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Let $R$ be a ring with identity. When we study almost relative injective modules, the following problem is essential: Assume that an $R$-module $V$ is almost $U_j$-injective for $R$-modules $U_j (j=1, 2, \ldots, n)$, then under what conditions is $V$ also almost $\Sigma_j U_j$-injective?

This problem is true without any assumptions, provided $V$ is $U_j$-injective [2]. Y. Baba [3] gave an answer to the problem, when all $V, U_j$ are uniform modules with finite length, and the author [6] generalized it to a case where the $U_j$ are artinian indecomposable modules. Extending and utilizing the arguments given in [6], we shall drop the assumption "artinian" in this short note.

The proof will be completed by following the arguments given in [6]. Hence we shall explain only how we should modify the original proof in [6].

1. Preliminaries

Let $R$ be a ring with identity. Every module in this paper is a right unitary $R$-module. We shall follow [3] and [6] for the terminologies. In [6], Theorem 2 we assumed that every module contained the non-zero socle. In this note we shall drop this assumption. Let $W_1$ and $W_2$ be $R$-modules. Take a diagram with $V_2$ a submodule of $W_2$:

\[
\begin{array}{c}
W_2 \\ \\
V_2 \\ \\
W_1
\end{array}
\xleftarrow{i} 0 \\
\downarrow{g}
\]

Consider the following two conditions:

1) There exists $\tilde{g}: W_2 \rightarrow W_1$ such that $\tilde{g}|V_2 = g$.

2) There exist a non-zero direct summand $W$ of $W_2$: $W_2 \cong W \oplus W'$ and $\tilde{g}: W_1 \rightarrow W$ such that $\tilde{g}|V_2 = \pi|V_2$, where $\pi$ is the projection of $W_2$ onto $W$.

If either 1) or 2) holds true for any diagram (1), then we say that $W_1$ is almost $W_2$-injective (if 1) always holds true, then we say that $W_1$ is $W_2$-injective [2]).

We assume in the above that $W_2$ is indecomposable. If $W_1$ is almost
we always obtain (1), provided \( g \) is not a monomorphism.

**Lemma 1.** The above (1) is equivalent to the following fact: \( W_1 \) is \( W_2/W \)-injective for any non-zero submodule \( W \) of \( W_2 \).

**Proof.** If \( g \) is not a monomorphism, then taking \( g^{-1}(0) = W \), from (1) we obtain the diagram:

\[
\begin{array}{ccc}
W_2 & 
\xleftarrow{\nu} & V_2 \\
\downarrow \nu & & \downarrow \nu \\
W_2/W & 
\xleftarrow{\nu} & V_2/W \\
\downarrow g & & \\
W_1 &
\end{array}
\]

where \( \bar{g} \) is the induced map from \( g \) and \( \nu : W_2 \to W_2/W \) is the natural epimorphism. Hence if \( W_1 \) is \( W_2/W \)-injective, we have \( \bar{g} : W_2/W \to W_1 \) such that \( \bar{g} = \bar{g}' | V_2/W \). Putting \( \bar{g} = \bar{g}' \nu, \bar{g} | V_2 = g \). The converse is also clear from the above diagram.

**Lemma 2.** Let \( U \) be an \( R \)-module and \( U_1 \) an indecomposable \( R \)-module. Assume that \( U \) is almost \( U_1 \)-injective. If \( U \) is not \( U_1 \)-injective, then there exist a non-zero submodule \( T \) of \( U_1 \) and a monomorphism \( g : T \to U \), which is not extendible to an element in \( \text{Hom}_R(U_1, U_1) \). In this case we obtain the same situation for any non-zero submodule \( T' \) in \( T \) and \( g | T' \).

