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ELASTICA IN A RIEMANNIAN SUBMANIFOLD

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For a curve $\gamma(s)$ in a riemannian manifold $M$ we define two quantities: the length $L(\gamma)$ and the total square curvature $E(\gamma)$. A curve $\gamma$ is called an elastica if it is a critical point of the functional $E$ restricted to the space of curves of a fixed length $L_0$. The notion of elastica is quite old. But modern approaches to it in differential geometry are rather new. J. Langer and D.A. Singer classified all closed elasticae in the euclidean space ([1]), and showed that Palais-Smale's condition (C) holds for the space of curves in a riemannian manifold ([2]).

In this paper we consider elasticae restricted in a submanifold. For example, let $M$ be a compact surface of the euclidean space and $C$ the set of all closed curves of given length in the surface. Is there a closed curve in $C$ which minimizes the elastic energy $E$ (defined as curves of the euclidean space)?

We will affirmatively answer to the question in a more general situation.

**Theorem.** Let $\bar{M}$ be a riemannian manifold, $M$ a compact submanifold of $\bar{M}$ and $L_0$ a positive real number. Let $C$ be the space of all closed regular curves of length $L_0$ in $M$. For each $\gamma \in C$, we measure its (exterior) elastic energy $E(\gamma)$ as a curve in the manifold $\bar{M}$. Then the infimum $E_0$ of the energy $E$ on the set $C$ is attained by a $C^\infty$ curve in $C$.

Let $\gamma$ be a curve in $M$ of unit speed. We denote by $\tau$ and $\tau'$ its curvature vector as a curve of $\bar{M}$ and $M$, respectively. The difference of these two curvature is measured by the second fundamental form $\alpha$ of $M$ in $\bar{M}$. That is,

$$\tau = \tau + \alpha(\dot{\gamma}, \dot{\gamma}),$$

$$|\tau|^2 = |\tau|^2 + |\alpha(\dot{\gamma}, \dot{\gamma})|^2.$$

This formula leads us to a more general situation: elastic energy with "potential". That is what we treat in the following. Let $M$ be a riemannian manifold and $\phi$ a $C^\infty$-function defined on the unit tangent vector bundle over $M$. For a (closed) regular curve $\gamma$ in $M$, we reparametrize it by the arc length, and define the energy density $f(\gamma)$ and the energy $F(\gamma)$ by

$$f(\gamma) = |\tau|^2 + \phi(\dot{\gamma}),$$

$$F(\gamma) = \int f(\gamma) ds.$$
Since the curvature vector $\tau$ is given by
\[
\tau = \frac{d^2}{ds^2}\gamma^i + \Gamma^i_{jk} \frac{d}{ds}\gamma^j \frac{d}{ds}\gamma^k
\]
for a unit speed curve $\gamma$, where $\Gamma$ is the Christoffel's symbol, the functional $F$ is defined on the space of all regular $H^2$-curves. We will prove the following proposition, which implies our theorem as a corollary.

**Proposition.** Let $M$ be a compact riemannian manifold and $L_0$ a positive real number. Let $C$ be the space of all closed $C^\infty$ regular curves of length $L_0$, and $C_0$ a $C^1$ connected component of $C$. Then the infimum $F_0$ of the energy $F$ on the set $C_0$ is attained by a $C^\infty$ curve.

**Remark.** The following proof also applies to equivariant situations. I.e., if a compact Lie group $G$ acts on the manifold $M$ as isometries and if it invariantly preserves the function $\phi$, then we will get a $G$-invariant minimizer of $F$.

To prove this, we take a minimizing sequence $\{\gamma^p\}$ in $C_0$ of the functional $F$. We may assume that every curve $\gamma^p$ has unit speed. Since the manifold $M$ is compact, all curves of unit speed and of bounded length are bounded in $C^1$-topology. Moreover, since the function $\phi$ is bounded, the boundedness of values $F(\gamma^p)$ implies the boundedness of the family $\{\gamma^p\}$ in $H^2$-topology. Therefore a subsequence converges into some $C^1$ curve $\gamma_\infty$ in $C^1$-topology. Thus we may assume that the sequence $\{\gamma^p\}$ itself converges into $\gamma_\infty$ in $C^1$-topology.

