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**ON CONTIGUITY RELATIONS OF  
 JACKSON'S BASIC HYPERGEOMETRIC SERIES  
 $\Upsilon_1(a; b; c; x, y, 1/2)$   
 AND ITS GENERALIZATIONS**

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**1. Introduction.** Our object is the following  $q$ -hypergeometric series of confluent type

$$(1) \quad \sum_{v_1, \dots, v_m=1}^{\infty} \frac{(\alpha : \sum_{i=1}^m v_i)_q (\beta_1 : v_1)_q \cdots (\beta_{m-1} : v_{m-1})_q}{(\gamma : \sum_{i=1}^m v_i)_q (1 : v_1)_q \cdots (1 : v_{m-1})_q (1 : v_m)_q} y_1^{v_1} \cdots y_m^{v_m} q^{v_m(v_m-1)/2},$$

where  $q$  is a complex number satisfying  $|q| < 1$ . We have used the following notation  $(a : n)_q = (a)_q (a+1)_q \cdots (a+n-1)_q$ ,  $(a)_q = \frac{1-q^a}{1-q}$ . When  $m=1$ , this series gives a  $q$ -analog of Kummer's hypergeometric series. This series (1) coincides with Jackson's basic double hypergeometric series  $\Upsilon_1(\alpha; \beta_1; \gamma; y_1, y_2, 1/2)$  [7] when  $m=2$ . Two series of this form are said to be contiguous if parameters  $\alpha, \beta_i, \gamma$  and  $\alpha', \beta'_i, \gamma'$  corresponding to them differ at most 1 for each pair. We also say that two such series are contiguous to each other. For later convenience we introduce new parameters

$$(2) \quad \alpha = \mu_2 + 1, \gamma = \mu_2 + \mu_3 + 2, \beta_i = -\mu_i \quad (4 \leq i < n), \sum_{i=1}^{n-1} \mu_i = -2.$$

We also rename independent variables as  $y_i = x_{i+3}$  and set  $n = m + 3$  to make formulas appear later simple. In these new variables and parameters the series (1) looks as

$$(3) \quad \sum_{v_4, \dots, v_n=1}^{\infty} \frac{(\mu_2 + 1 : \sum_{i=4}^n v_i)_q (-\mu_4 : v_4)_q \cdots (-\mu_{n-1} : v_{n-1})_q}{(\mu_2 + \mu_3 + 2 : \sum_{i=4}^n v_i)_q (1 : v_4)_q \cdots (1 : v_{n-1})_q (1 : v_n)_q} x_4^{v_4} \cdots x_n^{v_n} q^{v_n(v_n-1)/2}.$$

We shall describe  $q$ -difference operators which increase one of the  $\mu_i$ s and decrease one of the  $\mu_i$ s. We call such operators raising and/or lowering operators.

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Commutation relations among raising and/or lowering operators are called contiguity relations.

Contiguity relations for Gauß hypergeometric series are well known for long time. In the theory of general hypergeometric systems by Gelfand et. al. [1], [2], [3] the Gauß hypergeometric equation can be regarded as the general hypergeometric system on the Grassmannian  $G(2,4)$ . Horikawa [5] studies contiguity relations for Heine’s basic hypergeometric series which is a  $q$ -deformation of Gauß hypergeometric series exploiting this view point and showed that they constitute a representation of the quantum algebra  $U_q(\mathfrak{sl}_4)$ . This result has been generalized to  $q$ -analogs of Lauricella’s hypergeometric series by Horikawa [6]. In this case the corresponding algebra is  $U_q(\mathfrak{sl}_n)$ . Noumi [10] rederived this result by using a  $q$ -analog of the function ring of the Grassmanian  $G(2,n)$  and Casimir like elements of  $U_q(\mathfrak{sl}_n)$ .

