We have been studying many interesting properties of small submodules. W.W. Leonard [8] and M. Rayar [12] defined small modules and gave elementary properties of them. Recently, the author has studied non-small modules and given a class of rings which are concerned with non-small modules and located between QF-rings and QF-3 rings [4] and [5].

In this note we shall consider two conditions (*) and (*)* in [4] and [5] (see §1) and study a semi-primary ring whose every factor ring satisfies either (*) or (*)*. We shall show such a ring with condition (QS) (see §1) coincides with a generalized uni-serial ring of the first category in the sense of Murase [9].

1. The main theorem

Let $R$ be a ring with identity. We always assume that $R$ is a semi-primary ring, namely the Jacobson radical $J$ of $R$ is nilpotent and $R/J$ is artinian, and every $R$-module is an unitary right $R$-module unless otherwise stated. Let $M$ be an $R$-module. By $E(M)$ and $J(M)$ we denote an injective hull and the Jacobson radical of $M$, respectively. If $M$ is a small submodule in $E(M)$, we say $M$ is a small module [8], [12] and if $M$ is not a small module, we say $M$ is non-small module [5]. As the dual concept to the above, we define a non-cosmall module $N$ as follows: there exist a projective module $P$ and an epimorphism $f: P \rightarrow N$ such that ker $f$ is not essential in $P$.

In [4] and [5] we have introduced two conditions:

(*) Every non-small module contains a non-zero injective module.

(*)* Every non-cosmall module contains a non-zero projective direct summand.

We have shown that if $R$ satisfies either (*) or (*)*, then $R$ is a right QF-3 ring [13] ($E(R)$ is projective by [7]) and every QF-ring satisfies both (*) and (*)*. Thus, a class of rings satisfying either (*) or (*)* is located between a class of QF-rings and one of QF-3 rings when $R$ is a left and right artinian ring. If $R$ is left and right artinian and $eR, Re$ have unique composition series for every
primitive idempotent \( e \), we call \( R \) a \textit{generalized uni-serial ring} [10]. It is easily seen that every generalized uni-serial ring satisfies both (\( * \)) and (\( * \)) (Corollary 1 to Lemma 1 below).

Following Murase [9] we say a two-sided indecomposable generalized uni-serial ring is in \textit{the first category}, if there exists a primitive idempotent \( e \) such that \( eR \) is simple. In order to show that some rings in the new class coincide with the above rings, we introduce the conditions:

(F\( * \)) (resp. (F\( * \)*)) Every factor ring of \( R \) satisfies (\( * \)) (resp. (\( * \))*)

(FQF-3) Every factor ring of \( R \) is right QF-3. And

(QS) If a factor ring of \( R \) is a QF-ring, then it is semi-simple.

Now, we can state our theorem.

\textbf{Theorem.} Let \( R \) be a semi-primary ring. Then the following statements are equivalent.

1) \( R \) satisfies (F\( * \)) and (QS).
2) \( R \) satisfies (F\( * \)*) and (QS).
3) \( R \) satisfies (FQF-3) and (QS).
4) \( R \) is isomorphic to a factor ring of QF-3 and hereditary ring. And
5) \( R \) is a direct sum of generalized uni-serial rings of the first category.

We know from [2], Theorem 2 and [9], Theorems 17 and 18 that the ring \( R \) in the theorem is a direct sum of factor rings of rings of triangular matrices over division rings when \( R \) is basic. Hence, it has a perspective form.

We shall give remarks on the above conditions.

\textbf{Remarks} 1. If \( R \) is a generalized uni-serial ring of the second category [9], \( R \) satisfies (F\( * \)), (F\( * \)*) and (FQF-3) but not (QS) (see §2).

2. If \( R \) is a left and right artinian, then \( R \) is a generalized uni-serial ring if and only if \( R \) satisfies (FQF-3) [6].

3. Let \( K \subseteq L \) be fields with \( [L:K]<\infty \) and

\[
R = \begin{pmatrix} K & L \\ 0 & K \end{pmatrix}.
\]

Then \( R \) satisfies (QS) but not any of (F\( * \)), (F\( * \)*) and (FQF-3).

4. If \( R \) is a commutative artinian ring and satisfies (QS), then \( R \) is a direct sum of fields.

Because, we may assume \( R \) is a local ring with maximal ideal \( M \). If \( M \neq 0 \), we could find a maximal one \( M' \) among ideals contained in \( M \). Then \( R/M' \) is a QF-ring and so \( M/M'=0 \).

