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ALMOST HEREDITARY RINGS

MANABU HARADA

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A. Tozaki and the author defined almost relative projective modules in [5], and the author has given a new concept of almost projective modules in [6]. On the other hand, we know that an artinian ring $R$ is hereditary if and only if the Jacobson radical of $R$ is projective, namely the Jacobson radical of $P$ is projective for any finitely generated $R$-projective module $P$. Analogously we call $R$ a right almost hereditary ring if the Jacobson radical of $P$ is almost projective. In the first section, we shall show the following two theorems: 1) $R$ is right almost hereditary if and only if $R$ is a direct sum of i) hereditary rings, ii) serial (two-sided Nakayama) rings and iii) special tri-angular matrix rings over hereditary rings and serial rings in the first category; 2) $R$ is two-sided almost hereditary if and only if $R$ is a direct sum of hereditary rings and serial rings.

We shall give a proof of the second theorem in the third section. In the fourth section, we shall study more strong rings such that every submodule of $P$ is again almost projective (resp. the Jacobson radidal of $Q$ is almost projective for any finitely generated and almost projective module $Q$).

1. Main theorems

In this paper every ring $R$ is an artinian ring with identity and every module $M$ is a unitary right $R$-module. By $|M|$, $J(M)$, $E(M)$ and $\text{Soc}_k(M)$ we denote the length, the Jacobson radical, the injective hull and the \textit{k}th-lower Loewy series of $M$, respectively. $\bar{M}$ means $M/J(M)$. We shall denote $J(R)$ by $J$. As is well known, if $J$ is $R$-projective, then $R$ is called a hereditary ring [1]. Analogously if $J$ is almost projective as a right $R$-module [5], then we call $R$ a right almost hereditary ring. We can define similarly a left almost hereditary ring. The above definition is equivalent to the following: $J(P)$ is almost projective for any finitely generated projective module $P$. Therefore the definition of almost hereditary ring is Morita equivalent, and hence we may assume that $R$ is a basic artinian ring, when we study the structure of $R$.

If $R$ is hereditary, every submodule of $P$ is again projective. However if $R$ is right almost hereditary, then every submodule of $P$ is not necessarily almost projective (see § 4).
From now on, \( R \) is a basic right artinian ring and \( \{ e_i \} \) is a set of mutually orthogonal primitive idempotents with \( 1 = \sum e_i \). In this paper following [8], we call two-sided Nakayama rings serial rings. Consider a sequence \( \{ e_1, e_2, \ldots, e_s \} \) (or \( \{ e_1R, e_2R, \ldots, e_sR \} \)). If \( \bar{e}_i \bar{J} \sim \bar{e}_{i+1} \bar{R} \) for \( 1 \leq i \leq s-1 \) (and \( \bar{e}_s \bar{J} \sim \bar{e}_1 \bar{R} \)), then we call this sequence a (cyclic) Kupisch series. Let \( D \) be a division ring. Consider a factor ring of triangular matrix ring over \( D \) by an ideal:

\[
\begin{pmatrix}
D & D & \cdots & D & 0 & \cdots & 0 \\
D & D & \cdots & D & 0 & \cdots & 0 \\
\vdots & \ddots & \cdots & \ddots & \cdots & \cdots & \cdots \\
O & & \cdots & \cdots & \cdots & \cdots & D
\end{pmatrix}
\]

By \( T(n_1, n_2, \ldots, n_s; D) \) we denote the above ring. It is well known that the above ring is a serial (two-sided Nakayama) ring. We call this ring a serial ring in the first category and a serial ring with cyclic Kupisch series is called a serial ring in the second category [8].

**Lemma 1.** Let \( R \) be a right almost hereditary ring. 1): Assume that \( e_1R \) is not injective and \( e_1J \neq 0 \). Then, \( J \) does not contain a direct summand \( X \) isomorphic to \( e_1R/A \) for any \( A \subset e_2R \). 2): Assume that \( e_2R \) is injective and \( 0 \neq e_2J \sim e_iR/A_i \) for some \( e_i \) and some \( A_i \) in \( e_1R \). Then \( e_2J \) does not contain a direct summand \( Y \) isomorphic to \( e_iR/B_i \) for any \( B_i \) in \( e_1R \), provided \( e_3R \sim e_2R \).

Proof. The first half is clear from [6], Theorem 1, for \( |e_1R| > |e_1J| \). Assume that \( e_2J \) contains a direct summand \( Y \) isomorphic to \( e_3R/B_i \). Then \( B_i \supset A_i \) or \( B_i \subset A_i \) by [6], Theorem 1. If \( B_i \supset A_i \), \( e_iR/A_i \sim e_2J \) is injective by [6], Theorem 1, a contradiction. Similarly we obtain the same result for \( B_i \subset A_i \), since \( e_2R (\supset e_3J) \) is indecomposable. Hence \( B_i = A_i \). Consider a diagram

\[
\begin{array}{c}
e_3R \\
\downarrow p \\
Y \\
\downarrow \overline{e_2J} \\
e_2R
\end{array}
\]

where \( p \) is the projection.

Since \( e_2R \) is injective, there exists \( g: e_2R \rightarrow e_2R \), which makes the above
diagram commutative. Since \( e_2 R \sim e_2 R \), \( g(e_3 R) = e_2 J = g(e_2 J) \). Hence \( e_2 R \) being local, \( e_2 R = e_2 J \), a contradiction.

The following two lemmas are well known (cf. [4], Proposition 1).

**Lemma 2.** Let \( R \) be a basic ring and \( 1 = g_1 + g_2 + \cdots + g_t + f_1 + f_2 + \cdots + f_s \), where \( \{g_i, f_j\} \) is a set of mutually orthogonal primitive idempotents. Assume \( g_i J = 0 \) or \( f_j J = 0 \) for all \( i \leq t \) and \( f_j J = 0 \) or \( f_j J = \Sigma (g_i J) \) for all \( j \leq s \). Then \( R = (g_1 + \cdots + g_t)R(g_1 + \cdots + g_t) \oplus (f_1 + \cdots + f_s)R(f_1 + \cdots + f_s) \) as rings, where \( \{g_i J\} \subset \{g_i\} \) and \( \{f_j J\} \subset \{f_j\} \).

Proof. We assume that \( g_i J = \Sigma \oplus g_{j(i,p)} R \), where \( g_{j(i,p)} \subset \{g_i, \cdots, g_s\} \). Then \( g_i J = a_i R + a_2 R + \cdots + a_n R + g_i J^{p+1} \), where \( a_i = a_i g_{j(i,p)} \). Then \( g_i J = a_i g_{j(i,p)} J + \cdots + a_n g_{j(i,p)} J + g_i J^{p+1} \). Hence \( g_i J^{p+1} \) is a homomorphic image of \( \Sigma J \oplus g_{j(i)} J \) by assumption. As a consequence \( g_i J^{p+1} = \Sigma J \oplus g_{j(i+1)} J \) and \( g_{j(i+1)} J \subset \{g_1, \cdots, g_t\} \). Thus any simple factor module in the composition series of \( g_i R \oplus \cdots \oplus g_i R \) is isomorphic to some \( g_i R \). Therefore \( R = (g_1 + \cdots + g_t) \oplus (f_1 + \cdots + f_s)R(f_1 + \cdots + f_s) \) as rings.

