Factor rings of a hereditary and QF-3 ring

Harada, Manabu

Osaka Journal of Mathematics. 17(1) P.1-P.8

1980

Publisher

https://doi.org/10.18910/3808

10.18910/3808

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We have been studying many interesting properties of small submodules. W.W. Leonard [8] and M. Rayar [12] defined small modules and gave elementary properties of them. Recently, the author has studied non-small modules and given a class of rings which are concerned with non-small modules and located between QF-rings and QF-3 rings [4] and [5].

In this note we shall consider two conditions (*) and (*)* in [4] and [5] (see §1) and study a semi-primary ring whose every factor ring satisfies either (*) or (*)*. We shall show such a ring with condition (QS) (see §1) coincides with a generalized uni-serial ring of the first category in the sense of Murase [9].

1. The main theorem

Let $R$ be a ring with identity. We always assume that $R$ is a semi-primary ring, namely the Jacobson radical $J$ of $R$ is nilpotent and $R/J$ is artinian, and every $R$-module is an unitary right $R$-module unless otherwise stated. Let $M$ be an $R$-module. By $E(M)$ and $J(M)$ we denote an injective hull and the Jacobson radical of $M$, respectively. If $M$ is a small submodule in $E(M)$, we say $M$ is a small module [8], [12] and if $M$ is not a small module, we say $M$ is non-small module [5]. As the dual concept to the above, we define a non-cosmall module $N$ as follows: there exist a projective module $P$ and an epimorphism $f: P \rightarrow N$ such that $\ker f$ is not essential in $P$.

In [4] and [5] we have introduced two conditions:

(*) Every non-small module contains a non-zero injective module.

(*)* Every non-cosmall module contains a non-zero projective direct summand.

We have shown that if $R$ satisfies either (*) or (*)*, then $R$ is a right QF-3 ring [13] ($E(R)$ is projective by [7]) and every QF-ring satisfies both (*) and (*)*. Thus, a class of rings satisfying either (*) or (*)* is located between a class of QF-rings and one of QF-3 rings when $R$ is a left and right artinian ring. If $R$ is left and right artinian and $eR, Re$ have unique composition series for every
primitive idempotent $e$, we call $R$ a generalised uni-serial ring [10]. It is easily seen that every generalised uni-serial ring satisfies both (*) and (*) (Corollary 1 to Lemma 1 below).

Following Murase [9] we say a two-sided indecomposable generalised uni-serial ring is in the first category, if there exists a primitive idempotent $e$ such that $eR$ is simple. In order to show that some rings in the new class coincide with the above rings, we introduce the conditions:

- $(F*)$ (resp. $(F**)$) Every factor ring of $R$ satisfies $(*)$ (resp. $(*)^*$).
- $(FQF-3)$ Every factor ring of $R$ is right QF-3. And
- $(QS)$ If a factor ring of $R$ is a QF-ring, then it is semi-simple.

Now, we can state our theorem.

**Theorem.** Let $R$ be a semi-primary ring. Then the following statements are equivalent.

1) $R$ satisfies $(F*)$ and $(QS)$.
2) $R$ satisfies $(F**)$. and $(QS)$.
3) $R$ satisfies $(FQF-3)$ and $(QS)$.
4) $R$ is isomorphic to a factor ring of QF-3 and hereditary ring. And
5) $R$ is a direct sum of generalised uni-serial rings of the first category.

We know from [2], Theorem 2 and [9], Theorems 17 and 18 that the ring $R$ in the theorem is a direct sum of factor rings of rings of tri-angular matrices over division rings when $R$ is basic. Hence, it has a perspective form.

We shall give remarks on the above conditions.

**Remarks 1.** If $R$ is a generalised uni-serial ring of the second category [9], $R$ satisfies $(F*)$, $(F**)$ and $(FQF-3)$ but not $(QS)$ (see §2).

2. If $R$ is a left and right artinian, then $R$ is a generalised uni-serial ring if and only if $R$ satisfies $(FQF-3)$ [6].

3. Let $K \subseteq L$ be fields with $[L:K]<\infty$ and

$$R = \begin{pmatrix} K & L \\ 0 & K \end{pmatrix}.$$ 

Then $R$ satisfies $(QS)$ but not any of $(F*)$, $(F**)$ and $(FQF-3)$.

4. If $R$ is a commutative artinian ring and satisfies $(QS)$, then $R$ is a direct sum of fields.

Because, we may assume $R$ is a local ring with maximal ideal $M$. If $M \neq 0$, we could find a maximal one $M'$ among ideals contained in $M$. Then $R/M'$ is a QF-ring and so $M/M'=0$.

