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SPLITTING A 4-MANIFOLD WITH INFINITE CYCLIC FUNDAMENTAL GROUP

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1. Introduction.

Throughout the paper, n-dimensional connected orientable topological manifolds will be understood as n-manifolds, unless otherwise stated. In this paper we show the following splitting theorem for closed 4-manifolds with infinite cyclic fundamental groups:

Theorem 1.1. *Every closed connected orientable* 4*-manifold* M *with* $\pi_1(M)\cong Z$ *is homeomorphic to the connected sum* $S^1\times S^3\# M_1$ *for a closed simply connected* 4*-manifold* M_1.

The idea of the proof is to investigate when a homology cobordism between closed 4-manifolds with infinite cyclic fundamental groups is an h-cobordism which is always a product cobordism by Freedman [**2**], for we can construct a homology cobordism between M and $S^1\times S^3\# M_1$ by a method similar to Kervaire's surgery argument [**11**]. Freedman showed in [**1**] that any two closed oriented simply connected 4-manifolods M_1, M'_1 are orientation-preservingly homeomorphic if and only if the intersection forms on $H_2(M_1;Z)$, $H_2(M'_1;Z)$ are isomorphic and the Kirby-Siebenmann invariants $\mathrm{ks}(M_1)$, $\mathrm{ks}(M'_1)(\in Z_2)$ are equal, and in this case there is an orientation-preserving homeomorphism $M_1\cong M'_1$ inducing the isomorphism of the intersection forms. By combining this classification of Freedman with the above splitting theorem, we have a similar characterization for closed oriented 4-manifolds with infinite cyclic fundamental groups:

Corollary 1.2. *Two closed connected oriented* 4*-manifolds* M, M' *with* $\pi_1(M)\cong\pi_1(M')\cong Z$ *are orientation-preservingly homeomorphic if and only if the intersection forms on* $H_2(M;Z)$, $H_2(M';Z)$ *are isomorphic and* $\mathrm{ks}M=\mathrm{ks}M'$, *and in this case there is an orientation-preserving homeomorphism* $M\cong M'$ *inducing the isomorphism of the intersection forms.*

As a next observation, let $\tilde{M}$ be the infinite cyclic(= universal) covering space of a 4-manifold M with $\pi_1(M)\cong Z$. Let $\Lambda = Z[Z] = Z[t, t^{-1}]$ be the group ring of Z. In [**3**], Freedman-Quinn showed that every nonsingular Hermitian Λ-form on a free Λ-module of finite rank is realizable as a Λ-intersection form on $H_2(\tilde{M}; Z)$(which is a free Λ-module) of a closed 4-manifold M with $\pi_1(M)\cong Z$. Applying the splitting theorem to this M, we obtain the following result in algebra:

Corollary 1.3. *Every non-singular Hermitian Λ-form on a free Λ-module of finite rank is Λ-isomorphic to a trivial Λ-extension of a non-singular symmetric bilinear form on a free abelian group of finite rank.*

It is known that a (null-homotopic) locally flat 2-knot K in a simply connected 4-manifold M_1 has a tubular neighborhood $N(K)\cong S^2\times D^2$ (cf. [**3**, 9.3]), and it is trivial(i.e., bounds a bicollared 3-disk in M_1) if and only if the exterior $E = M_1 - \mathrm{int}N(K)$ is homeomorphic to a connected sum $S^1\times D^3\# M_1$ (cf. Gluck [**5**, (17.1)]). The following unknotting criterion for a 2-knot is proved by Freedman [**2**] when $M_1 = S^4$, and a weaker but more general result is given by Matumoto [**12**]:

Corollary 1.4. *A locally flat 2-knot K in a closed simply connected 4-manifold M_1 is trivial if and only if $\pi_1(M_1 - K)\cong Z$.*

This follows by applying the splitting theorem to a closed 4-manifold M with $\pi_1(M)\cong Z$, obtained from M_1 by a surgery replacing $N(K)$ with $D^3\times S^1$, because any two simple loops representing a generator of $\pi_1(M)$ and their tubular neighborhoods are ambient isotopic in M, respectively. Proof of the splitting theorem (Theorem 1.1) is given in Section 2. In Section 3, we note a smooth structure on $M = S^1\times S^3\# M_1$ for a simply connected 4-manifold M_1. Splitting a closed 4-manifold with infinite cyclic fundamental group is a prelude to the topological classification problem for bounded 4-manifolds with infinite cyclic fundamental groups. The typical problem is the unknotting problem for a closed connected orientable surface in S^4, which is solved in a joint work with Hillman [**6**].

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2. Proof of the splitting theorem.