From the above argument we obtain

**Lemma 3.** Assume \( e_1 J \sim e_2 R \) and \( e_2 J \sim e_3 R \). Then \( e_1 R \) and \( e_2 R \) are uniserial and the simple factor modules in the composition series of \( e_1 R \) (resp. \( e_2 R \)) is \( \{e_1 R, e_2 R, e_3 R, \cdots, e_n R\} \). We obtain the similar result for a cyclic Kupisch series \( \{e_1, e_2, \cdots, e_n\} \).

**Lemma 4.** If \( R \) is a hereditary ring or a serial ring, then \( R \) is a (right) almost hereditary ring.

Proof. If \( R \) is hereditary, then \( R \) is clearly right almost hereditary. We assume that \( R \) is serial. We take a cyclic Kupisch series \( \{e_1, e_2, \cdots, e_n\} \). Then \( e_i J \sim e_{i+1} R/A_{i+1} \) for some \( A_i \), provided \( e_i J = 0 \). Hence

\[
|e_i R| - 1 \leq |e_{i+1} R|,
\]

Further

\[
|e_i R| \sim |e_j R| \leq n - 1 \quad \text{for any } i \text{ and } j.
\]

Put \( m = |e_i R| \). If \( |e_{i+1} R| = m - 1 \), \( e_i J \sim e_{i+1} R \), and hence \( e_i J \) is projective. Assume \( |e_{i+1} R| \geq m \), and take any \( k \) such that \( |e_{i+1} R/\text{Soc}_k(e_{i+1} R)| \geq m \). Let \( \text{Soc}_k(e_{i+1} R)/\text{Soc}_k(e_{i+1} R) \sim e_q R \) for some \( q \), and put \( E = e_q R \). Since \( R \) is serial, \( E \sim e_i R/A_s \) and \( \text{Soc}(e_i R/A_s) \sim e_q R \) for some \( s \) and \( A_s \). Further

\[
|e_i R/A_s| \text{ is the largest one among } |e_{i'} R/A_{i'}| \text{, with}
\]

\[
\text{Soc}(e_{i'} R/A_{i'}) \sim e_q R.
\]

On the other hand, by the structure of \( e_i R \) (see Figer (2) below) \( |e_{i+1} R/\text{Soc}_k(e_{i+1} R)| \) is a unique largest one among \( |e_i R/A_{i'}| \) with \( \text{Soc}(e_i R/A_{i'}) \sim e_q R \). Hence \( e_{i+1} R/\text{Soc}_k(e_{i+1} R) \) is injective. Furthermore \( J \text{ Soc}_k(e_{i+1} R) = 0 \) from (2).
Hence $e_1 J$ is almost projective by [6], Theorem 1. We obtain the similar result for a non cyclic Kupisch series.

\[ e: \begin{array}{cccc}
  & i & i+1 &  \\
 1 & q & & \\
 0 & q & & \\
 0 & q & & \\
 0 & q & & \\
 0 & q & & \\
 0 & q & & \\
 \end{array} \]

\[ (2) \]

\[ \begin{array}{cccc}
  & 1 & i & i+1 & n \\
 e: & q & & & \\
 m & q & & & \\
 0 & q & & & \\
 0 & q & & & \\
 0 & q & & & \\
 \end{array} \]

\[ \begin{array}{ccc}
 1 & M_{12} & \cdots & M_{1n} \\
 0 & M_2 & \cdots & M_{2n} \\
 \vdots & & \ddots & \vdots \\
 0 & & & M_n \\
 \end{array} \]

**Lemma 5.** Let $R$ be a basic ring. Assume that $e_1 R$ is injective and \{ $e_1, e_2, \ldots, e_n$ \} is a Kupisch series such that $0 = e_1 J \sim e_{i+1} R$ for $i \leq n-1$, $e_n J = 0$ and $\sum e_i = 1$. Then $R \sim T(n; D)$.

Proof. $R$ is right Nakayama by Lemma 2 and its proof, and $R$ is hereditary. Hence $R$ has the following form:

where the $D_i$ are division rings and the $M_{ij}$ are $D_i - D_j$ bimodules (cf. [3], Theorem 1). We denote the above ring by $T(D_1, D_2, \ldots, D_n)$. We shall show the lemma by induction on $n$. Let \{ $e_{ij}$ \} be the set of matrix units. Then we may suppose $e_i = e_{ii}$. Now $e_i R$ is a two-sided ideal in $R$. Put $\bar{R} = R/e_i R$. Since $e_1 J$ is characteristic in $e_1 R$ and $e_1 J \sim e_2 R$, $\bar{e}_2 \bar{R}$ is injective as an $\bar{R}$-module by Bare’s criterion. Furthermore \{ $\bar{e}_2, \ldots, \bar{e}_n$ \} is a Kupisch series in $\bar{R}$, which satisfies the condition in the lemma. Put $E = e_2 + \cdots + e_n$. Then $\tau: \bar{R} \sim ERE$ and $\theta: T(n-1; D) \sim \bar{R}$ as rings by induction. Let $u_j = \theta(g_{jj})$, where the $g_{jj}$ are the matrix units in $T(n-1; D)$. Then $u_j \sim e_j$, $E = \sum_{j=2}^n u_j$, and hence \{ $e_1, u_2, \ldots, u_n$ \} is a set of mutually orthogonal primitive idempotents in $R$ with $1 = e_1 + \sum_{j=2}^n u_j$. Hence we can suppose that $R$ is of the form.
Since \( e_1 \sim e_2 R \), \( M_{ij} = v_{ii} D \). Consider any element \( d \) in \( D \sim \text{Hom}_k(e_2 R, e_1 R) \). Since \( e_1 R \) is injective and \( v_{12} \): \( e_2 R \sim e_1 J \) is given by a left multiplication of an element \( d_1 \) in \( D_1 \). Then \( d_1 v_{12} = v_{12} d_1 \) and hence \( d_1 v_{12} = v_{12} d_1 \). On the other hand, \( D_1 v_{12} \subset v_{12} \). Hence there exists an isomorphism \( g \) of \( D \) to \( D_1 \) such that \( g(d_1 v_{12}) = v_{12} d_1 \). We may assume \( v_{ii} = v_{12} e_{2i} \). Then \( g(d_1 v_{12}) = v_{12} d_1 \). Hence we obtain an isomorphism \( G \) of \( T(n; D) \) to \( R \):

\[
G 
\begin{pmatrix}
    d_1 x_{12} & \cdots & x_{1n} \\
    d_2 x_{22} & \cdots & x_{2n} \\
    \vdots & \ddots & \vdots \\
    d_n & \cdots & x_{nn}
\end{pmatrix}
= 
\begin{pmatrix}
    g(d_1) v_{12} x_{12} & \cdots & v_{1n} x_{1n} \\
    d_2 x_{22} & \cdots & x_{2n} \\
    \vdots & \ddots & \vdots \\
    d_n & \cdots & x_{nn}
\end{pmatrix}
\]

where the \( d_i \) and the \( x_{ij} \) are elements in \( D \) (cf. [3], Lemma 13).