2. **Proof of Theorem**

We always assume that $R$ is a semi-primary ring with identity and every
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Let $M$ be a unitary right $R$-module. We shall denote the Jacobson radical and the injective hull by $J(M)$ and $E(M)$, respectively. Let $R$ be as above and $1 = \sum_{i=1}^{\infty} \sum_{j=1}^{n} g_{ij}$, where $\{g_{ij}\}$ is a set of mutually orthogonal primitive idempotents such that $g_{ij}R \cong g_{ii}R$ for any $j$ and $g_{ij}R \cong g_{ji}R$ for $i \neq i'$. We put $g = \sum_{i=1}^{\infty} g_{ii}$ and $R_0 = gRg$. $gRg$ is the basic ring of $R$. It is well known that the category of right $R$-modules is Morita equivalent to one of right $R_0$-modules. We have a one to one mapping between the set of two-sided ideals $A$ in $R$ and one of those $A_0$ in $R_0$ such that $A_0 = gAg$ and $A = RA_0R$.

**Lemma 1.** Let $A$ be a two-sided ideal. We put $\bar{R} = R/A$ and $A_0 = gAg$. Then $\bar{R}_0 = R_0/A_0$ is the basic ring of $\bar{R}$.

Proof. It is clear that $1 = \sum_{i=1}^{\infty} \sum_{j=1}^{n} g_{ij}$ and $g_{ij}\bar{R} \cong g_{ii}\bar{R}$. If $g_{ij} = 0$, $g_{ij}$ is also a primitive idempotent and $g_{ij}g_{i'j'} = \delta_{ij}\delta_{ij'}g_{ij'}$. We assume $g_{ii}\bar{R} = g_{ii}\bar{R}$ for $i \neq j$. Then there exists $x$ in $g_{ii}Rg_{ii}$ such that $xg_{ii}R + g_{ii}A = g_{ii}R$. Since $g_{ii}R \cong g_{ii}R$, $xg_{ii}R \subseteq g_{ii}J(R)$. Hence, $g_{ji}A = g_{ii}R$ by Nakayama’s Lemma and so $g_{ii}A \subseteq A$ for any $k$. Thus, $\bar{R}_0$ is the basic ring of $\bar{R}$.

**Corollary.** $R$ satisfies one of $(F*)$, $(F**)$, $(FQF-3)$ and $(QS)$ if and only if $\bar{R}$ does the basic ring of $\bar{R}$.

**Lemma 2.** Let $R$ be a generalized uni-serial ring. Then every indecomposable non-small (resp. non-coflatMap) module is injective (resp. projective).

Proof. Every indecomposable module is uni-serial by [10]. Hence, the lemma is trivial from the definitions.

**Corollary 1.** Every generalized uni-serial ring satisfies $(F*)$, $(F**)$ and $(FQF-3)$.

**Corollary 2.** Let $R$ be left and right artinian. Then the following statements are equivalent.

1) $R$ satisfies $(FQF-3)$.
2) $R$ satisfies $(F*)$.
3) $R$ satisfies $(F**)$.
4) $R$ is a generalized uni-serial ring.

Proof. 1)$\iff$4) is proved in [6]. Corollary 1 gives 4)$\iff$2) and 3). We know 2)$\iff$1) and 3)$\iff$1) from [5], Propositions 2.5 and 3.4.

In order to prove the theorem, we may always assume from Lemma 1 that $R$ is basic and $g_{ii}Rg_{ii}g_{ii}Jg_{ii} = \Delta_i$ is a division ring. Let $M_{ij}$ be a $\Delta_i - \Delta_j$ bimodule ($i < j$). We defined the ring of generalized upper tri-angular ma-
traces $T_n(\Delta_i; M_{ij})$ [3]. When $\Delta_i=\Delta$ for all $i$ and $M_{ij}=\Delta$, we shall denote the usual upper tri-angular matrix ring by $T_n(\Delta)$ and the set of matrix units by $\{e_{ij}\}_{i<j}$.

**Lemma 3.** Let $\Delta_i$ be division rings and $R=T_n(\Delta_i; M_{ij})$. 1) We assume $e_{ii}R$ is injective and $M_{ik}=0$ and $M_{it}=0$ for all $t>k$. Then $\text{Hom}_{\Delta_i}(Re_{ii}/(M_{i-1k} \oplus M_{i-2k} \oplus \cdots \oplus M_{1k}), \Delta_i)$ is isomorphic to $e_{ii}R$ by multiplications of elements in $e_{ii}R$ from the left side. Hence, $M_{ik}=0$ if and only if $M_{pk}=0$.

