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DIRECT SUMS OF ALMOST RELATIVE INJECTIVE MODULES

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Let R be a ring with identity. When we study almost relative injective modules, the following problem is essential: Assume that an R -module V is almost U_j -injective for R -modules U_j ($j=1, 2, \dots, n$), then *under what conditions is V also almost $\Sigma_j \oplus U_j$ -injective?*

This problem is true without any assumptions, provided V is U_j -injective [2]. Y. Baba [3] gave an answer to the problem, when all V, U_j are uniform modules with finite length, and the author [6] generalized it to a case where the U_j are artinian indecomposable modules. Extending and utilizing the arguments given in [6], we shall drop the assumption “*artinian*” in this short note.

The proof will be completed by following the arguments given in [6]. Hence we shall explain only how we should modify the original proof in [6].

1. Preliminaries

Let R be a ring with identity. Every module in this paper is a right unitary R -module. We shall follow [3] and [6] for the terminologies. In [6], Theorem 2 we assumed that every module contained the non-zero socle. In this note we shall drop this assumption. Let W_1 and W_2 be R -modules. Take a diagram with V_2 a submodule of W_2 :

$$(1) \quad \begin{array}{c} W_2 \xleftarrow{i} V_2 \xleftarrow{\quad} 0 \\ \quad \downarrow g \\ \quad W_1 \end{array}$$

Consider the following two conditions:

- 1) There exists $\tilde{g}: W_2 \rightarrow W_1$ such that $\tilde{g}|V_2 = g$.
 - 2) There exist a non-zero direct summand W of W_2 : $W_2 = W \oplus W'$ and $\tilde{g}: W_1 \rightarrow W$ such that $\tilde{g}g = \pi|V_2$, where π is the projection of W_2 onto W .
- If either 1) or 2) holds true for any diagram (1), then we say that W_1 is *almost W_2 -injective* (if 1) always holds true, then we say that W_1 is *W_2 -injective* [2]).

We assume in the above that W_2 is indecomposable. If W_1 is almost

W_2 -injective,

(#) we always obtain 1), provided g is not a monomorphism.

Lemma 1. *The above (#) is equivalent to the following fact: W_1 is W_2/W -injective for any non-zero submodule W of W_2 .*

Proof. If g is not a monomorphism, then taking $g^{-1}(0)=W$, from (1) we obtain the diagram:

$$\begin{array}{ccccc} W_2 & \longleftarrow & V_2 & \longleftarrow & 0 \\ \downarrow \nu & & \downarrow \nu & & \\ W_2/W & \longleftarrow & V_2/W & \longleftarrow & 0 \\ & & \downarrow \bar{g} & & \\ & & W_1 & & \end{array}$$

where \bar{g} is the induced map from g and $\nu: W_2 \rightarrow W_2/W$ is the natural epimorphism. Hence if W_1 is W_2/W -injective, we have $\bar{g}': W_2/W \rightarrow W_1$ such that $\bar{g} = \bar{g}'|V_2/W$. Putting $\tilde{g} = \bar{g}'\nu$, $\tilde{g}|V_2 = g$. The converse is also clear from the above diagram.

Lemma 2. *Let U be an R -module and U_1 an indecomposable R -module. Assume that U is almost U_1 -injective. If U is not U_1 -injective, then there exist a non-zero submodule T of U_1 and a monomorphism $g: T \rightarrow U$, which is not extendible to an element in $\text{Hom}_R(U_1, U_1)$. In this case we obtain the same situation for any non-zero submodule T' in T and $g|T'$.*

Proof. The first half is clear from definition. Consider a diagram for a non-zero submodule T' in T ;

$$\begin{array}{ccc} U_1 & \xleftarrow{i} & T' \longleftarrow 0 \\ & \downarrow g|T' & \\ & U & \end{array}$$

Assume that there exists $\tilde{g}: U_1 \rightarrow U$ such that $\tilde{g}|T' = g|T'$. Put $g^* = g - (\tilde{g}|T): T \rightarrow U$. Then $g^{*-1}(0) \supset T' \neq 0$. Then from (#) there exists $\tilde{g}^*: U_1 \rightarrow U$ such that $\tilde{g}^*|T = g^* = g - (\tilde{g}|T)$. Hence $\tilde{g}^* + \tilde{g}$ is an extension of g , a contradiction.

