad h = \exp H\in {}_-A_\Lambda \end{cases} \tag{3.4}$$

where $s_0$ denotes the Weyl reflexion $H\to H-2(\alpha_0(H)/\alpha_0(H_{\alpha_0}))H_{\alpha_0}$ on $\mathfrak{h}_1^C$. We put

$$\xi_\Lambda^{(1)} = \xi_{\Lambda+\rho}. \tag{3.5}$$

For any $s\in W$, we put

$$\begin{aligned} \varepsilon_1(s) &= \varepsilon(s), \\ \varepsilon_2(s) &= \begin{cases} -\varepsilon(s) & \text{if} \quad \langle s\rho, \alpha_0\rangle>0, \\ \varepsilon(s) & \text{if} \quad \langle s\rho, \alpha_0\rangle<0, \end{cases} \end{aligned} \tag{3.6}$$

where $\varepsilon(s)$ denotes the constant which is uniquely determined by

$$\Delta_1(\exp sH) = \varepsilon(s)\Delta_1(\exp H) \qquad \text{for all} \quad H\in\mathfrak{h}_1.$$

For any $\Lambda\in\mathfrak{F}_0$ we define

$$\eta_\Lambda^{(k)}(h) = \sum_{s\in W}\varepsilon_k(s)\xi_{\Lambda^s}^{(k)}(h) \qquad \text{for} \quad h\in A_k \quad (k=1, 2), \tag{3.7}$$

where $\Lambda^s=s(\Lambda+\rho)-\rho$.

**Proposition 2.** *Let*

$$T_\Lambda(f) = (-1)^q\sum_{k=1}^{2}\int_{A_k}\eta_\Lambda^{(k)}(h)F_f^{(k)}(h)dh \qquad (f\in C_c^\infty(G)). \tag{3.8}$$

*Then $T_\Lambda$ is an invariant distribution on $G$. Moreover $T_\Lambda$ is an eigendistribution of $\mathfrak{Z}$ on $G'$.*

Proof. Since $\eta_\Lambda^{(k)}$ is continuous on $A_k$, making use of Lemma 2, we can easily show that $T_\Lambda$ is a distribution. Moreover, from the definition of $F_f^{(k)}$, we have

$$F_{f^g}^{(k)} = F_f^{(k)} \qquad (g \in G),$$

for all $f \in C_c^\infty(G)$.

Therefore from (3.8) we have

$$T_\Lambda(f^g) = T_\Lambda(f) \qquad (g \in G),$$

for all $f \in C_c^\infty(G)$. Hence $T_\Lambda$ is an invariant distribution. For any $\Delta \in \mathfrak{Z}$ and $f \in C_c^\infty(G')$, we have

$$\begin{aligned} (3.9) \qquad (\Delta T_\Lambda)(f) &= T_\Lambda(\Delta^* f) \\ &= (-1)^q \sum_{k=1}^{2} \int_{A_k} \eta_\Lambda^{(k)}(h) F_{\Delta^* f}^{(k)}(h) dh \\ &= (-1)^q \sum_{k=1}^{2} \int_{A_k} \eta_\Lambda^{(k)}(h) F_f^{(k)}(h\,;\ \gamma_k(\Delta^*)) dh \\ &= (-1)^q \sum_{k=1}^{2} \int_{A_k} \eta_\Lambda^{(k)}(h) F_f^{(k)}(h\,;\ \gamma_k(\Delta)^*) dh \\ &= (-1)^q \sum_{k=1}^{2} \int_{A_k} \eta_\Lambda^{(k)}(h\,;\ \gamma_k(\Delta)) F_f^{(k)}(h) dh . \end{aligned}$$

In the above deduction, we made use of Lemma 3 and 4. From Lemma 4, we can easily deduce that

$$(3.10) \qquad \gamma_k(\Delta) \xi_{\Lambda^s}^{(k)} = \langle \Lambda + \rho, \gamma_1(\Delta) \rangle \xi_{\Lambda^s}^{(k)} \quad \text{for all} \quad s \in W .$$

Since $\langle \tilde{\Lambda} + \tilde{\rho}, \gamma_2(\Delta) \rangle = \langle \Lambda + \rho, \gamma_1(\Delta) \rangle$ and $s\gamma_1(\Delta) = \gamma_1(\Delta)$ $(s \in W)$. It follows from (3.7), (3.9) and (3.10) that

$$(3.11) \qquad \Delta T_\Lambda(f) = \langle \Delta + \rho, \gamma_1(\Delta) \rangle T_\Lambda(f) .$$

Hence $T_\Lambda$ is an eigendistribution on $G'$.

This completes the proof of the proposition.

Now we assume that $G$ satisfies the additional condition (A.5) (see Introduction).

**Proposition 3.** *The series*

$$\sum_{\Lambda \in \mathfrak{F}_0} \langle \Lambda + \rho, \pi \rangle T_\Lambda(f) \qquad (f \in C_c^\infty(G))$$

*converges absolutely.*

Proof. By the assumption (A.5), there exists for any $\Lambda \in \mathfrak{F}_0$ a finite number of irreducible unitary representations $\omega_\Lambda^{(1)}, \cdots, \omega_\Lambda^{(s)}$ of $G$ such that

$$T_\Lambda = \sum_{k=1}^{s} T_\Lambda^{(k)}, \tag{3.12}$$

where $T_\Lambda^{(k)}$ denotes the character of $\omega_\Lambda^{(k)}$ $(k=1, \cdots, s)$. We may assume that $\omega_\Lambda^{(1)}, \cdots, \omega_\Lambda^{(s)}$ are not mutually equivalent.

Take any $f \in C_c^\infty(G')$. Then from Proposition 2, there exists a homomorphism $\chi_\Lambda$ of $\mathfrak{Z}$ in $\boldsymbol{C}$ such that

$$T_\Lambda(\Delta^* f) = \chi_\Lambda(\Delta) T_\Lambda(f) \qquad (\Delta \in \mathfrak{Z}). \tag{3.13}$$

On the other hand, we have

$$T_\Lambda^{(k)}(\Delta^* f) = \chi_\Lambda^{(k)}(\Delta) T_\Lambda^{(k)}(f) \qquad (\Delta \in \mathfrak{Z}), \tag{3.14}$$

where $\chi_\Lambda^{(k)}$ is the infinitesimal character of $\omega_\Lambda^{(k)}$.

From (3.12), (2.13) and (3.14) we get

$$\sum_{k=1}^{s} (\chi_\Lambda^{(k)}(\Delta) - \chi_\Lambda(\Delta)) T_\Lambda^{(k)}(f) = 0 \qquad (\Delta \in \mathfrak{Z}).$$

Therefore from Lemma 1, we can show (see [3 (c)]) that

$$\chi_\Lambda^{(k)}(\Delta) = \chi_\Lambda(\Delta) \qquad (\Delta \in \mathfrak{Z}) \quad (k=1, \cdots, s).$$

Now let $f$ be an element of $C_c^\infty(G)$. Then we have

$$\begin{aligned} \Delta T_\Lambda(f) = T_\Lambda(\Delta^* f) &= \sum_{k=1}^{s} T_\Lambda^{(k)}(\Delta^* f) = \sum_{k=1}^{s} \chi_\Lambda^{(k)}(\Delta) T_\Lambda^{(k)}(f) \\ &= \sum_{k=1}^{s} \chi_\Lambda(\Delta) T_\Lambda^{(k)}(f) = \chi_\Lambda(\Delta) T_\Lambda(f). \end{aligned} \tag{3.15}$$

Hence $T_\Lambda$ is an eigendistribution of $\mathfrak{Z}$ on $G$.

Let $C$ be the Casimir operator on $G$. Then from (3.11) and (3.15) we have

$$\chi_\Lambda(C) = \langle \Lambda + \rho, \gamma_1(C) \rangle. \tag{3.16}$$

On the other hand from (3.8) we have

$$\begin{aligned} |T_\Lambda(f)| &\leq \sum_{k=1}^{2} \int_{A_k} |\eta_\Lambda^{(k)}(h)| \cdot |F_f^{(k)}(h)| dh \\ &\leq w \sum_{k=1}^{2} \int_{A_k} |F_f^{(k)}(h)| dh < \infty, \end{aligned}$$

where $w$ is the order of the Weyl group $W$.

The convergence of the integral follows from (1) in Lemma 3. Put

$$M_f = w \sum_{k=1}^{2} \int_{A_k} |F_f^{(k)}(h)| dh,$$

Then $M_f$ is independent of $\Lambda$ and

$$|T_\Lambda(f)| \leqslant M_f . \tag{3.17}$$

Since $\gamma_1(C)$ is a differential operator of elliptic type, we can find integers $l$, $m$ such that

$$|\langle\Lambda+\rho, \pi\rangle| \leqslant |\langle\Lambda+\rho, \gamma_1(C)|^l \quad \text{and} \tag{3.18}$$

$$\sum_{\Lambda\in\tilde{\mathfrak{F}}} \frac{1}{|\langle\Lambda+\rho, \gamma_1(C)\rangle|^m} < \infty \tag{3.19}$$

where $\tilde{\mathfrak{F}} = \{\Lambda\in\mathfrak{F}_0 : \langle\Lambda+\rho, \gamma_1(C)\rangle \neq 0\}$ .

Therefore, from (3.15)~(3.18) we have

$$\begin{aligned} |\langle\Lambda+\rho, \pi\rangle| \cdot |T_\Lambda(f)| &= |\langle\Lambda+\rho, \pi\rangle| \cdot \left|\frac{T_\Lambda(C^{l+m}f)}{\chi_\Lambda(C^{l+m})}\right| \\ &\leqslant \frac{|\langle\Lambda+\rho, \pi\rangle|}{|\langle\Lambda+\rho, \gamma_1(C)\rangle|^l} \cdot \frac{|T_\Lambda(C^{l+m}f)|}{|\langle\Lambda+\rho, \gamma_1(C)\rangle|^m} \\ &\leqslant \frac{M_{C^{l+m}f}}{|\langle\Lambda+\rho, \gamma_1(C)\rangle|^m} . \end{aligned} \tag{3.20}$$

It is easy to see that $\mathfrak{F}_0 - \tilde{\mathfrak{F}}$ is a finite set. It follows from (3.19) and (3.20) that

$$\sum_{\Lambda\in\mathfrak{F}_0} \langle\Lambda+\rho, \pi\rangle |T_\Lambda(f)| < \infty .$$

Thus the proposition is proved.

Now we shall define $T_\Lambda$ also for any $\Lambda\in\mathfrak{F}$ as follows.
First we define $\xi_\Lambda^{(1)}$, $\eta_\Lambda^{(1)}$ also by (3.5), (3.7) respectively.
Then we have

$$\eta_\Lambda^{(1)}(h) = 0 \quad \text{for all} \quad h\in A_1 \quad \text{if} \quad \Lambda\in\mathfrak{F}-\mathfrak{F}' .$$

Put

$$\begin{aligned} {}_+A_2 &= \{h = \exp H\in A_2 : \ |e^{\alpha_0(H)}| \geqslant 1\} , \\ {}_-A_2 &= \{h = \exp H\in A_2 : \ |e^{\alpha_0(H)}| < 1\} . \end{aligned}$$

For any $\Lambda\in\mathfrak{F}$ such that $\langle\Lambda+\rho, \alpha_0\rangle = 0$, we define

$${}_+A_\Lambda = {}_+A_2 \quad \text{and} \quad {}_-A_\Lambda = {}_-A_2 .$$

On the other hand, for any $\Lambda\in\mathfrak{F}$ such that $\langle\Lambda+\rho, \alpha_0\rangle \neq 0$, we define ${}_+A_\Lambda$, ${}_-A_\Lambda$ again by (3.2), (3.3) respectively. We define $\xi_\Lambda^{(2)}$ also by (3.4). For any $\Lambda\in\mathfrak{F}$ such that $\Lambda+\rho\in\mathfrak{F}_0$ we define $\eta_\Lambda^{(2)}$ and $T_\Lambda$ again by (3.7), (3.8) respectively. Finally for any $\Lambda\in\mathfrak{F}$ we define

$$T_\Lambda = T_{\Lambda_1}$$

where $\Lambda_1$ is a unique element of $\mathfrak{F}$ such that

$$\Lambda_1+\rho\in\mathfrak{F}_0 \quad \text{and} \quad \Lambda^s=\Lambda_1 \quad \text{for some} \quad s\in W.$$

It is easy to see that

$$T_{\Lambda^s} = T_\Lambda \quad \text{for all} \quad s\in W \quad \text{and} \quad \Lambda\in\mathfrak{F}.$$

The following proposition is a direct consequence of Proposition 3.

