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## DIRECT SUMS OF ALMOST RELATIVE INJECTIVE MODULES

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Let  $R$  be a ring with identity. When we study almost relative injective modules, the following problem is essential: Assume that an  $R$ -module  $V$  is almost  $U_j$ -injective for  $R$ -modules  $U_j$  ( $j=1, 2, \dots, n$ ), then *under what conditions is  $V$  also almost  $\Sigma_j \oplus U_j$ -injective?*

This problem is true without any assumptions, provided  $V$  is  $U_j$ -injective [2]. Y. Baba [3] gave an answer to the problem, when all  $V, U_j$  are uniform modules with finite length, and the author [6] generalized it to a case where the  $U_j$  are artinian indecomposable modules. Extending and utilizing the arguments given in [6], we shall drop the assumption “*artinian*” in this short note.

The proof will be completed by following the arguments given in [6]. Hence we shall explain only how we should modify the original proof in [6].

### 1. Preliminaries

Let  $R$  be a ring with identity. Every module in this paper is a right unitary  $R$ -module. We shall follow [3] and [6] for the terminologies. In [6], Theorem 2 we assumed that every module contained the non-zero socle. In this note we shall drop this assumption. Let  $W_1$  and  $W_2$  be  $R$ -modules. Take a diagram with  $W_2$  a submodule of  $W_1$ :

$$(1) \quad \begin{array}{ccc} & i & \\ W_2 & \xleftarrow{\quad} & V_2 \xleftarrow{\quad} 0 \\ & \downarrow g & \\ & & W_1. \end{array}$$

Consider the following two conditions:

- 1) There exists  $\tilde{g}: W_2 \rightarrow W_1$  such that  $\tilde{g}|V_2 = g$ .
- 2) There exist a non-zero direct summand  $W$  of  $W_2$ :  $W_2 = W \oplus W'$  and  $\tilde{g}: W_1 \rightarrow W$  such that  $\tilde{g}g = \pi|V_2$ , where  $\pi$  is the projection of  $W_2$  onto  $W$ .

If either 1) or 2) holds true for any diagram (1), then we say that  $W_1$  is *almost  $W_2$ -injective* (if 1) always holds true, then we say that  $W_1$  is  *$W_2$ -injective* [2]).

We assume in the above that  $W_2$  is indecomposable. If  $W_1$  is almost

$W_2$ -injective,

(#) we always obtain 1), provided  $g$  is not a monomorphism.

**Lemma 1.** *The above (#) is equivalent to the following fact:  $W_1$  is  $W_2/W$ -injective for any non-zero submodule  $W$  of  $W_2$ .*

Proof. If  $g$  is not a monomorphism, then taking  $g^{-1}(0)=W$ , from (1) we obtain the diagram:

$$\begin{array}{ccccccc} W_2 & \longleftarrow & V_2 & \longleftarrow & 0 \\ \downarrow \nu & & \downarrow \nu & & \\ W_2/W & \longleftarrow & V_2/W & \longleftarrow & 0 \\ & & \downarrow \bar{g} & & \\ & & W_1 & , & \end{array}$$

where  $\bar{g}$  is the induced map from  $g$  and  $\nu: W_2 \rightarrow W_2/W$  is the natural epimorphism. Hence if  $W_1$  is  $W_2/W$ -injective, we have  $\tilde{g}' : W_2/W \rightarrow W_1$  such that  $\bar{g} = \tilde{g}'|_{V_2/W}$ . Putting  $\tilde{g} = \tilde{g}'\nu$ ,  $\tilde{g}|V_2 = g$ . The converse is also clear from the above diagram.