From the above proof we obtain

Corollary. *Consider the diagram (1). Assume that there exists a non-zero submodule V in V_2 such that $g|V$ is extendible to an element in $\text{Hom}_R(W_2, W_1)$ and W_1 is W_2/V -injective. Then g is extendible.*

Lemma 3 ([6], Proposition 2). *Let U , U_2 be R -modules and U_1 an in-*

decomposable R -module. Assume that U is almost U_1 -injective, but not U_1 -injective. Under those assumptions 1): if U is U_2 -injective, then U_1 is U_2 -injective. 2): Assume that U_2 is indecomposable. If U is almost U_2 -injective, but not U_2 -injective, then we obtain the following fact: i); U_1 is U_2/V_2 -injective for any non-zero submodule V_2 of U_2 and hence ii); if U_1 and U_2 do not contain isomorphic submodules, U_1 is U_2 -injective. iii); Assume that U_2 (resp. U_1) contains non-zero submodule T_1 (resp. T_2) such that $g: T_1 \approx T_2$. Then we have the following equivalent conditions:

- a) U_1 (resp. U_2) is almost U_2 -(resp. U_1 -) injective.
- b) Either g or g^{-1} is extendible to an element in $\text{Hom}_R(U_1, U_2)$ or in $\text{Hom}_R(U_2, U_1)$ for every pair (T_2, T_1) .

Proof. The first half and 1), 2) are dual to [7], Proposition 1. However we shall give a proof for the sake of completeness.

1) By Lemma 2 there exist a submodule V_1 of U_1 , a monomorphism $g: V_1 \rightarrow U$ and $f: U \rightarrow U_1$ such that $fg = 1_{V_1}$. Put $E_i = E(U_i)$, the injective hull of U_i . Then there exist $\lambda: E_1 \rightarrow E_0$ and $\sigma: E_0 \rightarrow E_1$, which are extensions of g and f , respectively. Since U_1 is uniform from [6], Theorem 1, $\sigma\lambda$ is an automorphism of E_1 and hence $E_0 = E'_1 \oplus \ker \sigma$, where $E'_1 = \lambda(E_1)$. Further since $\sigma|_{E'_1}$ is an isomorphism, we can take a submodule U'_1 in E'_1 with $\sigma(U'_1) = U_1$. On the other hand $\sigma(U) = f(U) \subset U_1 = \sigma(U'_1)$. Hence $U \subset U'_1 \oplus \ker \sigma$. Now we may show that U'_1 is U_2 -injective. Let s be any element in $\text{Hom}_R(U_2, E'_1) \subset \text{Hom}_R(U_2, E_0)$. Since U is U_2 -injective $s(U_2) \subset U \subset U'_1 \oplus \ker \sigma \subset E'_1 \oplus \ker \sigma$ by [1], Proposition 1.4 (cf. [4], Lemma 9). Hence $s(U_2) \subset E'_1 \cap (U'_1 \oplus \ker \sigma) = U'_1$, and so U'_1 is U_2 -injective again by [1], Proposition 2.5.

2), i-ii) Since U is U_2/V_2 -injective by Lemma 1 for any (non-zero) submodule V_2 of U_2 , we can see from the above argument that U_1 is U_2/V_2 -injective.

2), iii) a) implies b) from definition. Assume b). Take a diagram with V_2 a submodule of U_2

$$\begin{array}{ccc} U_2 & \xleftarrow{i} & V_2 \leftarrow 0 \\ & \downarrow g & \\ & U_1 & \end{array}$$

If g is not a monomorphism, then there exists $\tilde{g}: U_2 \rightarrow U_1$ with $\tilde{g}|_{V_2} = g$ from 2), i) and Lemma 1. Hence we can assume that g is a monomorphism. As a consequence U_1 is almost U_2 -injective by b).