**Proposition 3′.** *The series*

$$\sum_{\Lambda\in\mathfrak{F}} |\langle\Lambda+\rho, \pi\rangle| T_\Lambda(f) \qquad (f\in C_c^\infty(G))$$

*converges absolutely.*

For the proof, we have only to notice that $\mathfrak{F}'=\{\Lambda^s : \Lambda\in\mathfrak{F}_0, s\in W\}$ and that $\langle\Lambda+\rho, \pi\rangle=0$ for all $\Lambda\in\mathfrak{F}-\mathfrak{F}'$.

Now for any $\Lambda\in\mathfrak{F}$, we define

$$\xi_\Lambda^{(2)}(h) = \begin{cases} \tilde{\xi}_\Lambda^{(2)}(h) & \text{if} \quad h\in {}_+A_2, \\ -\tilde{\xi}_\Lambda^{(2)}(h) & \text{if} \quad h\in {}_-A_2, \end{cases} \tag{3.21}$$

and

$$\tilde{\eta}_\Lambda^{(k)}(h) = \sum_{s\in W} \xi_{\Lambda^s}^{(k)}(h) \qquad \text{for} \qquad h\in A_k \quad (k=1, 2), \tag{3.22}$$

where
$$\xi_\Lambda^{(1)} = \xi_{\Lambda+\rho}.$$

**Theorem 1.** *For any* $\Lambda\in\mathfrak{F}$,

$$|\langle\Lambda+\rho, \pi\rangle| T_\Lambda(f) = (-1)^{n+q}\sum_{k=1}^{2}\int_{A_k} \tilde{\eta}_\Lambda^{(k)}(h) F_f^{(k)}(h\,;\, \partial(\pi_k))dh. \tag{3.23}$$

A proof of this theorem will be given in § 7.

We shall now prove some formulas which will be needed in the next section.

**Proposition 4.** *The notation being as in Lemma* 6, *we have*

$$\sum_{\Lambda\in\mathfrak{F}}\int_{A_1} \tilde{\eta}_\Lambda^{(1)}(h) F_f^{(1)}(h\,;\, \partial(\pi))dh = wc(-1)^{n+q}f(e)$$
$$\textit{for all} \quad f\in C_c^\infty(G),$$

*where $w$ is the order of the group $W$.*

Proof. In the proof of Proposition 1, we have shown that $\mathfrak{h}_1/\Gamma\cong A_1$, the isomorphism being induced by the exponential mapping. Hence if we put

$$F(\theta_1, \theta_2, \cdots, \theta_l) = F_f^{(1)}(\exp(\sum_{i=1}^{l}\sqrt{-1}\theta_i H_i)\,;\ \partial(\pi))\,,$$

then $F$ is periodic in each of the variables with period $2\pi$. Therefore if we put $\Lambda=\sum_{i=0}^{l} m_i\Lambda_i$, then we have

$$\int_{A_1}\xi_\Lambda(h)F_f^{(1)}(h\,;\ \partial(\pi))dh$$
$$= \frac{1}{(2\pi)^l}\int_0^{2\pi}\cdots\int_0^{2\pi} F(\theta_1, \theta_2, \cdots, \theta_l)e^{\sum_{i=1}^{l}\sqrt{-1}m_i\theta_i}d\theta_1 d\theta_2\cdots d\theta_l\,.$$

From (1) in Lemma 3, we easily see that $F$ is piecewise smooth in each of the variables $\theta_i$ $(i=1, 2, \cdots, l)$. It follows from the theory of Fourier series and Lemma 6 that the series

$$\sum_{\Lambda\in\mathfrak{F}}\int_{A_1}\xi_\Lambda(h)F_f^{(1)}(h:\ \partial(\pi))dh \tag{3.24}$$
$$= \sum_{m_1,\cdots,m_l}\frac{1}{(2\pi)^l}\int_0^{2\pi}\cdots\int_0^{2\pi}F(\theta_1, \cdots, \theta_l)e^{\sum_{i=1}^{l}\sqrt{1-}m_i\theta_i}d\theta_1\cdots d\theta_l$$

converges absolutely to

$$\lim_{(\theta_1,\cdots\theta_l)\to(0,\cdots,0)} F(\theta_1, \cdots, \theta_l) = \lim_{h\to e}F_f^{(1)}(h\,;\ \partial(\pi))$$
$$= c(-1)^{n+q}f(e)\,.$$

Therefore from the absolute convergence of (3.24) we have

$$\sum_{\Lambda\in\mathfrak{F}}\int_{A_1}\tilde{\eta}_\Lambda^{(1)}(h)F_f^{(1)}(h\,;\ \partial(\pi))dh$$
$$= \sum_{\Lambda\in\mathfrak{F}}\sum_{s\in W}\int_{A_1}\xi_{\Lambda^s}^{(1)}(h)F_f^{(1)}(h\,;\ \partial(\pi))dh$$
$$= \sum_{s\in W}\left\{\sum_{\Lambda\in\mathfrak{F}}\int_{A_1}\xi_{s(\Lambda+\rho)}(h)F_f^{(1)}(h\,;\ \partial(\pi))\,dh\right\}$$
$$= \sum_{s\in W}\left\{\sum_{\Lambda\in\mathfrak{F}}\int_{A_1}\xi_\Lambda(h)F_f^{(1)}(h\,;\ \partial(\pi))\,dh\right\}$$
$$= wc(-1)^{n+q}f(e)\,.$$

In the above deduction, we made use of the fact that $\rho$ is a integral form and that for any $s\in W$ the mapping $\Lambda\to s\Lambda$ $(\Delta\in\mathfrak{F})$ is a bijection from $\mathfrak{F}$ onto itself. Thus Proposition 4 is proved.

Now we consider the series

$$\sum_{\Lambda\in\mathfrak{F}}\int_{A_2}\tilde{\eta}_\Lambda^{(2)}(h)F_f^{(2)}(h\,;\ \partial(\pi_2))dh\,, \tag{3.25}$$

It follows from Proposition 3′, Theorem 1 and the absolute convergence of (3.24) that the series (3.25) converges absolutely. Since $A_2$ is connected, if we put $A=\exp\mathfrak{a}$ and $A^-=\exp\mathfrak{h}^-$, then we have $A_2=AA^-$. Put $a_t=\exp\sqrt{-1}tU_{\alpha_0}$. Then any $h\in A_2$ is written uniquely in the form $h=a_th^-$ ($t\in R$, $h^-\in A^-$). Put $\mathfrak{h}^+=\nu\mathfrak{a}$. Then we have $\mathfrak{h}_1=\mathfrak{h}^++\mathfrak{h}^-$ (direct) (see § 1).

Let $\hat{\mathfrak{h}}^+$, $\hat{\mathfrak{h}}^-$, $\hat{\mathfrak{h}}_1$ denote the vector spaces of all pure imaginary valued linear forms on $\mathfrak{h}^+$, $\mathfrak{h}^-$, $\mathfrak{h}_1$ respectively. Since $\mathfrak{h}_1=\mathfrak{h}^++\mathfrak{h}^-$ (direct), we can consider $\hat{\mathfrak{h}}^+$ and $\hat{\mathfrak{h}}^-$ as subspaces of $\hat{\mathfrak{h}}_1$ so that $\hat{\mathfrak{h}}_1=\hat{\mathfrak{h}}^++\hat{\mathfrak{h}}^-$ (direct). For any $\Lambda\in\hat{\mathfrak{h}}_1$ we denote by $\Lambda^+$, $\Lambda^-$ the $\hat{\mathfrak{h}}^+$, $\hat{\mathfrak{h}}^-$–component of $\Lambda$ respectively. We put $A^+=\exp\mathfrak{h}^+$. Then it is clear that $A_1=A^+A^-$ and that $D=A^+\cap A^-$ is a finite group. Let $A_0$ be a closed subgroup of $A_1$. Then it is well known that the character group $\hat{A}_0$ of $A_0$ is given by

$$\hat{A}_0 = \{\xi\,|\,A_0 : \xi\in\hat{A}_1\}\,.$$

where $\xi|A_0$ is the restriction of $\xi$ to $A_0$.

For any $\xi\in\hat{A}_1$ it is clear that

$$(\xi\,|\,A^+)\,|\,D = (\xi\,|\,A^-)\,|\,D.$$

Conversely for $\xi^+\in\hat{A}^+$, $\xi^-\in\hat{A}^-$ such that $\xi^+|D=\xi^-|D$, there exists a unique element $\xi\in\hat{A}_1$ such that

$$\xi\,|\,A^+ = \xi^+ \quad\text{and}\quad \xi\,|\,A^-=\xi^-\,.$$

Put

$$\mathfrak{F}^+ = \{\Lambda^+ : \Lambda\in\mathfrak{F}\} \quad\text{and}\quad \mathfrak{F}^- = \{\Lambda^- : \Lambda\in\mathfrak{F}\}\,.$$

Then

$$\hat{A}^+ = \{\xi_{\Lambda^+} : \Lambda^+\in\mathfrak{F}^+\} \quad\text{and}\quad \hat{A}^- = \{\xi_{\Lambda^-} : \Lambda^-\in\mathfrak{F}^-\}\,.$$

Since $A^+$ is a one dimensional torus, $D$ is a cyclic group. Let $m$ be the order of $D$ and let $\gamma$ be a generator of $D$. Select $\Gamma^+\in\mathfrak{h}^+$ and $\Gamma^-\in\mathfrak{h}^-$ such that $\gamma=\exp\Gamma^+=\exp\Gamma^-$. For any integer $k$ ($0\leqslant k<m$), we put

$$\mathfrak{F}_k^+ = \{\Lambda^+\in\mathfrak{F}^+ : e^{(\Lambda^++\rho^+)(\Gamma^+)} = e^{\sqrt{-1}\,2\pi k/m}\}\,,$$
$$\mathfrak{F}_k^- = \{\Lambda^-\in\mathfrak{F}^- : e^{(\Lambda^-+\rho^-)(\Gamma^-)} = e^{\sqrt{-1}\,2\pi k/m}\}\,.$$

Then we have

$$\mathfrak{F} = \bigcup_{k=0}^{m-1}\mathfrak{F}_k \qquad \text{(disjoint sum)}\,,$$

where
$$\mathfrak{F}_k = \{\Lambda^++\Lambda^- : \Lambda^+\in\mathfrak{F}_k^+,\ \Lambda^-\in\mathfrak{F}_k^-\}\,.$$

For any $\Lambda\in\mathfrak{F}$, there exists an integer $r$ such that $\Lambda^++\rho^+=r\Lambda_0^+$, where $\Lambda_0=\frac{1}{2}\alpha_0$. It is clear that

$$ {}_{+}A_2 = \{a_t h^- \in A_2 : t \geqslant 0,\ h^- \in A^-\}\,, $$
$$ {}_{-}A_2 = \{a_t h^- \in A_2 : t < 0,\ h^- \in A^-\}\,. $$

For any $h = a_t h^- \in {}_{+}A_2$,

$$ \xi_\Lambda^{(2)}(a_t h^-) = \begin{cases} e^{rt + (\Lambda^- + \rho^-)(H^-)} & \text{if } r < 0\,, \\ e^{-rt + (\Lambda^- + \rho^-)(H^-)} & \text{if } r \geqslant 0\,. \end{cases} $$

Hence

$$ \xi_\Lambda^{(2)}(a_t h^-) = e^{-|r|t + (\Lambda^- + \rho^-)(H^-)} \qquad (a_t h^- \in {}_{+}A_2)\,. \tag{3.26} $$

Similarly we have

$$ \xi_\Lambda^{(2)}(a_t h^-) = -e^{|r|t + (\Lambda^- + \rho^-)(H^-)} \qquad (a_t h^- \in {}_{-}A_2)\,. \tag{3.27} $$

Now we need the following Lemma 7, a proof of which will be given at the end of the present section.

