Date, E. and Horiuchi, E.
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ON CONTIGUITY RELATIONS OF JACKSON'S BASIC HYPERGEOMETRIC SERIES Υ_1 (*a*; *b*; *c*; *x*, *y*, 1/2) AND ITS GENERALIZATIONS

ETSURO DATE and EMI HORIUCHI

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1. Introduction. Our object is the following q-hypergeometric series of confluent type

$$(1)\qquad \sum_{\nu_1,\cdots,\nu_m=1}^{\infty} \frac{(\alpha:\Sigma_{i=1}^{m}\nu_i)_q(\beta_1:\nu_1)_q\cdots(\beta_{m-1}:\nu_{m-1})_q}{(\gamma:\Sigma_{i=1}^{m}\nu_i)_q(1:\nu_1)_q\cdots(1:\nu_{m-1})_q(1:\nu_m)_q} y_1^{\nu_1}\cdots y_m^{\nu_m} q^{\nu_m(\nu_m-1)/2},$$

where q is a complex number satisfying $|q|<1$. We have used the following notation $(a:n)_q=(a)_q(a+1)_q\cdots(a+n-1)_q$, $(a)_q=\dfrac{1-q^a}{1-q}$. When $m=1$, this series gives a q-analog of Kummer's hypergeometric series. This series (1) coincides with Jackson's basic double hypergeometric series $\Upsilon_1(\alpha;\beta_1;\gamma;y_1,y_2,1/2)$ [7] when $m=2$. Two series of this form are said to be contiguous if parameters α, β_i, γ and α', β_i', γ' corresponding to them differ at most 1 for each pair. We also say that two such series are contiguous to each other. For later convenience we introduce new parameters

$$(2)\qquad \alpha=\mu_2+1,\ \gamma=\mu_2+\mu_3+2,\ \beta_i=-\mu_i\ (4\le i<n),\ \sum_{i=1}^{n-1}\mu_i=-2.$$

We also rename independent variables as $y_i=x_{i+3}$ and set $n=m+3$ to make formulas appear later simple. In these new variables and parameters the series (1) looks as

$$(3)\qquad \sum_{\nu_4,\cdots,\nu_n=1}^{\infty} \frac{(\mu_2+1:\Sigma_{i=4}^{n}\nu_i)_q(-\mu_4:\nu_4)_q\cdots(-\mu_{n-1}:\nu_{n-1})_q}{(\mu_2+\mu_3+2:\Sigma_{i=4}^{n}\nu_i)_q(1:\nu_4)_q\cdots(1:\nu_{n-1})_q(1:\nu_n)_q} x_4^{\nu_4}\cdots x_n^{\nu_n} q^{\nu_n(\nu_n-1)/2}.$$

We shall describe q-difference operators which increase one of the μ_is and decrease one of the μ_is. We call such operators raising and/or lowering operators.

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Commutation relations among raising and/or lowering operators are called contiguity relations.

Contiguity relations for Gauβ hypergeometric series are well known for long time. In the theory of general hypergeometric systems by Gelfand et. al. [1], [2], [3] the Gauβ hypergeometric equation can be regarded as the general hypergeometric system on the Grassmannian $G(2,4)$. Horikawa [5] studies contiguity relations for Heine's basic hypergeometric series which is a q-deformation of Gauβ hypergeometric series exploiting this view point and showed that they constitute a representation of the quantum algebra $U_q(\mathfrak{sl}_4)$. This result has been generalized to q-analogs of Lauricella's hypergeometric series by Horikawa [6]. In this case the corresponding algebra is $U_q(\mathfrak{sl}_n)$. Noumi [10] rederived this result by using a q-analog of the function ring of the Grassmanian $G(2,n)$ and Casimir like elemets of $U_q(\mathfrak{sl}_n)$.

Our motivation was to see what happens when we consider hypergeometric series of confluent type. The series (3) is one of the simplest among confluent hypergeometric series obtained from the Lauricella hypergeometric series. We shall see contiguity relations for (3) is a representation of a q-deformation of enveloping algebra of semi direct product of $\mathfrak{sl}_{n-2}$ and a finite Heisenberg algebra with $2n-2$ generators. This seems to be a new feature of confluent series. In relabelling the parameters (2) we consulted the theory of general confluent hypergeometric systems by Kimura-Haraoka-Takano [8], [9], who studied in detail generalized confluent hypergeometric systems in the line of Gelfand-Retakh-Serganova [4].