Lemma 4. *Let U be an R -module and U_1, U_2 LE R -modules. Assume that 1): U is almost U_1 -injective, but not U_1 -injective, 2): there exist submodules T_1, T_2 as in Lemma 3 and 3): U is almost $U_1 \oplus U_2$ -injective. Then either g or g^{-1}*

is extendible, and hence U_1 is almost U_2 -injective.

Proof. Since U is almost $U_1 \oplus U_2$ -injective, U is almost U_2 -injective. We show that U_1 is almost U_2 -injective. If U is U_2 -injective, U_1 is (almost) U_2 -injective by Lemma 3-1). Hence we assume that U is not U_2 -injective. Now there exist a non-zero submodule V_1 and $h: V_1 \rightarrow U$ given in Lemma 2. Since U_1 is uniform by [6], Theorem 1, we may assume $V_1 \subset T_1$ from the last part of Lemma 2. Take a diagram

$$\begin{array}{ccc} U_1 \oplus U_2 & \xleftarrow{i} & V_1 \oplus g(V_1) \leftarrow 0 \\ & & \downarrow h + hg^{-1} \\ & & U \end{array}.$$

Since h is not extendible, by assumption there exists an indecomposable direct summand Y of $U_1 \oplus U_2$ and $\tilde{h}: U \rightarrow Y$ such that $\tilde{h}(h + hg^{-1}) = \pi|(V_1 \oplus g(V_1))$, where π is the projection. Then either $g|V_1$ or $(g|V_1)^{-1}$ is extendible (cf. the proof of [5], Proposition 5). If $g|V_1$ is extendible, so does g from Corollary to Lemma 2, since U_2 is U_1/V_1 -injective by Lemma 3, 2)-i). Finally assume that $(g|V_1)^{-1}$ is extendible. Consider the diagram

$$\begin{array}{ccc} U_2 & \xleftarrow{i} & T_2 \supset g(V_1) \leftarrow 0 \\ & & \downarrow g^{-1} \\ & & U_1 \end{array}.$$

Since U_1 is $U_2/g(V_1)$ -injective by Lemma 3, 2)-i), we obtain an extension $\tilde{g}_2: U_2 \rightarrow U_1$ of g^{-1} from Corollary to Lemma 2. Therefore U_1 is almost U_2 -injective by Lemma 3-2), iii).

2. Main Theorem

In this section we shall give the desired theorem related to [3] and [6]. First we show the first half of the main theorem.

Lemma 5. *Let $\{U_i\}_{i=1}^m$ be a set of uniform R -modules and U an R -module. Assume that U_i and U_j are mutually almost relative injective for any pair (i, j) and U is almost U_i -injective for all $i > 0$. Then U is almost $\Sigma_{i=1}^m U_i$ -injective.*

Proof. Put $W = \Sigma_{i=1}^m U_i$, and consider a diagram with V a submodule of W :

$$\begin{array}{ccc} W & \xleftarrow{i} & V \leftarrow 0 \\ & & \downarrow h \\ & & U \end{array}.$$

In order to show the lemma, we may assume that

(*) V is essential in W (see [3] or [6], (#)).

Putting $V_j = V \cap U_j$ and $h_j = h|V_j$, we obtain the derived diagram:

$$(2) \quad \begin{array}{ccc} U_j & \xleftarrow{i_j} & V_j \longleftarrow 0 \\ & \downarrow h_j & \\ & U & \end{array}$$

Since U is almost U_j -injective, there exists

a) $\tilde{h}'_j: U_j \rightarrow U$ with $\tilde{h}'_j i_j = h_j$ or

b) $\tilde{h}_j: U \rightarrow U_j$ with $i_j = \tilde{h}_j h_j$.

We quote here the arguments given in [6]. From the argument in Step 3 in [6], namely from [3], Lemma C, (*) and induction on m , we know

if we obtain a) for all i , then there exists $\tilde{h}: W \rightarrow U$ with $\tilde{h}|V = h$.