For a compact n-manifold W

with $H_1(W;Z)\cong Z$, we have $H_{n-1}(W,\partial W;Z)\cong Z$ by Poincaré duality. A bicollared proper connected $(n-1)$-submanifold U of W is simply called a *leaf* in W if U represents a generator of $H_{n-1}(W,\partial W;Z)$. Note that U lifts trivially to the infinite cyclic cover $\tilde{W}$ of W. We use the following sufficient condition for a homology cobordism between closed 4-manifolds with infinite cyclic fundamental groups to be an h-cobordism:

Lemma 2.1. *Let W be a homology cobordism with $\pi_1(W)\cong Z$ between closed orientable 4-manifolds M and M' with $\pi_1(M)\cong\pi_1(M')\cong Z$. If there is a leaf U in W with $V=U\cap M$ a leaf in M such that a lift $\tilde{I}$: $(U,V)\to(\tilde{W},\tilde{M})$ of the inclusion I: $(U,V)\subset(W,M)$ to the infinite cyclic covering $(\tilde{W},\tilde{M})$ of (W,M) induces the trivial homomorphism $\tilde{I}_*=0$: $H_2(U,V;Q)\to H_2(\tilde{W},\tilde{M};Q)$, then W is an h-cobordism and hence a product cobordism.*

Proof. We use the dualities on the infinite cyclic covering, stated in [**9**], where we denote $\mathrm{Ext}^q(\ ,\Lambda)$ by E^q. Since $H_*(W,M;Z)=H_*(W,M';Z)=0$, we see from the Wang exact sequence that $\tilde{H}_*=H_*(\tilde{W},\tilde{M};Z)$ and $\tilde{H}'_*=H_*(\tilde{W},\tilde{M}';Z)$ are finitely generated torsion Λ-modules whose integral torsion parts are finite, for t-1 induces an automorphism on $\tilde{H}_*$ and $\tilde{H}'_*$ (cf. [**9**,§3]). Then $\tilde{H}_*\otimes Q$ *and* $\tilde{H}'_*\otimes Q$ are finitely generated over Q and we see from [**8**], [**10**] that $\tilde{I}_*$: $H_2(U,V;Q)\to H_2(\tilde{W},\tilde{M};Q)=\tilde{H}_2\otimes Q$ is onto. Since $\tilde{I}_*=0$, we have $\tilde{H}_2\otimes Q=0$. Then $E^1E^1\tilde{H}_2=0$, for the integral torsion part of $\tilde{H}_2$ is finite (cf. [**9**,§3]). By the first duality of [**9**], we have a non-singular Λ-sesquilinear pairing $E^1E^1\tilde{H}_p\times E^1E^1\tilde{H}'_r\to Q(\Lambda)/\Lambda$ for all p, r with $p+r=4$. Noting that $\tilde{H}_p=\tilde{H}'_p=0$ for $p\leq 1$ and $E^1E^1\tilde{H}_2=0$, we see that $E^1E^1\tilde{H}_*=E^1E^1\tilde{H}'_*=0$. Next, since $\tilde{H}_*$ and $\tilde{H}'_*$ are torsion Λ-modules, the second duality of [**9**] implies that there is a non-singular t-isometric pairing $E^2E^2\tilde{H}_p\times E^2E^2\tilde{H}'_s\to Q/Z$ for all p, s with $p+s=3$. Nothing that $\tilde{H}_p=\tilde{H}'_p=0$ for $p\leq 1$, we see that $E^2E^2\tilde{H}_*=E^2E^2\tilde{H}'_*=0$. *By* [**9**,3.7], we have $\tilde{H}_*=\tilde{H}'_*=0$. Hence $(W;M,M')$ is an h-cobordism which is a product cobordism by [**2**]. This completes the proof.

The following lemma is more or less known in principle, but for convenience we give a proof:

Lemma 2.2. *Let f be a bicollared embedding of $S^1\times D^2$ into a (possibly non-compact) orientable 4-manifold M such that $f(S^1\times 0)$ is null-homotopic in M. When M is spin, we assume that $f(S^1\times p)$ for a point $p\in\partial D^2=S^1$ is a longitude of the solid torus $f(S^1\times D^2)$ induced from a spin structure on M. When M is not spin, we assume that there is a spherical element in $H_2(M;Z)$ with odd self-intersection number. Then f extends to a bicollared embedding f^+: $D^2\times D^2\to M$.*