2) If $R$ is a right QF-3, $e_{ii}R$ is injective.

**Proof.** 1) Since $M_{ik}$ is the socle of $e_{ii}R$, $[M_{ik}: \Delta_i]=1$. We have the natural homomorphism $\varphi: e_{ii}R \to \text{Hom}_{\Delta_i}(Re_{ii}/(M_{i-1k} \oplus \cdots \oplus M_{1k}), \Delta_i)$. Since $\varphi(M_{ik}) \neq 0$, $\varphi$ is monomorphic. Let $f$ be in $\text{Hom}_{\Delta_i}(Re_{ii}/(M_{i-1k} \oplus \cdots \oplus M_{1k}), \Delta_i) = \sum_{i=1}^{\Delta} \oplus \text{Hom}_{\Delta_i}(M_{jk}, \Delta_i)$ and $f=\sum f_{ij} f_{ij} \in \text{Hom}_{\Delta_i}(M_{jk}, \Delta_i)$. Put $F_{ij}(M_{jk})=0$ for $i>k$. Then $F_{ij}(M_{jk})\in \text{Hom}_R(M_{jk}R, e_{ii}R)$, since $M_{it}=0$ for $t>k$. Hence, there exists an element $x_{ij}$ in $e_{ii}R$ such that $F_{ij}(m_{jk})=x_{ij} m_{jk} (x_{ij} \in M_{jk})$ for every $m_{jk}$, since $e_{ii}R$ is injective. Therefore, $f=\varphi(\sum x_{ij})$. Hence, $\varphi$ is isomorphic. 2) If $R$ is right QF-3, $E(R) \approx \sum \oplus (e_{ii}R)^*$, since $R$ is semi-primary. Being $e_{ii}R e_{ii}=0$ for $i>1$, the index set $K$ must contain 1. Hence, $e_{ii}R$ is injective.

Let $R=T_n(\Delta)$ and $A$ a two-sided ideal. It is clear [9]

$$R/A = \begin{pmatrix} \Delta & 0 \\ \Delta & \Delta \\ \vdots & \ddots & \Delta \\ 0 & \cdots & \Delta & \Delta \end{pmatrix} \quad (2.1).$$

We call such a form the *standard form* of $R/A$. It is easily seen that $R/A$ is a generalized uni-serial ring of the first category. Hence, from Lemmas 1 and 2 and [2], Theorem 2 (consider $e_{ii}R$) we have

**Lemma 4.** Let $R$ be a factor ring of a QF-3 and hereditary ring. Then $R$ satisfies $(F*)$, $(F*^*)$, $(FQF-3)$ and $(QS)$.

Now, we shall consider the converse case.

**Lemma 5.** If $R$ satisfies one of $(F*)$, $(F*^*)$, $(FQF-3)$ and $(QS)$, then so does every factor ring of $R$.

It is clear.

**Lemma 6.** Let $R=T_2(\Delta_1, R_2; M_{12})$. If $R$ is two-sided indecomposable and $e_{ii}R$ is injective, then $R_2$ is indecomposable, where $\Delta_1$ is a division ring and $R_2$ is a
semi-primary ring.

Proof. From the assumptions \( e_i R \) contains a unique minimal submodule. Hence, \( R_2 \) is indecomposable if so is \( R \).

**Lemma 7.** Let \( R \) be a semi-primary, two-sided indecomposable and basic ring. We assume \( J^2 = 0 \). If \( R \) satisfies (FQF-3) and (QS), then \( R \) is isomorphic to \( T_\alpha(\Delta)[J(T_\alpha(\Delta))]^2 \), where \( \Delta \) is a division ring.