2. \textbf{Proof of Theorem}

We always assume that \( R \) is a semi-primary ring with identity and every
FACTOR RINGS OF A HEREDITARY AND QF-3 RING

An R-module M is an unitary right R-module. We shall denote the Jacobson radical and the injective hull by J(M) and E(M), respectively. Let R be as above and \(1 = \sum \sum g_{ij}\), where \(\{g_{ij}\}\) is a set of mutually orthogonal primitive idempotents such that \(g_{ij}R \cong g_{ii}R\) for any \(j\) and \(g_{ij}R \cong g_{jj}R\) for \(i \neq i'\). We put \(g = \sum g_{ii}\) and \(R_0 = gRg\) i.e. \(gRg\) is the basic ring of \(R\) [11] and [2]. It is well known that the category of right R-modules is Morita equivalent to one of right \(R_0\)-modules. We have a one to one mapping between the set of two-sided ideals \(A\) in \(R\) and one of those \(A_0\) in \(R_0\) such that \(A_0 = gAg\) and \(A = RA_0R\).

**Lemma 1.** Let \(A\) be a two-sided ideal. We put \(\bar{R} = R/A\) and \(A_0 = gAg\). Then \(\bar{R}_0 = R_0/A_0\) is the basic ring of \(\bar{R}\).

Proof. It is clear that \(1 = \sum \sum g_{ij}\) and \(g_{ij}\bar{R} \cong g_{ii}\bar{R}\). If \(g_{ij} \neq 0, g_{ij}\) is also a primitive idempotent and \(g_{ij}g_{jj'} = \delta_{ii'}\delta_{jj'}g_{ij}\). We assume \(g_{nn}R \cong g_{jj}R\) for \(i \neq j\). Then there exists \(x\) in \(g_{nn}Rg_{ll}\) such that \(xg_{nn}R + g_{jj}A = g_{jj}R\). Since \(g_{nn}R \cong g_{jj}R\), \(xg_{nn}R \subseteq g_{nn}J(R)\). Hence, \(g_{jj}A = g_{jj}R\) by Nakayama's Lemma and so \(g_{nn} \subseteq A\) for any \(k\). Thus, \(\bar{R}_0\) is the basic ring of \(\bar{R}\).

**Corollary.** \(R\) satisfies one of \((F*), (F**), (FQF-3)\) and \((QS)\) if and only if so does the basic ring of \(R\).

**Lemma 2.** Let \(R\) be a generalized uni-serial ring. Then every indecomposable non-small (resp. non-cosmall) module is injective (resp. projective).

Proof. Every indecomposable module is uni-serial by [10]. Hence, the lemma is trivial from the definitions.

**Corollary 1.** Every generalized uni-serial ring satisfies \((F*), (F**)\) and \((FQF-3)\).

**Corollary 2.** Let \(R\) be left and right artinian. Then the following statements are equivalent.
1) \(R\) satisfies \((FQF-3)\).
2) \(R\) satisfies \((F*)\).
3) \(R\) satisfies \((F**)\). And
4) \(R\) is a generalized uni-serial ring.

Proof. 1)\(\leftrightarrow\)4) is proved in [6]. Corollary 1 gives 4)\(\rightarrow\)2) and 3). We know 2)\(\rightarrow\)1) and 3)\(\rightarrow\)1) from [5], Propositions 2.5 and 3.4.

In order to prove the theorem, we may always assume from Lemma 1 that \(R\) is basic and \(g_{nn}Rg_{ll}g_{nn}J_{g_{nn}} = \Delta_i\) is a division ring. Let \(M_{ij}\) be a \(\Delta_i - \Delta_j\) bimodule \((i < j)\). We defined the ring of generalized upper tri-angular ma-
traces $T_n(\Delta_i; M_{ij})$ [3]. When $\Delta_i = \Delta$ for all $i$ and $M_{ij} = \Delta$, we shall denote the usual upper tri-angular matrix ring by $T_n(\Delta)$ and the set of matrix units by $\{e_{ij}\}_{i<j}$.

**Lemma 3.** Let $\Delta_i$ be division rings and $R = T_n(\Delta_i; M_{ij})$. 1) We assume $e_i R$ is injective and $M_{ik} \neq 0$ and $M_{ij} = 0$ for all $t > k$. Then $\text{Hom}_{\Delta_k}(R e_k/ \bigoplus_{i \neq k} M_{ik}, \Delta_k)$ is isomorphic to $e_i R$ by multiplications of elements in $e_i R$ from the left side. Hence, $M_{ij} \neq 0$ if and only if $M_{ik} \neq 0$.