2. Raising and lowering operators. Let T_i be the q-shift operator acting on the i-th variable

$$(T_i f)(x_4,\cdots,x_n)=f(x_4,\cdots,x_{i-1},qx_i,x_{i+1},\cdots,x_n).$$

Define $(\vartheta_i)_q=\dfrac{1-T_i}{1-q}$. This is a q-analog of the Euler derivative $x_i\dfrac{\partial}{\partial x_i}$. Define further $(\vartheta_i+a)_q=\dfrac{1-q^aT_i}{1-q}$ for $a\in \mathbf{C}$. Let us denote by F the series (3) with parameters α, β_i and γ. The series F satisfies the following system of q-difference equations

$$\left\{(\sum_{i=4}^n \vartheta_i+\alpha)_q-T_n^{-1}\frac{1}{x_n}(\vartheta_n)_q(\sum_{i=4}^n \vartheta_i+\gamma-1)_q\right\}F=0, \tag{4}$$

$$\left\{(\sum_{i=4}^n \vartheta_i+\alpha)_q(\vartheta_k+\beta_k)_q-\frac{1}{x_k}(\vartheta_k)_q(\sum_{i=4}^n \vartheta_i+\gamma-1)_q\right\}F=0, \qquad 4\le k<n. \tag{5}$$

The system of equations (4), (5) can also be obtained as a q-analog of confluent hypergeometric system for the series obtained from Lauricella's hypergeometric series by the confluence of the variable x_n.

We shall denote contiguous series by attaching increased (resp. decreased) parameters as superfixes (resp. suffices). For example F_{γ}^{α} denotes the series (3) with parameters $\alpha+1$, β_i, $\gamma-1$. Such changes of parameters are rephrased by changes of μ_is in view of (2). Let us denote by C_{ij} the operator which increases μ_i by 1 and decreases μ_j by 1. Let us introduce an extra parameter μ_n by the relation $\mu_n=\mu_1$. The identification of two parameters μ_1, μ_n is a consequence of this process of confluence.

By direct calculations we have the following relations:

$$
\begin{aligned}
&C_{23} && (\sum_{i=4}^{n} \vartheta_i+\alpha)_q F=(\alpha)_q F^{\alpha},\\
&C_{1k} && -q^{-\beta_k}(\vartheta_k+\beta_k)_q F=(-\beta_k)_q F^{\beta_k}, \quad 4\leq k<n,\\
(6)\quad &C_{13} && -q^{1-\gamma}(\sum_{i=4}^{n} \vartheta_i+\gamma-1)_q F=(1-\gamma)_q F_{\gamma},\\
&C_{2k} && q^{\gamma-\beta_k}\frac{1}{x_k}(\vartheta_k)_q F=\frac{(\alpha)_q(-\beta_k)_q}{(-\gamma)_q}F^{\alpha\beta_k\gamma}, \quad 4\leq k<n,\\
&C_{2n} && -q^{\gamma}T_n^{-1}\frac{1}{x_n}(\vartheta_n)_q F=\frac{(\alpha)_q}{(-\gamma)_q}F^{\alpha\gamma}.
\end{aligned}
$$

We have chosen indices for the operators C_{1j} or C_{in} to make the structure of contiguity relations simpler (see Theorem 2). We can also recover the system of q-difference equations (4), (5) for (3) from these relations. The compatibility conditions of the system (4), (5) are

$$
(7)\qquad \left\{T_n^{-1}\frac{1}{x_n}(\vartheta_n)_q(\vartheta_k+\beta_k)_q-\frac{1}{x_k}(\vartheta_k)_q\right\}F=0,
$$

$$
(8)\qquad \left\{(\vartheta_k+\beta_k)_q\frac{1}{x_l}(\vartheta_l)_q-(\vartheta_l+\beta_l)_q\frac{1}{x_k}(\vartheta_k)_q\right\}F=0.
$$

Now let us derive the operator which lowers the parameter α by 1. Applying x_nT_n to the both hand sides of (4), we obtain

$$
(9)\qquad \left\{x_nT_n(\sum_{i=4}^{n} \vartheta_i+\alpha)_q-(\vartheta_n)_q(\sum_{i=4}^{n} \vartheta_i+\gamma-1)_q\right\}F=0.
$$

Usig the identity

$$(\sum_{i=4}^{n} \vartheta_i+\gamma-1)_q=q^{\gamma-\alpha-1}(\sum_{i=4}^{n} \vartheta_i+\alpha)_q+(\gamma-\alpha-1)_q,$$

we have

$$\left[\{x_nT_n-q^{\gamma-\alpha-1}(\vartheta_n)_q\}(\sum_{i=4}^{n} \vartheta_i+\alpha)_q-(\gamma-\alpha-1)_q(\vartheta_n)_q\right]F=0. \tag{10}$$