**Lemma 7.** $F_f^{(2)}(a_{-t}h^-\,;\,\partial(\pi_2)) = -F_f^{(2)}(a_t h^-\,;\,\partial(\pi_2))$ $(h^- \in A^-)$.

Making use of this lemma and (3.27) we have

$$ \begin{aligned} &\int_{-A_2} \xi_\Lambda^{(2)}(h) F_f^{(2)}(h\,;\,\partial(\pi_2))dh \\ &= -\int_{-\infty}^{0}\int_{A^-} e^{|r|t + (\Lambda^- + \rho^-)(H^-)} F_f^{(2)}(a_t h^-\,;\,\partial(\pi_2))dtdh^- \\ &= -\int_{0}^{\infty}\int_{A^-} \bar{e}^{|r|t + (\Lambda^- + \rho^-)(H^-)} F_f^{(2)}(a_{-t} h^-\,;\,\partial(\pi_2))dtdh^- \\ &= \int_{0}^{\infty}\int_{A^-} e^{-|r|t + (\Lambda^- + \rho^-)(H^-)} F_f^{(2)}(a_t h^-\,;\,\partial(\pi_2))dtdh^-\,. \end{aligned} $$

In the above formula, $dh^-$ denotes the Haar measure of $A^-$ such that $dh = dt\, dh^-$, where $dt$ is the canonical volume element in $\boldsymbol{R}^1$. The above formula and (3.26) give the equality

$$ \begin{aligned} &\int_{A_2} \xi_\Lambda^{(2)}(h) F_f^{(2)}(h\,;\,\partial(\pi_2))\,dh \\ &= 2\int_{0}^{\infty}\int_{A^-} e^{-|r|t + (\Lambda^- + \rho^-)(H^-)} F_f^{(2)}(a_t h^-\,;\,\partial(\pi_2))dtdh^-\,. \end{aligned} \tag{3.28} $$

It is easy to see that

$$ \mathfrak{F}_k^+ = \{(lm+k)\Lambda_0^+ - \rho^+ : l \in \boldsymbol{Z}\}\,, $$

where
$$ \Lambda_0 = \frac{1}{2}\alpha_0\,. $$

Hence from (3.28) we have

$$(3.29)\quad \sum_{\Lambda^+\in\mathfrak{F}_k^+}\int_{A_2}\xi_\Lambda^{(2)}(h)F_f^{(2)}(h\,;\,\partial(\pi_2))dh$$
$$=2\lim_{\substack{p\to\infty\\ q\to\infty}}\sum_{l=-(p-1)}^{q-1}\int_0^\infty\int_{A^-}e^{-|lm+k|t+(\Lambda^-+\rho^-)(H)}F_f^{(2)}(a_th^-\,;\,\partial(\pi_2))dtdh^-$$
$$=2\lim_{\substack{p\to\infty\\ q\to\infty}}\int_0^\infty\int_{A^-}\frac{e^{-kt}+e^{-(m-k)t}-e^{-(pm-k)t}-e^{-(qm-k)t}}{1-e^{-mt}}e^{(\Lambda^-+\rho^-)(H^-)}$$
$$\times F_f^{(2)}(a_th^-\,;\,\partial(\pi_2))dtdh^-$$
$$=2\int_0^\infty\int_{A^-}\frac{\mathrm{ch}((m-2k)t/2)}{\mathrm{sh}(mt/2)}e^{(\Lambda^-+\rho^-)(H^-)}F_f^{(2)}(a_th^-\,;\,\partial(\pi_2))dtdh^-\,.$$

In the above deduction, we used the following fact.

**Lemma 8.** $\frac{1}{t}F_f^{(2)}(a_th^-\,;\,\partial(\pi_2))$ *can be extended to a $C^\infty$-function with the compact support on $A_2$.*

This lemma is an immediate consequence of (2) in Lemma 3 and Lemma 7.

From the absolute convergence of the series (3.25), using (3.22) and (3.29), we have

$$\sum_{\Lambda\in\mathfrak{F}}\int_{A_2}\eta_\Lambda^{(2)}(h)F_f^{(2)}(h\,;\,\partial(\pi_2))dh$$
$$=\sum_{s\in W}\left\{\sum_{\Lambda\in\mathfrak{F}}\int_{A_2}\xi_{\Lambda^s}^{(2)}(h)F_f^{(2)}(h\,;\,\partial(\pi_2))dh\right\}$$
$$=w\sum_{k=0}^{m-1}\sum_{\Lambda^-\in\mathfrak{F}_k^-}\left\{\sum_{\Lambda^+\in\mathfrak{F}_k^+}\int_{A_2}\xi_\Lambda^{(2)}(h)F_f^{(2)}(h\,;\,\partial(\pi_2))dh\right\}$$
$$=2w\sum_{k=0}^{m-1}\sum_{\Lambda^-\in\mathfrak{F}_k^-}\int_0^\infty\int_{A^-}\frac{\mathrm{ch}((m-2k)t/2)}{\mathrm{sh}(mt/2)}e^{(\Lambda^-+\rho^-)(H^-)}F_f^{(2)}(a_th^-\,;\,\partial(\pi_2))dtdh^-$$
$$=2w\sum_{k=0}^{m-1}\int_0^\infty\left\{\sum_{\Lambda^-\in\mathfrak{F}_k^-}\int_{A^-}e^{(\Lambda^-+\rho^-)(H^-)}F_f^{(2)}(a_th^-\,;\,\partial(\pi_2))dh^-\right\}\frac{\mathrm{ch}((m-2k)t/2)}{\mathrm{sh}(mt/2)}dt\,.$$

In the above, we used Lemma 8 again. Thus we have obtained the following result.

**Proposition 5.**

$$\sum_{\Lambda\in\mathfrak{F}}\int_{A_2}\eta_\Lambda^{(2)}(h)F_f^{(2)}(h\,;\,\partial(\pi_2))dh$$
$$=2w\sum_{k=0}^{m-1}\int_0^\infty\left\{\sum_{\Lambda^-\in\mathfrak{F}_k^-}\int_{A^-}e^{(\Lambda^-+\rho^-)(H^-)}F_f^{(2)}(a_th^-\,;\,\partial(\pi_2))dh^-\right\}\frac{\mathrm{ch}((m-2k)t/2)}{\mathrm{sh}(mt/2)}dt,$$

*where $m$ is the order of the group $D=A^+\cap A^-$.*

Now we come to the proof of Lemma 7. We put $k_0 = \exp((\pi/2)\sqrt{-1}H_0)$. Then from (1. 1), (1. 2), (1. 5) and (1. 7) it follows easily that

$$k_0 \in K, \quad \mathrm{Ad}(k_0)\,|\,\mathfrak{h}_2 = \tilde{s}_0 . \tag{3. 30}$$

On the other hand, from (2. 2) we have $(F_f^{(2)})^{k_0^{-1}} = F_f^{(2)}$. From this and (3. 10) we have

$$\begin{aligned} F_f^{(2)}(a_t h^- ; \partial(\pi_2)) &= (F_f^{(2)})^{k_0^{-1}}(k_0 a_t h^- k_0^{-1} ; \partial(s_0\pi_2)) \\ &= -F_f^{(2)}(a_{-t}h^- ; \partial(\pi_2)) . \end{aligned}$$

Thus Lemma 7 is proved.

REMARK 1. It is plausible that a simple Lie group $G$ satisfies always the condition (A. 5) whenever it satisfies the conditions (A. 1)$\sim$(A. 4).

REMARK 2. We shall prove in § 5 that the order $m$ of the cyclic group $D = A^+ \cap A^-$ is equal to 2.

## 4. Main theorem

Let $\mathfrak{m}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$ and let $M$ be the corresponding analytic subgroup of $G$. Then it is easy to see that $\mathfrak{h}^-$ is a Cartan subalgebra of $\mathfrak{m}$ and that $A^-$ is a Cartan subgroup of $M$. $P_2^-$ is naturally identified with the set of all positive roots of $\mathfrak{m}^C$ with respect to $(\mathfrak{h}^-)^C$ under some linear order.

For any $\Lambda^- \in \mathfrak{F}^-$ we put

$$\varepsilon(\Lambda^-) = \begin{cases} 1 & \text{if } \langle \Lambda^- + \rho^-, \pi^- \rangle \geqslant 0 , \\ -1 & \text{if } \langle \Lambda^- + \rho^-, \pi^- \rangle < 0 \end{cases}$$

where $\rho^- = \frac{1}{2}\sum_{\alpha \in P_2^-} \alpha$ and $\pi^- = \prod_{\alpha \in P_2^-} \alpha$. Take a $\Lambda^- \in \mathfrak{F}^-$ such that $\langle \Lambda^- + \rho^-, \pi^- \rangle \neq 0$. Then it is known that there exists an irreducible unitary representation $\delta_{\Lambda^-}$ of $M$ whose character $\zeta_{\Lambda^-}$ is given by the following formula;

$$\zeta_{\Lambda^-}(h^-) = \frac{\varepsilon(\Lambda^-)}{\Delta_-(h^-)} \sum_{s \in W_-} \varepsilon(s) e^{s(\Lambda^- + \rho^-)(H^-)} \qquad (h^- = \exp H^- \in (A^-)')$$

where

$$W_- = \{ s \in W : s\alpha_0 = \alpha_0 \} ,$$
$$\Delta_-(h^-) = \prod_{\alpha \in P_2^-} (e^{\alpha(H)/2} - e^{-\alpha(H)/2}) .$$

For each $\alpha \in P_2$ let $X_\alpha$ be an element of $\mathfrak{g}^C$ such that $X_\alpha \neq 0$ and $[H, X_\alpha] = \alpha(H) X_\alpha$ for all $H \in \mathfrak{h}_2^C$. Then from the assumption for the order intronduced in $\Sigma_2$, we can show that

$$\mathfrak{n} = \Big(\sum_{\alpha \in P_2^+} CX_\alpha\Big) \cap \mathfrak{g}$$

is a nilpotent Lie algebra. Let $N$ be the analytic group corresponding to $\mathfrak{n}$. Then Iwasawa has shown that

$$G = KAN, \tag{4.1}$$

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n} \qquad \text{(direct sum as a vector space)}. \tag{4.2}$$

It is easy to show that $MAN$ is an analytic subgroup of $G$. For a non zero real number $\lambda$ and $\Lambda^- \in \mathfrak{F}^-$ such that $\langle \Lambda^- + \rho^-, \pi^- \rangle \neq 0$, we define the irreducible unitary representation $L_{\lambda, \Lambda^-}$ of the group $MAN$ by

$$L_{\lambda, \Lambda^-}(ma_t n) = e^{\sqrt{-1}\lambda t} \delta_{\Lambda^-}(m).$$

Let $\omega_{\lambda, \Lambda^-}$ denote the unitary representation of $G$ induced by the representation $L_{\lambda, \Lambda^-}$ of $MAN$ (see [2]). The following formula for the character $T_{\lambda, \Lambda^-}$ of the representation $\omega_{\lambda, \Lambda^-}$ is due to Harish-Chandra (see [3 (b)]).

**Lemma 9.** *$T_{\lambda, \Lambda^-}(f) = \int_G f(g) \chi_{\lambda, \Lambda^-}(g) dg$ for all $f \in C_c^\infty(G)$, where $\chi_{\lambda, \Lambda^-}$ is the invariant analytic function on $G'$ defined by*

$$\chi_{\lambda, \Lambda^-}(h) = \begin{cases} 0 \quad \text{if} \quad h \in A_1', \\ \dfrac{2\varepsilon(\Lambda^-) \cos \lambda t}{\Delta_2(h)} \displaystyle\sum_{s \in W_-} \varepsilon(s) e^{s(\Lambda^- + \rho^-)(H^-)} \\ \quad \text{if} \quad h = a_t \exp H^- \in A_2'. \end{cases}$$

To prove this lemma, we have only to notice that $\zeta_{(\Lambda^-)^{s_0}} = \zeta_{\Lambda^-}$ and $c=1$ in Theorem 2 of [3 (b)].

For any real number $\lambda$ we put

$$\operatorname{sgn}(\lambda) = \begin{cases} 1 & \text{if} \quad \lambda \geqslant 0, \\ -1 & \text{if} \quad \lambda < 0. \end{cases}$$

We define $T_{\lambda, \Lambda^-}(f) = 0$ for all $f \in C_c^\infty(G)$ if $\langle \sqrt{-1}\lambda \tilde{\Lambda}_0 + \Lambda^- + \rho^-, \pi \rangle = 0$.

**Proposition 6.** *For any real number $\lambda$,*

$$\sum_{\Lambda^- \in \mathfrak{F}_k^-} |\langle \sqrt{-1}\lambda \tilde{\Lambda}_0 + \Lambda^- + \rho^-, \pi \rangle| \, T_{\lambda, \Lambda^-}(f)$$
$$= 4w_-(-1)^{n+q-1} \operatorname{sgn}(\lambda) \int_0^\infty \Big\{ \sum_{\Lambda^- \in \mathfrak{F}_k^-} \int_{A^-} e^{(\Lambda^- + \rho^-)(H^-)} F_f^{(2)}(a_t h^-; \partial(\pi_2)) dh^- \Big\}$$
$$\times \sin \lambda t dt \qquad \text{for all} \quad f \in C_c^\infty(G),$$

*where $w_-$ is the order of the group $W_-$.*

*Moreover this series converges absolutely and uniformly with respect to* $\lambda$.

Proof. Put $W_\sigma=\{s\in W_2: s\mathfrak{h}_2=\mathfrak{h}_2\}$. Then it is easy to see that $W_\sigma=\{1, \tilde{s}_0\}\tilde{W}_-$ (direct). For any $s\in W_\sigma$ we have

$$\Delta_2(\exp sH) = \varepsilon_0(s)\Delta_2(\exp H)$$

where $\varepsilon_0(s)=1$ or $-1$.