Our motivation was to see what happens when we consider hypergeometric series of confluent type. The series (3) is one of the simplest among confluent hypergeometric series obtained from the Lauricella hypergeometric series. We shall see contiguity relations for (3) is a representation of a  $q$ -deformation of enveloping algebra of semi direct product of  $\mathfrak{sl}_{n-2}$  and a finite Heisenberg algebra with  $2n-2$  generators. This seems to be a new feature of confluent series. In relabelling the parameters (2) we consulted the theory of general confluent hypergeometric systems by Kimura-Haraoka-Takano [8], [9], who studied in detail generalized confluent hypergeometric systems in the line of Gelfand-Retakh-Serganova [4].

**2. Raising and lowering operators.** Let  $T_i$  be the  $q$ -shift operator acting on the  $i$ -th variable

$$(T_i f)(x_4, \dots, x_n) = f(x_4, \dots, x_{i-1}, qx_i, x_{i+1}, \dots, x_n).$$

Define  $(\vartheta_i)_q = \frac{1-T_i}{1-q}$ . This is a  $q$ -analog of the Euler derivative  $x_i \frac{\partial}{\partial x_i}$ . Define further  $(\vartheta_i + a)_q = \frac{1-q^a T_i}{1-q}$  for  $a \in \mathbb{C}$ . Let us denote by  $F$  the series (3) with parameters  $\alpha, \beta_i$  and  $\gamma$ . The series  $F$  satisfies the following system of  $q$ -difference equations

$$(4) \quad \left\{ \left( \sum_{i=4}^n \vartheta_i + \alpha \right)_q - T_n^{-1} \frac{1}{x_n} (\vartheta_n)_q \left( \sum_{i=4}^n \vartheta_i + \gamma - 1 \right)_q \right\} F = 0,$$

$$(5) \quad \left\{ \left( \sum_{i=4}^n \vartheta_i + \alpha \right)_q (\vartheta_k + \beta_k)_q - \frac{1}{x_k} (\vartheta_k)_q \left( \sum_{i=4}^n \vartheta_i + \gamma - 1 \right)_q \right\} F = 0, \quad 4 \leq k < n.$$

The system of equations (4), (5) can also be obtained as a  $q$ -analog of confluent hypergeometric system for the series obtained from Lauricella's hypergeometric series by the confluence of the variable  $x_n$ .

We shall denote contiguous series by attaching increased (resp. decreased) parameters as superfixes (resp. suffices). For example  $F_7^\alpha$  denotes the series (3) with parameters  $\alpha + 1, \beta_i, \gamma - 1$ . Such changes of parameters are rephrased by changes of  $\mu_i$ s in view of (2). Let us denote by  $C_{ij}$  the operator which increases  $\mu_i$  by 1 and decreases  $\mu_j$  by 1. Let us introduce an extra parameter  $\mu_n$  by the relation  $\mu_n = \mu_1$ . The identification of two parameters  $\mu_1, \mu_n$  is a consequence of this process of confluence.

By direct calculations we have the following relations:

$$\begin{aligned}
 C_{23} & \quad \left(\sum_{i=4}^n \vartheta_i + \alpha\right)_q F = (\alpha)_q F^\alpha, \\
 C_{1k} & \quad -q^{-\beta_k} (\vartheta_k + \beta_k)_q F = (-\beta_k)_q F^{\beta_k}, \quad 4 \leq k < n, \\
 (6) \quad C_{13} & \quad -q^{1-\gamma} \left(\sum_{i=4}^n \vartheta_i + \gamma - 1\right)_q F = (1-\gamma)_q F_\gamma, \\
 C_{2k} & \quad q^{\gamma-\beta_k} \frac{1}{x_k} (\vartheta_k)_q F = \frac{(\alpha)_q (-\beta_k)_q}{(-\gamma)_q} F^{\alpha\beta_k\gamma}, \quad 4 \leq k < n, \\
 C_{2n} & \quad -q^\gamma T_n^{-1} \frac{1}{x_n} (\vartheta_n)_q F = \frac{(\alpha)_q}{(-\gamma)_q} F^{\alpha\gamma}.
 \end{aligned}$$

