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DIRECT SUMS OF ALMOST RELATIVE INJECTIVE MODULES

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Let R be a ring with identity. When we study almost relative injective modules, the following problem is essential: Assume that an R-module V is almost U_j-injective for R-modules U_j ($j=1, 2, \cdots, n$), then *under what conditions is V also almost $\Sigma_j \oplus U_j$-injective?*

This problem is true without any assumptions, provided V is U_j-injective [2]. Y. Baba [3] gave an answer to the problem, when all V, U_j are uniform modules with finite length, and the author [6] generalized it to a case where the U_j are artinian indecomposable modules. Extending and utilizing the arguments given in [6], we shall drop the assumption "*artinian*" in this short note.

The proof will be completed by following the arguments given in [6]. Hence we shall explain only how we should modify the original proof in [6].

1. Preliminaries

Let R be a ring with identity. Every module in this paper is a right unitary R-module. We shall follow [3] and [6] for the terminologies. In [6], Theorem 2 we assumed that every module contained the non-zero socle. In this note we shall drop this assumption. Let W_1 and W_2 be R-modules. Take a diagram with V_2 a submodule of W_2:

$$
\begin{array}{ccccc}
W_2 & \overset{i}{\leftarrow} & V_2 & \leftarrow & 0 \\
 & & \downarrow g & & \\
 & & W_1 \ . & &
\end{array}
\tag{1}
$$

Consider the following two conditions:

1) There exists $\tilde{g}$: $W_2 \to W_1$ such that $\tilde{g} | V_2 = g$.

2) There exist a non-zero direct summand W of W_2: $W_2 = W \oplus W'$ and $\tilde{g}$: $W_1 \to W$ such that $\tilde{g}g = \pi | V_2$, where π is the projection of W_2 onto W.

If either 1) or 2) holds true for any diagram (1), then we say that W_1 *is almost W_2-injective* (if 1) always holds true, then we say that W_1 *is W_2-injective* [2]).

We assume in the above that W_2 is indecomposable. If W_1 is almost

W_2-injective,
(♯) we always obtain 1), provided g is not a monomorphism.

Lemma 1. *The above* (♯) *is equivalent to the following fact: W_1 is W_2/W-injective for any non-zero subomdule W of W_2.*

Proof. If g is not a monomorphism, then taking $g^{-1}(0)=W$, from (1) we obtain the diagram:

$$\begin{tikzcd} W_2 \arrow[d, "\nu"] & V_2 \arrow[l] \arrow[d, "\nu"] & 0 \arrow[l] \\ W_2/W & V_2/W \arrow[l] \arrow[d, "\bar{g}"] & 0 \arrow[l] \\ & W_1\ , & \end{tikzcd}$$

where $\bar{g}$ is the induced map from g and $\nu\colon W_2 \to W_2/W$ is the natural epimorphism. Hence if W_1 is W_2/W-injective, we have $\tilde{g}'\colon W_2/W \to W_1$ such that $\bar{g}=\tilde{g}' \mid V_2/W$. Putting $\tilde{g}=\tilde{g}'\nu$, $\tilde{g} \mid V_2=g$. The converse is also clear from the above diagram.

Lemma 2. *Let U be an R-module and U_1 an indecomposable R-module. Assume that U is almost U_1-injective. If U is not U_1-injective, then there exist a non-zero submodule T of U_1 and a monomorphism $g\colon T \to U$, which is not extendible to an element in* $\mathrm{Hom}_R(U_1, U_1)$. *In this case we obtain the same situation for any non-zero submodule T' in T and g/T'.*

Proof. The first half is clear from definition. Consider a diagram for a non-zero submodule T' in T;

$$\begin{tikzcd} U_1 & T' \arrow[l, "i"'] \arrow[d, "g \mid T'"] & 0 \arrow[l] \\ & U\ . & \end{tikzcd}$$

Assume that there exists $\tilde{g}\colon U_1 \to U$ such that $\tilde{g} \mid T'=g \mid T'$. Put $g^*=g-(\tilde{g} \mid T)\colon T \to U$. Then $g^{*-1}(0) \supset T' \neq 0$. Then from (♯) there exists $\tilde{g}^*\colon U_1 \to U$ such that $\tilde{g}^* \mid T=g^*=g-(\tilde{g} \mid T)$. Hence $\tilde{g}^*+\tilde{g}$ is an extension of g, a contradiction.