We define $\varepsilon_0(s)$ $(s\in W_\sigma)$ by this formula. From Lemma 5 and 9 we have

$$\begin{aligned} T_{\lambda,\Lambda^-}(f) &= \int_G f(g)\chi_{\lambda,\Lambda^-}(g)dg \\ &= \int_{A_2} \chi_{\lambda,\Lambda^-}(h)|\Delta_2(h)|^2 dh \int_{G/A_2} f(h^{x^{(2)}})dx^{(2)} \qquad (4.3)\\ &= \varepsilon(\Lambda^-)\int_{A_2} \eta_{\lambda,\Lambda^-}(h)F_f^{(2)}(h)dh\,. \end{aligned}$$

In this formula we put

$$\eta_{\lambda,\Lambda^-}(h) = \sum_{s\in W_\sigma} \varepsilon_0(s)e^{s(\Lambda+\rho_2)(H)} \qquad (h=\exp H\in A_2') \qquad (4.4)$$

where
$$\Lambda+\rho_2 = \sqrt{-1}\lambda\tilde{\Lambda}_0+\Lambda^-+\rho^-\,.$$

Making use of (2) in Lemma 3, for each $\alpha\in P_2$ we get

$$\begin{aligned} &\langle\Lambda+\rho_2, \alpha\rangle\int_{A_2} e^{s(\Lambda+\rho_2)(H)}F_f^{(2)}(h)dh \\ &= \int_{A_2} \langle s(\Lambda+\rho_2), s\alpha\rangle e^{s(\Lambda+\rho_2)(H)}F_f^{(2)}(h)dh \\ &= \int_{A_2} \{\partial(s\alpha)e^{s(\Lambda+\rho_2)(H)}\}F_f^{(2)}(h)dh \\ &= -\int_{A_2} e^{s(\Lambda+\rho_2)(H)}F_f^{(2)}(h\,;\,\partial(s\alpha))dh\,. \end{aligned}$$

Applying this formula repeatedly we conclude that

$$\begin{aligned} &\langle\Lambda+\rho_2, \pi_2\rangle\int_{A_2} e^{s(\Lambda+\rho_2)(H)}F_f^{(2)}(h)dh \\ &= (-1)^n\int_{A_2} e^{s(\Lambda+\rho_2)(H)}F_f^{(2)}(h\,;\,\partial(s\pi_2))dh \\ &= (-1)^n\varepsilon(s)\int_{A_2} e^{s(\Lambda+\rho_2)(H)}F_f^{(2)}(h\,;\,\partial(\pi_2))dh\,. \end{aligned}$$

Hence from (4.3) and (4.4) we have

$$\langle\Lambda+\rho_2, \pi_2\rangle T_{\lambda,\Lambda^-}(f) = (-1)^n\varepsilon(\Lambda^-)\int_{A_2} \tilde{\eta}_{\lambda,\Lambda^-}(h)F_f^{(2)}(h\,;\,\partial(\pi_2))dh \qquad (4.5)$$

where $$\tilde{\eta}_{\lambda,\Lambda^-}(h) = \sum_{s\in W_\sigma} \varepsilon_0(s)\varepsilon(s)e^{s(\Lambda+\rho_2)(H)} \qquad (h=\exp H\in A_2),$$
$$= 2\sqrt{-1}\sin\lambda t \sum_{s\in W_-} e^{s(\Lambda^-+\rho^-)(H^-)} \quad (h=a_t\exp H^-\in A_2).$$

By a complex root $\alpha$, we mean a root $\alpha$ in $\Sigma_2$ such that $\alpha(\mathfrak{a})\neq(0)$ and $\alpha(\mathfrak{h}^-)\neq(0)$. It is easy to see that for any $\alpha\in P_2^+$, ${}^t\sigma\alpha=\alpha$ if and only if $\alpha=\tilde{\alpha}_0$. Since ${}^t\sigma P_2^+=P_2^+$, it follows that the number $r$ of all complex roots in $P_2$ is even. Moreover there exists a subset $P_2'$ of $P_2^+-\{\tilde{\alpha}_0\}$ such that ${}^t\sigma P_2'\cap P_2'=\phi$ and $P_2^+-\{\tilde{\alpha}_0\}={}^t\sigma P_2'\cap P_2'$.

Now we shall prove that

$$1+\frac{1}{2}r=q. \tag{4.6}$$

Clearly we have

$$\dim\mathfrak{g}=l+2n, \quad \dim\mathfrak{k}=l+2(n-q)$$
$$\dim\mathfrak{a}=1 \quad \text{and} \quad \dim\mathfrak{n}=1+r.$$

It follows from (4.2) that

$$l+2n=l+2(n-q)+1+(1+r).$$

Hence we have $1+\frac{1}{2}r=q$. Thus our assertion is proved. Since ${}^t\sigma(\sqrt{-1}\lambda\tilde{\Lambda}_0+\Lambda^-+\rho^-)=\sqrt{-1}\lambda\tilde{\Lambda}_0-\Lambda^--\rho^-$, we have

$$\begin{aligned}
&\langle\Lambda+\rho_2,\pi_2\rangle\\
&=\langle\sqrt{-1}\lambda\tilde{\Lambda}_0+\Lambda^-+\rho^-,\tilde{\alpha}_0\rangle\prod_{\alpha\in P_2^-}\langle\sqrt{-1}\lambda\tilde{\Lambda}_0+\Lambda^-+\rho^-,\alpha\rangle\\
&\times\prod_{\alpha\in P_2'}\{\langle\sqrt{-1}\tilde{\Lambda}_0+\Lambda^-+\rho^-,\alpha\rangle\langle\sqrt{-1}\tilde{\Lambda}_0+\Lambda^-+\rho^-,{}^t\sigma\alpha\rangle\}\\
&=\sqrt{-1}\lambda\frac{\alpha_0(H_{\alpha_0})}{2}\langle\Lambda^-+\rho^-,\pi_-\rangle\prod_{\alpha\in P_2'}\{-|\langle\sqrt{-1}\lambda\tilde{\Lambda}_0+\Lambda^-+\rho^-,\alpha\rangle|^2\}\\
&=\sqrt{-1}(-1)^{r/2}\lambda\frac{\alpha_0(H_{\alpha_0})}{2}\langle\Lambda^-+\rho^-,\pi_-\rangle\prod_{\alpha\in P_2'}|\langle\sqrt{-1}\tilde{\Lambda}_0+\Lambda^-+\rho^-,\alpha\rangle|^2.
\end{aligned}$$

It follows that

$$\langle\Lambda+\rho_2,\pi_2\rangle=\sqrt{-1}(-1)^{r/2}\operatorname{sgn}(\lambda)\varepsilon(\Lambda^-)|\langle\Lambda+\rho_2,\pi_2\rangle|. \tag{4.7}$$

From (4.5)~(4.7), noticing that $\langle\Lambda+\rho_2,\pi_2\rangle=\langle\sqrt{-1}\lambda\tilde{\Lambda}_0+\Lambda^-+\rho^-,\pi\rangle$, we get

$$\begin{aligned}
&|\langle\sqrt{-1}\lambda\tilde{\Lambda}_0+\Lambda^-+\rho^-,\pi\rangle|\,T_{\lambda,\Lambda^-}(f)\\
&\quad=2(-1)^{n+q-1}\operatorname{sgn}(\lambda)\int_{-\infty}^{\infty}\int_{A^-}\sin\lambda t\sum_{s\in W_-}e^{s(\Lambda^-+\rho^-)(H^-)}\\
&\quad\times F_f^{(2)}(a_th^-;\partial(\pi_2))dtdh^-.
\end{aligned}\tag{4.8}$$

From Lemma 7, the right hand side of this equation is equal to

$$4(-1)^{n+q-1}\operatorname{sgn}(\lambda)\int_0^\infty\int_{A^-}\sin\lambda t\sum_{s\in W_-}e^{s(\Lambda^-+\rho^-)(H^-)}F_f^{(2)}(a_th^-;\partial(\pi_2))dtdh^-.$$

Since $F_f^{(2)}(a_th^-;\partial(\pi_2))$ is clearly invariant by the operation of any $s\in W_-$, we have

$$\begin{aligned}&\int_{A^-}e^{s(\Lambda^-+\rho^-)(H^-)}F_f^{(2)}(a_th^-;\partial(\pi_2))dh^-\\ &=\int_{A^-}e^{(\Lambda^-+\rho^-)(s^{-1}H^-)}F_f^{(2)}(a_th^-;\partial(\pi_2))dh^-\\ &=\int_{A^-}e^{(\Lambda^-+\rho^-)(H^-)}F_f^{(2)}(a_th^-;\partial(\pi_2))dh^-.\end{aligned}\tag{4.9}$$

From the well known fact about Fourier series, making use of (2) in Lemma 3 we can show that the series

$$\sum_{\Lambda^-\in\mathfrak{F}_k^-}\int_{A^-}e^{s(\Lambda^-+\rho^-)(H^-)}F_f^{(2)}(a_th^-;\partial(\pi_2))dh^-\tag{4.10}$$

is convergent absolutely and uniformly with respect to the variable $t$. Therefore from (4.9) we get

$$\begin{aligned}&\sum_{\Lambda^-\in\mathfrak{F}_k^-}\int_0^\infty\int_{A^-}\sin\lambda t\sum_{s\in W_-}e^{s(\Lambda^-+\rho^-)(H^-)}F_f^{(2)}(a_th^-;\partial(\pi_2))dtdh^-\\ &=\sum_{s\in W_-}\int_0^\infty\left\{\sum_{\Lambda^-\in\mathfrak{F}_k^-}\int_{A^-}e^{s(\Lambda^-+\rho^-)(H^-)}F_f^{(2)}(a_th^-;\partial(\pi_2))\,dh^-\right\}\sin\lambda tdt\\ &=\sum_{s\in W_-}\int_0^\infty\left\{\sum_{\Lambda^-\in\mathfrak{F}_k^-}\int_{A^-}e^{(\Lambda^-+\rho^-)(H^-)}F_f^{(2)}(a_th^-;\partial(\pi_2))dh^-\right\}\sin\lambda tdt\\ &=w_-\int_0^\infty\left\{\sum_{\Lambda^-\in\mathfrak{F}_k^-}\int_{A^-}e^{(\Lambda^-+\rho^-)(H^-)}F_f^{(2)}(a_th^-;\partial(\pi_2))dh^-\right\}\sin\lambda tdt.\end{aligned}$$

From this and (4.8) our proposition follows immediately.

**Lemma 10.** *Let $a$ be a real number such that $0<a<m$. Then*

$$\int_0^\infty\frac{\operatorname{ch}((m-2a)t/2)}{\operatorname{sh}(mt/2)}\sin\lambda tdt=\frac{\pi}{m}\frac{\operatorname{sh}(2\pi\lambda/m)}{\operatorname{ch}(2\pi\lambda/m)-\cos(2\pi a/m)}.$$

For the proof, see [5] (p. 147).