We have chosen indices for the operators  $C_{1j}$  or  $C_{in}$  to make the structure of contiguity relations simpler (see Theorem 2). We can also recover the system of  $q$ -difference equations (4), (5) for (3) from these relations. The compatibility conditions of the system (4), (5) are

$$(7) \quad \left\{ T_n^{-1} \frac{1}{x_n} (\vartheta_n)_q (\vartheta_k + \beta_k)_q - \frac{1}{x_k} (\vartheta_k)_q \right\} F = 0,$$

$$(8) \quad \left\{ (\vartheta_k + \beta_k)_q \frac{1}{x_l} (\vartheta_l)_q - (\vartheta_l + \beta_l)_q \frac{1}{x_k} (\vartheta_k)_q \right\} F = 0.$$

Now let us derive the operator which lowers the parameter  $\alpha$  by 1. Applying  $x_n T_n$  to the both hand sides of (4), we obtain

$$(9) \quad \left\{ x_n T_n \left(\sum_{i=4}^n \vartheta_i + \alpha\right)_q - (\vartheta_n)_q \left(\sum_{i=4}^n \vartheta_i + \gamma - 1\right)_q \right\} F = 0.$$

Usig the identity

$$\left(\sum_{i=4}^n \vartheta_i + \gamma - 1\right)_q = q^{\gamma - \alpha - 1} \left(\sum_{i=4}^n \vartheta_i + \alpha\right)_q + (\gamma - \alpha - 1)_q,$$

we have

$$(10) \quad \left[ \{x_n T_n - q^{\gamma - \alpha - 1}(\vartheta_n)_q\} \left(\sum_{i=4}^n \vartheta_i + \alpha\right)_q - (\gamma - \alpha - 1)_q (\vartheta_n)_q \right] F = 0.$$

Applying  $T_4 \cdots T_{n-1}$  to (10) and using the relation  $x_i T_i = q^{-1} T_i x_i$ , we get

$$(11) \quad \left[ \{q^{-1} T_4 \cdots T_n x_n - q^{\gamma - \alpha - 1} T_4 \cdots T_{n-1}(\vartheta_n)_q\} \left(\sum_{i=4}^n \vartheta_i + \alpha\right)_q - (\gamma - \alpha - 1)_q T_4 \cdots T_{n-1}(\vartheta_n)_q \right] F = 0.$$

Multiplying  $x_k$  to the both hand sides of (5) and rewriting it in a similar way as above, we have

$$(12) \quad \left[ \{x_k(\vartheta_k + \beta_k)_q - q^{\gamma - \alpha - 1}(\vartheta_k)_q\} \left(\sum_{i=4}^n \vartheta_i + \alpha\right)_q - (\gamma - \alpha - 1)_q (\vartheta_k)_q \right] F = 0,$$

$k = 4, \dots, n - 1.$

Let us denote the eqation (12) with  $k$  as  $(12)_k$ . Taking the sum  $(11) + (12)_4 + T_4(12)_5 + \cdots + T_4 \cdots T_{n-2}(12)_{n-1}$ , we obtain

$$(13) \quad \left[ \left\{ q^{-1} T_4 \cdots T_n x_n + \sum_{k=5}^{n-1} T_4 \cdots T_{k-1} x_k (\vartheta_k + \beta_k)_q - q^{\gamma - \alpha - 1} \left(\sum_{i=4}^n \vartheta_i\right)_q \right\} \left(\sum_{i=4}^n \vartheta_i + \alpha\right)_q - (\gamma - \alpha - 1)_q \left(\sum_{i=4}^n \vartheta_i\right)_q \right] F = 0.$$

