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## FACTOR RINGS OF A HEREDITARY AND QF-3 RING

Dedicated to Professor Goro Azumaya on his 60th birthday

MANABU HARADA

(Received April 19, 1979)

We have been studying many interesting properties of small submodules. W.W. Leonard [8] and M. Rayar [12] defined small modules and gave elementary properties of them. Recently, the author has studied non-small modules and given a class of rings which are concerned with non-small modules and located between QF-rings and QF-3 rings [4] and [5].

In this note we shall consider two conditions (\*) and (\*)<sup>\*</sup> in [4] and [5] (see §1) and study a semi-primary ring whose every factor ring satisfies either (\*) or (\*)<sup>\*</sup>. We shall show such a ring with condition (QS) (see §1) coincides with a generalized uni-serial ring of the first category in the sense of Murase [9].

### 1. The main theorem

Let  $R$  be a ring with identity. We always assume that  $R$  is a semi-primary ring, namely the Jacobson radical  $J$  of  $R$  is nilpotent and  $R/J$  is artinian, and every  $R$ -module is an unitary right  $R$ -module unless otherwise stated. Let  $M$  be an  $R$ -module. By  $E(M)$  and  $J(M)$  we denote an injective hull and the Jacobson radical of  $M$ , respectively. If  $M$  is a small submodule in  $E(M)$ , we say  $M$  is a *small module* [8], [12] and if  $M$  is not a small module, we say  $M$  is a *non-small module* [5]. As the dual concept to the above, we define a *non-cosmall module*  $N$  as follows: there exist a projective module  $P$  and an epimorphism  $f: P \rightarrow N$  such that  $\ker f$  is not essential in  $P$ .

In [4] and [5] we have introduced two conditions:

(\*) *Every non-small module contains a non-zero injective module.*

(\*)<sup>\*</sup> *Every non-cosmall module contains a non-zero projective direct summand.*

We have shown that if  $R$  satisfies either (\*) or (\*)<sup>\*</sup>, then  $R$  is a right QF-3 ring [13] ( $E(R)$  is projective by [7]) and every QF-ring satisfies both (\*) and (\*)<sup>\*</sup>. Thus, a class of rings satisfying either (\*) or (\*)<sup>\*</sup> is located between a class of QF-rings and one of QF-3 rings when  $R$  is a left and right artinian ring. If  $R$  is left and right artinian and  $eR, Re$  have unique composition series for every

primitive idempotent  $e$ , we call  $R$  a *generalized uni-serial ring* [10]. It is easily seen that every generalized uni-serial ring satisfies both  $(*)$  and  $(*)^*$  (Corollary 1 to Lemma 1 below).

Following Murase [9] we say a two-sided indecomposable generalized uni-serial ring is in *the first category*, if there exists a primitive idempotent  $e$  such that  $eR$  is simple. In order to show that some rings in the new class coincide with the above rings, we introduce the conditions:

(F\*) (resp. (F\*)) *Every factor ring of  $R$  satisfies  $(*)$  (resp.  $(*)^*$ ).*

(FQF-3) *Every factor ring of  $R$  is right QF-3. And*

(QS) *If a factor ring of  $R$  is a QF-ring, then it is semi-simple.*

Now, we can state our theorem.

**Theorem.** *Let  $R$  be a semi-primary ring. Then the following statements are equivalent.*

- 1)  $R$  satisfies (F\*) and (QS).
- 2)  $R$  satisfies (F\*) and (QS).
- 3)  $R$  satisfies (FQF-3) and (QS).
- 4)  $R$  is isomorphic to a factor ring of QF-3 and hereditary ring. And
- 5)  $R$  is a direct sum of generalized uni-serial rings of the first category.

We know from [2], Theorem 2 and [9], Theorems 17 and 18 that the ring  $R$  in the theorem is a direct sum of factor rings of rings of triangular matrices over division rings when  $R$  is basic. Hence, it has a perspective form.

We shall give remarks on the above conditions.