Making use of this lemma and Proposition 6, from the well known theory of Fourier transforms we have

$$(4.11)\quad \int_0^\infty \left\{ \sum_{\Lambda^-\in\mathfrak{F}_k^-} \int_{A^-} e^{(\Lambda^-+\rho^-)(H^-)} F_f^{(2)}(a_t h^-;\ \partial(\pi_2))dh^- \right\} \frac{\mathrm{ch}\,((m-2a)/t)}{\mathrm{sh}\,(mt/2)}\,dt$$
$$= \frac{1}{2mw_-}(-1)^{n+q-1}\int_0^\infty \frac{\mathrm{sh}\,(2\pi\lambda/m)}{\mathrm{ch}\,(2\pi\lambda/m)-\cos(2\pi a/m)}$$
$$\times\left\{ \sum_{\Lambda^-\in\mathfrak{F}_k^-} |\langle\sqrt{-1}\lambda\Lambda_0^+ +\Lambda^- +\rho^-,\ \pi\rangle|\, T_{\lambda,\Lambda^-}(f)\right\} d\lambda\,.$$

This formula is valid for a real number $a$ such that $0<a<m$, in particular for $a=k$ $(0<k<m)$. Since $T_{\lambda,\Lambda^-}(f)$ is a Fourier transform of a function of class $C^\infty$ with the compact support, it follows that the integral on the right hand side in (4.11) is convergent uniformly for sufficiently small $a$. In view of Lemma 8 similar statements hold for the left hand side in (4.11). Tending $a$ to zero, we conclude that the equality (4.11) is valid also for $a=0$. Hence the equality (4.11) is valid for all integers $a=k$ such that $0\leqslant k<m$. Therefore from Proposition 5 we have

$$\sum_{\Lambda\in\mathfrak{F}} \int_{A_2} \tilde{\eta}_\Lambda^{(2)}(h) F_f^{(2)}(h;\ \partial(\pi_2))dh$$
$$= \frac{w}{mw_-}(-1)^{n+q-1}\sum_{k=0}^{m-1}\sum_{\Lambda^-\in\mathfrak{F}_k^-}\int_0^\infty \frac{\mathrm{sh}\,(2\pi\lambda/m)}{\mathrm{ch}\,(2\pi\lambda/m)-\cos(2\pi k/m)}$$
$$\times |\langle\sqrt{-1}\lambda\Lambda_0^+ +\Lambda^- +\rho^-,\ \pi\rangle|\, T_{\lambda,\Lambda^-}(f)d\lambda\,.$$

From this, Theorem 1, Proposition 3′ and Proposition 4, we have

$$(4.12)\quad \sum_{\Lambda\in\mathfrak{F}} |\langle\Lambda+\rho,\ \pi\rangle|\, T_\Lambda(f)$$
$$= wcf(e)-\frac{w}{mw_-}\sum_{k=0}^{m-1}\sum_{\Lambda^-\in\mathfrak{F}_k^-}\int_0^\infty \frac{\mathrm{sh}\,(2\pi\lambda/m)}{\mathrm{ch}\,(2\pi\lambda/m)-\cos(2\pi k/m)}$$
$$\times |\langle\overline{-1}\lambda\Lambda_0^+ +\Lambda^- +\rho^-,\ \pi\rangle|\, T_{\lambda,\Lambda^-}(f)d\lambda\,.$$

We put

$$\mathfrak{F}_0^- = \{\Lambda^-\in\mathfrak{F}^- :\ s\Lambda^-\leqslant\Lambda^- \quad \text{for all} \quad s\in W_-\} \quad \text{and} \quad (\mathfrak{F}_k^-)_0 = \mathfrak{F}_k^-\cap\mathfrak{F}_0^-.$$

Then clearly $\mathfrak{F}_0^- = \bigcup_{k=0}^{m-1}(\mathfrak{F}_k^-)_0$ (disjoint sum). It is easy to see that

$$|\langle\Lambda^s+\rho,\ \pi\rangle|\, T_{\Lambda^s}(f) = |\langle\Lambda+\rho,\ \pi\rangle|\, T_\Lambda(f) \qquad (s\in W)\,,$$
$$|\langle\sqrt{-1}\lambda\Lambda_0+(\Lambda^-)^s+\rho^-,\ \pi\rangle|\, T_{\lambda,(\Lambda^-)^s}(f) = |\langle\sqrt{-1}\lambda\Lambda_0+\Lambda^-+\rho^-,\ \pi\rangle|\, T_{\lambda,\Lambda^-}(f)$$
$$(s\in W_-)\,.$$

Therefore noticing that $|\langle\Lambda+\rho,\ \pi\rangle| = \langle\Lambda+\rho,\ \pi\rangle$ $(\Lambda\in\mathfrak{F}_0)$, from (4.12) we have

$$
\begin{aligned}
cf(e) &= \sum_{\Lambda\in\mathfrak{F}_0} \langle \Lambda+\rho,\ \pi\rangle T_\Lambda(f) \\
&\quad + \frac{1}{m}\sum_{k=0}^{m-1} \sum_{\Lambda^-\in(\mathfrak{F}_k^-)_0} \int_0^\infty \frac{\operatorname{sh}(2\pi\lambda/m)}{\operatorname{ch}(2\pi\lambda/m)-\cos 2\pi k/m} \\
&\quad \times |\langle \sqrt{-1}\lambda\Lambda_0^+ + \Lambda^- + \rho^-,\ \pi\rangle|\, T_{\lambda,\Lambda^-}(f)d\lambda\,.
\end{aligned}
\tag{4.13}
$$

Now let $\Omega$ denote the set of all equivalence classes of irreducible unitary representations of $M$. Then it is well known that the correspondence $\Lambda^-\to\delta_{\Lambda^-}$ $(\Lambda^-\in\mathfrak{F}_0^-)$ is a bijection of $\mathfrak{F}_0^-$ onto $\Omega$.

We put

$$\Omega_k = \{\delta_{\Lambda^-}\in\Omega :\ \Lambda^-\in(\mathfrak{F}_k^-)_0\}\,.$$

Then $\Omega=\bigcup_{k=0}^{m-1}\Omega_k$ (disjoint sum). When $\Lambda^-\in\mathfrak{F}_0^-$ corresponds to $\delta\in\Omega$, we put

$$
\begin{aligned}
T_{\lambda,\delta} &= T_{\lambda,\Lambda^-} \\
d_{\lambda,\delta} &= \frac{|\langle \sqrt{-1}\lambda\Lambda_0 + \Lambda^- + \rho^-,\ \pi\rangle|}{\langle\rho,\ \pi\rangle}\,.
\end{aligned}
\tag{4.14}
$$

And we also put

$$d_\Lambda = \frac{\langle\Lambda+\rho,\ \pi\rangle}{\langle\rho,\ \pi\rangle} \qquad (\Lambda\in\mathfrak{F}_0)\,. \tag{4.15}$$

Since $c$ is a positive constant and $\langle\rho,\ \pi\rangle>0$, we can normalize the Haar measure of $G$ such that

$$c = \langle\rho,\ \pi\rangle\,. \tag{4.16}$$

Hence from (4.13) we finally get

$$
\begin{aligned}
f(e) &= \sum_{\Lambda\in\mathfrak{F}_0} d_\Lambda T_\Lambda(f) \\
&\quad + \frac{1}{m}\sum_{k=0}^{m-1} \sum_{\delta\in\Omega_k} \int_0^\infty \frac{\operatorname{sh}(2\pi\lambda/m)}{\operatorname{ch}(2\pi\lambda/m)-\cos(2\pi k/m)}\, d_{\lambda,\delta}T_{\lambda,\delta}(f)d\lambda
\end{aligned}
$$

under the above defined normalization of the Haar measure of $G$. Thus we have obtained the following result.

**Theorem 2.** *The Haar measure of $G$ can be so normalized that*

$$
\begin{aligned}
f(e) &= \sum_{\Lambda\in\mathfrak{F}_0} d_\Lambda T_\Lambda(f) \\
&\quad + \frac{1}{m}\sum_{k=0}^{m-1} \sum_{\delta\in\Omega_k} \int_0^\infty \frac{\operatorname{sh}(2\pi\lambda/m)}{\operatorname{ch}(2\pi\lambda/m)-\cos(2\pi k/m)}\, d_{\lambda,\delta}T_{\lambda,\delta}(f)d\lambda
\end{aligned}
$$

*for all $f\in C_c^\infty(G)$.*

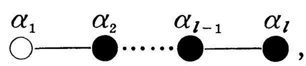
REMARK 3. With a slight modification, our method can be applied also to $SL(2, \boldsymbol{R})$ which does not satisfy the condition (A. 4) (see § 5).

REMARK 4. It is instructive to compare (4. 14), (4. 15) with the formula for the formal degree given by Harish-Chandra (see [3 (e)] p. 612).

## 5. Some consequences of (A. 1), (A. 2) and (A. 3)

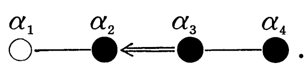
In this section we shall make use of the notation of § 1 and [4] without further comment.

Let $G$ be a simple Lie group which satisfies the conditions (A. 1), (A. 2) and (A. 3). It is known that the Lie algebra $\mathfrak{g}$ of such a group is one of the types $\mathfrak{su}(l, 1)$ $(l \geqslant 1)$, $\mathfrak{so}(2l, 1)$ $(l \geqslant 2)$, $\mathfrak{sp}(l-1, 1)$ $(l \geqslant 2)$ and FII (see [8]). The diagrams of the complexifications $\mathfrak{g}^C$ of these Lie algebras are as follows;

(I) $\mathfrak{su}(l, 1)$ 

(II) $\mathfrak{so}(2l, 1)$ 

(III) $\mathfrak{sp}(l-1, 1)$ 

(IV) FII 

In these diagrams the white vertex ○ denotes the unique non compact simple root of $\mathfrak{g}^C$ with respect to $\mathfrak{h}_1^C$ under a certain linear order. From these diagrams we see easily that

$$\alpha_1(H_2) = -1, \tag{5. 1}$$

if rank $G \geqslant 2$.

Therefore the value of $\alpha_1$ at $H = 2\pi\sqrt{-1}\sum_{i=1}^{l} a_i H_i$ in $\mathfrak{h}_1$ is equal to $\alpha_1(H) = 2\pi\sqrt{-1}(2a_1 - a_2)$. It follows that in the case of rank $G \geqslant 2$,

$$H \in \tilde{\mathfrak{h}}^- \text{ if and only if } 2a_1 = a_2, \tag{5. 2}$$

where $$\tilde{\mathfrak{h}}^- = \{H \in \mathfrak{h}_1 : \alpha_1(H) = 0\}.$$

Put $\tilde{\mathfrak{h}}_2 = \sqrt{-1}\boldsymbol{R}U_{\alpha_1} + \tilde{\mathfrak{h}}^-$. Then $\tilde{\mathfrak{h}}_2$ is a Cartan subalgebra of $\mathfrak{g}$ which is not conjugate to $\mathfrak{h}_1$ (c.f. § 1). Let $\tilde{A}_2$ be the Cartan subgroup corresponding to $\tilde{\mathfrak{h}}_2$. Then from (A. 3) $A_2$ is connected if and only if $\tilde{A}_2$ is connected.

**Proposition 7.** *Let* $\mathfrak{g}$ *be one of the above types* (I), (II) *and* (III).

*Then every Cartan subgroup of $G$ is connected if and only if rank $G \geqslant 2$.*

Proof. In view of (A. 1), in the cases (I), (III) we have $G = \boldsymbol{SU}(l, 1)$ or $\boldsymbol{S_p}(l-1, 1)$. The proposition is verified immediately in these cases. In the case (II), $G$ is a proper covering group of the identity component $\boldsymbol{SO}_0(2l, 1)$ of $\boldsymbol{SO}(2l, 1)$ and

$$G/Z \cong \boldsymbol{SO}_0(2l, 1) \qquad \text{(isomorphic)} \tag{5.3}$$

where $Z$ is the center of $G$.

We remark that every Cartan subgroup of $\boldsymbol{SO}_0(2l, 1)$ $(l \geqslant 2)$ is connected. On the other hand it is clear that $\tilde{A}^- = \exp \mathfrak{h}^- \subset (\tilde{A}_2)_0$, where $(\tilde{A}_2)_0$ is the connected component of $\tilde{A}_2$. Therefore, if we show

$$Z \subset \tilde{A}^-, \tag{5.4}$$

it follows from (5. 3) that $\tilde{A}_2 = (\tilde{A}_2)_0 Z = (\tilde{A}_2)_0$, which proves that $\tilde{A}_2$ is connected. Since compact Cartan subgroups are connected (see § 1), this will prove Proposition 7.

Now we come to the proof of (5. 4). Let $z \in Z$. Then from the definition of $A_1$ we have $z \in A_1$. Suppose $H = 2\pi\sqrt{-1}\sum_{i=1}^{l} c_i H_i \in \mathfrak{h}_1$ be such that $z = \exp H$. Then $\alpha_1(H)$ is an integral multiple of $2\pi\sqrt{-1}$, that is, there exists an integer $l$ such that $2c_1 - c_2 = l$ (see (5. 1)). Since $\exp(2\pi\sqrt{-1}\,H_2)$ is the identity element of $G$ (see § 1), it follows that $\exp H = \exp(2\pi\sqrt{-1}(c_1H_1 + 2c_1H_2 + \sum_{i=3}^{l} c_iH_i))$. In view of (5. 2), the right hand side belongs to $\tilde{A}^- = \exp \mathfrak{h}^-$. This proves (5. 4) and so Proposition 7.

**Proposition 8.** *Under the assumptions* (A. 1), (A. 2) and (A. 3) *put* $\mathfrak{h}^+ = \sqrt{-1}\boldsymbol{R}H_1$, *and* $A^+ = \exp \mathfrak{h}^+$. *Then the order $m$ of the cyclic group $D = A^+ \cap A^-$ is equal to 2 if and only if rank $G \geqslant 2$. Moreover, $D$ consists of the elements* $\exp(k\pi\sqrt{-1}H_1)$ $(k = 0, 1)$.