Using the identity

$$\left(\sum_{i=4}^n \vartheta_i\right)_q = q^{-\alpha} \left(\sum_{i=4}^n \vartheta_i + \alpha\right)_q - q^{-\alpha}(\alpha)_q,$$

we rewrite (13) as

$$\left[ \left\{ q^{-1} T_4 \cdots T_n x_n + \sum_{k=5}^{n-1} T_4 \cdots T_{k-1} x_k (\vartheta_k + \beta_k)_q - q^{\gamma - \alpha - 1} \left(\sum_{i=4}^n \vartheta_i\right)_q - q^{-\alpha}(\gamma - \alpha - 1)_q \right\} \left(\sum_{i=4}^n \vartheta_i + \alpha\right)_q \right] F = -q^{-\alpha}(\gamma - \alpha - 1)_q (\alpha)_q F.$$

Multiplying  $-q^\alpha$  to the both hand sides and using  $(\sum_{i=4}^n \vartheta_i + \alpha)_q F = (\alpha)_q F^\alpha$ , we get

$$\left\{ -q^{\alpha-2}T_4 \cdots T_n x_n - q^{\alpha-1} \sum_{k=5}^{n-1} T_1 \cdots T_{k-1} x_k (\vartheta_k + \beta_k)_q + q^{\gamma-1} \left( \sum_{i=4}^n \vartheta_i \right)_q + (\gamma - \alpha)_q \right\} F = (\gamma - \alpha)_q F_\alpha$$

after dividing the resulting equation by  $(\alpha)_q$  and changing  $\alpha$  to  $\alpha - 1$ . Similar calculations based on (4), (5), (7), (8) give the following

$$(14) \quad C_{32} \quad \left\{ -q^{\alpha-2}T_4 \cdots T_n x_n - q^{\alpha-1} \sum_{k=5}^{n-1} T_1 \cdots T_{k-1} x_k (\vartheta_k + \beta_k)_q + q^{\gamma-1} \left( \sum_{i=4}^n \vartheta_i \right)_q + (\gamma - \alpha)_q \right\} F = (\gamma - \alpha)_q F_\alpha,$$

$$C_{3n} \quad \left\{ q^{2\gamma-\alpha} T_n^{-1} \frac{1}{x_n} (\vartheta_n)_q - q^\gamma \right\} F = \frac{(\gamma - \alpha)_q}{(-\gamma)_q} F^\gamma,$$

$$C_{kn} \quad \left\{ 1 - q^{\beta_k-1} x_k T_n^{-1} \frac{1}{x_n} (\vartheta_n)_q \right\} F = F_{\beta_k}, \quad 4 \leq k < n,$$

$$C_{k2} \quad \left\{ q^{-\gamma} T_k^{-1} x_n + q^{\alpha-\gamma} T_k^{-1} \sum_{j=5}^{n-1} T_4 \cdots T_{j-1} x_j (\vartheta_j + \beta_j)_q + q^{1-\gamma} T_k^{-1} (\alpha - 1)_q - q^{1-\gamma} T_n^{-1} T_k^{-1} \left( \sum_{i=4}^n \vartheta_i + \gamma - 1 \right)_q \right\} F = (1 - \gamma)_q F_{\alpha\beta_k\gamma}, \quad 4 \leq k < n,$$

$$C_{12} \quad \left\{ q^{\alpha-\gamma-1} T_4 \cdots T_n x_n - q^{1-\gamma} \left( \sum_{i=4}^n \vartheta_i + \gamma - 1 \right)_q + q^{\alpha-\gamma} \sum_{j=5}^{n-1} T_4 \cdots T_{j-1} x_j (\vartheta_j + \beta_j)_q \right\} F = (1 - \gamma)_q F_{\alpha\gamma},$$

$$C_{3k} \quad \left\{ q^{\gamma-\beta_k} (\vartheta_k + \beta_k)_q - q^{2\gamma-\alpha-\beta_k} \frac{1}{x_k} (\vartheta_k)_q \right\} F = \frac{(-\beta_k)_q (\gamma - \alpha)_q}{(-\gamma)_q} F^{\beta_k\gamma}, \quad 4 \leq k < n,$$