**Proof.** The first half is clear from definition. Consider a diagram for a non-zero submodule \( T' \) in \( T \);

\[
\begin{array}{ccc}
U_1 & 
\xleftarrow{i} & T' \\
\downarrow g | T' & & \\
U &
\end{array}
\]

Assume that there exists \( \bar{g} : U_1 \to U \) such that \( \bar{g} | T' = g | T' \). Put \( g^* = g | T \); \( T \to U \). Then \( g^* | T' = g | T' \neq 0 \). Then from (1) there exists \( \bar{g}^* : U_1 \to U \) such that \( \bar{g}^* | T = g^* = g \). Hence \( \bar{g}^* + \bar{g} \) is an extension of \( g \), a contradiction.

From the above proof we obtain

**Corollary.** Consider the diagram (1). Assume that there exists a non-zero submodule \( V \) in \( V_2 \) such that \( g | V \) is extendible to an element in \( \text{Hom}_R(W_2, W_1) \) and \( W_1 \) is \( W_2/V \)-injective. Then \( g \) is extendible.

**Lemma 3** ([6], Proposition 2). Let \( U, U_2 \) be \( R \)-modules and \( U_1 \) an in-
decomposable R-module. Assume that U is almost $U_1$-injective, but not $U_1$-injective. Under those assumptions 1): if U is $U_2$-injective, then $U_1$ is $U_2$-injective. 2): Assume that $U_2$ is indecomposable. If U is almost $U_2$-injective, but not $U_2$-injective, then we obtain the following fact: i); $U_1$ is $U_2[1]/V_2$-injective for any non-zero submodule $V_2$ of $U_2$ and hence ii); if $U_1$ and $U_2$ do not contain isomorphic submodules, $U_1$ is $U_2$-injective. iii); Assume that $U_2$ (resp. $U_1$) contains non-zero submodule $T_1$ (resp. $T_2$) such that $g: T_1 \cong T_2$. Then we have the following equivalent conditions:

- a) $U_1$ (resp. $U_2$) is almost $U_2$ (resp. $U_1$)-injective.
- b) Either g or $g^{-1}$ is extendible to an element in $\text{Hom}_R(U_1, U_2)$ or in $\text{Hom}_R(U_2, U_1)$ for every pair $(T_2, T_1)$.

Proof. The first half and 1), 2) are dual to [7], Proposition 1. However we shall give a proof for the sake of completeness.

1) By Lemma 2 there exist a submodule $V_1$ of $U_1$, a monomorphism $g: V_1 \rightarrow U$ and $f: U \rightarrow U_1$ such that $fg = 1_{V_1}$. Put $E_1 = E(U_1)$, the injective hull of $U_1$. Then there exist $\lambda: E_1 \rightarrow E_0$ and $\sigma: E_0 \rightarrow E_1$, which are extensions of $g$ and $f$, respectively. Since $U_1$ is uniform from [6], Theorem 1, $\sigma \lambda$ is an automorphism of $E_1$ and hence $E_0 = E_1 \oplus \ker \sigma$, where $E_1 = \lambda(E_0)$. Further since $\sigma|E_1$ is an isomorphism, we can take a submodule $U_1'$ in $E_1$ with $\sigma(U_1') = U_1$. On the other hand $\sigma(U) = f(U) \subset U_1 = \sigma(U_1')$. Hence $U \subset U_1 \oplus \ker \sigma$. Now we may show that $U_1'$ is $U_2$-injective. Let $s$ be any element in $\text{Hom}_R(U_2, E_1) \subset \text{Hom}_R(U_2, E_0)$. Since $U$ is $U_2$-injective $s(U_2) \subset U \subset U_1 \oplus \ker \sigma \subset E_1 \oplus \ker \sigma$ by [1], Proposition 1.4 (cf. [4], Lemma 9). Hence $s(U_2) \subset E_1 \cap (U'_1 \oplus \ker \sigma) = U_1'$, and so $U_1'$ is $U_2$-injective again by [1], Proposition 2.5.

2) i-ii) Since U is $U_2/V_2$-injective by Lemma 1 for any (non-zero) submodule $V_2$ of $U_2$, we can see from the above argument that $U_1$ is $U_2/V_2$-injective.