Proof. Since $\alpha_0$ was arbitrary fixed non compact root in $\Sigma_1$, we may assume $\alpha_0 = \alpha_1$ by changing the order introduced $\Sigma_1$ if necessary. "Only if" part of the proposition is trivial. So we assume that rank $G \geqslant 2$. Suppose that $\exp H^+ = \exp H^-$ $(H^+ \in \mathfrak{h}^+, H^- \in \mathfrak{h}^-)$. Then $H^+ - H^- \in \Gamma$ (see § 1). On the other hand it follows from (5. 2) that

$$H^+ = 2\pi\sqrt{-1}\,aH_1\,, \quad H^- = 2\pi\sqrt{-1}\,(a_1H_1 + 2a_1H_2 + \sum_{i=3}^{l} a_iH_i)\,,$$

where $a, a_1, a_3, \cdots, a_l$ are real numbers (see § 3). Therefore, in order that $H^+ - H^- \in \Gamma$, it is necessary and sufficient that $a - a_1$, $2a_1$, $a_3, \cdots, a_l$ are integers. From this, it follows that $D$ consists of the elements

$$\exp(k\pi\sqrt{-1}\,H_1) \qquad (k=0,1).$$

Thus Proposition 8 is proved.

Making use of Proposition 8, we obtain the following improved version of Theorem 2.

**Theorem 2′.** *Let $G$ be a simple Lie group which satisfies the conditions* (A. 1)~(A. 5).

*Then the Haar measure of $G$ can be so normalized that*

$$f(e) = \sum_{\Lambda\in\mathfrak{F}_0} d_\Lambda T_\Lambda(f) + \frac{1}{2}\sum_{k=0}^{1}\sum_{\delta\in\Omega_{k+1}}\int_0^\infty \mathrm{th}\,(\pi(\lambda+k\sqrt{-1})/2)\, d_{\lambda,\delta}T_{\lambda,\delta}(f)d\lambda\,.$$

For the proof of this theorem, we have only to notice that

$$\frac{\mathrm{sh}\,\pi\lambda}{\mathrm{ch}\,\pi\lambda-\cos\pi(k+1)} = \mathrm{th}\,(\pi(\lambda+k\sqrt{-1})/2) \qquad (k=0,1).$$

## 6. Universal covering group of De Sitter group

In this section we shall prove that the formula of Theorem 2 actually gives the explicit Plancherel formula for the universal covering group $G$ of De Sitter group. Let $\boldsymbol{Q}$ be the usual quaternion field. For any $q\in\boldsymbol{Q}$ let $\bar{q}$ denote the conjugate quaternion of $q$. The field of complex numbers $\boldsymbol{C}$ can be canonically identified with a subfield of $\boldsymbol{Q}$. Let $G$ be the group of all matrices $g$ of degree 2 with coefficients in the quaternion field, satisfying the condition;

$$g\sigma g^* = \sigma\,,$$

where $\sigma=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ and $g^*=\begin{pmatrix}\bar{a}&\bar{c}\\\bar{b}&\bar{d}\end{pmatrix}$ if $g=\begin{pmatrix}a&b\\c&d\end{pmatrix}$. Then $G$ is isomorphic to the universal covering group of De Sitter group (see [9 (b)]). It follows that $G$ satisfies the conditions (A. 1)~(A. 4) (see § 5). We put

$$a_t = \begin{pmatrix}\mathrm{ch}\,t/2 & \mathrm{sh}\,t/2\\ \mathrm{sh}\,t/2 & \mathrm{ch}\,t/2\end{pmatrix},\quad u_\theta = \begin{pmatrix}e^{i\theta/2} & 0\\ 0 & e^{-i\theta/2}\end{pmatrix} \text{ and } m_\varphi = \begin{pmatrix}e^{i\varphi/2} & 0\\ 0 & e^{i\varphi/2}\end{pmatrix},$$

where $i$ is the imaginary unit of $\boldsymbol{C}$. Let

$$A_1 = \{u_\theta m_\varphi :\ \theta,\ \varphi\in\boldsymbol{R}\}\,,$$
$$A_2 = \{a_t m_\varphi :\ t,\ \varphi\in\boldsymbol{R}\}\,.$$

Then $A_1$ and $A_2$ are the non conjugate Cartan subgroups of $G$. Every Cartan subgroup of $G$ is conjugate with either $A_1$ or $A_2$ (see [8]). The Lie algebra of $A_1$ is

$$\mathfrak{h}_1 = \left\{H(\theta, \varphi) = \begin{pmatrix} i(\varphi+\theta)/2 & 0 \\ 0 & i(\varphi-\theta)/2 \end{pmatrix} : \quad \theta, \varphi \in \boldsymbol{R}\right\}.$$

We define two linear forms $\Lambda_1$ and $\Lambda_2$ on $\mathfrak{h}_1$ by

$$\Lambda_1(H(\theta, \varphi)) = i(\varphi+\theta)/2\,,$$
$$\Lambda_2(H(\theta, \varphi)) = i\theta\,.$$

Then we can show that

$$\mathfrak{F}_0 = \{l\Lambda_1+m\Lambda_2 : \quad l \geqslant 0,\ m \geqslant 0,\ l, m \in \boldsymbol{Z}\}$$

under a certain linear order in the dual space of $\sqrt{-1}\mathfrak{h}_1$.

Suppose that $\qquad n = m+\dfrac{l}{2}+1 \quad \text{and} \quad p = \dfrac{l}{2}+1\,.$

Then if $l$ and $m$ are both non negative integers, $n$ and $p$ are half integers such that $n \geqslant p \geqslant 1$ and $n-p \in \boldsymbol{Z}$. Moreover we have

$$(l\Lambda_1+m\Lambda_2)\,(H(\theta, \varphi)) = i(2n+1)\theta/2+i(2p-1)\varphi/2\,.$$

Let $U_{n,\nu}$, $T_{n,p}$ denote the characters of the representations $U^{n,3/2+i\nu}$, $T^{n,0,p} \oplus T^{0,n,p}$ defined in [9 (b)] respectively and let $\chi^{(1)}_{n,\nu}$, $\chi^{(2)}_{n,p}$ be the locally summable functions on $G$ which coincide with $U_{n,\nu}$, $T_{n,p}$ as distributions respectively.

For any $\Lambda = l\Lambda_1+m\Lambda_2 \in \mathfrak{F}_0$, we define

$$T_\Lambda = T_{n,p} \quad \text{and} \quad d_{n,p} = d_\Lambda$$

where $\qquad n = m+\dfrac{l}{2}+1 \quad \text{and} \quad p = \dfrac{l}{2}+1\,.$

The following character formulas are due to T. Hirai;

$$\chi^{(1)}_{n,\nu}(u_\theta m_\varphi) = 0\,,$$

$$\chi^{(1)}_{n,\nu}(a_t m_\varphi) = \frac{(e^{i\nu t}+e^{-i\nu t})\,(e^{i(2n+1)\varphi/2}-e^{-i(2n+1)\varphi/2})}{\Delta_2(h)}\,, \tag{6.1}$$

$$\chi^{(2)}_{n,p}(u_\theta m_\varphi) =$$

$$\frac{1}{\Delta_1(h)}\{(e^{i(2n+1)\theta/2}-{}^{-i(2n+1)\theta/2})\,(e^{i(2p-1)\varphi/2}-e^{-i(2p-1)\varphi/2})-(e^{i(2p-1)\theta/2}-e^{-i(2p-1)\theta/2}) \tag{6.2}$$
$$\times(e^{i(2n+1)\varphi/2}-e^{-i(2n+1)\varphi/2})\}\,,$$

$$\chi^{(2)}_{n,p}(a_t m_\varphi) =$$
$$\frac{-2\{e^{-(2n+1)|t|/2}(e^{i(2p-1)\varphi/2}-e^{-i(2p-1)\varphi/2})-e^{-(2p-1)|t|/2}(e^{i(2n+1)\varphi/2}-e^{-i(2n+1)\varphi/2})\}}{\Delta_2(h)}.$$

It follows immediately that $G$ satisfies also the assumption (A. 5). Therefore we can apply Theorem 2′ to $G$. It is easy to see that

$$D = A_1 \cap A_2 = \left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\right\},$$
$$M = \left\{\begin{pmatrix}u & 0\\ 0 & u\end{pmatrix}:\ |u| = 1\right\}.$$

Let $\rho^n$ be the irreducible unitary representation of $U = \{u \in Q : |u| = 1\}$ of dimension $2n+1$ defined in [9 (b)], where $n$ is a half integer i.e. $2n \in \boldsymbol{Z}$. We define the irreducible unitary representation $\tilde{\rho}^n$ of $M$ by

$$\tilde{\rho}^n(m) = \rho^n(u) \quad \text{for} \quad m = \begin{pmatrix}u & 0\\ 0 & u\end{pmatrix} \in M.$$

Then we have

$$\Omega_0 = \{\tilde{\rho}^n:\ \ 2n \equiv 1 \pmod 2\},$$
$$\Omega_1 = \{\tilde{\rho}^n:\ \ 2n \equiv 0 \pmod 2\}.$$

It is easy to see that

$$d_{n,p} = (2n+1)(2p-1)(n+p)(n-p+1)/6,$$
$$d_{\lambda,\tilde{\rho}^n} = |(2n+1)\lambda[(2n+1)^2+\lambda^2]|/24.$$

Since $\lambda = 2\nu$ and $T_{n,p} = T_{n,0,p} + T_{0,n,p}$, it follows from Theorem 2′ that

$$6\ f(e) = \sum_{n\geqslant 1}(2n+1)\sum_{n\geqslant p\geqslant 1}(2p-1)(n+p)(n-p+1)\{T_{n,0,p}(f)+T_{0,n,p}(f)\}$$
$$+2\sum_{n\geqslant 0}(2n+1)\int_0^\infty \operatorname{th}(\pi(\nu+\sqrt{-1}n)t)\nu\left[\left(n+\frac{1}{2}\right)^2+\nu^2\right]U_{n,\nu}(f)d\nu$$

under the normalization of the Haar measure of $G$ such that (4. 16) holds.

Let $d^{n,0,p}$ be the formal degree of $T^{n,0,p}$ (see [3 (e)]). From Remark 5. 2 in [9 (b)] (p. 431), we have

$$d^{n,0,p} = (2n+1)(2p-1)(n+p)(n-p+1)/16\pi^2,$$

under the normalization of the Haar measure of $G$ introduced in [9 (b)].

Therefore from the uniqueness of the Plancherel measure, we have the following result.

**Theorem 3.** *Let* $T_{n,0,p}$, $T_{0,n,p}$ *and* $U_{n,\nu}$ *be the characters of the* *representations* $T^{n,0,p}$, $T^{0,n,p}$ *and* $U^{n,3/2+i\nu}$ *respectively. Then*

$$f(e)=\frac{1}{16\pi^2}\sum_{n\geqslant 1}(2n+1)\sum_{n\geqslant p\geqslant 1}(2p-1)(n+p)(n-p+1)\{T_{n,0,p}(f)+T_{0,n,p}(f)\}$$
$$+\frac{1}{8\pi^2}\sum_{n\geqslant 0}(2n+1)\int_0^\infty \mathrm{th}(\pi(\nu+\sqrt{-1}n))\nu\left[\left(n+\frac{1}{2}\right)^2+\nu^2\right]U_{n,\nu}(f)d\nu\,,$$

*under the normalization of the Haar mearure of G that is introduced in* [9 (b)].

This formula was conjectured by R. Takahashi in [9 (b)] (p. 432).

REMARK 5. As far as the author knows, the explicit character formulas of the representations $T^{n,0,p}$ and $T^{0,n,p}$ are not known although the character of the representation $T^{n,0,p}\oplus T^{0,n,p}$ is known (see (6.1), (6.2)). As we saw above, in order to obtain the explicit Plancherel formula, it is sufficient for us only to know the character of the representation $T^{n,0,p}\oplus T^{0,n,p}$. These facts suggest that it is natural to consider the character $T_\Lambda$ which is the sum of the characters of irreducible unitary representations having the same infinitesimal and central character (see the proof of Proposition 3).

REMARK 6. There are some misprints in the character formulas given in [6]. The correct formulas should be (6.1) and (6.2). Other misprints in [6] are as follows; p. 24, line 12, read "$+\frac{1}{4}\cdots$" instead of "$-\frac{1}{4}\cdots$"; p. 26, line 16, read "Lemma 2" instead of "Theorem 1", line 19, read "$-H_0H_2(H_0^2-H_2^2)$" instead of "$H_0H_2(H_0^2-H_2^2)$".

## 7. Proof of Theorem 1

In this section we shall give a proof of Theorem 1. In view of the definition of $T_\Lambda$, it is sufficient to prove the theorem when $\Lambda+\rho\in\mathfrak{F}_0$. Let $\alpha_1,\alpha_2,\cdots,\alpha_n$ be all the distinct roots in $P_1$. Then $P_2=\{\tilde{\alpha}_1,\tilde{\alpha}_2,\cdots,\tilde{\alpha}_n\}$. Put

$$p_r=\prod_{i=1}^r\alpha_i\,,\quad \tilde{p}_r=\prod_{i=1}^r\tilde{\alpha}_i\,.$$

Then we have

$$sp_r=\prod_{i=1}^r s\alpha_i\,,\quad \tilde{s}\tilde{p}_r=\prod_{i=1}^r\tilde{s}\tilde{\alpha}_i\qquad\text{for all}\quad s\in W\,.$$

Fix an element $\Lambda\in\mathfrak{F}$ such that $\Lambda+\rho\in\mathfrak{F}_0$. We shall prove the following by the induction on $r$ $(0\leqslant r\leqslant n)$.