Applying $T_4\cdots T_{n-1}$ to (10) and using the relation $x_iT_i=q^{-1}T_ix_i$, we get

$$\begin{aligned}\Big[\{q^{-1}T_4\cdots T_nx_n-q^{\gamma-\alpha-1}T_4\cdots T_{n-1}(\vartheta_n)_q\}(\sum_{i=4}^{n} \vartheta_i+\alpha)_q \\ -(\gamma-\alpha-1)_qT_4\cdots T_{n-1}(\vartheta_n)_q\Big]F=0.\end{aligned} \tag{11}$$

Multiplying x_k to the both hand sides of (5) and rewriting it in a similar way as above, we have

$$\left[\{x_k(\vartheta_k+\beta_k)_q-q^{\gamma-\alpha-1}(\vartheta_k)_q\}(\sum_{i=4}^{n} \vartheta_i+\alpha)_q-(\gamma-\alpha-1)_q(\vartheta_k)_q\right]F=0, \tag{12}$$

$$k=4,\cdots,n-1.$$

Let us denote the eqation (12) with k as $(12)_k$. Taking the sum $(11)+(12)_4$ $+T_4(12)_5+\cdots+T_4\cdots T_{n-2}(12)_{n-1}$, we obtain

$$\begin{aligned}\Big[\Big\{q^{-1}T_4\cdots T_nx_n+\sum_{k=5}^{n-1} T_4\cdots T_{k-1}x_k(\vartheta_k+\beta_k)_q-q^{\gamma-\alpha-1}(\sum_{i=4}^{n} \vartheta_i)_q\Big\}(\sum_{i=4}^{n} \vartheta_i+\alpha)_q \\ -(\gamma-\alpha-1)_q(\sum_{i=4}^{n} \vartheta_i)_q\Big]F=0.\end{aligned} \tag{13}$$

Using the identity

$$(\sum_{i=4}^{n} \vartheta_i)_q=q^{-\alpha}(\sum_{i=4}^{n} \vartheta_i+\alpha)_q-q^{-\alpha}(\alpha)_q,$$

we rewrite (13) as

$$\begin{aligned}\Big[\Big\{q^{-1}T_4\cdots T_nx_n+\sum_{k=5}^{n-1} T_4\cdots T_{k-1}x_k(\vartheta_k+\beta_k)_q-q^{\gamma-\alpha-1}(\sum_{i=4}^{n} \vartheta_i)_q \\ -q^{-\alpha}(\gamma-\alpha-1)_q\Big\}(\sum_{i=4}^{n} \vartheta_i+\alpha)_q\Big]F=-q^{-\alpha}(\gamma-\alpha-1)_q(\alpha)_qF.\end{aligned}$$

Multiplying $-q^{\alpha}$ to the both hand sides and using $(\Sigma_{i=4}^{n}\vartheta_i+\alpha)_qF=(\alpha)_qF^{\alpha}$, we get

$$\left\{-q^{\alpha-2}T_4\cdots T_nx_n-q^{\alpha-1}\sum_{k=5}^{n-1}T_1\cdots T_{k-1}x_k(\vartheta_k+\beta_k)_q\right.$$
$$\left.+q^{\gamma-1}(\sum_{i=4}^{n}\vartheta_i)_q+(\gamma-\alpha)_q\right\}F=(\gamma-\alpha)_qF_\alpha$$

after dividing the resulting equation by $(\alpha)_q$ and changing α to $\alpha-1$.

Similar calculations based on (4), (5), (7), (8) give the following

(14) C_{32}
$$\left\{-q^{\alpha-2}T_4\cdots T_nx_n-q^{\alpha-1}\sum_{k=5}^{n-1}T_1\cdots T_{k-1}x_k(\vartheta_k+\beta_k)_q\right.$$
$$\left.+q^{\gamma-1}(\sum_{i=4}^{n}\vartheta_i)_q+(\gamma-\alpha)_q\right\}F=(\gamma-\alpha)_qF_\alpha,$$

C_{3n}
$$\left\{q^{2\gamma-\alpha}T_n^{-1}\frac{1}{x_n}(\vartheta_n)_q-q^\gamma\right\}F=\frac{(\gamma-\alpha)_q}{(-\gamma)_q}F^\gamma,$$

C_{kn}
$$\left\{1-q^{\beta_k-1}x_kT_n^{-1}\frac{1}{x_n}(\vartheta_n)_q\right\}F=F_{\beta_k},\quad 4\le k<n,$$

C_{k2}
$$\left\{q^{-\gamma}T_k^{-1}x_n+q^{\alpha-\gamma}T_k^{-1}\sum_{j=5}^{n-1}T_4\cdots T_{j-1}x_j(\vartheta_j+\beta_j)_q\right.$$
$$\left.+q^{1-\gamma}T_k^{-1}(\alpha-1)_q-q^{1-\gamma}T_n^{-1}T_k^{-1}(\sum_{i=4}^{n}\vartheta_i+\gamma-1)_q\right\}F$$
$$=(1-\gamma)_qF_{\alpha\beta_k\gamma},\quad 4\le k<n,$$