REMARKS 1. If  $R$  is a generalized uni-serial ring of the second category [9],  $R$  satisfies (F\*), (F\*) and (FQF-3) but not (QS) (see §2).

2. If  $R$  is a left and right artinian, then  $R$  is a generalized uni-serial ring if and only if  $R$  satisfies (FQF-3) [6].

3. Let  $K \cong L$  be fields with  $[L:K] < \infty$  and

$$R = \begin{pmatrix} K & L \\ 0 & K \end{pmatrix}.$$

Then  $R$  satisfies (QS) but not any of (F\*), (F\*) and (FQF-3).

4. If  $R$  is a commutative artinian ring and satisfies (QS), then  $R$  is a direct sum of fields.

Because, we may assume  $R$  is a local ring with maximal ideal  $M$ . If  $M \neq 0$ , we could find a maximal one  $M'$  among ideals contained in  $M$ . Then  $R/M'$  is a QF-ring and so  $M/M' = 0$ .

## 2. Proof of Theorem

We always assume that  $R$  is a semi-primary ring with identity and every

$R$ -module  $M$  is an unitary right  $R$ -module. We shall denote the Jacobson radical and the injective hull by  $J(M)$  and  $E(M)$ , respectively. Let  $R$  be as above and  $1 = \sum_{i=1}^n \sum_{j=1}^{p(i)} g_{ij}$ , where  $\{g_{ij}\}$  is a set of mutually orthogonal primitive idempotents such that  $g_{ij}R \approx g_{i1}R$  for any  $j$  and  $g_{ij}R \not\approx g_{i'j'}R$  for  $i \neq i'$ . We put  $g = \sum_{i=1}^n g_{i1}$  and  $R_0 = gRg$  i.e.  $gRg$  is the basic ring of  $R$  [11] and [2]. It is well known that the category of right  $R$ -modules is Morita equivalent to one of right  $R_0$ -modules. We have a one to one mapping between the set of two-sided ideals  $A$  in  $R$  and one of those  $A_0$  in  $R_0$  such that  $A_0 = gAg$  and  $A = RA_0R$ .

**Lemma 1.** *Let  $A$  be a two-sided ideal. We put  $\bar{R} = R/A$  and  $A_0 = gAg$ . Then  $\bar{R}_0 = R_0/A_0$  is the basic ring of  $\bar{R}$ .*

Proof. It is clear that  $\bar{1} = \sum_{i=1}^n \sum_{j=1}^{p(i)} \bar{g}_{ij}$  and  $\bar{g}_{ij}\bar{R} \approx \bar{g}_{i1}\bar{R}$ . If  $\bar{g}_{ij} \neq \bar{0}$ ,  $\bar{g}_{ij}$  is also a primitive idempotent and  $\bar{g}_{ij}\bar{g}_{i'j'} = \delta_{ii'}\delta_{jj'}\bar{g}_{ij}$ . We assume  $\bar{g}_{i1}\bar{R} \approx \bar{g}_{j1}\bar{R}$  for  $i \neq j$ . Then there exists  $x$  in  $g_{i1}Rg_{i1}$  such that  $xg_{i1}R + g_{j1}A = g_{j1}R$ . Since  $g_{i1}R \not\approx g_{j1}R$ ,  $xg_{i1}R \subseteq g_{j1}J(R)$ . Hence,  $g_{j1}A = g_{j1}R$  by Nakayama's Lemma and so  $g_{ik} \in A$  for any  $k$ . Thus,  $\bar{R}_0$  is the basic ring of  $\bar{R}$ .

**Corollary.**  *$R$  satisfies one of (F\*), (F\*\*), (FQF-3) and (QS) if and only if so does the basic ring of  $R$ .*

**Lemma 2.** *Let  $R$  be a generalized uni-serial ring. Then every indecomposable non-small (resp. non-cosmall) module is injective (resp. projective).*

Proof. Every indecomposable module is uni-serial by [10]. Hence, the lemma is trivial from the definitions.