First, we consider the case that $\gamma_\infty$ is a geodesic of $M$. Then the energy $F(\gamma^p)$ is greater or equal to the potential term $\int \phi(\gamma^p)ds$, which converges to $F(\gamma_\infty) = \int \phi(\gamma_\infty)ds$. Thus, we see that $F_0 \geq F(\gamma_\infty)$, hence $\gamma_\infty$ is the desired curve.

Next, suppose that $\gamma_\infty$ is not a geodesic. Note that even when $\gamma_\infty$ has a self-intersection, we can take a covering of a neighbourhood of the image of $\gamma_\infty$, and may assume that $\gamma_\infty$ has no self-intersections. Thus we can take a $C^\infty$ vector field $V$ on a neighbourhood of the image of $\gamma_\infty$ so that
\[
\frac{d}{dt} \big|_{t=0} L(\exp tV \circ \gamma_\infty) = 1,
\]
because $\gamma_\infty$ is not a critical point of the functional $L$.

For two points $x$ and $y$ which are sufficiently close in $M$ we denote by $\mu(x,y)$ their mid point on the distance geodesic joining $x$ and $y$. Using it we define the mean curve $\gamma_{pq}$ of two curves $\gamma_p$ and $\gamma_q$ which are sufficiently close to $\gamma_\infty$ by
\[
\gamma_{pq}(s) = \mu(\gamma_p(s), \gamma_q(s)).
\]
Since the length $L(\gamma_{pq})$ may differ from $L_0$, we normalize it by a diffeomorphism
exp \( tV \) as follows: Note that the curve \( \gamma_{pq} \) is close to \( \gamma_\infty \) in \( C^1 \)-topology. Therefore we can take a (small) real number \( t_{pq} \) so that

\[ L((\exp t_{pq}V)\circ \gamma_{pq}) = L_0. \]

The desired normalized curve \( \tilde{\gamma}_{pq} \) is defined by

\[ \tilde{\gamma}_{pq}(s) = (\exp t_{pq}V)(\gamma_{pq}(s)). \]

Note that \( t_{pq} \) converges into 0 and that the sequences \( \{\gamma_{pq}\} \) and \( \{\tilde{\gamma}_{pq}\} \) converge into \( \gamma_\infty \) in \( C^1 \)-topology.

To simplify the following calculation, we introduce the potential term \( T(\gamma) \) and the modified energies \( \tilde{E}(\gamma) \) and \( \tilde{F}(\gamma) \) by

\[ T(\gamma) = \int \phi(|\dot{\gamma}|^{-1}\dot{\gamma})ds, \]
\[ \tilde{E}(\gamma) = \int |D\dot{\gamma}|^2 ds, \]
\[ \tilde{F}(\gamma) = \tilde{E}(\gamma) + T(\gamma). \]

Note that \( \tilde{F}(\gamma) \) coincides with \( F(\gamma) \) when \( \gamma \) is so of unit speed. We set \( T_0 = T(\gamma_\infty) \) and \( E_0 = F_0 - T_0 \). We also denote by \( \varepsilon \) a positive quantity which converges into 0 when \( p \) (and \( q \)) \( \to \infty \). For example, \( |T(\gamma_p) - T(\gamma_q)| \leq \varepsilon. \)

From the inequality

\[ |\tau|^2 = |\dot{\gamma}|^{-4} |D_\gamma \dot{\gamma}|^2 - |\dot{\gamma}|^{-6} (\dot{\gamma}, D_\gamma \dot{\gamma})^2 \leq |\dot{\gamma}|^{-4} |D_\gamma \dot{\gamma}|^2, \]

we see that

\[ \tilde{E}(\gamma) = \int |D_\gamma \dot{\gamma}|^2 ds \geq \int |\dot{\gamma}|^4 |\tau|^2 ds = \int |\dot{\gamma}|^2 |\tau|^2 |\dot{\gamma}| ds. \]