Proof. If necessary, by replacing M with a punctured submanifold of M, we can assume that M is a smooth 4-manifold (cf. [**3**]). Since $f(S^1\times 0)$ is null-homotopic in M, we have an immersed disk D in M such that $D\cap f(S^1\times D^2)=f(S^1\times 0)=\partial D$. Let D_p be an immersed disk in M with $D_p\cap f(S^1\times D^2)=f(S^1\times p)=\partial D_p$, obtained from the singular disk $D\cup f(S^1\times[0,p])$ by pushing the part $f(S^1\times[0,p))$ in the positive normal direction of $f(S^1\times D^2)$ in M, where $[0,p]$ denotes the interval in D^2 with endpoints $0,p$. The intersection number of D and D_p in M is denoted by $I_f(D)$ since it depends only on D and f. We first show that we can take $I_f(D)$ to be even. When M is spin, $I_f(D)$ is always even by our assumption on the solid torus $f(S^1\times D^2)$. When M is not spin and $I_f(D)$ is odd, we take a connected sum in M of D and an immersed 2-sphere with odd self-intersection number. The resulting immersed disk, written again as D, has that $I_f(D)$ is even. Note that the singularity of D consists of double points, say p_i, $i=1, 2,\cdots,s$, each of which is locally represented as the vertex of a cone over a Hopf link. We choose mutually disjoint simple arcs a_i, $i=1, 2,\cdots,s$, in D such that a_i joints p_i and a point in ∂D and then take a small disk neighborhood Δ_i of a_i in D. By sliding the arc $\Delta_i\cap\partial D$ to the arc $\partial\Delta_i-\mathrm{int}(\Delta_i\cap\partial D)$ along Δ_i for each i, the embedding f is ambient isotopic to an embedding f': $S^1\times D^2\to M$ such that there exists a smoothly embedded disk D' in M with $f'(S^1\times D^2)\cap D'=f'(S^1\times 0)=\partial D'$. Hence there is a smoothly embedded disk in M, written again as D, such that $f(S^1\times D^2)\cap D=f(S^1\times 0)=\partial D$. Since the slide of $f(S^1\times 0)$ along each Δ_i contributes $I_f(D)$ by ± 2, we see that $I_f(D)$ is even for the embedded disk D. For each integer m, let f_m: $S^1\times D^2\to M$ be a smooth embedding with $f_m(S^1\times D^2)=f(S^1\times D^2)$ such that $f_m(S^1\times p)$ is homologous to $f(S^1\times p)+mf(p\times S^1)$ in $H_1(f(S^1\times S^1));Z)$. Then we have $I_{f_m}(D)=0$ for some even integer m. Since there is an embedded $S^1\times S^2$ in M such that $f(S^1\times D^2)\subset S^1\times S^2$ and $f(S^1\times 0)=S^1\times *$ for a point $*\in S^2$, we see from Gluck's observation [**5**] that f_m is ambient isotopic to f in $S^1\times S^2$ and hence in M for any even integer m. Thus, we have a smoothly embedded disk D with $D\cap f(S^1\times D^2)=f(S^1\times 0)=\partial D$ and $I_f(D)=0$. This means that f extends to a bicollared embedding $f^+: D^2\times D^2\to M$. This completes the proof.

We are in a position to prove Theorem 1.1 (the splitting theorem).

Proof of Theorem 1.1: Let $S^1\times D^3$ be a neighborhood of a loop representing a generator of $\pi_1(M)$ in M. Let M_1 be the manifold obtained from M by the surgery replacing $S^1\times D^3$ with $D^2\times S^2$. Let $K=0\times S^2\subset M_1$ be a 2-knot and $X=M_1-K$. Then M_1 is simply connected and $\pi_1(X)\cong Z$. Let V be a Seifert hypersurface for K in M_1. We obtain a 3-disk D^3 from V by surgery on 2-handles $h_s^2, s=1,2, \cdots, n$, disjointedly attached to $\mathrm{int}V$, where when M_1 is spin, this surgery should be taken the spin surgery on a spin structure on V induced from a spin structure on M_1 (cf. [7]). Let A_s be the attaching solid torus of h_s^2 to $\mathrm{int}V$. Note that each core of A_s is null-homotopic in X since it is null-homologous in X and $\pi_1(X)$ is abelian. Since there is a natural isomorphism $H_2(X;Z)\cong H_2(M_1;Z)$, we have that X is spin if and only if M_1 is spin. When M_1 is not spin, there is an element in $H_2(X;Z)$ with odd self-intersection number by the universal coefficient theorem, which is spherical by the Hopf theorem, for $\pi_1(X)\cong Z$. Applying Lemma 2.2 inductively, we can embed the 2-handles $h_s^2, s=1,2, \cdots, n$, into X disjointedly by embeddings extending the inclusions $A_s\subset \mathrm{int}V\subset X, s=1,2, \cdots, n$. By an isotopic deformation of the h_s^2's relative to the A_s's, we can assume that $T_s=V\cap h_s^2-A_s$ is a union of solid tori in $\mathrm{int}V$. Let D_0^3 be a 3-disk in $\mathrm{int}V-\bigcup_{s=1}^n(A_s\cup T_s)$ and $K_0=\partial D_0^3\subset M_1$. For any numbers $t_i, i=1,2$, with $0<t_1<t_2<1$, the 4-manifold