**Corollary 1.** *Every generalized uni-serial ring satisfies (F\*), (F\*\*) and (FQF-3).*

**Corollary 2.** *Let  $R$  be left and right artinian. Then the following statements are equivalent.*

- 1)  $R$  satisfies (FQF-3).
- 2)  $R$  satisfies (F\*).
- 3)  $R$  satisfies (F\*\*). And
- 4)  $R$  is a generalized uni-serial ring.

Proof. 1)  $\leftrightarrow$  4) is proved in [6]. Corollary 1 gives 4)  $\rightarrow$  2) and 3). We know 2)  $\rightarrow$  1) and 3)  $\rightarrow$  1) from [5], Propositions 2.5 and 3.4.

In order to prove the theorem, we may always assume from Lemma 1 that  $R$  is basic and  $g_{i1}Rg_{i1}/g_{i1}Jg_{i1} = \Delta_i$  is a division ring. Let  $M_{ij}$  be a  $\Delta_i - \Delta_j$  bimodule ( $i < j$ ). We defined the ring of generalized upper tri-angular ma-

trices  $T_n(\Delta_i; M_{ij})$  [3]. When  $\Delta_i = \Delta$  for all  $i$  and  $M_{ij} = \Delta$ , we shall denote the usual upper triangular matrix ring by  $T_n(\Delta)$  and the set of matrix units by  $\{e_{ij}\}_{i < j}$ .

**Lemma 3.** *Let  $\Delta_i$  be division rings and  $R = T_n(\Delta_i; M_{ij})$ . 1) We assume  $e_{ii}R$  is injective and  $M_{ik} \neq 0$  and  $M_{it} = 0$  for all  $t > k$ . Then  $\text{Hom}_{\Delta_k}(Re_{kk}/(M_{i-1k} \oplus \cdots \oplus M_{1k}), \Delta_k)$  is isomorphic to  $e_{ii}R$  by multiplications of elements in  $e_{ii}R$  from the left side. Hence,  $M_{ip} \neq 0$  if and only if  $M_{pk} \neq 0$ .*

2) *If  $R$  is a right QF-3,  $e_{11}R$  is injective.*

*Proof.* 1) Since  $M_{ik}$  is the socle of  $e_{ii}R$ ,  $[M_{ik}: \Delta_k] = 1$ . We have the natural homomorphism  $\varphi: e_{ii}R \rightarrow \text{Hom}_{\Delta_k}(Re_{kk}/(M_{i-1k} \oplus \cdots \oplus M_{1k}), \Delta_k)$ . Since  $\varphi(M_{ik}) \neq 0$ ,  $\varphi$  is monomorphic. Let  $f$  be in  $\text{Hom}_{\Delta_k}(Re_{kk}/(M_{i-1k} \oplus \cdots \oplus M_{1k}), \Delta_k) = \sum_{p=i}^k \oplus \text{Hom}_{\Delta_k}(M_{pk}, M_{ik})$  and  $f = \sum f_p; f_p \in \text{Hom}_{\Delta_k}(M_{pk}, M_{ik})$ . Put  $F_p | M_{pk} = f_p$

$F_p(M_{pt}) = 0$  for  $t > k$ . Then  $F_p \in \text{Hom}_R(M_{pk}R, e_{ii}R)$ , since  $M_{it} = 0$  for  $t > k$ . Hence, there exists an element  $x_p$  in  $e_{ii}R$  such that  $F_p(m_{pk}) = x_p m_{pk}$  ( $x_p \in M_{ip}$ ) for every  $m_{pk}$ , since  $e_{ii}R$  is injective. Therefore,  $f = \varphi(\sum x_p)$ . Hence,  $\varphi$  is isomorphic.

2) If  $R$  is right QF-3,  $E(R) \approx \sum_{i \in K} \oplus (e_{ii}R)^n$  since  $R$  is semi-primary. Being  $e_{ii}Re_{11} = 0$  for  $i > 1$ , the index set  $K$  must contain 1. Hence,  $e_{11}R$  is injective.