**Lemma 2.** *Let  $U$  be an  $R$ -module and  $U_1$  an indecomposable  $R$ -module. Assume that  $U$  is almost  $U_1$ -injective. If  $U$  is not  $U_1$ -injective, then there exist a non-zero submodule  $T$  of  $U_1$  and a monomorphism  $g: T \rightarrow U$ , which is not extendible to an element in  $\text{Hom}_R(U_1, U)$ . In this case we obtain the same situation for any non-zero submodule  $T'$  in  $T$  and  $g|T'$ .*

Proof. The first half is clear from definition. Consider a diagram for a non-zero submodule  $T'$  in  $T$ ;

$$\begin{array}{ccc} & i & \\ U_1 & \xleftarrow{i} & T' \leftarrow 0 \\ & & \downarrow g|T' \\ & & U . \end{array}$$

Assume that there exists  $\tilde{g}: U_1 \rightarrow U$  such that  $\tilde{g}|T' = g|T'$ . Put  $g^* = g - (\tilde{g}|T): T \rightarrow U$ . Then  $g^{*-1}(0) \supset T' \neq 0$ . Then from (#) there exists  $\tilde{g}^*: U_1 \rightarrow U$  such that  $\tilde{g}^*|T = g^* = g - (\tilde{g}|T)$ . Hence  $\tilde{g}^* + \tilde{g}$  is an extension of  $g$ , a contradiction.

From the above proof we obtain

**Corollary.** *Consider the diagram (1). Assume that there exists a non-zero submodule  $V$  in  $V_2$  such that  $g/V$  is extendible to an element in  $\text{Hom}_R(W_2, W_1)$  and  $W_1$  is  $W_2/V$ -injective. Then  $g$  is extendible.*

**Lemma 3** ([6], Proposition 2). *Let  $U, U_2$  be  $R$ -modules and  $U_1$  an in-*

decomposable  $R$ -module. Assume that  $U$  is almost  $U_1$ -injective, but not  $U_1$ -injective. Under those assumptions 1): if  $U$  is  $U_2$ -injective, then  $U_1$  is  $U_2$ -injective. 2): Assume that  $U_2$  is indecomposable. If  $U$  is almost  $U_2$ -injective, but not  $U_2$ -injective, then we obtain the following fact: i);  $U_1$  is  $U_2/V_2$ -injective for any non-zero submodule  $V_2$  of  $U_2$  and hence ii); if  $U_1$  and  $U_2$  do not contain isomorphic submodules,  $U_1$  is  $U_2$ -injective. iii); Assume that  $U_2$  (resp.  $U_1$ ) contains non-zero submodule  $T_1$  (resp.  $T_2$ ) such that  $g: T_1 \approx T_2$ . Then we have the following equivalent conditions:

- a)  $U_1$  (resp.  $U_2$ ) is almost  $U_2$ - (resp.  $U_1$ -) injective.
- b) Either  $g$  or  $g^{-1}$  is extendible to an element in  $\text{Hom}_R(U_1, U_2)$  or in  $\text{Hom}_R(U_2, U_1)$  for every pair  $(T_2, T_1)$ .

Proof. The first half and 1), 2) are dual to [7], Proposition 1. However we shall give a proof for the sake of completeness.

1) By Lemma 2 there exist a submodule  $V_1$  of  $U_1$ , a monomorphism  $g: V_1 \rightarrow U$  and  $f: U \rightarrow U_1$  such that  $fg = 1_{V_1}$ . Put  $E_i = E(U_i)$ , the injective hull of  $U_i$ . Then there exist  $\lambda: E_1 \rightarrow E_0$  and  $\sigma: E_0 \rightarrow E_1$ , which are extensions of  $g$  and  $f$ , respectively. Since  $U_1$  is uniform from [6], Theorem 1,  $\sigma\lambda$  is an automorphism of  $E_1$  and hence  $E_0 = E'_1 \oplus \ker \sigma$ , where  $E'_1 = \lambda(E_1)$ . Further since  $\sigma|E'_1$  is an isomorphism, we can take a submodule  $U'_1$  in  $E'_1$  with  $\sigma(U'_1) = U_1$ . On the other hand  $\sigma(U) = f(U) \subset U_1 = \sigma(U'_1)$ . Hence  $U \subset U'_1 \oplus \ker \sigma$ . Now we may show that  $U'_1$  is  $U_2$ -injective. Let  $s$  be any element in  $\text{Hom}_R(U_2, E'_1) \subset \text{Hom}_R(U_2, E_0)$ . Since  $U$  is  $U_2$ -injective  $s(U_2) \subset U \subset U'_1 \oplus \ker \sigma \subset E'_1 \oplus \ker \sigma$  by [1], Proposition 1.4 (cf. [4], Lemma 9). Hence  $s(U_2) \subset E'_1 \cap (U'_1 \oplus \ker \sigma) = U'_1$ , and so  $U'_1$  is  $U_2$ -injective again by [1], Proposition 2.5.