**Lemma 6.** Let \( R \) be right almost hereditary. If any \( e_i R \) is never injective or all \( i \), then \( R \) is hereditary.

**Proof.** This is clear from by [6], Theorem 1.

Let \( R \) be a basic and (right) artinian ring and \( 1 = \sum_{i=1}^l e_i \) as before. We assume that \( R \) is right almost hereditary. If \( R \) is not hereditary, then there exists an injective module \( e_i R \) for some \( j \) from Lemma 6. From now on we shall denote an injective module \( e_i R \) by \( f_i R \) and a non-injective module \( e_i R \) by \( g_i R \). Hence \( 1 = \sum f_i + \sum g_i \). We start from \( f_i \). If \( f_i J \neq 0 \), then since \( f_i J \) is uniform, \( f_i J \sim f_i R/A_2 \) or \( f_i J \sim g_i R \) by [6], Theorem 1. In either case \( f_i R \) and \( g_i R \) are uniform. Hence \( g_i J \sim g_i R \) or \( g_i J \sim f_i R/A_2 \), provided \( g_i J \neq 0 \) (resp. \( f_i J \sim g_i R \) or \( f_i J \sim f_i R/A_3 \), provided \( f_i J \neq 0 \)). Furthermore we have monomorphisms \( \theta_2: g_{12} R \rightarrow g_{11} R \) and \( \theta_1: g_{11} R \rightarrow f_1 R \). Repeating this procedure, we can get a Kupisch series \( f_1 R, g_{11} R, \ldots, g_{i_1} R, f_{i_2} R, \ldots, f_{i_r} R, \ldots, g_{w_1} R, g_{w_2} R, \ldots \) each of which is uniform.

**Lemma 7.** Let \( R \) be a basic and right almost hereditary ring. We assume that there exists a cyclic Kupisch series \( f_i R, g_{i_1} R, \ldots, g_{i_r} R, f_{i_s} R, \ldots, f_{i_t} R, \ldots, g_{w_1} R, g_{w_2} R, \ldots \) such that \( 1 = \sum f_i + \sum g_i \). Then \( R \) is serial.

**Proof.** By \( n_i \) (resp. \( n_{ij} \)) we denote \( |f_i R| \) (resp. \( |g_{ij} R| \)). Then \( n_{ip} = n_j - r_j \) for any \( j \). Since \( g_i j \sim f_{i+1} R/A_{i+1}, f_{i+1} R/(\text{Soc} e_{i+1}(f_{i+1} R), f_{i+1} R/\text{Soc} e_{i+1}(f_{i+1} R), f_{i+1} R/\text{Soc} e_{i+1}(f_j R)) \), \( f_{i+1} R/\text{Soc} e_{i+1}(f_j R) \) are injective by [6], Theorem 1. Clearly \( f_j R/\text{Soc} e_{i}(f_j R) \sim f_j R/\text{Soc} e_{i}(f_j R) \) for \( j \neq k \). Therefore there exist
\[ \Sigma(n_{j+1}-n_j+r_j+1)=s+\Sigma r_j=n \] distinct injective modules. Since the \( f_iR \) the \( g_{ij}R \) are uniserial by Lemma 3, \( R \) is right Nakayama and right co-Nakayama. As a consequence \( R \) is serial by [2], Theorem 5.4.

**Lemma 8.** Let \( R \) be as above. Assume that \( f_iR, g_{11}R, \ldots, g_{1r}R, f_2R, \ldots, g_{2p}R, \ldots, f_iR, \ldots, g_{nR} \) is a cyclic Kupisch series. Then \( R=FRF \oplus (1-F)R(1-F) \) and \( FRF \) is serial, where \( F=\Sigma f_i+\Sigma g_{ij} \).

Proof. From the structure of \( \{f_iR, g_{jk}R\} \), every simple factor module in the composition series of \( FR \) is isomorphic to some \( f_iR \) or \( g_{jk}R \). Let \( h \) be an idempotent in \( \{e\} - \{f_i, g_{jk}\} \). Suppose that \( hJ \) contains a direct summand \( X \) isomorphic to \( g_{jk}R \). Since \( g_{jk}R \subset f_iR \) (isomorphically), there exists a homomorphism \( \theta: hR \rightarrow f_iR \) since \( f_iR \) injective, which is a contradiction from the above observation. Next assume \( X \sim f_iR/A_i \). Then \( f_{i-1}R \supset g_{i-1} \sim f_iR/A_i \) by [6], Theorem 1 (cf. the proof of Lemma 1). Hence we have the same result. Accordingly \( hJ \) does not contain a simple component isomorphic to \( f_iR \) or \( g_{jk}R \) again by [6], Theorem 1. Therefore \( R=FRF \oplus (1-F)R(1-F) \) by Lemma 2, and \( FRF \) is serial by Lemma 7.

**Lemma 9.** Let \( R \) be a basic and right almost hereditary ring. Assume that \( R \) is two-sided indecomposable, not hereditary and \( e_iJ \neq 0 \) for all \( i \). Then \( R \) is serial.

Proof. Since \( R \) is not hereditary, there exists some injective module \( e_iR \) by Lemma 6. Let \( \{f_iR\} \) be the set of injective modules \( e_iR \) and \( \{g_{ij}R\} \) the set of non-injective modules \( e_iR \) as before. Since \( e_iJ \neq 0 \) for all \( k \), extending a Kupisch series from \( f_iR \) as long as possible, we obtain finally a Kupisch series \( \{f_iR, g_{11}R, \ldots, f_iR, g_{1r}R, \ldots, g_{at}R, e_sR\} \) such that

i) \( \{f_iR, g_{at}R, \ldots, e_sR\} \) is cyclic.

ii) \( \text{Soc}(f_aR) \sim \text{Soc}(e_sR) = \text{Soc}(g_{at}R) \sim \text{Soc}(f_sR) \), and hence \( s=a \). Then \( \{g_{at}(R, g_{at+1}R, \ldots, g_{ar}R) \) is a cyclic Kupisch series. Since \( g_{sj}R \sim g_{sj+1}R \) by [6], Theorem 1, \( |g_{sj}R| > |g_{sj+1}R| \), which is a contradiction to a cyclic series. Hence we obtain always a cyclic Kupisch series in i). Therefore by Lemmas 7 and 8 \( a=1 \) and \( R \) is serial, since \( R \) is two-sided indecomposable.