Hence we assume that we have b) for some i , say $i=1$, i.e.

$$(3) \quad \begin{array}{ccc} U_1 & \xleftarrow{i_1} & V_1 \longleftarrow 0 \\ & \swarrow \tilde{h}_1 & \downarrow h_1 \\ & & U \end{array}$$

is commutative, which corresponds to (4') in [6]. Before proceeding the proof, we note the following fact from the argument in Steps 7 and 8 in [6]: We assume

(3) and there exists $\tilde{h}'_j: U_j \rightarrow U_1$ for all $j \neq 1$ such that

$$(4) \quad \begin{array}{ccc} U_j & \xleftarrow{i_j} & V_j \longleftarrow 0 \\ & \downarrow h_j & \\ & U & \\ \tilde{h}'_j \swarrow & & \downarrow \tilde{h}_1 \\ & & U_1 \end{array}$$

is commutative, which corresponds to (8) and step 7 in [6]. Then we obtain a new decomposition of $W := U_1 \oplus U'_2 \oplus \cdots \oplus U'_m$ and $h^*: U \rightarrow U_1$ such that $U'_i \approx U_{\rho(i)}$ (ρ is a permutation on $\{2, \dots, m\}$) and

$$(5) \quad \begin{array}{ccc} U_1 \oplus U'_2 \cdots \oplus U'_m & \xleftarrow{i} & V \longleftarrow 0 \\ \downarrow \pi_1 & & \downarrow h \\ U_1 & \xleftarrow{h^*} & U \end{array}$$

is commutative, which corresponds to (7) and step 8 in [6], where π_1 is the projection.

(In [6] we needed the assumption "artinian" to get the above (4). We note that the above (5) is shown by induction on m and the argument given after (10) in [6].)

Now we resume the proof of the lemma. Put $W_k = \Sigma_{i \leq k} U_i$ and hence $W = W_m$. We shall show by induction on k that there exist a new decomposition $W_k = U'_1 \oplus U'_2 \oplus \cdots \oplus U'_k$ and $\tilde{h}^{(k)}: U \rightarrow U'_1$ such that $U'_i \approx U_{\rho'(i)}$ (ρ' is a permutation on $\{1, \dots, k\}$) and

$$(5,k) \quad \begin{array}{ccc} U'_1 \oplus \cdots \oplus U'_k = W_k & \xleftarrow{i} & W_k \cap V \longleftarrow 0 \\ \downarrow \pi'_1 & & \downarrow h|(W_k \cap V) \\ U'_1 & \xleftarrow{\tilde{h}^{(k)}} & U \end{array}$$

is commutative, which implies

$$(6) \quad \begin{aligned} &\tilde{h}^{(k)} h'_1 = 1_{(V \cap U'_1)} \text{ and} \\ &\tilde{h}^{(k)} h'_j = \tilde{h}^{(k)} h|(V \cap U'_j) = \pi'_1(V \cap U'_j) = 0 \text{ for } j \neq 1, \text{ where} \\ &h'_j = h|(V \cap U'_j) \text{ for all } j. \end{aligned}$$

(3) is nothing but $k=1$ in (6). We assume that W_k has the above decomposition and $\tilde{h}^{(k)}: U \rightarrow U'_1$. $W_{k+1} = W_k \oplus U_{k+1} = U'_1 \oplus U'_2 \oplus \cdots \oplus U'_k \oplus U_{k+1}$. Take the diagram:

$$\begin{array}{ccc} U_{k+1} & \xleftarrow{i_{k+1}} & V_{k+1} \longleftarrow 0 \\ & & \downarrow h_{k+1} \\ & & U \\ & & \downarrow g \\ & & U'_1 \end{array}$$

Put $g = \tilde{h}^{(k)} h_{k+1}$. Since U'_1 is almost U_{k+1} -injective, we obtain either

i) there exists $\tilde{h}_{k+1}: U_{k+1} \rightarrow U'_1$ with $\tilde{h}_{k+1} i_{k+1} = g$,

or

ii) $\tilde{g}: U'_1 \rightarrow U_{k+1}$ with $i_{k+1} = \tilde{g}g$.