2), i-ii) Since  $U$  is  $U_2/V_2$ -injective by Lemma 1 for any (non-zero) submodule  $V_2$  of  $U_2$ , we can see from the above argument that  $U_1$  is  $U_2/V_2$ -injective.

2), iii) a) implies b) from definition. Assume b). Take a diagram with  $V_2$  a submodule of  $U_2$

$$\begin{array}{ccc} & i & \\ U_2 & \xleftarrow{\quad} & V_2 \xleftarrow{\quad} 0 \\ & & \downarrow g \\ & & U_1 \end{array}$$

If  $g$  is not a monomorphism, then there exists  $\tilde{g}: U_2 \rightarrow U_1$  with  $\tilde{g}|V_2 = g$  from 2), i) and Lemma 1. Hence we can assume that  $g$  is a monomorphism. As a consequence  $U_1$  is almost  $U_2$ -injective by b).

**Lemma 4.** Let  $U$  be an  $R$ -module and  $U_1, U_2$  LE  $R$ -modules. Assume that 1):  $U$  is almost  $U_1$ -injective, but not  $U_1$ -injective, 2): there exist submodules  $T_1, T_2$  as in Lemma 3 and 3):  $U$  is almost  $U_1 \oplus U_2$ -injective. Then either  $g$  or  $g^{-1}$

is extendible, and hence  $U_1$  is almost  $U_2$ -injective.

Proof. Since  $U$  is almost  $U_1 \oplus U_2$ -injective,  $U$  is almost  $U_2$ -injective. We show that  $U_1$  is almost  $U_2$ -injective. If  $U$  is  $U_2$ -injective,  $U_1$  is (almost)  $U_2$ -injective by Lemma 3-1). Hence we assume that  $U$  is not  $U_2$ -injective. Now there exist a non-zero submodule  $V_1$  and  $h: V_1 \rightarrow U$  given in Lemma 2. Since  $U_1$  is uniform by [6], Theorem 1, we may assume  $V_1 \subset T_1$  from the last part of Lemma 2. Take a diagram

$$\begin{array}{ccccc} U_1 \oplus U_2 & \xleftarrow{i} & V_1 \oplus g(V_1) & \xleftarrow{0} & \\ & & \downarrow h+hg^{-1} & & \\ & & U & & \end{array} .$$

Since  $h$  is not extendible, by assumption there exists an indecomposable direct summand  $Y$  of  $U_1 \oplus U_2$  and  $\tilde{h}: U \rightarrow Y$  such that  $\tilde{h}(h+hg^{-1}) = \pi|_{(V_1 \oplus g(V_1))}$ , where  $\pi$  is the projection. Then either  $g|_{V_1}$  or  $(g|_{V_1})^{-1}$  is extendible (cf. the proof of [5], Proposition 5). If  $g|_{V_1}$  is extendible, so does  $g$  from Corollary to Lemma 2, since  $U_2$  is  $U_1/V_1$ -injective by Lemma 3, 2)-i). Finally assume that  $(g|_{V_1})^{-1}$  is extendible. Consider the diagram

$$\begin{array}{ccccc} U_2 & \xleftarrow{i} & T_2 \supset g(V_1) & \xleftarrow{0} & \\ & & \downarrow g^{-1} & & \\ & & U_1 & & \end{array} .$$

Since  $U_1$  is  $U_2/g(V_1)$ -injective by Lemma 3, 2)-i), we obtain an extension  $\tilde{g}_2: U_2 \rightarrow U_1$  of  $g^{-1}$  from Corollary to Lemma 2. Therefore  $U_1$  is almost  $U_2$ -injective by Lemma 3-2), iii).