$(P_r)$
$$\begin{aligned}&\langle\Lambda+\rho, p_r\rangle T_\Lambda(f)\\&=(-1)^{r+q}\sum_{s\in W}\Big\{\varepsilon(s)\int_{A_1} e^{s(\Lambda+\rho)(H)}F_f^{(1)}(h\,;\,\partial(sp_r))\,dh\\&+2\varepsilon_2(s)\int_{+A_{\Lambda^s}} e^{\tilde{s}(\tilde{\Lambda}+\tilde{\rho})(H)}F_f^{(2)}(h\,;\,\partial(\tilde{s}\tilde{p}_r))dh\Big\}\,.\end{aligned}$$

**Lemma 11.**
$$\int_{-A_{\Lambda^s}} e^{\tilde{s}_0\tilde{s}(\tilde{\Lambda}+\tilde{\rho})(H)}F_f^{(2)}(h\,;\,\partial(\tilde{s}_0\tilde{s}\tilde{p}_r))\,dh=\int_{+A_{\Lambda^s}} e^{\tilde{s}(\tilde{\Lambda}+\tilde{\rho})(H)}F_f^{(2)}(h\,;\,\partial(\tilde{s}\tilde{p}_r))\,dh\,.$$

Proof. We use the notation of the proof of Lemma 7. Since $Adk_0({}_+A_{\Lambda^s})={}_-A_{\Lambda^s}$, we have
$$\begin{aligned}&\int_{-A_{\Lambda^s}} e^{\tilde{s}_0\tilde{s}(\tilde{\Lambda}+\tilde{\rho})(H)}F_f^{(2)}(h\,;\,\partial(\tilde{s}_0\tilde{s}\tilde{p}_r))\,dh\\&=\int_{+A_{\Lambda^s}} e^{\tilde{s}_0\tilde{s}(\tilde{\Lambda}+\tilde{\rho})(\tilde{s}_0H)}F_f^{(2)}(k_0hk_0^{-1}\,;\,\partial(\tilde{s}_0\tilde{s}\tilde{p}_r))\,dh\\&=\int_{+A_{\Lambda^s}} e^{\tilde{s}(\tilde{\Lambda}+\tilde{\rho})(H)}(F_f^{(2)})^{k_0^{-1}}(h\,;\,\partial(\tilde{s}\tilde{p}_r))\,dh\\&=\int_{+A_{\Lambda^s}} e^{\tilde{s}(\tilde{\Lambda}+\tilde{\rho})(H)}F_f^{(2)}(h\,;\,\partial(\tilde{s}\tilde{p}_r))\,dh\,.\end{aligned}$$

Thus the lemma is proved.

Supposing $(P_n)$ is valid for a moment, we shall prove Theorem 1. In view of Lemma 11, $(P_n)$ implies that
$$\begin{aligned}&\langle\Lambda+\rho, \pi\rangle T_\Lambda(f)\\&=(-1)^{n+q}\sum_{s\in W}\Big\{\int_{A_1} e^{s(\Lambda+\rho)(H)}F_f^{(1)}(h\,;\,\partial(\pi))\,dh\\&+\varepsilon_2(s)\varepsilon(s)\int_{+A_{\Lambda^s}} e^{\tilde{s}(\tilde{\Lambda}+\tilde{\rho})(H)}F_f^{(2)}(h\,;\,\partial(\pi_2))\,dh+\varepsilon_2(s)\varepsilon(s)\\&\times\int_{-A_{\Lambda^s}} e^{\tilde{s}_0\tilde{s}(\tilde{\Lambda}+\tilde{\rho})(H)}F_f^{(2)}(h\,;\,\partial(\pi_2))\,dh\Big\}\,,\end{aligned}$$

since $p_n=\pi$, $\varepsilon(s_0)=-1$, $\partial(sp_n)=\varepsilon(s)\partial(\pi)$ and $\partial(\tilde{s}\tilde{p}_n)=\varepsilon(s)\partial(\pi_2)$ for all $s\in W$. Moreover, clearly we have the followings (see § 3).

$$\begin{aligned}&\varepsilon_2(s)\varepsilon(s)=-1\,,\quad {}_+A_{\Lambda^s}={}_-A_2\quad\text{and}\quad {}_-A_{\Lambda^s}={}_+A_2\qquad\text{if}\quad\langle s\rho, \alpha_0\rangle>0\,,\\&\varepsilon_2(s)\varepsilon(s)=1\,,\quad {}_+A_{\Lambda^s}={}_+A_2\quad\text{and}\quad {}_-A_{\Lambda^s}={}_-A_2\qquad\text{if}\quad\langle s\rho, \alpha_0\rangle<0\,.\end{aligned}$$

Therefore from the above formula we can derive the following;
$$\begin{aligned}&\langle\Lambda+\rho, \pi\rangle T_\Lambda(f)\\&=(-1)^{n+q}\sum_{s\in W}\Big\{\int_{A_1}\xi_{\Lambda^s}^{(1)}(h)F_f^{(1)}(h\,;\,\partial(\pi))dh\\&+\int_{+A_2}\xi_{\Lambda^s}^{(2)}(h)F_f^{(2)}(h\,;\,\partial(\pi_2))dh-\int_{-A_2}\xi_{\Lambda^s}^{(2)}(h)F_f^{(2)}(h\,;\,\partial\pi_2))\,dh\Big\}\,,\end{aligned}$$

(see (§ 3)). From the definition of $\tilde{\eta}_{\Lambda}^{(k)}$ $(k=1, 2)$, Theorem 1 is now obvious.

Now we come to the proof of $(P_r)$ $(0\leqslant r\leqslant n)$. Again making use of Lemma 11, we can show that $(P_0)$ is equivalent to

$$T_{\Lambda}(f) = (-1)^q \sum_{k=1}^{2} \int_{A_k} \eta_{\Lambda}^{(k)}(h) F_f^{(k)}(h) dh .$$

Since this is valid from the definition of $T_{\Lambda}$, our assertion follows immediately for $r=0$. Assume now that $(P_r)$ is valid for some $r$ $(0\leqslant r<n)$. It is easy to see that $\langle \Lambda+\rho, \alpha_{r+1}\rangle e^{s(\Lambda+\rho)(H)} = \langle s(\Lambda+\rho), s\alpha_{r+1}\rangle e^{s(\Lambda+\rho)(H)} = \partial(s\alpha_{r+1}) e^{s(\Lambda+\rho)(H)}$, $\langle \Lambda+\rho, \alpha_{r+1}\rangle e^{\tilde{s}(\tilde{\Lambda}+\tilde{\rho})(H)} = \langle \tilde{s}(\tilde{\Lambda}+\tilde{\rho}), \tilde{s}\tilde{\alpha}_{r+1}\rangle e^{\tilde{s}(\tilde{\Lambda}+\tilde{\rho})(H)} = \partial(\tilde{s}\tilde{\alpha}_{r+1}) e^{\tilde{s}(\tilde{\Lambda}+\tilde{\rho})(H)}$. Hence multiplying the both sides of $(P_r)$ by $\langle \Lambda+\rho, \alpha_{r+1}\rangle$, we get

$$\begin{aligned}
(7.1)\quad & \langle \Lambda+\rho, p_{r+1}\rangle T_{\Lambda}(f) \\
&= (-1)^{r+q} \sum_{s\in W} \Big\{ \varepsilon(s) \int_{A_1} [\partial(s\alpha_{r+1}) e^{s(\Lambda+\rho)(H)}] F_f^{(1)}(h\,;\, \partial(sp_r))\, dh \\
&+ 2\varepsilon_2(s) \int_{+A_{\Lambda}^s} [\partial(\tilde{s}\tilde{\alpha}_{r+1}) e^{\tilde{s}(\tilde{\Lambda}+\tilde{\rho})(H)}] F_f^{(2)}(h\,;\, \partial(\tilde{s}\tilde{p}_r))\, dh \Big\} \\
&= (-1)^{r+1+q} \sum_{s\in W} \Big\{ \varepsilon(s) \int_{A_1} e^{s(\Lambda+\rho)(H)} F_f^{(1)}(h\,;\, \partial(sp_{r+1}))\, dh \\
&+ 2\varepsilon_2(s) \int_{+A_{\Lambda}^s} e^{\tilde{s}(\tilde{\Lambda}+\tilde{\rho})(H)} F_f^{(2)}(h\,;\, \partial(\tilde{s}\tilde{p}_{r+1}))\, dh \Big\} \\
&+ (-1)^{r+q} \sum_{s\in W} \Big\{ \varepsilon(s) \int_{A_1} \partial(s\alpha_{r+1}) [e^{s(\Lambda+\rho)(H)} F_f^{(1)}(h\,;\, \partial(sp_r))]\, dh \\
&+ 2\varepsilon_2(s) \int_{+A_{\Lambda}^s} \partial(\tilde{s}\tilde{\alpha}_{r++1}) [e^{\tilde{s}(\tilde{\Lambda}+\tilde{\rho})(H)} F_f^{(2)}(h\,;\, \partial\tilde{s}\tilde{p}_r))]\, dh \Big\} .
\end{aligned}$$

Put

$$J_{\Lambda}^{(1)} = \sum_{s\in W} \varepsilon(s) \int_{A_1} \partial(s\alpha_{r+1}) [e^{s(\Lambda+\rho)(H)} F_f^{(1)}(h\,;\, \partial(sp_r))]\, dh ,$$

$$J_{\Lambda}^{(2)} = \sum_{s\in W} 2\varepsilon_2(s) \int_{+A_{\Lambda}^s} \partial(\tilde{s}\tilde{\alpha}_{r+1}) [e^{\tilde{s}(\tilde{\Lambda}+\tilde{\rho})(H)} F_f^{(2)}(h\,;\, \partial(\tilde{s}\tilde{p}_r))]\, dh .$$

Then the validity of $(P_{r+1})$ is equivalent to $J_{\Lambda}^{(1)}+J_{\Lambda}^{(2)}=0$. In order to prove that $J_{\Lambda}^{(1)}+J_{\Lambda}^{(2)}=0$, we need more precise informations about $J_{\Lambda}^{(1)}$ and $J_{\Lambda}^{(2)}$.

First we consider $J_{\Lambda}^{(1)}$. For any $\alpha\in P_1^0$, put $\sigma_{\alpha} = \{h=\exp H\in A_1 : \alpha(H)\in 2\pi\sqrt{-1}\boldsymbol{Z}\}$. It is easy to see that $P_1^0$ is exactly the set of all positive singular roots in $\Sigma_1$ (see [3 (h)] for the definition of a singular root).