From Lemma 9 we may suppose that there do not exist any cyclic Kupisch series. Hence we study the structure of \( R \) in case of there exists a simple module \( e_jR \), i.e. \( e_jJ=0 \) for some \( j \). We note that if \( f_iR, g_{11}R, \ldots, g_{1r}R, f_2R, \ldots, g_{at}R, g_{2p}R \) is a Kupisch series with \( g_{at}R \) simple, then \( g_{1k}R \neq g_{2k}R \) for any \( k \) and \( t \), for \( f_iR \sim f_zR \). Now we obtain Kupisch series
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\[ f_1 R, g_{11} R, \ldots, f_2 R, \ldots, f_\nu R, \ldots, g_{\nu^*} R \text{ and } g_{\nu^*} J = 0, \]
\[ (4) \]
\[ f'_1 R, g'_{11} R, \ldots, f'_2 R, \ldots, f'_\nu R, g'_{\nu^*'} R \text{ and } g'_{\nu^*'} J = 0, \]

If \( g_{ij} = g'_{i'j'} \), for some \((t, j)\) and \((t', j')\), then \( \text{Soc}(f_i R) \sim \text{Soc}(f'_i R) \) and hence \( f_i = f'_i \). If \( t = 1 \), or \( t' = 1 \) one series is a part of the other. Hence we assume \( t > 1 \), \( t' > 1 \). Since \( g_{i-1} R \approx f_i R / A_i \), \( g'_{i-1} R \approx f'_i R / A'_i \), \( A_i = A'_i \) by [6], Theorem 1. Accordingly \( \text{Soc}(g_{i-1} R) \approx \text{Soc}(g'_{i-1} R) \), and hence \( f_{i-1} = f'_{i-1} \). If \( f_i = f'_{i'}, \{ f_{i'}, g_{i'} \} \approx \{ f'_{i'}, g'_{i'} \} \). Therefore \( \{ f_1, g_{11}, \ldots, g_{\nu^*} \} \subset \{ f'_{1'}, \ldots, g'_{\nu^*'} \} \) or \( \{ f_1, g_{11}, \ldots, g_{\nu^*} \} \sim \{ f'_{1'}, \ldots, g'_{\nu^*'} \} \), provided they have a common component. Hence we may assume that the Kupisch series in (4) are the longest series and they are disconnected and contains all \( f_i R \). We put \( \{ h_1, h_2, \ldots, h_r \} = \{ f_i, g_{11}, \ldots, g_{\nu^*}, f'_1, \ldots, g'_{\nu^*'} \} \) in (4) \( \text{and } H = h_1 + h_2 + \ldots + h_r. \)

**Lemma 10.** Let \( R \) and \( H \) be as above. Then \( HRH \) is a hereditary ring, \( (1-H)RH=0 \) and \( (1-H)R(1-H) = \Sigma \oplus_i T(n_{i(1)}, n_{i(2)}, \ldots, n_{i(q)} : D_i). \)

Proof. From the proof of Lemma 8 we know that \( h_i J \) does not contain a direct summand \( X \) isomorphic to \( g_{ji} R \) (or \( f_{ji} R / A_i \) \( i = 1 \)), where \( f_{ji} R \) and \( g_{ji} R \) are in (4). Since \( \{ h_{i'}, j \} \cap \{ f_{ji}^{(k)}, g_{ji}^{(k)} \} = \emptyset, (1-H)RH=0 \) and

\[ f_{ji}^{(k)} R = f_{ji}^{(k)} (1-H) R (1-H), g_{ji}^{(k)} R = g_{ji}^{(k)} (1-H) R (1-H) \]

from Lemma 2 and the structure of \( \{ f_{ji}^{(k)} R, g_{ji}^{(k)} R \} \). On the other hand, from the above we have

\[ h_i J \sim \Sigma \oplus_{i(\neq)} (h_i R)^{(\sigma, i)} \oplus (f_i R / A_i)^{(\sigma, i)} \oplus (f'_i R / A'_i)^{(\sigma, i')}, A_i = 0, A'_i = 0 \ldots \]

Accordingly \( h_i JH \sim \Sigma \oplus_{i(\neq)} (h_i HRH)^{(\sigma, i)}, \) and hence \( HRH \) is hereditary. Next we shall show \( (1-H) R (1-H) \sim T(n_{i}, n_{2}, \ldots, n_{q} : D) \) by induction on \( n = \# \{ f_{ji}^{(1)}, g_{ji}^{(1)} \ldots \} \) from (5) we may replace \( R \) by \( (1-H) R (1-H) \). Since \( e_i R f_i = 0 \) for all \( e_i \sim f_i, f_i R \) is a two-sided ideal. Put \( \tilde{R} = R / f_i R \). Then \( \tilde{g}_{ij} \tilde{R} \) is injective and \( \{ \tilde{g}_{ij}, \ldots, \tilde{f}_2, \ldots, \tilde{g}_2^2, \ldots \} \) is a Kupisch series with the same property as \( \{ f_{ji}, \ldots, g_{ji}, \ldots \} \) (cf. the proof of Lemma 5). Hence \( R \sim T(n', n_2, \ldots, n_q : D) \) by induction. Further \( f_i R \) is injective and \( f_i J \sim \tilde{g}_{ij} R \), and hence we can show in the manner given in the proof of Lemma 5 that \( R \sim T(n' + 1, n_2, \ldots, n_q : D) \).

We shall discuss the structure of \( HR(1-H) \). Since \( HRH \) is hereditary, \( HRH \) is of the form (3). We may assume \( h_i = e_i \) in (3). We shall first rewrite (4) in more detail.

\[ f_{11} R, g_{11,11} R, \ldots, g_{11,2} R, \ldots, f_{12} R, \ldots, f_{22} R, \ldots, g_{22} R, f_{23} R, \ldots, f_{32} R, \ldots, g_{32} R, \ldots, g_{33} R \]
\[ \ldots \]
\[ f_{n_1} R, g_{n_1,n_1} R, \ldots, g_{n_1,n_2} R, f_{n_2} R, \ldots, f_{n_2} R, \ldots, g_{n_2} R, \ldots, g_{n_2} R, \ldots, g_{n_2} R \]
Since $HRH$ is hereditary, if $e_{ii}JH \sim (e_{i+1,i+1}RH)^{m(i,i+1)} \oplus (e_{i+2,i+2}RH)^{m(i,i+2)} \oplus \oplus (e_{nn}RH)^{m(i,n)}$, then we have from (6)

$$h_i J \sim (h_{i+1}RH)^{m(i,i+1)} \oplus (h_{i+2}RH)^{m(i,i+2)} \oplus \oplus (h_{m}RH)^{m(i,m)} \oplus (f_{1,1}R/A_{1,1})^{m(i,1)} \oplus (f_{2,1}R/A_{2,1})^{m(i,2)} \oplus \cdots ,$$

where $m(i, a), m(i, b)$ are non-negative integers, i.e.,

$$h_i R(1-H) \sim (h_{i+1}R(1-H))^{m(i,i+1)} \oplus (h_{i+2}R(1-H))^{m(i,i+2)} \oplus \cdots \oplus (f_{1,1}R/A_{1,1})^{m(i,1)} \oplus (f_{2,1}R/A_{2,1})^{m(i,2)} \oplus \cdots ,$$

where the $A_{i,i} = 0, \{m(i, i+1), m(i, i+2), \cdots \}$ are the integers given in the above and $h_i R(1-H) = (f_{1,1}R/A_{1,1})^{m(i,1)} \oplus (f_{2,1}R/A_{2,1})^{m(i,2)} \oplus \cdots$.