Let  $R = T_n(\Delta)$  and  $A$  a two-sided ideal. It is clear [9]

$$R/A = \begin{pmatrix} \Delta & & & & \\ & \Delta & & & \\ & & \Delta & & \\ & & & \ddots & \\ 0 & & & & \Delta \\ & & & & & \Delta \end{pmatrix} \quad (2.1).$$

We call such a form the *standard form* of  $R/A$ . It is easily seen that  $R/A$  is a generalized uni-serial ring of the first category. Hence, from Lemmas 1 and 2 and [2], Theorem 2 (consider  $e_{nn}R$ ) we have

**Lemma 4.** *Let  $R$  be a factor ring of a QF-3 and hereditary ring. Then  $R$  satisfies (F\*), (F\*\*), (FQF-3) and (QS).*

Now, we shall consider the converse case.

**Lemma 5.** *If  $R$  satisfies one of (F\*), (F\*\*), (FQF-3) and (QS), then so does every factor ring of  $R$ .*

It is clear.

**Lemma 6.** *Let  $R = T_2(\Delta_1, R_2; M_{12})$ . If  $R$  is two-sided indecomposable and  $e_{11}R$  is injective, then  $R_2$  is indecomposable, where  $\Delta_1$  is a division ring and  $R_2$  is a*

*semi-primary ring.*

Proof. From the assumptions  $e_{11}R$  contains a unique minimal submodule. Hence,  $R_2$  is indecomposable if so is  $R$ .

**Lemma 7.** *Let  $R$  be a semi-primary, two-sided indecomposable and basic ring. We assume  $J^2=0$ . If  $R$  satisfies (FQF-3) and (QS), then  $R$  is isomorphic to  $T_n(\Delta)/J(T_n(\Delta))^2$ , where  $\Delta$  is a division ring.*

Proof. Let  $R = \sum_{i=1}^n \oplus e_i R \oplus \sum_{j=1}^m \oplus f_j R$  be a decomposition of  $R$  with indecomposable modules  $e_i R$  and  $f_j R$ , where the  $e_i R$  is injective and the  $f_j R$  is small (see [5], Theorem 1.3). We quote here the argument in [6], Lemma in pp. 404–405. We know  $\sum \oplus e_i R$  is faithful. Let  $x \neq 0$  be in  $f_j R$ . Then  $(\sum \oplus e_i R)x \neq 0$  and so there exists  $e_i r$  such that  $0 \neq e_i r x = e_i r f_j x \in Jx$ . Hence,  $x \notin f_j J$  since  $J^2=0$ . Therefore,  $f_j R$  is simple if  $f_j R \neq 0$ . Since  $e_i R$  is injective and  $J^2=0$ ,  $e_i R$  is uni-serial. Accordingly,  $R$  is right artinian. First, we assume  $m=0$ . Then  $R$  is self-injective and so a QF-ring (see [1], Theorem 1). Therefore,  $R$  is a division ring by (QS). Thus, we may assume  $m \neq 0$ . We know from the above that  $f_1 R$  is simple. Hence,  $f_1 R g = 0$  for any primitive idempotent  $g$  ( $\not\approx f_1$ ) and  $f_1 R f_1 = \Delta$  is a division ring. Thus, we have

$$R = \begin{pmatrix} R_1 & FRf_1 \\ 0 & \Delta \end{pmatrix} \quad (2.2),$$

where  $F=1-f_1$  and  $R_1=FRF$  satisfies (QS) and (FQF-3). We first assume  $s=n+m=2$ . Then  $n=m=1$ . Hence,  $R_1$  is a division ring from the case  $m=0$ . Therefore,  $R \approx T_2(\Delta)$  by [2], Theorem 2 and [3], Theorem 1. Now, we shall prove the lemma by induction on  $s=s(R)$  (we assume  $m \neq 0$ ). We have done it when  $s \leq 2$ . Since  $s(R) > s(R_1)$ ,  $R_1 \approx \sum \oplus T_{n_i}(\Delta_i)/J(T_{n_i}(\Delta_i))^2$  by the induction, where the  $\Delta_i$  is a division ring. Hence, we obtain  $R = T_s(\Delta_1, \Delta_2, \dots, \Delta_{s-1}, \Delta; M_{ij})$ . Lemma 3,2) shows that  $e_{11}R$  is injective. It is clear  $e_{kk}R e_{11} = 0$  for  $k \neq 1$ . We put  $F' = 1 - e_{11}$  and  $R_1' = F' R F'$ . Then we have