## 2. Main Theorem

In this section we shall give the desired theorem related to [3] and [6]. First we show the first half of the main theorem.

**Lemma 5.** *Let  $\{U_i\}_{i=1}^m$  be a set of uniform  $R$ -modules and  $U$  an  $R$ -module. Assume that  $U_i$  and  $U_j$  are mutually almost relative injective for any pair  $(i, j)$  and  $U$  is almost  $U_i$ -injective for all  $i > 0$ . Then  $U$  is almost  $\sum_{i=1}^m \oplus U_i$ -injective.*

Proof. Put  $W = \sum_{i=1}^m \oplus U_i$ , and consider a diagram with  $V$  a submodule of  $W$ :

$$\begin{array}{ccccc} W & \xleftarrow{i} & V & \xleftarrow{0} & \\ & & \downarrow h & & \\ & & U & & \end{array} .$$

In order to show the lemma, we may assume that

(\*)  $V$  is essential in  $W$  (see [3] or [6], (#)).

Putting  $V_j = V \cap U_j$  and  $h_j = h|V_j$ , we obtain the derived diagram:

$$(2) \quad \begin{array}{ccccc} & & i_j & & \\ & U_j & \xleftarrow{\quad} & V_j & \xleftarrow{\quad} 0 \\ & & \downarrow h_j & & \\ & & U & & \end{array}$$

Since  $U$  is almost  $U_j$ -injective, there exists

- a)  $\tilde{h}'_j: U_j \rightarrow U$  with  $\tilde{h}'_j i_j = h_j$  or
- b)  $\tilde{h}_j: U \rightarrow U_j$  with  $i_j = \tilde{h}_j h_j$ .

We quote here the arguments given in [6]. From the argument in Step 3 in [6], namely from [3], Lemma C, (\*) and induction on  $m$ , we know

if we obtain a) for all  $i$ , then there exists  $\tilde{h}: W \rightarrow U$  with  $\tilde{h}|V = h$ .

Hence we assume that we have b) for some  $i$ , say  $i=1$ , i.e.

$$(3) \quad \begin{array}{ccccc} & & i_1 & & \\ & U_1 & \xleftarrow{\quad} & V_1 & \xleftarrow{\quad} 0 \\ & \tilde{h}_1 & \swarrow & \downarrow h_1 & \\ & & U & & \end{array}$$

is commutative, which corresponds to (4') in [6]. Before proceeding the proof, we note the following fact from the argument in Steps 7 and 8 in [6]: We assume

(3) and there exists  $\tilde{h}'_j: U_j \rightarrow U_1$  for all  $j \neq 1$  such that

$$(4) \quad \begin{array}{ccccc} & & i_j & & \\ & U_j & \xleftarrow{\quad} & V_j & \xleftarrow{\quad} 0 \\ & \tilde{h}'_j & \searrow & \downarrow h_j & \\ & & U & & \downarrow \tilde{h}_1 \\ & & & & U_1 \end{array}$$

is commutative, which corresponds to (8) and step 7 in [6]. Then we obtain a new decomposition of  $W := U_1 \oplus U'_2 \oplus \cdots \oplus U'_m$  and  $h^*: U \rightarrow U_1$  such that  $U'_i \approx U_{\rho(i)}$  ( $\rho$  is a permutation on  $\{2, \dots, m\}$ ) and

$$(5) \quad \begin{array}{ccccc} & & i & & \\ & U_1 \oplus U'_2 \oplus \cdots \oplus U'_m & \xleftarrow{\quad} & V & \xleftarrow{\quad} 0 \\ \downarrow \pi_1 & & h^* & & \downarrow h \\ U_1 & \xleftarrow{\quad} & U & \xleftarrow{\quad} & \end{array}$$

is commutative, which corresponds to (7) and step 8 in [6], where  $\pi_1$  is the projection.