$$C_{k3} \quad \left\{ q^{-\gamma} x_k T_k^{-1} \left( \sum_{i=4}^n \vartheta_i + \alpha \right)_q - q^{1-\gamma} T_k^{-1} \left( \sum_{i=4}^n \vartheta_i + \gamma - 1 \right)_q \right\} F = (1 - \gamma)_q F_{\beta_k\gamma}, \quad 4 \leq k < n,$$

$$C_{kl} \quad \left\{ q^{-\beta_l-1} x_k T_k^{-1} \frac{1}{x_l} (\vartheta_l)_q - q^{-\beta_l} (\vartheta_l + \beta_l)_q T_k^{-1} \right\} F$$

$$= (-\beta_l)_q F_{\beta_k}^{\beta_l}, \quad 4 \leq k \neq l < n.$$

Let us denote by  $S(\alpha; \beta_4; \dots; \beta_{n-1}; \gamma)$  the formal solution space of the system of equations (4), (5).

By direct calculations we have,

**Theorem 1.** *The operator  $C_{ij}$ ,  $1 \leq i, j \leq n$  defined by the left hand sides of (6), (14) acts on the space  $\bigoplus_{a, b_k, c \in \mathbf{Z}} S(\alpha + a; \beta_4 + b_4; \dots; \beta_{n-1} + b_{n-1}; \gamma + c)$ , and increases the parameter  $\mu_i$  by 1 and decrease the parameter  $\mu_j$  by 1.*

**3. Contiguity relations.** The raising and/or lowering operators  $C_{ij}$  (6), (14) satisfy the following commutation relations on the space  $\bigoplus_{a, b_k, c \in \mathbf{Z}} S(\alpha + a; \beta_4 + b_4; \dots; \beta_{n-1} + b_{n-1}; \gamma + c)$ .

**Theorem 2.**

$$(15) \quad \begin{aligned} [C_{1k}, C_{kn}]_q &= 1, \quad 2 \leq k < n, \\ [C_{ik}, C_{kj}]_q &= C_{ij}, \quad (i, j) \neq (1, n) \text{ and } i < k < j \text{ or } i > k > j, \\ [C_{ij}, C_{ji}] &= q^{\mu_i - \mu_j} \end{aligned}$$

where  $[a, b]_q = ab - qba$  and  $[a, b]$  denotes usual commutator. All the other pairs are commutative with respect to the usual commutator.

In order to see the structure of above commutation relations (15), let us consider the limit  $q \rightarrow 1$ . In this limit the above commutation relations give a representation of the following Lie subalgebra of  $\mathfrak{sl}_n$ :

$$(16) \quad \mathfrak{G} = \left\{ \begin{pmatrix} 0 & a_{12} & \cdots & a_{1,n-1} & a_{1n} \\ 0 & a_{22} & \cdots & a_{2,n-1} & a_{2n} \\ \cdot & \dots & \dots & \dots & \dots \\ 0 & a_{n-1,2} & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \right\}.$$

Let us denote by  $E_{ij}$   $n \times n$  matrix units. The correspondence  $E_{ij} \rightarrow C_{ij}|_{q=1}$  for  $i \neq j$  and  $E_{ii} - E_{jj} \rightarrow \mu_i - \mu_j$  give a representation of  $\mathfrak{G}$ . Note that in  $\mathfrak{G}$ , elements  $E_{1j}, E_{jn}$   $2 \leq j < n$  and  $E_{1n}$  form a finite Heisenberg algebra with  $E_{1n}$  as a central element and constitute an ideal of  $\mathfrak{G}$ . Therefore we see that contiguity relations for (3) give a representation of a  $q$ -deformation of the Lie algebra  $\mathfrak{G}$  (16).

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