$$R = \begin{pmatrix} \Delta_1 & e_{11} R F' \\ 0 & R_1' \end{pmatrix} \quad (2.3).$$

Here  $R_1'$  is two-sided indecomposable by Lemma 6. Hence,  $R_1' \approx T_{s-1}(\Delta')/J(T_{s-1}(\Delta'))^2$  by the hypothesis of induction. Now  $R$  is of the form

$$\begin{pmatrix} \Delta_1 & A_2 & \dots & A_s \\ & \Delta' & \Delta' & \\ & & \ddots & 0 \\ & & & \ddots & \Delta' \\ & 0 & & & \Delta' \end{pmatrix} \quad (2.4)$$

Since  $e_{11}R$  contains a unique (minimal) submodule, only one  $A_i$  is not zero. If  $i \neq 2$ ,  $A_2=0$  implies  $M_{i-1i}=\Delta'=0$  by Lemma 3. Hence,  $A_i=0$  for  $i > 2$ . Since  $s \geq 3$ , we have  $\Delta' \approx \Delta_1$  and  $A_2=\Delta_1$  by the induction (cf. [3], Lemma 13).

**Lemma 8.** *If  $R$  satisfies (FQF-3) and (QS), then  $R$  is isomorphic to a factor ring of a semi-primary hereditary ring  $R'$  such that  $R/J(R) \approx R'/J(R')$ .*

*Proof.* We know  $R/J^2 \approx \sum \oplus T_{n_i}(\Delta_i)/J(T_{n_i}(\Delta_i))^2$  by Lemmas 5 and 7. Hence,  $\text{gl. dim } R/J^2 < \infty$  by [3], Theorem 3. Therefore, we obtain the lemma by [3], Theorem 5 and its proof.

Since  $R/J(R) \approx R'/J(R')$ ,  $R'$  is basic and  $R' \approx T_n(\Delta_i; M_{ij})$  by [3], Theorem 4'. Let  $\{f_{ij}\}$  be the usual matrix units in  $R'$ . Then  $gR'f_{11}=0$  for any primitive idempotent  $g$  with  $gR' \not\approx f_{11}R'$ . Let  $\varphi: R' \rightarrow R$  be the ring epimorphism. Then  $J(R') = \varphi^{-1}(J(R))$  and  $\{e_{ii} = \varphi(f_{ii})\}$  is a complete set of mutually orthogonal primitive idempotents in  $R$ . If  $0 \neq e_{jj}Re_{11} = \varphi(f_{jj}R'f_{11})$  implies  $j=1$ . Furthermore,  $e_{11}J(R)e_{11} = \varphi(f_{11}J(R')f_{11}) = 0$ . From now on, we shall denote  $e_{ii}$  by  $e_i$ . Then  $\Delta_1 = e_1Re_1$  is a division ring from the above.

**Lemma 9.** *If  $R$  satisfies (FQF-3) and (QS), then  $R$  is isomorphic to  $\sum \oplus T_{n_i}(\Delta_i)/C_i$ , where  $C_i$  is a two-sided ideal in  $T_{n_i}(\Delta_i)$ .*