(In [6] we needed the assumption “artinian” to get the above (4). We note that the above (5) is shown by induction on  $m$  and the argument given after (10) in [6].)

Now we resume the proof of the lemma. Put  $W_k = \sum_{i \leq k} U_i$  and hence  $W = W_m$ . We shall show by induction on  $k$  that there exist a new decomposition  $W_k = U'_1 \oplus U'_2 \oplus \cdots \oplus U'_k$  and  $\tilde{h}^{(k)}: U \rightarrow U'_1$  such that  $U'_i \approx U_{\rho'(i)}$  ( $\rho'$  is a permutation on  $\{1, \dots, k\}$ ) and

$$(5,k) \quad \begin{array}{ccccc} & U'_1 \oplus \cdots \oplus U'_k = W_k & \xleftarrow{\quad} & W_k \cap V & \xleftarrow{\quad} 0 \\ \downarrow \pi'_1 & & \tilde{h}^{(k)} & & \downarrow h|(W_k \cap V) \\ U'_1 & \xleftarrow{\quad} & U & \xleftarrow{\quad} & \end{array}$$

is commutative, which implies

$$(6) \quad \begin{aligned} \tilde{h}^{(k)} h'_1 &= 1_{(V \cap U'_1)} \text{ and} \\ \tilde{h}^{(k)} h'_j &= \tilde{h}^{(k)} h|(V \cap U'_j) = \pi'_1(V \cap U'_j) = 0 \text{ for } j \neq 1, \text{ where} \\ h'_j &= h|(V \cap U'_j) \text{ for all } j. \end{aligned}$$

(3) is nothing but  $k=1$  in (6). We assume that  $W_k$  has the above decomposition and  $\tilde{h}^{(k)}: U \rightarrow U'_1$ .  $W_{k+1} = W_k \oplus U_{k+1} = U'_1 \oplus U'_2 \oplus \cdots \oplus U'_k \oplus U_{k+1}$ . Take the diagram:

$$\begin{array}{ccccc} & U_{k+1} & \xleftarrow{i_{k+1}} & V_{k+1} & \xleftarrow{\quad} 0 \\ & \downarrow h_{k+1} & & \downarrow & \\ & U & & g & \\ & \downarrow \tilde{h}^{(k)} & & \downarrow & \\ U'_1 & & & & \end{array} .$$

Put  $g = \tilde{h}^{(k)} h_{k+1}$ . Since  $U'_1$  is almost  $U_{k+1}$ -injective, we obtain either

- i) there exists  $\tilde{h}_{k+1}: U_{k+1} \rightarrow U'_1$  with  $\tilde{h}_{k+1} i_{k+1} = g$ ,
- or
- ii)  $\tilde{g}: U'_1 \rightarrow U_{k+1}$  with  $i_{k+1} = \tilde{g} g$ .

Case i) By taking  $\tilde{h}_1 = \tilde{h}^{(k)}$ ,  $\tilde{h}'_j = 0$  ( $1 < j \leq k$ ) and  $\tilde{h}_{k+1}' = \tilde{h}_{k+1}$ , the condition

(4) on  $W_{k+1}$  is satisfied from (6). Hence we obtain a new decomposition  $W_{k+1} = U'_1 \oplus U''_2 \oplus \cdots \oplus U''_k \oplus U''_{k+1}$  ( $U''_i \simeq U'_{\rho(i)}$ ) and  $\tilde{h}^{(k+1)}: U \rightarrow U'_1$ , which satisfies (5,  $k+1$ ).