Summarizing the above we have

**Theorem 1.** Let $R$ be a (basic) artinian ring. Then $R$ is right almost hereditary if and only if $R$ is a direct sum of the following rings:

1. Hereditary rings.
2. Serial rings.
3. 

\[
\begin{pmatrix}
T_1 & X \\
0 & T_2
\end{pmatrix}
\]

where $T_1$ is a hereditary ring and $T_2$ is a serial ring in the first category and $X$ is a $T_1 - T_2$ bimodule given by (8) (see the example below).

Proof. We have shown the first half. Assume that $R$ is of the form (9). Then for each primitive idempotent $e$ in $T_2$, any simple module in the composition series of $eR$ is never isomorphic to other one. Hence we can easily show $f_{1,1}R/Soc_1(f_{1,1}R), f_{1,1}R/Soc_{i-1}(f_{1,1}R), \cdots, f_{1,1}R$ are injective by Baer’s criterion and $JA_{11} = 0$, where $Soc_{i+1}(f_{1,1}R) = A_{1,1}$. Accordingly $f_{1,1}R/A_{11}$ is almost projective by [6], Theorem 1. We have the same results for $f_{j,k}R$ and $g_{j,k}R$. Further $f_{j,k}R$ and $g_{j,k}R$ are projective from the structure of $T_2$. Therefore $R$ in (9) is right almost hereditary.

**Theorem 2.** Let $R$ be as above. $R$ is right and left almost hereditary if and only if $R$ is a direct sum of hereditary rings and serial rings.

We shall give a proof in the third section.

2. Corollaries

In this section we shall give some corollaries to Theorem 1.

**Corollary 1.** Let $R$ be a basic ring with $1 = \sum_{i=1}^{n} e_i$. $R$ is right almost hereditary if and only if
Almost Hereditary Rings

R is a direct sum of hereditary rings and serial rings, provided \( n \leq 2 \).

If \( n=3 \), R is either a direct sum of hereditary rings and serial rings or the following form:

\[
\begin{pmatrix}
D_1 & X & 0 \\
0 & D_2 & D_2 \\
0 & 0 & D_2
\end{pmatrix}
\]

where \( D_1, D_2 \) are division rings and \( X \) is a \( D_1-D_2 \) bimodule.

Proof. This is clear from Theorem 1.

We note that the above ring is right almost hereditary, but not left almost hereditary if \( [X:D_1]>1 \).

Corollary 2. If \( R \) is right almost hereditary ring, then \( R \) is a direct sum of serial rings and factor rings of hereditary rings.

Proof. This is clear from Theorem 1 and [3], Theorem 5.

Corollary 3. Let \( R \) be a hereditary and two-sided indecomposable ring. If \( e_iR \) is injective for some \( i \), then \( R \) is serial, i.e. \( R=T(n: D) \). Hence there do not exist two non-isomorphic injective modules \( e_iR \) and \( e_jR \).

Proof. From Lemma 8 and [3], Theorem 1 we obtain a Kupisch series \( g_1R, g_2R, \ldots, g_sR \) and \( g_sJ=0 \), where \( g_1=e_i \). Let \( h \) be a primitive idempotent not in \( \{g_i\} \). Since \( g_1R \) is injective, \( hJ \) does not contain a direct summand \( X \) isomorphic to \( g_1R \). If \( X \cong g_iR \) for some \( i \neq 1 \), \( hR \cong g_kR \) for some \( k \) as in the proof of Lemma 8. Hence \( (g_1+\cdots+g_s)R(g_1+\cdots+g_s) \) is a direct summand of \( R \) as rings by Lemma 2. Therefore \( R=(g_1+\cdots+g_s)R(g_1+\cdots+g_s)=T(n: D) \) by Lemma 5.

Corollary 4 ([8]). Let \( R \) be a serial and two-sided indecomposable ring. If \( e_iR \) is simple for some \( i \), then \( R=T(n_1, n_2, \ldots, n_r: D) \).

Proof. Taking a Kupisch series, we obtain the corollary from the proof of Lemma 10.

It is well known for a hereditary ring \( R \) that \( R \) is a QF-ring if and only if \( R \) is semisimple. If we replace "hereditary" by "almost hereditary", we obtain

Corollary 5. Let \( R \) be a (non-semisimple) two-sided indecomposable basic and artinian ring. Let \( \{e_i\}_{i=1}^k \) be a set of mutually orthogonal primitive idempotents with \( 1=\sum e_i \). Consider the following conditions:

1) \( R \) is a QF-ring.
2) \( R \) is a right almost hereditary ring.
3) $R$ is a (right Nakayama) ring with the following properties: i) $e_iJ \sim e_{i+1}R$ for $1 \leq i \leq n-1$ and $e_nJ \sim e_1R$, i.e., $\{e_1, e_2, \ldots, e_n\}$ is a cyclic Kupisch series. ii) $|e_iR| = |e_jR|$ for all $i$.

Then any two of the above conditions imply the remainder.

Proof. This is clear from Theorem 1, [2], Theorem 5.4 and [6], Theorem 1.

3. Proof of Theorem 2

We shall give here a proof of Theorem 2. “If” part is clear from Lemma 4. Assume that $R$ is a two-sided almost hereditary ring.

Let $T_1$, $T_2$ be the hereditary ring and the serial ring in (9), respectively. We assume first that $T_1$ and $T_2$ are two-sided indecomposable rings. We put $T_1 = T(H_1, H_2, \ldots, H_n)$ and we may assume $T_2 = T(m: D)$ (cf. the proof of Lemma 10), where the $H_i$ and the $D$ are division rings. Further we set $T_1 = \sum_{i=1}^{n} h_i R$, $T_2 = \sum_{i=1}^{m} d_j R$ and $E = \sum d_j$, where $\{h_i\}$ and $\{d_j\}$ are sets of orthogonal primitive idempotents.