Proof. Let \( R = \sum_{i=1}^n e_i R \oplus \sum_{j=1}^m f_j R \) be a decomposition of \( R \) with indecomposable modules \( e_i R \) and \( f_j R \), where the \( e_i R \) is injective and the \( f_j R \) is small (see [5], Theorem 1.3). We quote here the argument in [6], Lemma in pp. 404-405. We know \( \sum \oplus e_i R \) is faithful. Let \( x \neq 0 \) be in \( f_j R \). Then \( (\sum \oplus e_i R)x \neq 0 \) and so there exists \( e_i R \) such that \( 0 \neq e_i x = \phi f_j x \in J x \). Hence, \( x \notin f_j J \) since \( J^2 = 0 \). Therefore, \( f_j R \) is simple if \( f_j R \neq 0 \). Since \( e_i R \) is injective and \( J^2 = 0 \), \( e_i R \) is uni-serial. Accordingly, \( R \) is right artinian. First, we assume \( m = 0 \). Then \( R \) is self-injective and so a QF-ring (see [1], Theorem 1). Therefore, \( R \) is a division ring by (QS). Thus, we may assume \( m \neq 0 \). We know from the above that \( f_j R \) is simple. Hence, \( f_j R_g = 0 \) for any primitive idempotent \( g \) (\( \neq f_i \)) and \( f_i R f_i = \Delta \) is a division ring. Thus, we have

\[
R = \begin{pmatrix} R_1 & FRf_i \\ 0 & \Delta \end{pmatrix}
\]

where \( F = 1 - f_1 \) and \( R_1 = FRF \) satisfies (QS) and (FQF-3). We first assume \( s = n + m = 2 \). Then \( n = m = 1 \). Hence, \( R_1 \) is a division ring from the case \( m = 0 \). Therefore, \( R \approx T_2(\Delta) \) by [2], Theorem 2 and [3], Theorem 1. Now, we shall prove the lemma by induction on \( s = s(R) \) (we assume \( m = 0 \)). We have done it when \( s \leq 2 \). Since \( s(R) > s(R_1) \), \( R_1 \approx \sum \oplus T_\alpha(\Delta_i)[J(T_\alpha(\Delta_i))]^2 \) by the induction, where the \( \Delta_i \) is a division ring. Hence, we obtain \( R = T_\alpha(\Delta_1, \Delta_2, \ldots, \Delta_{s-1}, \Delta; M_{ij}) \). Lemma 3,2) shows that \( e_{ii} R \) is injective. It is clear \( e_{kk} R e_{kk} = 0 \) for \( k \neq 1 \). We put \( F' = 1 - e_{ii} \) and \( R_1' = F' RF' \). Then we have

\[
R = \begin{pmatrix} \Delta_1 & e_{ii} R F' \\ 0 & R_1' \end{pmatrix}
\]

(2.3)

Here \( R_1' \) is two-sided indecomposable by Lemma 6. Hence, \( R_1' \approx T_{s-1}(\Delta') \) by the hypothesis of induction. Now \( R \) is of the form

\[
\begin{pmatrix}
\Delta_1 & A_2 & \cdots & A_s \\
\Delta' & \Delta' & & \\
& & \ddots & 0 \\
& & & 0 & \cdots & \Delta' \\
& & & & & \Delta'
\end{pmatrix}
\]

(2.4)
Since $e_{ij}R$ contains a unique (minimal) submodule, only one $A_i$ is not zero. If $i \neq 2$, $A_2 = 0$ implies $M_{i-1} = \Delta = 0$ by Lemma 3. Hence, $A_i = 0$ for $i > 2$. Since $s \geq 3$, we have $\Delta' = \Delta_i$ and $A_2 = \Delta_i$ by the induction (cf. [3], Lemma 13).

**Lemma 8.** If $R$ satisfies (FQF-3) and (QS), then $R$ is isomorphic to a factor ring of a semi-primary hereditary ring $R'$ such that $R/J(R) \approx R'/J(R')$.

**Proof.** We know $R/J^2 \approx \bigoplus T_n(\Delta_j)/J(T_n(\Delta_j))^2$ by Lemmas 5 and 7. Hence, gl. dim $R/J^2 < \infty$ by [3], Theorem 3. Therefore, we obtain the lemma by [3], Theorem 5 and its proof.

Since $R/J(R) \approx R'/J(R')$, $R'$ is basic and $R' \approx T_n(\Delta_i; M_i)$ by [3], Theorem 4'. Let $\{f_{ii}\}$ be the usual matrix units in $R'$. Then $gRf_{ii} = 0$ for any primitive idempotent $g$ with $gR' \cong f_{ii}R'$. Let $\varphi: R' \to R$ be the ring epimorphism. Then $J(R') = \varphi^{-1}(J(R))$ and $\{e_{ii} = \varphi(f_{ii})\}$ is a complete set of mutually orthogonal primitive idempotents in $R$. If $0 \neq e_{jj}Re_{ii} = \varphi(f_{jj}R'f_{ii})$ implies $j = 1$. Furthermore, $e_{ii}J(R)e_{ii} = \varphi(f_{ii}J(R')f_{ii}) = 0$. From now on, we shall denote $e_{ii}$ by $e_i$. Then $\Delta_i = e_iRe_i$ is a division ring from the above.