Case ii) If we put  $\tilde{h}_{k+1} = \tilde{g} \tilde{h}^{(k)}: U \rightarrow U_{k+1}$ , then from (6)

$$\begin{array}{ccccc}
 U'_1 & \longleftarrow & V'_1 & \longleftarrow & 0 \\
 \searrow & & \downarrow h'_1 & & \downarrow \\
 & \tilde{g} & & \tilde{h}^{(k)} & \tilde{h}_{k+1} \\
 \searrow & & \downarrow & & \downarrow \\
 U & \downarrow & U'_1 & \downarrow & U_{k+1} \\
 & & \tilde{g} & & \\
 & & U_{k+1} & &
 \end{array}$$

and for  $j \neq 1$

$$\begin{array}{ccccc}
 U'_j & \longleftarrow & V'_j & \longleftarrow & 0 \\
 \searrow & & \downarrow h'_j & & \downarrow \\
 & 0 & & \tilde{h}^{(k)} & \tilde{h}_{k+1} \\
 \searrow & & \downarrow & & \downarrow \\
 U & \downarrow & U'_1 & \downarrow & U_{k+1} \\
 & & \tilde{g} & & \\
 & & U_{k+1} & &
 \end{array}$$

are commutative. Therefore from (3) and (4) there exists a new decomposition  $W_{k+1} = U_{k+1} \oplus U''_1 \oplus \cdots \oplus U''_k$  such that

$$\begin{array}{ccccc}
 U_{k+1} \oplus U''_1 \oplus \cdots \oplus U''_k & = & W_{k+1} & \longleftarrow & W_{k+1} \cap V \longleftarrow 0 \\
 \downarrow \pi_{k+1}'' & & \tilde{h}_{k+1} & & \downarrow h \\
 U_{k+1} & \longleftarrow & U & \longleftarrow & U
 \end{array}
 \tag{5,  $k+1$ }$$

is commutative. Thus we have completed the proof.

In general let  $\{D_i\}_{i=1}^{\rho}$  be a set of indecomposable  $R$ -modules and  $U$  and  $R$ -module. Assume that  $U$  is almost  $\Sigma_i \oplus D_i$ -injective. Then  $U$  is almost  $D_i$ -injective for all  $i$ . We shall divide  $\{D_i\}$  into two disjoint parts  $\{D_i\} = \{U_i\} \cup \{I_k\}$  as follows:

- (7) 1)  $U$  is  $I_k$ -injective for all  $k$  and
- 2)  $U$  is almost  $U_j$ -injective, but not  $U_j$ -injective for all  $j$ .

Then we note that all  $U_j$  are uniform from [6], Theorem 1. Finally we give the

main theorem

**Theorem.** *Let  $U$  be an  $R$ -module. Further let  $\{U_j\}_{j=1}^m$  be a set of indecomposable  $R$ -modules and  $\{I_k\}_{k=1}^n$  a set of  $R$ -modules. We assume that  $\{U_j, I_k\}$  satisfy (7). Then if  $U_i, U_j$  are mutually almost relative injective, then  $U$  is almost  $\Sigma_{j=1}^m \oplus U_j \oplus \Sigma_{k=1}^n \oplus I_k$ -injective. Conversely if  $U$  is almost  $\Sigma_{i=1}^m \oplus U_j \oplus \Sigma_{k=1}^n \oplus I_k$ -injective and the  $U_j$  are LE modules, then  $U_i, U_j$  are mutually almost relative injective for any pair  $(i, j)$ .*

Proof. The second half is clear from Lemmas 3, 2-ii) and 4. We study the first half. From (7) and Lemma 3  $U_j$  is  $I_k$ -injective for any  $j$  and  $k$ . If  $U'_1$  is  $U_{k+1}$ -injective in the proof of Lemma 5, then we always obtain the case i). Therefore using Lemma 5, we can follow the proof in [6], Theorem 2 and get the theorem.

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