Put

$$R = \begin{pmatrix}
H_1 & M_{12} & \cdots & M_{1n} \\
H_2 & M_{23} & \cdots & M_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
H_n & \cdots & \cdots & M_{n1}
\end{pmatrix}
\begin{pmatrix}
N_{11} & N_{12} & \cdots & N_{1c} \\
N_{21} & N_{22} & \cdots & N_{2c} \\
\vdots & \vdots & \ddots & \vdots \\
N_{n1} & N_{n2} & \cdots & N_{nc}
\end{pmatrix}
\begin{pmatrix}
D_1 & D_1 & \cdots & D_1 \\
D_1 & D_1 & \cdots & D_1 \\
\vdots & \vdots & \ddots & \vdots \\
D_1 & D_1 & \cdots & D_1
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
\vdots \\
0
\end{pmatrix}
$$

where $(N_{k_1}, N_{k_2}, \ldots, N_{k_c}) \sim (d_i R/A_i)^t$ and $A_i = d_i T_2 d_{c+1} + d_i T_2 d_{c+2} + \cdots + d_1 T_2 d_m = (0, \ldots, 0, D_1, \ldots, D_1)$.

We note the following fact. Since $R$ is a generalized tri-angular matrix ring, $Rh_i \supseteq (M_{i1}, \ldots, M_{i-1})' \supseteq \cdots \supseteq (M_{i1}, 0, \ldots, 0)'$ is a chain of submodules in $R_{hi}$, where $\cdot'$ is the transpose matrix of $(\cdot)$. We have a similar result for $Rd_j$. From (10) $Jd_{c+1}$ is not projective $(X \neq 0)$. Hence since $R$ is left almost hereditary, we have by [6], Theorem 1

$$Rd_k/(N_{k_1}, \ldots, N_{k_c})'$$

is injective for any $k < n$, provided $N_{k'} \neq 0$ for some $k' > k$.

Now we suppose
(12) \[ X \sim (h_a^* R E)^{(a_1)} \oplus (h_a^* R E)^{(a_2)} \oplus \cdots \oplus (h_a^* R E)^{(a_p)} \]

where \( a_1 < a_2 < \cdots < a_i \) and \( x_i > 0 \), \( h_a^* R E \sim d_i T_i / A_i \) for all \( i \).

Suppose \( \Lambda f \beta \gamma \theta \) for some \( f \in A_i \). We note that \( M_{a_{ij}} = (M_{a_{ij}})^{(M_{a_{ij}})} \) and \( M_{a_{ij}} = (N_{a_{ij}})^{(N_{a_{ij}})} \) for all \( i \). Hence \( M_{a_{ij}} = 0 \) for all \( j \in Z \) and all \( a_i \).

Now \( R_{d_i} \) is injective from (11). If \( R_{d_i} \) is injective for some \( t < c \), then since \( N_{a_{ij}} = 0 \) and \( N_{a_{ij}} = 0 \), there exists a non-zero homomorphism of \( R_{h} \) to \( R_{h} \). However \( h Rh_a = 0 \) from (10). Accordingly \( R_{d_i} \) is not injective for all \( t < c \). Hence \( \{R_{d_i}, R_{d_1}, \ldots, R_{d_0}\} \) is a Kupisch series such that \( J_{d_i} \sim R_{d_{i-1}} \) and the \( R_{d_i} \) are uniform. Let \( \Lambda^\Phi \) for some \( k \). Then \( R_{h} \) is not injective by assumption. Hence.

\[ M_{a_{ij}} = 0 \quad \text{for all } j \in Z \text{ and all } a_i. \]

Therefore \( (a_1 + \cdots + a_i) T_i (a_1 + \cdots + a_i) \) is a direct summand of \( T_i \) as rings, and hence \( \{a_1, \ldots, a_i\} = \{1, 2, \ldots, n\} \) since \( T_i \) is indecomposable. Then \( \{R_{d_1}, \ldots, R_{d_i}, R_{h}, \ldots, R_{h_1}\} \) is a Kupisch series from the above. Hence

\[ (13) \quad R = T(n+c, (m-c); D) \]

by Lemma 10. Next suppose that

some \( R_{h_i} \) is injective.

Then \( i = n \) by Corollary 3, and \( T_i = T(n; D) \). We note as above that \( \{R_{d_1}, R_{d_1}, \ldots, R_{d_i}\} \) is a Kupisch series such that \( J_{d_i} \sim R_{d_{i-1}} \) and the \( R_{d_i} \) are uniform. Let \( N_{a_{ij}} = 0 \) for some \( k \). Then \( R_{h_i} (M_{a_{ij}})^{(M_{a_{ij}})} \) is isomorphic to a submodule of the injective module \( R_{d_i} (N_{a_{ij}})^{(N_{a_{ij}})} \) by (11). Hence \( N_{a_{ij}} = 0 \) for all \( k' \geq k \) and so \( J_{d_i} = (0, 0, 0, N_{a_{ij}}, \ldots, N_{a_{ij}}) \) for some \( r \) and \( N_{a_{ij}} = 0 \) for all \( q \geq r \). Then \( \{R_{d_1}, \ldots, R_{d_{i+1}}, R_{d_1}, \ldots, R_{d_1}, R_{h}, \ldots, R_{h_1}\} \) is a Kupisch series. Hence
Finally we study the general form in (9). Let $T_{11}, \ldots, T_{ls}$ be two-sided indecomposable and hereditary rings and $T_{21}, \ldots, T_{2s}$ two-sided indecomposable and serial rings. Then

$$R = \begin{pmatrix} T_{11} & 0 & X \\ 0 & T_{1s} & \\ T_{21} & 0 & \\ 0 & T_{2s} & \end{pmatrix}$$

Let $\{h_{ij}\}$ (resp. $\{d_{ij}\}$) be the matrix units in the diagonal of $T_{11}$ (resp. $T_{21}$), and $E_i = \Sigma_j d_{ij}$. Put $F_j = \Sigma_i h_{ji}$. Now

$$X = \Sigma \oplus X_{jk}, \text{ where } X_{jk} = F_jXE_k.$$  

First we consider the following ring:

$$\begin{pmatrix} T_{11} & 0 & X_{11} \\ 0 & T_{1s} & X_{s1} \\ & & T_{21} \end{pmatrix}$$

It is clear from the above structure and (9) that $Rd_{ls} = \Sigma \oplus \Sigma X_{pl}d_{ls}$ and $X_{pl} \neq 0$ if and only if $X_{pl}d_{ls} = 0$. On the other hand, since $R$ is left almost hereditary, $Rd_{ls}$ is injective. Therefore $X_{pl} \neq 0$ only for one $l$ and $X_{pl} = 0$ for all $j \neq l$. Similarly $X_{pl} \neq 0$ only for one $i$ and $X_{pl} = 0$ for all $j \neq l$. For any $k$. Next we consider the following ring:

$$\begin{pmatrix} T_{11} & N^{(1)}_{11} \cdots N^{(1)}_{1s} & N^{(2)}_{11} \cdots N^{(2)}_{1s} \\ \vdots & \ddots & \vdots \\ N^{(1)}_{n1} \cdots N^{(1)}_{ns} & N^{(2)}_{n1} \cdots N^{(2)}_{ns} \\ 0 & T_{21} & 0 \end{pmatrix}$$