Case i) By taking $\tilde{h}_1 = \tilde{h}^{(k)}$, $\tilde{h}'_j = 0$ ($1 < j \leq k$) and $\tilde{h}_{k+1}' = \tilde{h}_{k+1}$, the condition

(4) on W_{k+1} is satisfied from (6). Hence we obtain a new decomposition $W_{k+1} = U'_1 \oplus U'_2 \oplus \cdots \oplus U'_k \oplus U_{k+1}'' (U'_i \simeq U'_{\rho(i)})$ and $\tilde{h}^{(k+1)}: U \rightarrow U'_1$, which satisfies (5, $k+1$).

Case ii) If we put $\tilde{h}_{k+1} = \tilde{g}\tilde{h}^{(k)}: U \rightarrow U_{k+1}$, then from (6)

$$\begin{array}{ccccc}
 U'_1 & \longleftarrow & V'_1 & \longleftarrow & 0 \\
 & & \downarrow h'_1 & & \\
 & & U & & \\
 & \searrow \tilde{g} & \downarrow \tilde{h}^{(k)} & \downarrow \tilde{h}_{k+1} & \\
 & & U'_1 & & \\
 & & \downarrow \tilde{g} & & \\
 & & U_{k+1} & &
 \end{array}$$

and for $j \neq 1$

$$\begin{array}{ccccc}
 U'_j & \longleftarrow & V'_j & \longleftarrow & 0 \\
 & & \downarrow h'_j & & \\
 & & U & & \\
 & \searrow 0 & \downarrow \tilde{h}^{(k)} & \downarrow \tilde{h}_{k+1} & \\
 & & U'_1 & & \\
 & & \downarrow \tilde{g} & & \\
 & & U_{k+1} & &
 \end{array}$$

are commutative. Therefore from (3) and (4) there exists a new decomposition $W_{k+1} = U_{k+1} \oplus U'_1 \oplus \cdots \oplus U'_k$ such that

$$\begin{array}{ccccc}
 U_{k+1} \oplus U'_1 \oplus \cdots \oplus U'_k = W_{k+1} & \longleftarrow & W_{k+1} \cap V & \longleftarrow & 0 \\
 \downarrow \pi_{k+1}'' & & \tilde{h}_{k+1} & & \downarrow h \\
 U_{k+1} & \longleftarrow & & & U
 \end{array}$$

is commutative. Thus we have completed the proof.

In general let $\{D_i\}_{i=1}^l$ be a set of indecomposable R -modules and U and R -module. Assume that U is almost $\Sigma_i \oplus D_i$ -injective. Then U is almost D_i -injective for all i . We shall divide $\{D_i\}$ into two disjoint parts $\{D_i\} = \{U_i\} \cup \{I_k\}$ as follows:

- 1) U is I_k -injective for all k and
- (7) 2) U is almost U_j -injective, but not U_j -injective for all j .

Then we note that all U_j are uniform from [6], Theorem 1. Finally we give the

main theorem

Theorem. *Let U be an R -module. Further let $\{U_j\}_{j=1}^m$ be a set of indecomposable R -modules and $\{I_k\}_{k=1}^n$ a set of R -modules. We assume that $\{U_j, I_k\}$ satisfy (7). Then if U_i, U_j are mutually almost relative injective, then U is almost $\Sigma_{j=1}^m \oplus U_j \oplus \Sigma_{k=1}^n \oplus I_k$ -injective. Conversely if U is almost $\Sigma_{i=1}^m \oplus U_i \oplus \Sigma_{k=1}^n \oplus I_k$ -injective and the U_j are LE modules, then U_i, U_j are mutually almost relative injective for any pair (i, j) .*

Proof. The second half is clear from Lemmas 3, 2-ii) and 4. We study the first half. From (7) and Lemma 3 U_j is I_k -injective for any j and k . If U'_1 is U_{k+1} -injective in the proof of Lemma 5, then we always obtain the case i). Therefore using Lemma 5, we can follow the proof in [6], Theorem 2 and get the theorem.

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