2), iii) a) implies b) from definition. Assume b). Take a diagram with $V_2$ a submodule of $U_2$

$$
\begin{array}{ccc}
U_2 & \xrightarrow{i} & V_2 \\
\downarrow g & & \downarrow 0 \\
U_1 & \xrightarrow{} & \\
\end{array}
$$

If $g$ is not a monomorphism, then there exists $\tilde{g}: U_2 \rightarrow U_1$ with $\tilde{g}|V_2 = g$ from 2), i) and Lemma 1. Hence we can assume that $g$ is a monomorphism. As a consequence $U_1$ is almost $U_2$-injective by b).

**Lemma 4.** Let U be an R-module and $U_1$, $U_2$ LE R-modules. Assume that 1): U is almost $U_1$-injective, but not $U_1$-injective, 2): there exist submodules $T_1$, $T_2$ as in Lemma 3 and 3): U is almost $U_1 \oplus U_2$-injective. Then either g or $g^{-1}$
is extendible, and hence $U_1$ is almost $U_2$-injective.

Proof. Since $U$ is almost $U_1 \oplus U_2$-injective, $U$ is almost $U_2$-injective. We show that $U_1$ is almost $U_2$-injective. If $U$ is $U_2$-injective, $U_1$ is (almost) $U_2$-injective by Lemma 3-1). Hence we assume that $U$ is not $U_2$-injective. Now there exist a non-zero submodule $V_1$ and $h: V_1 \to U$ given in Lemma 2. Since $U_1$ is uniform by [6], Theorem 1, we may assume $V_1 \subseteq T_1$ from the last part of Lemma 2. Take a diagram

$$
\begin{array}{ccc}
U_1 \oplus U_2 & \xrightarrow{i} & V_1 \oplus g(V_1) \xleftarrow{0} \\
\downarrow & \downarrow h + hg^{-1} & \\
U & .
\end{array}
$$

Since $h$ is not extendible, by assumption there exists an indecomposable direct summand $Y$ of $U_1 \oplus U_2$ and $\overline{h}: U \to Y$ such that $\overline{h}(h + hg^{-1}) = \pi |(V_1 \oplus g(V_1))$, where $\pi$ is the projection. Then either $g | V_1$ or $(g | V_1)^{-1}$ is extendible (cf. the proof of [5], Proposition 5). If $g | V_1$ is extendible, so does $g$ from Corollary to Lemma 2, since $U_2$ is $U_1 | V_1$-injective by Lemma 3, 2)-i). Finally assume that $(g | V_1)^{-1}$ is extendible. Consider the diagram

$$
\begin{array}{ccc}
U_2 & \xleftarrow{i} & T_2 \xrightarrow{g(V_1)} \xleftarrow{0} \\
\downarrow g^{-1} & & \\
U_1 & .
\end{array}
$$

Since $U_1$ is $U_2 | g(V_1)$-injective by Lemma 3, 2)-i), we obtain an extension $\tilde{g}_2: U_2 \to U_1$ of $g^{-1}$ from Corollary to Lemma 2. Therefore $U_1$ is almost $U_2$-injective by Lemma 3-2), iii).

2. Main Theorem

In this section we shall give the desired theorem related to [3] and [6]. First we show the first half of the main theorem.

Lemma 5. Let $\{U_i\}_{i=1}^{n}$ be a set of uniform $R$-modules and $U$ an $R$-module. Assume that $U_i$ and $U_j$ are mutually almost relative injective for any pair $(i, j)$ and $U$ is almost $U_i$-injective for all $i > 0$. Then $U$ is almost $\Sigma_{i=1}^{n} \oplus U_i$-injective.

Proof. Put $W = \Sigma_{i=1}^{n} \oplus U_i$, and consider a diagram with $V$ a submodule of $W$:

$$
\begin{array}{ccc}
W & \xleftarrow{i} & V \xleftarrow{0} \\
\downarrow h & & \\
U & .
\end{array}
$$
In order to show the lemma, we may assume that

(*) \( V \) is essential in \( W \) (see [3] or [6], (#)).