2) If $R$ is a right QF-3, $e_i R$ is injective.

**Proof.**

1) Since $M_{ik}$ is the socle of $e_i R$, $[M_{ik}: \Delta_i] = 1$. We have the natural homomorphism $\phi: e_i R \to \text{Hom}_{\Delta_k}(R e_k/ \bigoplus_{i \neq k} M_{ik}, \Delta_k)$. Since $\phi(M_{ik}) \neq 0$, $\phi$ is monomorphic. Let $f$ be in $\text{Hom}_{\Delta_k}(R e_k/ \bigoplus_{i \neq k} M_{ik}, \Delta_k)$.

$$f = \sum_{p=1}^i f_p M_{ik}; f_p \in \text{Hom}_{\Delta_k}(M_{pk}, \Delta_k).$$

Put $F_p = f_p M_{ik} = 0$ for $i > k$. Then $F_p \in \text{Hom}_{\Delta_k}(M_{pk} R, e_i R)$, since $M_{it} = 0$ for $t > k$. Hence, there exists an element $x_p$ in $e_i R$ such that $F_p(m_{pk}) = x_p m_{pk} (x_p \in M_{ip})$ for every $m_{pk}$, since $e_i R$ is injective. Therefore, $f = \phi(\sum x_p)$. Hence, $\phi$ is isomorphic.

2) If $R$ is right QF-3, $E(R) \approx \sum_{i \in K} (e_i R)^*$, since $R$ is semi-primary. Being $e_i R e_{ii} = 0$ for $i > 1$, the index set $K$ must contain 1. Hence, $e_{ii} R$ is injective.

Let $R = T_n(\Delta)$ and $A$ a two-sided ideal. It is clear [9]

$$R/A = \begin{pmatrix} \Delta & 0 \\ \Delta & \Delta \\ \vdots & \vdots \\ 0 & \ldots & \Delta \end{pmatrix} \quad (2.1).$$

We call such a form the standard form of $R/A$. It is easily seen that $R/A$ is a generalized uni-serial ring of the first category. Hence, from Lemmas 1 and 2 and [2], Theorem 2 (consider $e_n R$) we have

**Lemma 4.** Let $R$ be a factor ring of a QF-3 and hereditary ring. Then $R$ satisfies $(F*)$, $(F^{*})$, $(FQF-3)$ and $(QS)$.

Now, we shall consider the converse case.

**Lemma 5.** If $R$ satisfies one of $(F*)$, $(F^{*})$, $(FQF-3)$ and $(QS)$, then so does every factor ring of $R$.

It is clear.

**Lemma 6.** Let $R = T_2(\Delta_1, R_2; M_{12})$. If $R$ is two-sided indecomposable and $e_{ii} R$ is injective, then $R_2$ is indecomposable, where $\Delta_1$ is a division ring and $R_2$ is a
semiprimary ring.

Proof. From the assumptions $e_iR$ contains a unique minimal submodule. Hence, $R_2$ is indecomposable if so is $R$.

**Lemma 7.** Let $R$ be a semiprimary, two-sided indecomposable and basic ring. We assume $J^2=0$. If $R$ satisfies (FQF-3) and (QS), then $R$ is isomorphic to $T_\lambda(\Delta)/J(T_\lambda(\Delta))^2$, where $\Delta$ is a division ring.

Proof. Let $R=\sum e_iR \oplus \sum f_jR$ be a decomposition of $R$ with indecomposable modules $e_iR$ and $f_jR$, where the $e_iR$ is injective and the $f_jR$ is small (see [5], Theorem 1.3). We quote here the argument in [6], Lemma in pp. 404-405. We know $\sum \oplus e_iR$ is faithful. Let $x \neq 0$ be in $f_jR$. Then $(\sum \oplus e_iR)x \neq 0$ and so there exists $e_iR$ such that $0 \neq e_ix = e_if_jx \in Jx$. Hence, $x \notin f_jJ$ since $J^2=0$. Therefore, $f_jR$ is simple if $f_jR \neq 0$. Since $e_iR$ is injective and $J^2=0$, $e_iR$ is uni-serial. Accordingly, $R$ is right artinian. First, we assume $m=0$. Then $R$ is self-injective and so a QF-ring (see [1], Theorem 1). Therefore, $R$ is a division ring by (QS). Thus, we may assume $m \neq 0$. We know from the above that $f_jR$ is simple. Hence, $f_jRg=0$ for any primitive idempotent $g \in \Theta$ and $f_jRf_j=A$ is a division ring. Thus, we have