From the above proof we obtain

Corollary. *Consider the diagram* (1). *Assume that there exists a non-zero submodule V in V_2 such that g/V is extendible to an element in* $\mathrm{Hom}_R(W_2, W_1)$ *and W_1 is W_2/V-injective. Then g is extendible.*

Lemma 3 ([6], Proposition 2). *Let U, U_2 be R-modules and U_1 an in-*

decomposable R-module. Assume that U is almost U_1-injective, but not U_1-injective. Under those assumptions 1): *if U is U_2-injective, then U_1 is U_2-injective.* 2): *Assume that U_2 is indecomposable. If U is almost U_2-injective, but not U_2-injective, then we obtain th following fact:* i); *U_1 is U_2/V_2-injective for any non-zero submodule V_2 of U_2 and hence* ii); *if U_1 and U_2 do not contain isomorphic submodules, U_1 is U_2-injective.* iii); *Assume that U_2 (resp. U_1) contains non-zero submodule T_1 (resp. T_2) such that g: $T_1 \approx T_2$. Then we have the following equivalent conditions:*

a) *U_1 (resp. U_2) is almost U_2-(resp. U_1-) injective.*

b) *Either g or g^{-1} is extendible to an element in* $\mathrm{Hom}_R(U_1, U_2)$ *or in* $\mathrm{Hom}_R(U_2, U_1)$ *for every pair (T_2, T_1).*

Proof. The first half and 1), 2) are dual to [7], Proposition 1. However we shall give a proof for the sake of completeness.

1) By Lemma 2 there exist a submodule V_1 of U_1, a monomorphism g: $V_1 \to U$ and f: $U \to U_1$ such that $fg = 1_{V_1}$. Put $E_i = E(U_i)$, the injective hull of U_i. Then there exist λ: $E_1 \to E_0$ and σ: $E_0 \to E_1$, which are extensions of g and f, respectively. Since U_1 is uniform from [6], Theorem 1, $\sigma\lambda$ is an automorphism of E_1 and hence $E_0 = E_1' \oplus \ker\sigma$, where $E_1' = \lambda(E_1)$. Further since $\sigma|E_1'$ is an isomorphism, we can take a submodule U_1' in E_1' with $\sigma(U_1') = U_1$. On the other hand $\sigma(U) = f(U) \subset U_1 = \sigma(U_1')$. Hence $U \subset U_1' \oplus \ker\sigma$. Now we may show that U_1' is U_2-injective. Let s be any element in $\mathrm{Hom}_R(U_2, E_1') \subset \mathrm{Hom}_R(U_2, E_0)$. Since U is U_2-injective $s(U_2) \subset U \subset U_1' \oplus \ker\sigma \subset E_1' \oplus \ker\sigma$ by [1], Proposition 1.4 (cf. [4], Lemma 9). Hence $s(U_2) \subset E_1' \cap (U_1' \oplus \ker\sigma) = U_1'$, and so U_1' is U_2-injective again by [1], Proposition 2.5.

2), i-ii) Since U is U_2/V_2-injective by Lemma 1 for any (non-zero) submodule V_2 of U_2, we can see from the above argument that U_1 is U_2/V_2-injective.

2), iii) a) implies b) from definition. Assume b). Take a diagram with V_2 a submodule of U_2

$$\begin{array}{ccccc} U_2 & \xleftarrow{i} & V_2 & \leftarrow & 0 \\ & & \downarrow g & & \\ & & U_1 & & \end{array}$$

If g is not a monomorphism, then there exists $\tilde{g}$: $U_2 \to U_1$ with $\tilde{g}|V_2 = g$ from 2), i) and Lemma 1. Hence we can assume that g is a monomorphism. As a consequence U_1 is almost U_2-injective by *b*).