Put $F^{(1)}(h:s) = F_f^{(1)}(h\,;\, \partial(sp_r))$. Then by making use of (1) in Lemma 3 we have the following (c.f. [3 (h)]).

$$(7.2)\qquad J_\Lambda^{(1)} = \sqrt{-1}\sum_{s\in W}\sum_{\alpha\in P_1^0}\frac{\varepsilon(s)\langle\alpha, s\alpha_{r+1}\rangle}{\sqrt{\alpha(H_\alpha)}}\int_{\sigma_\alpha} e^{s(\Lambda+\rho)(H)}F_\alpha^{(1)}(h:s)\,d\sigma_\alpha$$

where
$F_\alpha^{(1)}(h:s)=\lim_{\varepsilon\to+0}\{F^{(1)}(\exp(H+\varepsilon\sqrt{-1}H_\alpha):s)-F^{(1)}(\exp(H-\varepsilon\sqrt{-1}H_\alpha):s)\}$, (this limit always exists from (1) in Lemma 3). In this formula $d\sigma_\alpha$ denotes the canonical Lebesgue measure induced by $dh$ on $\sigma_\alpha$. Let $W_K$ denote the set of all elements $t\in W$ such that $\mathrm{Ad}(k)|\mathfrak{h}_1=t$ for some $k\in K$. Then for any $\alpha\in P_1^0$, there exists an element $t\in W_K$ such that $t\alpha_0=\alpha$. For each $\alpha\in P_1^0$, we fix such an element $t$ and denote it by $t_\alpha$. Then by definition, there exists an element $k\in K$ such that $\mathrm{Ad}(k)|\mathfrak{h}_1=t_\alpha$. We also fix such an element $k$ and denote it by $k_\alpha$. Then we have

$$\begin{aligned}
&F_f^{(1)}(\exp(t_\alpha H_t\pm\varepsilon\sqrt{-1}H_\alpha)\,;\,\partial(sp_r))\\
&= F_f^{(1)}(k_\alpha\exp(H\pm\varepsilon\sqrt{-1}t_\alpha^{-1}H_\alpha)k_\alpha^{-1}\,;\,\partial(sp_r))\\
&= (F_f^{(1)})^{k_\alpha^{-1}}(\exp((H\pm\varepsilon\sqrt{-1}H_{t_\alpha^{-1}\alpha})\,;\,\partial(t_\alpha^{-1}sp_r))\\
&= \varepsilon(t_\alpha)F_f^{(1)}(\exp(H\pm\varepsilon\sqrt{-1}H_{\alpha_0})\,;\,\partial(t_\alpha^{-1}sp_r))\,.
\end{aligned}$$

Hence

$$(7.3)\qquad F_\alpha^{(1)}(k_\alpha hk_\alpha^{-1}:s) = F_\alpha^{(1)}(\exp t_\alpha H:s) = \varepsilon(t_\alpha)F_{\alpha_0}^{(1)}(h:t_\alpha^{-1}s)\,.$$

Moreover it is easy to see that

$$(7.4)\qquad \alpha(H_\alpha)=\alpha_0(H_{\alpha_0})\,,\quad \langle\alpha, s\alpha_{r+1}\rangle = \langle\alpha_0, t_\alpha^{-1}s\alpha_{r+1}\rangle\,.$$

On the other hand, $d\sigma_{\alpha_0}$ goes to $d\sigma_\alpha$ under the mapping $h\to k_\alpha hk_\alpha^{-1}$ which maps $\sigma_{\alpha_0}$ onto $\sigma_\alpha$.

Therefore it follows from (7.2), (7.3) and (7.4) that

$$\begin{aligned}
J_\Lambda^{(1)} &= \sqrt{-1}\sum_{s\in W}\sum_{\alpha\in P_1^0}\frac{\varepsilon(s)\langle\alpha_0, t_\alpha^{-1}s\alpha_{r+1}\rangle}{\sqrt{\alpha_0(H_{\alpha_0})}}\int_{\sigma_{\alpha_0}} e^{s(\Lambda+\rho)(t_\alpha H)}\varepsilon(t_\alpha)F_{\alpha_0}^{(1)}(h:t_\alpha^{-1}s)\,d\sigma_{\alpha_0}\\
&= \sqrt{-1}\sum_{\alpha\in P_1^0}\left\{\sum_{s\in W}\frac{\varepsilon(t_\alpha^{-1}s)\langle\alpha_0, t_\alpha^{-1}s\alpha_{r+1}\rangle}{\sqrt{\alpha_0(H_{\alpha_0})}}\int_{\sigma_{\alpha_0}} e^{t_\alpha^{-1}s(\Lambda+\rho)(H)}F_{\alpha_0}^{(1)}(h:t_\alpha^{-1}s)\,d\sigma_{\alpha_0}\right.\\
&= \sqrt{-1}q\sum_{s\in W}\frac{\varepsilon(s)\langle\alpha_0, s\alpha_{r+1}\rangle}{\sqrt{\alpha_0(H_{\alpha_0})}}\int_{\sigma_{\alpha_0}} e^{s(\Lambda+\rho)(H)}F_{\alpha_0}^{(1)}(h:s)\,d\sigma_{\alpha_0}\,.
\end{aligned}$$

Put

$$(7.5)\qquad K_s^{(1)}(\Lambda) = \frac{\sqrt{-1}q\varepsilon(s)\langle\alpha_0, s\alpha_{r+1}\rangle}{\sqrt{\alpha_0(H_{\alpha_0})}}\int_{\sigma_{\alpha_0}} e^{s(\Lambda+\rho)(H)}F_{\alpha_0}^{(1)}(h:s)\,d\sigma_{\alpha_0}\,.$$

Then

$$(7.6)\qquad J_\Lambda^{(1)} = \sum_{s\in W}K_s^{(1)}(\Lambda)\,.$$

Now we come to $J_\Lambda^{(2)}$. Notice that $\tilde{\alpha}_0$ is the only one positive singular root in $\Sigma_2$. Then we get similarly as above

$$(7.7)\qquad J_\Lambda^{(2)} = \sum_{s\in W} \frac{2\varepsilon_2(s)\langle\alpha_0, s\alpha_{r+1}\rangle}{\sqrt{\alpha_0(H_{\alpha_0})}} \int_{A^-} e^{s(\Lambda+\rho)(H^-)} F^{(2)}(h^- : s)\, dh^-,$$

where $F_\Lambda^{(2)}(h^- : s) = \lim_{\varepsilon\to 0} F_f^{(2)}(a_\varepsilon h^- ; \partial(\tilde{s}\tilde{p}_r))$. It is easily seen from (2) in Lemma 3 that $F^{(2)}(h^- : s)$ is a function of class $C^\infty$. We put

$$(7.8)\qquad K_s^{(2)}(\Lambda) = \frac{2\varepsilon_2(s)\langle\alpha_0, s\alpha_{r+1}\rangle}{\sqrt{\alpha_0(H_{\alpha_0})}} \int_{A^-} e^{s(\Lambda+\rho)(H^-)} F^{(2)}(h^- : s)\, dh^-.$$

Then

$$(7.9)\qquad J_\Lambda^{(2)} = \sum_{s\in W} K_s^{(2)}(\Lambda).$$

Put $W_0 = \{s\in W : s\alpha_0 = \pm\alpha_0\}$.

Then from (7.6) and (7.9) we have

$$(7.10)\qquad \begin{aligned} J_\Lambda^{(1)} + J_\Lambda^{(2)} &= \sum_{s\in W} \{K_s^{(1)}(\Lambda) + K_s^{(2)}(\Lambda)\} \\ &= \sum_{s^*\in W_0\backslash W} \sum_{t\in W_0} \{K_{ts}^{(1)}(\Lambda) + K_{ts}^{(2)}(\Lambda)\} \end{aligned}$$

where $s^*$ denotes the coset in $W_0\backslash W$ which contains $s$. Hence in order to prove $J_\Lambda^{(1)} + J_\Lambda^{(2)} = 0$, it is sufficient to prove that $\sum_{t\in W_0} \{K_{ts}^{(1)}(\Lambda) + K_{ts}^{(2)}(\Lambda)\} = 0$ for all $s\in W$. For this purpose we need some additional lemmas.

Now fix a coset $s_1^*\in W_0\backslash W$ arbitrarily.

Put $\Lambda(l) = \Lambda + l s_1^{-1}\alpha_0$.

**Lemma 12.**

$$\lim_{l\to\infty} \langle\Lambda(l)+\rho, p_r\rangle T_{\Lambda(l)}(f) = 0 \qquad (l\in \boldsymbol{Z}).$$

Proof. Since $\Lambda$ is an integral form, $\Lambda(l) = \Lambda + l s_1^{-1}\alpha^0$ $(l\in\boldsymbol{Z})$ is again an integral form. Moreover if $l \neq l'$ $(l, l'\in\boldsymbol{Z})$ then $\Lambda(l)\neq\Lambda(l')$. Therefore since

$$|\langle\Lambda(l)+\rho, p_r\rangle| \leqslant |\langle\Lambda(l)+\rho, \pi\rangle|$$

for sufficiently large $l$, the lemma follows immediately from Proposition 3.

**Lemma 13.** *Let $s$ be any element of $W$. Then we have*

$$(1)\qquad \lim_{l\to\infty} \int_{A_1} e^{s(\Lambda(l)+\rho)(H)} F_f^{(1)}(h ; \partial(s p_{r+1}))\, dh = 0,$$

$$(2)\qquad \lim_{l\to\infty} \int_{+A_{\Lambda(l)^s}} e^{\tilde{s}(\tilde{\Lambda}(l)+\tilde{\rho})(H)} F_f^{(2)}(h ; \partial(\tilde{s}\tilde{p}_{r+1}))\, dh = 0.$$

Proof. In view of (1) in Lemma 3, (1) follows immediately from Riemann-Lebesgue theorem. When $s^* \neq s_1^*$ it is obvious that

$$\{H \in \mathfrak{h}^- : B(H, \mathfrak{h}_s) = (0)\} \neq (0)$$

where
$$\mathfrak{h}_s = \{H \in \mathfrak{h}^- : s s_1^{-1} \alpha_0(H) = 0\} .$$

Therefore in case $s^* \neq s_1^*$, (2) follows also from Riemann-Lebesgue theorem. Now we assume that $s^* = s_1^*$ in (2). When $l$ is sufficiently large $|e^{\tilde{s}_1(\tilde{\Lambda}(l)+\tilde{\rho})(H)}| < 1$ if and only if $|e^{\tilde{\alpha}_0(H)}| < 1$. Hence

$$\int_{+A_{\Lambda(l)}s} e^{\tilde{s}_1(\tilde{\Lambda}(l)+\tilde{\rho})(H)} F_f^{(2)}(h\,;\, \partial(\tilde{s}\tilde{p}_{r+1}))\,dh$$
$$= \int_0^\infty e^{-2lt} \left\{ \int_{A^-} e^{s_1(\Lambda+\rho)(tH_0+H^-)} F_f^{(2)}(a_t h^-\,;\, \partial(\tilde{s}\tilde{p}_{r+1}))\,dh^- \right\} dt$$

for sufficiently large $l$. Obviously the right hand side of the last equality tends to zero when $l \to \infty$. This proves the lemma.

**Lemma 14.**

(7.11) $$\lim_{l\to\infty} \{J_{\Lambda(l)}^{(1)} + J_{\Lambda(l)}^{(2)}\} = 0 .$$

This lemma is a direct consequence of Lemma 12, Lemma 13 and (7.1).

**Lemma 15.** *Let $s$ be any element of $W$ such that $s^* \neq s_1^*$. Then we have*

(7.12) $$\lim_{l\to\infty} K_{ts}^{(k)}(\Lambda(l)) = 0 \qquad (k=1, 2) ,$$

*for all $t \in W_0$.*

This lemma is proved in the same way as Lemma 13.

**Lemma 16.**

(7.13) $$K_{t\,s_1}^{(k)}(\Lambda(l)) = K_{t\,s_1}^{(k)}(\Lambda) \qquad (k=1, 2) ,$$

*for all $l \in \boldsymbol{Z}$.*

Proof. For $t \in W_0$ and $\exp H \in \sigma_{\alpha_0}$ we have

$$t\alpha_0(H) = \pm \alpha_0(H) \in 2\pi\sqrt{-1}\boldsymbol{Z} .$$

Hence
$$e^{t\,s_1(\Lambda + l s_1^{-1}\alpha_0 + \rho)(H)} = e^{t\,s_1(\Lambda+\rho)(H)}$$

for all $l \in \boldsymbol{Z}$.

It follows from (7.5) that

$$K_{t s_1}^{(1)}(\Lambda(l)) = K_{t s_1}^{(1)}(\Lambda)\,.$$

Similarly from (7. 8) we get

$$K_{t s_1}^{(2)}(\Lambda(l)) = K_{t s_1}^{(2)}(\Lambda)$$

for all $t \in W_0$ and $l \in \boldsymbol{Z}$ $(l \geqslant 0)$.

This proves the lemma.

When $l$ tends to infinity in (7. 10), we have from (7. 11), (7. 12) and (7. 13)

$$\begin{aligned}
&\sum_{t\in W_0} \{K_{t s_1}^{(1)}(\Lambda)+K_{t s_1}^{(2)}(\Lambda)\} \\
&\quad = \lim_{l\to\infty} \sum_{t\in W_0} \{K_{t s_1}^{(1)}(\Lambda(l))+K_{t s_1}^{(2)}(\Lambda(l))\} \\
&\quad = \lim_{l\to\infty} \{J_{\Lambda(l)}^{(1)}+J_{\Lambda(l)}^{(2)}-\sum_{\substack{s^*\in W_0\backslash W \\ s^*\neq s_1^*}} \sum_{t\in W} \{K_{t s}^{(1)}(\Lambda(l))+K_{t s}^{(2)}(\Lambda(l))\}\} \\
&\quad = 0\,.
\end{aligned}$$

Since $s_1^*$ was arbitrarily chosen, we have

$$\sum_{t\in W_0} \{K_{t s}^{(1)}(\Lambda)+K_{t s}^{(2)}(\Lambda)\} = 0$$

for all $s \in W$. Thus Theorem 1 is proved.

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*Added in proof.*

By the recent result of T. Hirai:

The characters of irreducible representations of the Lorenz group of $n$-th order, to appear.

Our assumptions (A. 1)$\sim$(A. 5) are satisfied also by the groups of type (II) in §5.