Assume $N^{(1)}_{k1} = 0$ and $N^{(2)}_{ks} = 0$ for some $k$. Then $Rd_{ls}^t(N^{(1)}_{11}, \ldots, N^{(1)}_{k1})^t = M^{(1)}$, $Rd_{ls}^t(N^{(2)}_{21}, \ldots, N^{(2)}_{k1})^t = M^{(2)}$ are injective. Hence there exists a
non-zero homomorphism of $M^{(2)}$ to $M^{(1)}$, a contradiction. As a consequence if $N^{(2)}_{ij} \neq 0$ for some $k$, then $N^{(2)}_{ij} = 0$ for all $i, j$ (if $N^{(2)}_{ij} \neq 0$, then $N^{(2)}_{kj} = 0$ for all $i, j$), because if $N^{(2)}_{ij} \neq 0$, $N^{(2)}_{ij} \neq 0$ from the argument to obtain (13) and (14). Thus we have shown

\[
\begin{pmatrix}
T_{11} & X_{11} \\
0 & T_{22}
\end{pmatrix}
\text{(resp. }egin{pmatrix}
T_{11} & X_{11} \\
0 & T_{22}
\end{pmatrix}\text{)}
\]

is a direct summand of $R$, provided $X_{11} \neq 0$ (resp. $X_{22} \neq 0$). We obtain the same result for any set $(T_{11}, T_{22}, T_{22})$. Therefore $R$ is serial from (13) and (14).

4. Strongly almost hereditary rings

Among right almost hereditary rings, we shall determine the structure of such rings with

(15) Every submodule of finitely generated projective module is again almost projective.

Lemma 11. If a simple module $eR/eJ$ is almost projective, then either $eR$ is uniserial and $eR/eJ^2, \ldots, eR/eJ^n, eR$ are injective or $eR$ is simple, where $eJ^{k+1} = 0$.

Proof. This is clear from [6], Theorem 1.

First we suppose that $R$ is a right almost hereditary ring with (15). Then we may study the following rings from Theorem 1:

α) Hereditary rings.

β) Serial rings in the first category.

Then $R = T(n_1, \ldots, n_r; D)$. Let $\{e^{(i)}_{j} \}_{i, j = 1}^{n_i}$ be matrix units in the diagonal of $R$. Take $\text{Soc}(e^{(r-1)}_{1}R)$. Then it is almost projective and isomorphic to $e^{(r)}_{k}R/e^{(r)}_{k}J$ for some $k$ from the structure of $R$. Hence $k = 1$ or $e^{(r)}_{k}R$ is simple from Lemma 11. However we do not have the latter. Hence $\text{Soc}(e^{(r-1)}_{1}R) \sim e^{(r)}_{k}R/e^{(r)}_{k}J$. Similarly we obtain $\text{Soc}(e^{(i)}_{1}R) \sim e^{(i+1)}_{k}R/e^{(i+1)}_{l}J$ for $n > i \geq 1$. (See the diagram below.)

\[
\begin{array}{c}
D_1 \\
D_2 \\
D_3 \\
D_4 \\
D_5
\end{array}
\]

(16)

γ) Serial rings in the second category.

Since $R$ has a cyclic Kupisch series, $eR/eJ \sim \text{Soc}(fR)$ for some $f$. Hence
$eR$ is injective by Lemma 11 for each $e$. Then $|eR|$ is same for all $e$ (cf. the proof of Lemma 4), say $|eR| = n$ and $eR/eJ^k$ is injective for all $k \geq 1$ by Lemma 11. Therefore there exist $m(n - 1)$ distinct injectives, where $1 = \sum_{i=1}^{n} e_i$. Accordingly $m(n - 1) \leq m$ ($R$ is basic), and hence $n \leq 2$.

\[ \delta \quad \left( \begin{array}{cc} T_1 & X \\ 0 & T_2 \end{array} \right) \]

From \( \gamma \) $T_2$ is a direct sum of serial rings in the first category as in \( \beta \). For the sake of simplicity, we study $R$ in which $T_2$ is indecomposable. Since $X = \Sigma \oplus f_{ii}R/A_{ii}$ as in (8), we have $A_{ii} = e_{ii}J$ by \( \beta \) and applying Lemma 11 to $\text{Soc}(f_{ii}R/A_{ii})$. Hence $X$ is semisimple.

Conversely we shall show that the above rings with stated properties satisfy the condition (15).

\( \alpha \) This is clear.

\( \beta \) and \( \gamma \) Let $A$ be a submodule of projective module $\Sigma_{i=1}^{n} e_i R$. If $n = 1$, we can easily see that $A$ is almost projective by the structure of $e_i R$ and [6], Theorem 1. Assume $n > 1$. Since $R$ is a serial ring, $A \sim \Sigma_{i} e_i' R/B_i$, $e_i' R/B_i$ being uniserial, $e_i' R/B_i$ is monomorphic to a submodule of some $e_i R$. Hence $A$ is almost projective by the initial remark.

\( \delta \) First we shall show that every submodule in $\Sigma_{i=1}^{X} h_i' R$ is almost projective, where the $h_i'$ are the $h_i$ in (8). Put

\[ C = R - \left( \begin{array}{cc} T_1 & X \\ 0 & f_{ii}R/e_{ii}J \end{array} \right). \]

(We may assume, from the proof below, that $T_2$ is indecomposable.) Then $C$ is a two-sided ideal. Set $R = R/C = \Sigma_{i=1}^{n} \oplus h_i' R \oplus D_i$, where $D_i = f_{ii}Rf_{ii} (\sim f_{ii}R/f_{ii}J)$. Then $A$ is an $R$-module. Since $X$ is semisimple, $R$ is hereditary by (8) and (9). Hence $A \sim \Sigma_{i} e_i' R/B_i$, $e_i' R/B_i$ being uniserial, $e_i' R/B_i$ is monomorphic to a submodule of some $e_i R$. Hence $A$ is almost projective by the initial remark.

Further $P_1 f_{ii} R \subset \text{Soc}(P_1)$. Hence $A_1/A_0$ is a semisimple module whose simple component is isomorphic to $f_{ii}R/e_{ii}J$. As a consequence $A_1 = X' \oplus A_0$, where $\text{Soc}(P_1) = X' \oplus X''$. Put $\tau: A_2^3 \sim A_1^3/A_0 = X'$. Let $P = (P_1 \oplus P_3)/A_{23} = \Sigma_{i} e_i' R/A_{23}, A_{23} \subset \Sigma_{i} e_i', R/A_{23}$. Since $(A_1/A_0) A_2^3/A_2$ is a semisimple module as
above, $A^p \subseteq \Sigma \oplus J' f_1' R f_1' J$. Since $f_1 R$ is uniserial, after taking a suitable decomposition $\Sigma_{J'} \oplus J' f_1' R = \Sigma_{J'} \oplus J' f_1' R (f_1' \sim f_1)$, we may assume that there exists a subset $J''$ of $J'$ such that $A_{p+1} = f_1'' J$ for $p \in J''$ and $A^p = \Sigma_{J''} \oplus J' f_1' R f_1' J$. Using the natural epimorphism $\theta: \Sigma \oplus J' f_1' R \to \Sigma \oplus J' f_1' R f_1' R$ we obtain $P = P_1 \oplus P_2 (\tau \theta) \oplus P_3$ and $A = A_1 \oplus P_2 (\tau \theta) \oplus A_2 \oplus \Sigma_{J''} \oplus e_4 A_3$, which is almost hereditary by the initial remark, where $P_2 = \Sigma_{J''} \oplus J' f_1' R$ and $P_3 = \Sigma_{J''} \oplus J' f_1' R$.