$$R = \begin{pmatrix} R_1 & FRf_1 \\ 0 & \Delta \end{pmatrix}$$

(2.2),

where $F=1-f_1$ and $R_1=FRF$ satisfies (QS) and (FQF-3). We first assume $s=n+m=2$. Then $n=m=1$. Hence, $R_1$ is a division ring from the case $m=0$. Therefore, $R \cong T_2(\Delta)$ by [2], Theorem 2 and [3], Theorem 1. Now, we shall prove the lemma by induction on $s=s(R)$ (we assume $m \neq 0$). We have done it when $s \leq 2$. Since $s(R)>s(R_1)$, $R_1 \cong \sum \oplus T_\lambda(\Delta_i)/J(T_\lambda(\Delta_i))^2$ by the induction, where the $\Delta_i$ is a division ring. Hence, we obtain $R=\sum \oplus T_\lambda(\Delta_1, \Delta_2, \ldots, \Delta_{s-1}, \Delta; M_{ij})$. Lemma 3,2) shows that $e_{11}R$ is injective. It is clear $e_{kk}R_{11}=0$ for $k \neq 1$. We put $F'=1-e_{11}$ and $R_1'=F'RF'$. Then we have

$$R = \begin{pmatrix} \Delta_1 & e_{11}R_{11}' \\ 0 & R_1' \end{pmatrix}$$

(2.3).

Here $R_1'$ is two-sided indecomposable by Lemma 6. Hence, $R_1' \cong T_{s-1}(\Delta')/J(T_{s-1}(\Delta'))^2$ by the hypothesis of induction. Now $R$ is of the form

$$\begin{pmatrix} \Delta_1 A_2 \cdots \cdots A_s \\ \Delta' \Delta' \\ \cdots \cdots \cdots \\ 0 & \cdots \cdots \cdots \\ 0 & \cdots \cdots \cdots \cdots \Delta' \end{pmatrix}$$

(2.4).
Since \( e_iR \) contains a unique (minimal) submodule, only one \( A_i \) is not zero. If \( i \neq 2 \), \( A_2=0 \) implies \( M_{i-1}=\Delta'=0 \) by Lemma 3. Hence, \( A_i=0 \) for \( i \neq 2 \). Since \( s \geq 3 \), we have \( \Delta ' \approx \Delta_i \) and \( A_2=\Delta_i \) by the induction (cf. [3], Lemma 13).

**Lemma 8.** If \( R \) satisfies (FQF-3) and (QS), then \( R \) is isomorphic to a factor ring of a semi-primary hereditary ring \( R' \) such that \( R/J(R) \approx R'/J(R') \).

**Proof.** We know \( R/J^2 \approx \sum T_n(\Delta) \) by Lemmas 5 and 7. Hence, gl. dim \( R/J^2 < \infty \) by [3], Theorem 3. Therefore, we obtain the lemma by [3], Theorem 5 and its proof.

Since \( R/J(R) \approx R'/J(R') \), \( R' \) is basic and \( R' \approx T_n(\Delta_m; M_{ii}) \) by [3], Theorem 4'. Let \( \{f_{ii}\} \) be the usual matrix units in \( R' \). Then \( gR'f_{ii}=0 \) for any primitive idempotent \( g \) with \( gR' \approx f_{ii}R' \). Let \( \varphi: R' \rightarrow R \) be the ring epimorphism. Then \( J(R')=\varphi^{-1}(J(R)) \) and \( \{e_{ii}=\varphi(f_{ii})\} \) is a complete set of mutually orthogonal primitive idempotents in \( R \). If \( 0 \neq e_j R e_i = \varphi(f_{ji}R'f_{ii}) \) implies \( j=1 \). Furthermore, \( e_{ii}J(R)e_{ii}=\varphi(f_{ii}J(R')f_{ii})=0 \). From now on, we shall denote \( e_{ii} \) by \( e_i \). Then \( \Delta_i = e_i R e_i \) is a division ring from the above.

**Lemma 9.** If \( R \) satisfies (FQF-3) and (QS), then \( R \) is isomorphic to \( \sum T_n(\Delta)/C_i \), where \( C_i \) is a two-sided ideal in \( T_n(\Delta_i) \).