Putting \( V_j = V \cap U_j \) and \( h_j = h \mid V_j \), we obtain the derived diagram:

\[
\begin{array}{c}
U_j \\
\downarrow h_j \\
U
\end{array}
\]

Since \( U \) is almost \( U_j \)-injective, there exists

a) \( \tilde{h}_j : U_j \to U \) with \( \tilde{h}_j i_j = h_j \) or

b) \( \bar{h}_j : U \to U_j \) with \( i_j = \bar{h}_j h_j \).

We quote here the arguments given in [6]. From the argument in Step 3 in [6], namely from [3], Lemma C, (*) and induction on \( m \), we know

if we obtain a) for all \( i \), then there exists \( \tilde{h} : W \to U \) with \( \tilde{h} \mid V = h \).

Hence we assume that we have b) for some \( i \), say \( i = 1 \), i.e.

\[
\begin{array}{c}
U_1 \\
\downarrow \bar{h}_1 \\
U
\end{array}
\]

is commutative, which corresponds to (4') in [6]. Before proceeding the proof, we note the following fact from the argument in Steps 7 and 8 in [6]: We assume

(3) and there exists \( \tilde{h}_j : U_j \to U_1 \) for all \( j \neq 1 \) such that

\[
\begin{array}{c}
U_j \\
\downarrow h_j \\
U
\end{array}
\]

is commutative, which corresponds to (8) and step 7 in [6]. Then we obtain a new decomposition of \( W := U \oplus U_2 \oplus \cdots \oplus U_m \) and \( h^* : U \to U_1 \) such that \( U'_i := U_{\rho(i)} \) (\( \rho \) is a permutation on \{2, \cdots, m\}) and
\[ U_1 \oplus U_2 \cdots \oplus U_m \overset{i}{\rightarrow} V \leftarrow 0 \]

(5)

\[
\begin{array}{c c c c}
\pi_1 & h^* & h \\
U_1 & \downarrow & \downarrow & \downarrow \\
& & U & U
\end{array}
\]

is commutative, which corresponds to (7) and step 8 in [6], where \( \pi_1 \) is the projection.

(In [6] we needed the assumption “artinian” to get the above (4). We note that the above (5) is shown by induction on \( m \) and the argument given after (10) in [6].)

Now we resume the proof of the lemma. Put \( W_k = \Sigma_{i \leq k} U_i \) and hence \( W = W_m \). We shall show by induction on \( k \) that there exist a new decomposition \( W_k = U_1' \oplus U_2' \oplus \cdots \oplus U_k' \) and \( h^{(k)}: U \rightarrow U_1' \) such that \( U_1' \approx U_{\rho'(k)} \) (\( \rho' \) is a permutation on \( \{1, \ldots, k\} \)) and

\[
U_1' \oplus \cdots \oplus U_k' = W_k \overset{i}{\rightarrow} W_k \cap V \leftarrow 0
\]

(5,k)

\[
\begin{array}{c c c c}
\pi_1' & h_k^{(k)} & h \\
U_1' & \downarrow & (W_k \cap V) & \downarrow \\
& & U & U
\end{array}
\]

is commutative, which implies

\[
\begin{align*}
\bar{h}_k^{(k)} h_i' & = 1_{(V \cap U'_i)} \\
\bar{h}_k^{(k)} h_j' & = \bar{h}_k^{(k)} h_j (V \cap U'_j) = \pi_i (V \cap U'_j) = 0 \quad \text{for} \ j \neq 1,
\end{align*}
\]

where \( h_i' = h_j (V \cap U'_j) \) for all \( j \).

(3) is nothing but \( k=1 \) in (6). We assume that \( W_k \) has the above decomposition and \( h^{(k)}: U \rightarrow U_1' \). \( W_{k+1} = W_k \oplus U_{k+1} = U_1' \oplus U_2' \oplus \cdots \oplus U_k' \oplus U_{k+1} \). Take the diagram:

\[
\begin{array}{c c c c}
U_{k+1} & i_{k+1} & V_{k+1} & \overset{0}{\rightarrow} \\
& \downarrow & h_{k+1} & \downarrow \\
& \downarrow & g & \downarrow \\
& \downarrow & \bar{h}_k^{(k)} & \downarrow \\
& \downarrow & U_1' & U
\end{array}
\]

Put \( g = \bar{h}_k^{(k)} h_{k+1} \). Since \( U_1' \) is almost \( U_{k+1} \)-injective, we obtain either

i) there exists \( \bar{h}_{k+1}: U_{k+1} \rightarrow U_1' \) with \( \bar{h}_{k+1} i_{k+1} = g \),

or

ii) \( \bar{g}: U_1' \rightarrow U_{k+1} \) with \( i_{k+1} = \bar{g} g \).