C_{12}
$$\left\{q^{\alpha-\gamma-1}T_4\cdots T_nx_n-q^{1-\gamma}(\sum_{i=4}^{n}\vartheta_i+\gamma-1)_q\right.$$
$$\left.+q^{\alpha-\gamma}\sum_{j=5}^{n-1}T_4\cdots T_{j-1}x_j(\vartheta_j+\beta_j)_q\right\}F=(1-\gamma)_qF_{\alpha\gamma},$$

C_{3k}
$$\left\{q^{\gamma-\beta_k}(\vartheta_k+\beta_k)_q-q^{2\gamma-\alpha-\beta_k}\frac{1}{x_k}(\vartheta_k)_q\right\}F$$
$$=\frac{(-\beta_k)_q(\gamma-\alpha)_q}{(-\gamma)_q}F^{\beta_k\gamma},\quad 4\le k<n,$$

C_{k3}
$$\left\{q^{-\gamma}x_kT_k^{-1}(\sum_{i=4}^{n}\vartheta_i+\alpha)_q-q^{1-\gamma}T_k^{-1}(\sum_{i=4}^{n}\vartheta_i+\gamma-1)_q\right\}F$$
$$=(1-\gamma)_qF_{\beta_k\gamma},\quad 4\le k<n,$$

C_{kl}
$$\left\{q^{-\beta_l-1}x_kT_k^{-1}\frac{1}{x_l}(\vartheta_l)_q-q^{-\beta_l}(\vartheta_l+\beta_l)_qT_k^{-1}\right\}F$$

$$=(-\beta_l)_q F^{\beta_l}_{\beta_k}, \quad 4\leq k\neq l<n.$$

Let us denote by $S(\alpha;\beta_4;\cdots;\beta_{n-1};\gamma)$ the formal solution space of the system of equations (4), (5).

By direct calculations we have,

Theorem 1. *The operator C_{ij}, $1\leq i, j\leq n$ defined by the left hand sides of* (6), (14) *acts on the space* $\bigoplus_{a,b_k,c\in \boldsymbol{Z}} S(\alpha+a;\beta_4+b_4;\cdots;\beta_{n-1}+b_{n-1};\gamma+c)$, *and increases the parameter μ_i by* 1 *and decrease the parameter μ_j by* 1.

3. Contiguity relations. The raising and/or lowering operators C_{ij} (6), (14) satisfy the following commutation relations on the space $\oplus_{a,b_k,c\in \boldsymbol{Z}} S(\alpha+a;\beta_4+b_4;\cdots;\beta_{n-1}+b_{n-1};\gamma+c)$.

Theorem 2.

$$[C_{1k}, C_{kn}]_q=1, \quad 2\leq k<n,$$

(15) $$[C_{ik}, C_{kj}]_q=C_{ij}, \quad (i,j)\neq(1,n) \text{ and } i<k<j \text{ or } i>k>j,$$

$$[C_{ij}, C_{ji}]=q^{\mu_j}(\mu_i-\mu_j)_q,$$

where $[a,b]_q=ab-qba$ and $[a,b]$ denotes usual commutator. All the other pairs are commutative with respect to the usual commutator.

In order to see the structure of above commutation relations (15), let us consider the limit $q\to 1$. In this limit the above commutation relations give a representation of the following Lie subalgebra of $\mathfrak{sl}_n$:

(16) $$\mathfrak{G}=\left\{\begin{pmatrix} 0 & a_{12} & \cdots & a_{1,n-1} & a_{1n} \\ 0 & a_{22} & \cdots & a_{2,n-1} & a_{2n} \\ \cdot & \cdots & \cdots & \cdots & \cdots \\ 0 & a_{n-1,2} & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}\right\}.$$

Let us denote by E_{ij} $n\times n$ matrix units. The correspondence $E_{ij}\to C_{ij}|_{q=1}$ for $i\neq j$ and $E_{ii}-E_{jj}\to \mu_i-\mu_j$ give a representation of $\mathfrak{G}$. Note that in $\mathfrak{G}$, elements E_{1j}, E_{jn} $2\leq j<n$ and E_{1n} form a finite Heisenberg algebra with E_{1n} as a central element and constitute an ideal of $\mathfrak{G}$. Therefore we see that contiguity relations for (3) give a representation of a q-deformation of the Lie ablebra $\mathfrak{G}$ (16).

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Etsuro Date
Department of Mathematics
Osaka University
Toyonaka 560, Japan

Emi Horiuchi
Information System Division
Hitachi Ltd.