Lemma 4. *Let U be an R-module and U_1, U_2* LE *R-modules. Assume that* 1): *U is almost U_1-injective, but not U_1-injective,* 2): *there exist submodules T_1, T_2 as in Lemma* 3 *and* 3): *U is almost $U_1 \oplus U_2$-injective. Then either g or g^{-1}*

is extendible, and hence U_1 is almost U_2-injective.

Proof. Since U is almost $U_1\oplus U_2$-injective, U is almost U_2-injective. We show that U_1 is almost U_2-injective. If U is U_2-injective, U_1 is (almost) U_2-injective by Lemma 3–1). Hence we assume that U is not U_2-injective. Now there exist a non-zero submodule V_1 and $h: V_1 \to U$ given in Lemma 2. Since U_1 is uniform by [6], Theorem 1, we may assume $V_1 \subset T_1$ from the last part of Lemma 2. Take a diagram

$$\begin{array}{ccc} U_1\oplus U_2 \overset{i}{\leftarrow} & V_1\oplus g(V_1) & \leftarrow 0 \\ & \downarrow\ h+hg^{-1} & \\ & U & \end{array}.$$

Since h is not extendible, by assumption there exists an indecomposable direct summand Y of $U_1\oplus U_2$ and $\tilde{h}: U \to Y$ such that $\tilde{h}(h+hg^{-1})=\pi|(V_1\oplus g(V_1))$, where π is the projection. Then either $g|V_1$ or $(g|V_1)^{-1}$ is extendible (cf. the proof of [5], Proposition 5). If $g|V_1$ is extendible, so does g from Corollary to Lemma 2, since U_2 is U_1/V_1-injective by Lemma 3, 2)-i). Finally assume that $(g|V_1)^{-1}$ is extendible. Consider the diagram

$$\begin{array}{ccc} U_2 \overset{i}{\leftarrow} T_2 \supset & g(V_1) & \leftarrow 0 \\ & \downarrow g^{-1} & \\ & U_1 & \end{array}.$$

Since U_1 is $U_2/g(V_1)$-injective by Lemma 3, 2)-i), we obtain an extension $\tilde{g}_2$: $U_2 \to U_1$ of g^{-1} from Corollary to Lemma 2. Therefore U_1 is almost U_2-injective by Lemma 3–2), iii).

2. Main Theorem

In this section we shall give the desired theorem related to [3] and [6]. First we show the first half of the main theorem.

Lemma 5. *Let $\{U_i\}_{i=1}^{m}$ be a set of uniform R-modules and U an R-module. Assume that U_i and U_j are mutually almost relative injective for any pair (i, j) and U is almost U_i-injective for all $i>o$. Then U is almost $\Sigma_{i=1}^{m}\oplus U_i$-injective.*

Proof. Put $W=\Sigma_{i=1}^{m}\oplus U_i$, and consider a diagram with V a submodule of W:

$$\begin{array}{ccc} W \overset{i}{\leftarrow} & V & \leftarrow 0 \\ & \downarrow h & \\ & U & \end{array}.$$

In order to show the lemma, we may assume that

(*) V is essential in W (see [3] or [6], (♯)).

Putting $V_j = V \cap U_j$ and $h_j = h \mid V_j$, we obtain the derived diagram:

$$\begin{tikzcd} U_j & V_j \arrow[l, "i_j"'] \arrow[d, "h_j"] & 0 \arrow[l] \\ & U & \end{tikzcd} \tag{2}$$

Since U is almost U_j-injective, there exists

a) $\tilde{h}'_j$: $U_j \rightarrow U$ with $\tilde{h}'_j i_j = h_j$ or

b) $\tilde{h}_j$: $U \rightarrow U_j$ with $i_j = \tilde{h}_j h_j$.

We quote here the arguments given in [6]. From the argument in Step 3 in [6], namely from [3], Lemma C, (*) and induction on m, we know

if we obtain a) for all i, then there exists $\tilde{h}$: $W \rightarrow U$ with $\tilde{h} \mid V = h$.