$$U=V\times[0,t_1]\cup\left(\bigcup_{s=1}^n A_s\times[t_1,t_2]\right)\cup\left(\bigcup_{s=1}^n h_s^2\times t_2\right)\cup D_0^3\times[t_1,1]$$

is a surgery trace from $V=V\times 0$ *to* $D^3=\partial U-\mathrm{int}V$, embedded in $M_1\times[0,1]$ with $M_1\times 0\cap U=V\times 0$ and $M_1\times 1\cap U=D_0^3\times 1$. Then $C=D^3-\mathrm{int}D_0^3\times 1\cong S^2\times[0,1]$ gives a knot cobordism in $M_1\times[0,1]$ between the 2-knot $\mathrm{K}\times 0\subset M_1\times 0$ and the trivial 2-knot $\mathrm{K}_0\times 1\subset M_1\times 1$. Let $Y=M_1\times[0,1]-C$. To see that $\pi_1(Y)\cong Z$, it suffices to show that $\pi_1(M_1\times[0,1]-U)=\{1\}$. Let l be a simple loop in $M_1\times(0,1)-U$, which bounds a disk D in $M_1\times(0,1)$ meeting U transversely. $L=D\cap U$ is a proper 1-submanifold of U with $\partial L\subset \mathrm{int}C$. Let l_L be a loop in L. Since $U-(L-l_L)$ is simply connected, l_L bounds a disk D_U in $\mathrm{int}U-(L-l_L)$. When we consider a disk D' obtained from D by replacing the disk in D bounded by l_L with D_U and then by pushing D_U into $M_1\times[0,1]-U$, we see that the number of loops in $D'\cap U$ is smaller than the number of loops in L. By induction on the number of loops in L, we can find a disk D with $\partial D=l$ such that L has only arc components. Let a be an arc component of L. Let a_S be a simple arc in $\mathrm{int}C$ with $\partial a_S=\partial a$ which does not meet $\partial(L-a)$. Since the loop $a\cup a_S$ bounds a disk D_a in U not meeting $L-a$, we can eliminate a from L by pushing out a bi-collar neighborhood of a in $\mathrm{int}D$ along D_a. By induction on the number of arc components of L, we can find a disk D with $\partial D=l$ and $L=\emptyset$, meaning that $\pi_1(M_1\times[0,1]-U)=\{1\}$. Hence $\pi_1(Y)\cong Z$. Then we have a homology cobordism W with $\pi_1(W)\cong Z$ between M and $M_0=S^1\times S^3\# M_1$ by a longitude surgery on $C\subset M_1\times[0,1]$. Note that $\pi_1(M)\cong\pi_1(M_0)\cong Z$. W has a leaf $\hat{U}$ homeomorphic to a union of U and a (3-disk)$\times[0,1]$ so that $\hat{U}\cap M$ is a closed leaf $\hat{V}$ in M closing V with a 3-disk and $\hat{U}\cap M_0$ is a factor $1\times S^3$ of the connected summand $S^1\times S^3$ of M_0. We show that a lift $\tilde{I}$: $(\hat{U},\hat{V})\to(\tilde{W},\tilde{M})$ of the inclusion I: $(\hat{U},\hat{V})\subset(W,M)$ induces a trivial homomorphism $\tilde{I}_*=0$: $H_2(\hat{U},\hat{V};Q)\to H_2(\tilde{W},\tilde{M};Q)$. Let d_s be a core disk of the 2-handle h_s^2 so that ∂d_s is a core circle of A_s and $k_s=\mathrm{int}\,d_s\cap V$ is a union of core circles of T_s. Let c: $V\times[-1,1]\to M_1$ be a bi-collar of V in M_1 with $c(x,0)=x$ for all $x\in V$. We may consider that $c(V\times[-1,1])\cap d_s$ $=c(\partial d_s\times[0,1])\cup c(k_s\times[-1,1])$. Let $e_s=\mathrm{cl}((d_s-c(k_s\times[-1,1]))$. Let $D_s=(\partial d_s)\times[0,t_2]\cup\ d_s\times t_2$ be a proper disk in U. Note that $H_2(\hat{U},\hat{V};Q)$ is generated by the disks $D_s, s=1,2,\cdots,n$. Let $E_s=(\partial d_s)\times[0,t_1]\cup e_s\times t_1$. For a sufficiently small $\varepsilon>0$, we consider the following two disks in $M_1\times[0,1]$:

$$D_s^{+\varepsilon}=E_s\cup c(k_s\times\{-1,1\})\times[t_1,t_1+\varepsilon]\cup c(k_s\times[-1,1])\times(t_1+\varepsilon)$$

and

$$D_s^{-\varepsilon}=E_s\cup c(k_s\times\{-1,1\})\times[t_1-\varepsilon,t_1]\cup c(k_s\times[-1,1])\times(t_1-\varepsilon).$$

Clearly, $(D_s^{\pm\varepsilon},\partial D_s^{\pm\varepsilon})\subset(Y,X)$. Let $N(C)\cong C\times D^2$ be a tubular neighborhood of C in $M_1\times[0,1]$ and $P=N(C)-C$. By the uniqueness of a tubular neighborhood, we may consider that $\mathrm{cl}(D_s^{\pm\varepsilon}-E_s)$ is contained in P. Let $\tilde{Y}$ be the infinite cyclic covering space of Y with $\tilde{X},\tilde{P}$ the lifts of X,P, respectively. Let $\tilde{D}_s^{+\varepsilon}$ and $\tilde{D}_s^{-\varepsilon}$ be lifting disks of $D_s^{+\varepsilon}$ and $D_s^{-\varepsilon}$ to $\tilde{Y}$, respectively, such that the lifting parts of E_s coincide. Then $[\tilde{D}_s^{+\varepsilon}]=[\tilde{D}_s^{-\varepsilon}]$ in $H_2(\tilde{Y},\tilde{X}\cup\tilde{P};Q)$. Since $H_2(\tilde{X}\cup\tilde{P},\tilde{X};Q)=0$, the natural homomorphism $H_2(\tilde{Y},\tilde{X};Q)\to H_2(\tilde{Y},\tilde{X}\cup\tilde{P};Q)$ is injective. Hence $[\tilde{D}_s^{+\varepsilon}]$ $=[\tilde{D}_s^{-\varepsilon}]$ in $H_2(\tilde{Y},\tilde{X};Q)$. By moves in $M_1\times[0,1]$ in the direction of the second factor, D_s is ∂-relatively isotopic to $D_s^{+\varepsilon}$ in Y and $D_s^{-\varepsilon}$ is ∂-relatively isotopic to $d_s\times 0\subset X$ in Y. This means that a lifting disk $\tilde{D}_s$ of D_s to $\tilde{Y}$ has $[\tilde{D}_s]=0$ in $H_2(\tilde{Y},\tilde{X};Q)$. This means that I_*: $H_2(\hat{U},\hat{V};Q)\to H_2(\tilde{W},\tilde{M};Q)$ is trivial. By Lemma 2.1, W is a product cobordism and there is a homeomorphism $M\cong M_0$. This completes the proof.

3. Note on smooth structure. The following proposition is worthy of note, though it is not hard to prove:

Proposition. *Let $S^1\times S^3$ have the standard smooth structure, and M_1, M_1' be simply connected (possibly non-compact) smooth 4-manifolds. If $M=S^1\times S^3\#M_1$ and $M'=S^1\times S^3\#M_1'$ are diffeomorphic, then M_1 and M_1' are diffeomorphic.*

Proof. The submanifold $S^1 \times D^3$ of $S^1 \times S^3$ for a 3-ball $D^3 \subset S^3$ is denoted by N or N' when we regard $S^1 \times S^3$ as the direct summand of M or M', respectively. Then any diffeomorphism f: $M \cong M'$ is smoothly isotopic to a diffeomorphism f': $M \cong M'$ sending N onto N' by the uniqueness of tubular neighborhoods, for $f(S^1 \times O)$ and $S^1 \times 0$ are smoothly isotopic in M'. Note that any 2-handle surgery of $S^1 \times S^3$ along $S^1 \times p (p \in S^3)$ produces S^4 (cf. [**5**, (17.1)], so that any surgery on M or M' replacing N or N' with $D^2 \times S^2$ produces M_1 or M'_1, respectively. Hence there is a diffeomorphism $M_1 \cong M'_1$. This completes the proof.

By this proposition, we see that when M_1 has infinitely many smooth structures (For example, we can take $M_1 = \mathrm{R}^4$ or $CP^2 \# 9 \bar{C}P^2$ by Gompf [**4**] or Okonek-Van de Ven [**13**]), M has also infinitely many smooth structures.

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