Summarizing the above, we have

**Theorem 3.** Let $R$ be an artinian ring. Then $R$ is a right almost hereditary ring with (15) if and only if $R$ is a direct sum of the following rings:

1. Hereditary rings.
2. Serial rings in the first category with the structure (16).
3. Serial rings in the second category with $J^2 = 0$.
4. Rings given in (9), where $T_2$ is a direct sum of serial rings in 2) and $A_N = e_1 J$ in (8).

Next we study the second stronger condition than that of almost hereditary rings.

(17) $J(Q)$ is almost projective for any finitely generated almost projective module $Q$.

(18) Every submodule of $Q$ is again almost projective.

**Lemma 12.** Let $R$ be a ring in (9) and $X \neq 0$. If $R$ is two-sided indecomposable and not serial, then $h_i R$ is never injective for any $h_i$ in (8).

Proof. $R$ is right almost hereditary by Theorem 1. First assume that $T_1 = T(D_1, D_2, \cdots, D_4)$ and $T_2$ are two-sided indecomposable. Further we may assume $T_2 = T(m: D)$ (see the proof below). Let $A_i = f_i T_2 f_{i+1} \oplus \cdots \oplus f_i T_2 f_m = (0, \cdots, 0, D, \cdots, D)$, where the $f$ are the idempotents in the diagonal of $T_2$. Put $E_i = 1_{T_1}$ and $C = R(f_{i+1} + \cdots + f_m)$. Then $C$ is a two-sided ideal. Set $\overline{R} = R/C$ (cf. the proof of Theorem 2), and $\overline{R}$ is hereditary by (8). Assume that $h_i R$ is injective for some $i$. Then $h_i R$ is also injective as an $\overline{R}$-module. Hence $\overline{R} = \overline{R} \oplus \overline{R}$ by the proof of Corollary 3, where $\overline{R} = T(n_s: D^s)$ is given from the Kupisch series $\{h_i, \cdots\}$. Now $T_1 = E_1 \overline{R} E_1 = E_1 \overline{R} E_1 \overline{R} E_1 \overline{R} E_1$, and hence $T = E_1 \overline{R} E_1$ for $E_i h_i E_i \neq 0$. As a consequence $T_1 \overline{R} \subseteq T(n_s: D_n) = \Sigma h_i(n) \overline{R} \oplus \Sigma f_i(n) \overline{R}$, where $\{h_i(n)\} \subset \{h_i\}$ and $\{f_i(n)\} \subset \{f_i\}$. Therefore $h_i \overline{R} = h_i R$ and $\{h_i, h_2, \cdots, h_n, f_1, \cdots, f_m\}$ is a Kupisch series in $R$ from the structure (9). Accordingly $R = T(n+k, m-k: D)$ is serial, a contradiction. Hence any $h_i$ is never injective. In general case we can use the same argument given in the proof of Theorem 2.

**Theorem 4.** Let $R$ be a basic artinian ring. Then $R$ satisfies (17) if and
only if \( R \) is a direct sum of the following rings:

1) Hereditary rings which are not serial.
2) Serial rings with \( f^2 = 0 \).
3) The ring in (9) with \( J(T_2)^2 = 0 \).

In this case \( R \) satisfies (18).

Proof. We may assume that \( R \) is two-sided indecomposable. The ring with (17) is almost hereditary.

i) \( R \) is hereditary. If some \( e_i R \) is injective, then \( R \) is a simple ring or \( R = T(n; D) \) by Corollary 3. Suppose \( R = T(n; D) \) \((n \geq 2)\). Then \( e_{ii} R / \text{Soc}(e_{ii} R) \) is almost projective by [6], Theorem 1, where the \( e_{ii} \) are matrix units in \( R \), and \( e_{ii} J / \text{Soc}(e_{ii} R) \sim e_{i2} R / \text{Soc}(e_{i2} R) \) by the structure of \( T(n; D) \), provided \( n > 2 \). Then \( e_{i2} R \) is injective by [6], Theorem 1, a contradiction. Therefore \( R = T(2; D) \). Next if any \( e_i R \) is not injective, then \( R \) is not serial.

ii) \( R \) is a serial ring in the first category. Then we can see \( f^2 = 0 \) in the above manner.

iii) \( R \) is a serial ring in the second category. Then \( f^2 = 0 \) from Theorem 3.

iv) \( R \) is the ring in (9), Then \( J(T_2)^2 = 0 \) as above. Conversely if \( R \) is non-serial and hereditary, then every almost projective module is projective by Corollary 3 and [6], Theorem 1. Hence (17) holds true. Next if \( R \) is a serial ring with \( f^2 = 0 \), then \( J(Q) = 0 \) for every non-projective, almost projective module \( Q \), and so (17) holds true from Theorem 3. Finally let \( R \) be of the form (3) and \( \bar{R} = R / C \) as in the proof of Theorem 2. Then \( \bar{R} \) is hereditary and any \( h_i R = \bar{h}_i \bar{R} \) is not injective by Lemma 12. Then there are no almost projective modules isomorphic to \( h_i R / A_i \) \((A_i \neq 0)\). Accordingly non-projective almost projective modules are given from \( T_2 \). Therefore \( Q' = \Sigma_i \oplus h_i R \oplus \Sigma_j \oplus f_j R \oplus \Sigma_k \oplus f_j R / f_j J \) for a finitely generated and almost projective module \( Q' \), where the \( f_j \) are primitive idempotents in \( T_2 \), and hence \( J(Q) \) is almost projective from the above. Thus we have obtained the equivalence in the theorem from Theorem 1. Noting that if \( h_i R \) is injective in the ring (3), then \( R \) is serial from the above, we can show (18) in the similar manner to the proof of Theorem 3.

We remain the last case:

Every submodule of \( Q \) is projective.

We can easily see that \( R \) has the above property if and only if \( R \) is hereditary and not serial. If we assume (15), (17) and (18) for left \( R \)-modules as well as right \( R \)-modules, then we obtain a characterization of hereditary rings and serial rings satisfying the conditions in Theorems 3 and 4, respectively.

Remark. In the above proof we used (17) and (18) only for indecomposable projective modules and indecomposable almost projective modules, respectively. Hence if every submodule of indecomposable and projective (resp. almost projective) module is almost projective, then (17) (resp. (18)) holds true.
References


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