Case i) By taking \( \bar{h}_1 = \bar{h}_k^{(k)} \), \( \bar{h}_j = 0 \) (\( 1 < j \leq k \)) and \( \bar{h}_{k+1} = \bar{h}_{k+1} \), the condition
(4) on $W_{k+1}$ is satisfied from (6). Hence we obtain a new decomposition $W_{k+1} = U_1' \oplus U_2' \oplus \cdots \oplus U_k' \oplus U_{k+1}'$ and $\bar{\eta}^{(k+1)}: U \to U'$, which satisfies (5, $k+1$).

Case ii) If we put $\bar{\eta}^{(k+1)} = \bar{\eta}^{(k)}: U \to U_{k+1}'$, then from (6)

\begin{align*}
U_1' & \quad V_1' \quad 0 \\
\bar{U}_1' & \quad \bar{U}_1' \\
\bar{U}_{k+1} & \quad U_{k+1} \\
\bar{\eta}^{(k)} & \quad \bar{\eta}^{(k)}
\end{align*}

and for $j \neq 1$

\begin{align*}
U_j' & \quad V_j' \quad 0 \\
\bar{U}_j' & \quad \bar{U}_j' \\
\bar{U}_{k+1} & \quad U_{k+1} \\
\bar{\eta}^{(k)} & \quad \bar{\eta}^{(k)}
\end{align*}

are commutative. Therefore from (3) and (4) there exists a new decomposition $W_{k+1} = U_{k+1} \oplus U_1' \oplus \cdots \oplus U_k'$ such that

\begin{align*}
U_{k+1} \oplus U_1' \oplus \cdots \oplus U_k' = W_{k+1} \leftarrow W_{k+1} \cap V \leftarrow 0 \\
(5, k+1)
\end{align*}

is commutative. Thus we have completed the proof.

In general let \{D_i\} be a set of indecomposable $R$-modules and $U$ and $R$-module. Assume that $U$ is almost $\Sigma_i \oplus D_i$-injective. Then $U$ is almost $D_i$-injective for all $i$. We shall divide \{D_i\} into two disjoint parts \{D_i\} = \{U_i\} \cup \{I_k\} as follows:

1) $U$ is $I_k$-injective for all $k$ and

2) $U$ is almost $U_j$-injective, but not $U_j$-injective for all $j$.

Then we note that all $U_j$ are uniform from [6], Theorem 1. Finally we give the
Theorem. Let $U$ be an $R$-module. Further let \( \{U_j\}_{j \in \mathbb{N}} \) be a set of indecomposable $R$-modules and \( \{I_k\}_{k \in \mathbb{N}} \) a set of $R$-modules. We assume that $\{U_j, I_k\}$ satisfy (7). Then if $U_i, U_j$ are mutually almost relative injective, then $U$ is almost $\sum \theta I_j \oplus \sum \theta I_k$-injective. Conversely if $U$ is almost $\sum \theta I_j \oplus \sum \theta I_k$-injective and the $U_j$ are LE modules, then $U_i, U_j$ are mutually almost relative injective for any pair $(i, j)$.

Proof. The second half is clear from Lemmas 3,2-ii) and 4. We study the first half. From (7) and Lemma 3 $U_j$ is $I_k$-injective for any $j$ and $k$. If $U_i$ is $U_{k+1}$-injective in the proof of Lemma 5, then we always obtain the case i). Therefore using Lemma 5, we can follow the proof in [6], Theorem 2 and get the theorem.

References


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