Hence we assume that we have b) for some i, say $i=1$, i.e.

$$\begin{tikzcd} U_1 & V_1 \arrow[l, "i_1"'] \arrow[d, "h_1"] & 0 \arrow[l] \\ & U \arrow[ul, "\tilde{h}_1"] & \end{tikzcd} \tag{3}$$

is commutative, which corresponds to (4') in [6]. Before proceeding the proof, we note the following fact from the argument in Steps 7 and 8 in [6]: We assume

(3) and there exists $\tilde{h}'_j$: $U_j \rightarrow U_1$ for all $j \neq 1$ such that

$$\begin{tikzcd} U_j \arrow[ddr, "\tilde{h}'_j"'] & V_j \arrow[l, "i_j"'] \arrow[d, "h_j"] & 0 \arrow[l] \\ & U \arrow[d, "\tilde{h}_1"] & \\ & U_1 & \end{tikzcd} \tag{4}$$

is commutative, which corresponds to (8) and step 7 in [6]. Then we obtain a *new decomposition of* $W := U_1 \oplus U'_2 \oplus \cdots \oplus U'_m$ *and* h^*: $U \rightarrow U_1$ *such that* $U'_i \approx U_{\rho(i)}$ (ρ *is a permutation on* $\{2, \cdots, m\}$) *and*

$$
(5)\qquad
\begin{tikzcd}
U_1\oplus U_2'\cdots\oplus U_m' \arrow[d, "\pi_1"] & V \arrow[l, "i"'] \arrow[d, "h"] & 0 \arrow[l] \\
U_1 & U \arrow[l, "h^*"'] &
\end{tikzcd}
$$

is commutative, which corresponds to (7) and step 8 in [6], where π_1 is the projection.

(In [6] we needed the assumption "artinian" to get the above (4). We note that the above (5) is shown by induction on m and the argument given after (10) in [6].)

Now we resume the proof of the lemma. Put $W_k=\Sigma_{i\leq k}\oplus U_i$ and hence $W=W_m$. We shall show by induction on k that there exist a new decomposition $W_k=U_1'\oplus U_2'\oplus\cdots\oplus U_k'$ and $\tilde{h}^{(k)}\colon U\to U_1'$ such that $U_i'\approx U_{\rho'(i)}$ (ρ' is a permutation on $\{1,\cdots,k\}$) and

$$
(5,\mathrm{k})\qquad
\begin{tikzcd}
U_1'\oplus\cdots\oplus U_k'=W_k \arrow[d, "\pi_1'"] & W_k\cap V \arrow[l, "i"'] \arrow[d, "h|(W_k\cap V)"] & 0 \arrow[l] \\
U_1' & U \arrow[l, "\tilde{h}^{(k)}"'] &
\end{tikzcd}
$$

is commutative, which implies

$$
(6)\qquad
\begin{aligned}
&\tilde{h}^{(k)}h_1' = 1_{(V\cap U_1')} \text{ and}\\
&\tilde{h}^{(k)}h_j' = \tilde{h}^{(k)}h|(V\cap U_j')=\pi_1'(V\cap U_j') = 0 \text{ for } j\neq 1, \text{ where}\\
&h_j' = h|(V\cap U_j') \text{ for all } j.
\end{aligned}
$$

(3) is nothing but $k=1$ in (6). We assume that W_k has the above decomposition and $\tilde{h}^{(k)}\colon U\to U_1'$. $W_{k+1}=W_k\oplus U_{k+1}=U_1'\oplus U_2'\oplus\cdots\oplus U_k'\oplus U_{k+1}$. Take the diagram:

$$
\begin{tikzcd}
U_{k+1} & V_{k+1} \arrow[l, "i_{k+1}"'] \arrow[d, "h_{k+1}"] & 0 \arrow[l] \arrow[dd, "g"] \\
& U \arrow[d, "\tilde{h}^{(k)}"] & \\
& U_1' &
\end{tikzcd}
.
$$

Put $g=\tilde{h}^{(k)}h_{k+1}$. Since U_1' is almost U_{k+1}-injective, we obtain either

i) there exists $\tilde{h}_{k+1}\colon U_{k+1}\to U_1'$ with $\tilde{h}_{k+1}i_{k+1}=g$,

or

ii) $\tilde{g}\colon U_1'\to U_{k+1}$ with $i_{k+1}=\tilde{g}g$.