**Proof.** We may assume \( R \) is a two-sided indecomposable. We shall use the notations above. Put \( F=1-e_1 \). Then

\[
R \approx \begin{pmatrix} \Delta_1 & A_2 & \cdots & A_n \\ 0 & R_1 & & \\ & & & \\ & & \end{pmatrix} \quad (2.5).
\]

We shall prove the lemma by induction on \( n \), where \( 1=\sum e_i \). If \( n \leq 2 \), the lemma is true by Lemma 7. We assume \( n \geq 3 \). Then since \( e_iR \) is injective by Lemma 3, 2), \( R_i \approx T_{n-i}(\Delta)/C \) by Lemma 6 and the induction. Thus, we obtain

\[
R = \begin{pmatrix} \Delta_1 & A_2 & \cdots & A_n \\ \Delta & \ddots & \cdots & \Delta \\ \vdots & \ddots & \ddots & \Delta \\ 0 & \cdots & \ddots & \Delta \\ \end{pmatrix} \quad (2.6).
\]

If we take a two-sided ideal \( Re_n \) and use the induction hypothesis, we know \( \Delta_i=\Delta \) and \( A_i(i<n) \) is equal to either zero or \( \Delta \) (cf. [3], Lemma 13). We assume \( A_n \neq 0 \). Since \( e_iR \) is injective and has a simple socle, \([A_n; \Delta]=1 \) as a right \( \Delta \)-module. Put \( A_n=u\Delta \). We know by Lemma 3 that every \( \Delta \)-endomorphism of \( u\Delta \) is given by a unique element of \( \Delta = e_iR e_i \). Let \( x \) be in \( e_iR e_i \).
FACTOR RINGS OF A HEREDITARY AND QE-3 RING

then \( xu = u \delta(x) \), where \( \delta \) is a ring homomorphism of \( \Delta \). Therefore, \( \delta(\Delta) = \Delta \) from the above and so \( A_s = \Delta \) as a two-sided \( \Delta \)-module, if \( A_s \neq 0 \). Now we may assume \( A_s = \Delta \) and \( A_{s+1} = \cdots = A_n = 0 \). We shall show \( A_s \neq 0 \). Assume \( A_2 = A_3 = \cdots = A_{s-1} = 0 \) and \( A_s = \Delta \) for some \( s \leq k \). We put \( D = \bigoplus_{s \leq k} Re_s \). Then \( \overline{R} = R/D \)

\[
R = \left( \begin{array}{cccc}
\Delta & 0 & 0 & \cdots & \Delta \\
\Delta & 0 & E_2 & \cdots & E_3 \\
\Delta & \cdots & \cdots & \cdots & \Delta \\
0 & \cdots & \cdots & \cdots & \cdot \\
\Delta & \cdots & \cdots & \cdots & \cdot \\
\end{array} \right) \tag{2.7}
\]

Since \( e_1 R \) is \( \overline{R} \)-injective, \( E_2 = \cdots = E_{s-1} = 0 \) by Lemma 3. However, \( R_1 \) is indecomposable and is of the standard form. Hence, \( E_{s-1} = 0 \), which is a contradiction. Accordingly, \( A_2 \neq 0 \) and \( e_2 R \neq 0 \) by Lemma 3. Again, since \( R_1 \) is of standard form, \( e_j R \neq 0 \) for \( j \leq k \). Therefore, \( R = T_n(\Delta)/C \).

**Lemma 10** ([9], Theorems 17 and 18). Let \( R \) be a two-sided indecomposable basic and generalized uni-serial ring. If there exists a primitive idempotent \( e \) such that \( eR \) is simple, then \( R \) is isomorphic to \( T_n(\Delta)/C \).

Proof. \( R \) satisfies (F*) by Corollary 1 to Lemma 2. First we assume \( J^2 = 0 \). We use the same notations in the proof of Lemma 7. We assume \( m = 0 \) and \( e_s R \) is simple. Since \( R \) is a QF-ring, \( e_i R \) is a two-sided ideal. Hence, \( R \) is a division ring. If \( m = 0 \), we obtain the form (2.2) and so (2.3). Hence, we can use the same argument. In general case, noting that \( e_i R \) is not simple in (2.3), we can use the induction. Therefore, Lemma 8 is true for the ring in the lemma. Again we can use the same argument in the proof of Lemma 9.

---

**References**


Department of Mathematics
Osaka City University
Sugimoto-cho, Sumiyoshi-ku
Osaka 558, Japan