*Proof.* We may assume  $R$  is a two-sided indecomposable. We shall use the notations above. Put  $F=1-e_1$ . Then

$$R \approx \begin{pmatrix} \Delta_1 & A \\ 0 & R_1 \end{pmatrix} \quad (2.5).$$

We shall prove the lemma by induction on  $n$ , where  $1 = \sum_{i=1}^n e_i$ . If  $n \leq 2$ , the lemma is true by Lemma 7. We assume  $n \geq 3$ . Then since  $e_{11}R$  is injective by Lemma 3, 2),  $R_1 \approx T_{n-1}(\Delta)/C$  by Lemma 6 and the induction. Thus, we obtain

$$R = \begin{pmatrix} \Delta_1 & A_2 & \cdots & \cdots & A_n \\ & \Delta & \begin{array}{c} \diagdown \\ \diagup \end{array} & & 0 \\ & & \ddots & \begin{array}{c} \diagdown \\ \diagup \end{array} & \\ 0 & & & \Delta & \\ & & & & \Delta \end{pmatrix} \quad (2.6).$$

If we take a two-sided ideal  $Re_n$  and use the induction hypothesis, we know  $\Delta_1 = \Delta$  and  $A_i (i < n)$  is equal to either zero or  $\Delta$  (cf. [3], Lemma 13). We assume  $A_n \neq 0$ . Since  $e_1R$  is injective and has a simple socle,  $[A_n: \Delta] = 1$  as a right  $\Delta$ -module. Put  $A_n = u\Delta$ . We know by Lemma 3 that every  $\Delta$ -endomorphism of  $u\Delta$  is given by a unique element of  $\Delta = e_1Re_1$ . Let  $x$  be in  $e_1Re_1$ ,

then  $xu = u\delta(x)$ , where  $\delta$  is a ring homomorphism of  $\Delta$ . Therefore,  $\delta(\Delta) = \Delta$  from the above and so  $A_n = \Delta$  as a two-sided  $\Delta$ -module, if  $A_n \neq 0$ . Now we may assume  $A_k = \Delta$  and  $A_{k+1} = \dots = A_n = 0$ . We shall show  $A_2 \neq 0$ . Assume  $A_2 = A_3 = \dots = A_{s-1} = 0$  and  $A_s = \Delta$  for some  $s \leq k$ . We put  $D = \sum_{p>s+1}^n \oplus Re_p$ . Then  $\bar{R} = R/D$

$$\approx \begin{pmatrix} \Delta & 0 & 0 & \dots & \Delta \\ & \Delta & \begin{matrix} \square \\ \vdots \\ \square \end{matrix} & 0 & E_2 \\ & & \Delta & \begin{matrix} \square \\ \vdots \\ \square \end{matrix} & E_3 \\ 0 & & & \ddots & \vdots \\ & & & & E_{s-1} \\ & & & & \Delta \end{pmatrix} \quad (2.7).$$

Since  $e_1\bar{R}$  is  $\bar{R}$ -injective,  $E_2 = \dots = E_{s-1} = 0$  by Lemma 3. However,  $R_1$  is indecomposable and is of the standard form. Hence,  $E_{s-1} \neq 0$ , which is a contradiction. Accordingly,  $A_2 \neq 0$  and  $e_2Re_k \neq 0$  by Lemma 3. Again, since  $R_1$  is of standard form,  $e_jRe_k \neq 0$  for  $j \leq k$ . Therefore,  $R \approx T_n(\Delta)/C$ .

**Lemma 10** ([9], Theorems 17 and 18). *Let  $R$  be a two-sided indecomposable basic and generalized uni-serial ring. If there exists a primitive idempotent  $e$  such that  $eR$  is simple, then  $R$  is isomorphic to  $T_n(\Delta)/C$ .*

Proof.  $R$  satisfies (F\*) by Corollary 1 to Lemma 2. First we assume  $J^2 = 0$ . We use the same notations in the proof of Lemma 7. We assume  $m = 0$  and  $e_nR$  is simple. Since  $R$  is a QF-ring,  $e_nR$  is a two-sided ideal. Hence,  $R$  is a division ring. If  $m \neq 0$ , we obtain the form (2.2) and so (2.3). Hence, we can use the same argument. In general case, noting that  $e_1R$  is not simple in (2.3), we can use the induction. Therefore, Lemma 8 is true for the ring in the lemma. Again we can use the same argument in the proof of Lemma 9.

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