Case i) By taking $\tilde{h}_1=\tilde{h}^{(k)}$, $\tilde{h}_j'=0$ $(1<j\leq k)$ and $\tilde{h}_{k+1}'=\tilde{h}_{k+1}$, the condition

(4) on W_{k+1} is satisfied from (6). Hence we obtain a new decomposition $W_{k+1}=U_1'\oplus U_2''\oplus\cdots\oplus U_k''\oplus U_{k+1}''(U_i''\simeq U'_{\rho(i)})$ and $\tilde{h}^{(k+1)}\colon U\to U_1'$, which satisfies $(5, k+1)$.

Case ii) If we put $\tilde{h}_{k+1}=\tilde{g}\tilde{h}^{(k)}\colon U\to U_{k+1}$, then from (6)

$$\begin{tikzcd}
U_1' \arrow[dddr, "\tilde{g}"'] & V_1' \arrow[l] \arrow[d, "h_1'"] & 0 \arrow[l] \\
& U \arrow[d, "\tilde{h}^{(k)}"] \arrow[dd, bend left=60, "\tilde{h}_{k+1}"] & \\
& U_1' \arrow[d, "\tilde{g}"] & \\
& U_{k+1} &
\end{tikzcd}$$

and for $j\neq 1$

$$\begin{tikzcd}
U_j' \arrow[dddr, "0"'] & V_j' \arrow[l] \arrow[d, "h_j'"] & 0 \arrow[l, dashed] \\
& U \arrow[d, "\tilde{h}^{(k)}"] \arrow[dd, bend left=60, "\tilde{h}_{k+1}"] & \\
& U_1' \arrow[d, "\tilde{g}"] & \\
& U_{k+1} &
\end{tikzcd}$$

are commutative. Therefore from (3) and (4) there exists a new decomposition $W_{k+1}=U_{k+1}\oplus U_1''\oplus\cdots\oplus U_k''$ such that

$$(5, k+1)\qquad \begin{tikzcd}
U_{k+1}\oplus U_1''\oplus\cdots\oplus U_k'' = W_{k+1} \arrow[d, "\pi_{k+1}''"] & W_{k+1}\cap V \arrow[l] \arrow[d, "h"] & 0 \arrow[l] \\
U_{k+1} & U \arrow[l, "\tilde{h}_{k+1}"] &
\end{tikzcd}$$

is commutative. Thus we have completed the proof.

In general let $\{D_i\}_{i=1}^{p}$ be a set of indecomposable R-modules and U and R-module. Assume that U is almost $\Sigma_i\oplus D_i$-injective. Then U is almost D_i-injective for all i. We shall divide $\{D_i\}$ into two disjoint parts $\{D_i\}=\{U_i\}\cup\{I_k\}$ as follows:

(7)
1) *U is I_k-injective for all k and*
2) *U is almost U_j-injective, but not U_j-injective for all j.*

Then we note that all U_j are uniform from [6], Theorem 1. Finally we give the main theorem

Theorem. *Let U be an R-module. Further let $\{U_j\}_{j=1}^m$ be a set of indecomposable R-modules and $\{I_k\}_{k=1}^n$ a set of R-modules. We assume that $\{U_j, I_k\}$ satisfy* (7). *Then if U_i, U_j are mutually almost relative injective, then U is almost $\Sigma_{j=1}^m \oplus U_j \oplus \Sigma_{k=1}^n \oplus I_k$-injective. Conversely if U is almost $\Sigma_{i=1}^m \oplus U_j \oplus \Sigma_{k=1}^n \oplus I_k$-injective and the U_j are* LE *modules, then U_i, U_j are mutually almost relative injective for any pair (i, j).*

Proof. The second half is clear from Lemmas 3, 2-ii) and 4. We study the first half. From (7) and Lemma 3 U_j is I_k-injective for any j and k. If $U_1^{\prime}$ is U_{k+1}-injective in the proof of Lemma 5, then we always obtain the case i). Therefore using Lemma 5, we can follow the proof in [6], Theorem 2 and get the theorem.

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