**Lemma 9.** If $R$ satisfies (FQF-3) and (QS), then $R$ is isomorphic to $\sum \oplus T_n(\Delta_i)/C_i$, where $C_i$ is a two-sided ideal in $T_n(\Delta_i)$.

**Proof.** We may assume $R$ is a two-sided indecomposable. We shall use the notations above. Put $F = 1 - e_i$. Then

$$R \approx \begin{pmatrix} \Delta_i & A \\ 0 & R_i \end{pmatrix}$$ (2.5).

We shall prove the lemma by induction on $n$, where $1 = \sum e_i$. If $n \leq 2$, the lemma is true by Lemma 7. We assume $n \geq 3$. Then since $e_{ij}R$ is injective by Lemma 3, 2), $R_i \approx T_{n-1}(\Delta)/C$ by Lemma 6 and the induction. Thus, we obtain

$$R = \begin{pmatrix} \Delta_i & A_2 & \cdots & A_n \\ \Delta & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \Delta \end{pmatrix}$$ (2.6).

If we take a two-sided ideal $Re_a$ and use the induction hypothesis, we know $\Delta_i = \Delta$ and $A_i (i < n)$ is equal to either zero or $\Delta$ (cf. [3], Lemma 13). We assume $A_n \neq 0$. Since $e_iR$ is injective and has a simple socle, $[A_n: \Delta] = 1$ as a right $\Delta$-module. Put $A_n = u\Delta$. We know by Lemma 3 that every $\Delta$-endomorphism of $u\Delta$ is given by a unique element of $\Delta = e_iRe_i$. Let $x$ be in $e_iRe_i$,
then \( xu = u\delta(x) \), where \( \delta \) is a ring homomorphism of \( \Delta \). Therefore, \( \delta(\Delta) = \Delta \) from the above and so \( A_s = \Delta \) as a two-sided \( \Delta \)-module, if \( A_s \neq 0 \). Now we may assume \( A_s = \Delta \) and \( A_{s+1} = \cdots = A_s = 0 \). We shall show \( A_s = 0 \). Assume \( A_s = A_s = \cdots = A_{s+k} = 0 \) and \( A_s = \Delta \) for some \( s \leq k \). We put \( D = \sum_{s=0}^{k} \oplus Re_s \). Then \( \overline{R} = R/D \)

\[
\begin{pmatrix}
\Delta & 0 & 0 & \cdots & \Delta \\
\Delta & \_ & 0 & \cdots & \_ \\
\_ & \Delta & \_ & \cdots & \_ \\
\_ & \_ & \_ & \cdots & \_ \\
0 & \_ & \_ & \cdots & \Delta
\end{pmatrix}
\]  

(2.7)

Since \( e_1 \overline{R} \) is \( \overline{R} \)-injective, \( E_s = \cdots = E_{s-1} = 0 \) by Lemma 3. However, \( R_1 \) is indecomposable and is of the standard form. Hence, \( E_{s-1} \neq 0 \), which is a contradiction. Accordingly, \( A_s = 0 \) and \( e_1R_1 \neq 0 \) by Lemma 3. Again, since \( R_1 \) is of standard form, \( e_jR_1 \neq 0 \) for \( j \leq k \). Therefore, \( R \cong T_n(\Delta)/C \).

**Lemma 10** ([9], Theorems 17 and 18). *Let \( R \) be a two-sided indecomposable basic and generalized uni-serial ring. If there exists a primitive idempotent \( e \) such that \( eR \) is simple, then \( R \) is isomorphic to \( T_n(\Delta)/C \).*

**Proof.** \( R \) satisfies (F*) by Corollary 1 to Lemma 2. First we assume \( J = 0 \). We use the same notations in the proof of Lemma 7. We assume \( m = 0 \) and \( e_nR \) is simple. Since \( R \) is a QF-ring, \( e_nR \) is a two-sided ideal. Hence, \( R \) is a division ring. If \( m \neq 0 \), we obtain the form (2.2) and so (2.3). Hence, we can use the same argument. In general case, noting that \( e_1R \) is not simple in (2.3), we can use the induction. Therefore, Lemma 8 is true for the ring in the lemma. Again we can use the same argument in the proof of Lemma 9.

**References**


Department of Mathematics
Osaka City University
Sugimoto-cho, Sumiyoshi-ku
Osaka 558, Japan