Combining it with inequalities \( |\dot{\gamma}_{pq}| - 1| \leq \varepsilon \) and \( E(\tilde{\gamma}_{pq}) + T(\tilde{\gamma}_{pq}) \geq F_0 \), we get

\[ \tilde{E}(\tilde{\gamma}_{pq}) \geq \int |\dot{\gamma}_{pq}|^2 |\tau(\tilde{\gamma}_{pq})|^2 |\dot{\gamma}_{pq}| ds \]
\[ \geq (1 - \varepsilon) \int |\tau(\tilde{\gamma}_{pq})|^2 |\dot{\gamma}_{pq}| ds = (1 - \varepsilon)E(\tilde{\gamma}_{pq}) \]
\[ \geq (1 - \varepsilon)(F_0 - T(\tilde{\gamma}_{pq})) \]
\[ = (1 - \varepsilon)(F_0 - T_0) - (1 - \varepsilon)(T(\tilde{\gamma}_{pq}) - T_0) \]
\[ \geq (1 - \varepsilon)E_0 - \varepsilon. \]

Now we take a sufficiently small geodesic coordinate system around each point \( \gamma_\infty(s) \) so that the metric tensor \( g_{ij} \) and the Christoffel's symbol \( \Gamma_{ij}^k \) are sufficiently close to \( \delta_{ij} \) and 0 respectively in \( C^0 \)-topology. Also we denote by \( |*|_e \) the euclidean norm defined by the coordinate functions. Then the expression
\((D\gamma)^i = \frac{d^2}{ds^2}\gamma^i + \Gamma^i_{jk}\frac{d}{ds}\gamma^j\frac{d}{ds}\gamma^k\)

leads the inequality

\[(1-\varepsilon)|\frac{d^2}{ds^2}\gamma|_\varepsilon - \varepsilon \leq |D\gamma| \leq (1+\varepsilon)|\frac{d^2}{ds^2}\gamma|_\varepsilon + \varepsilon\]

for a curve \(\gamma\) which is sufficiently close to \(\gamma^\omega\) in \(C^1\)-topology. Moreover, \(\hat{\gamma}_{pq}\) differs from \(\gamma_{pq}\) only by a \(C^{\infty}\)-diffeomorphism \(\exp_{pq}t\), where \(|t_{pq}| \leq \varepsilon\). Therefore we get

\[
|D_{l_{pq}}\hat{\gamma}_{pq}| \geq (1-\varepsilon)|\frac{d^2}{ds^2}\gamma_{pq}|_\varepsilon - \varepsilon
\]

\[
\geq (1-\varepsilon)((1-\varepsilon)|\frac{d^2}{ds^2}\tilde{\gamma}_{pq}|_\varepsilon - \varepsilon) - \varepsilon \geq (1-\varepsilon)|\frac{d^2}{ds^2}\tilde{\gamma}_{pq}|_\varepsilon - \varepsilon
\]

\[
\geq (1-\varepsilon)((1-\varepsilon)|D_{l_{pq}}\hat{\gamma}_{pq}| - \varepsilon) - \varepsilon \geq (1-\varepsilon)|D_{l_{pq}}\hat{\gamma}_{pq}| - \varepsilon.
\]

Combining it with (**), we see that

\((**)^*\) \[\|E(\gamma_{pq})\| \geq (1-\varepsilon)\|E(\tilde{\gamma}_{pq})\| - \varepsilon \geq (1-\varepsilon)E_0 - \varepsilon.\]

The relation \(z=\mu(x, y)\) is expressed by a geodesic \(\eta(\sigma)\):

\[
\frac{d^2}{d\sigma^2}\eta^i + \Gamma^i_{jk}\frac{d}{d\sigma}\eta^j\frac{d}{d\sigma}\eta^k = 0,
\]

\[
x = \eta(-1), \quad y = \eta(1), \quad z = \eta(0).
\]

Regarding \(\eta\) to depend on a parameter \(t\), we take second derivative with respect to \(t\) at the origin: \(x' = y' = 0, \Gamma^i_{jk} = 0\). Then we get

\[
\frac{d}{d\sigma}\frac{d}{dt}\eta^i = \frac{d^2}{d\sigma^2}\frac{d^2}{dt^2}\eta^j = 0,
\]

which implies that \(\mu\) is sufficiently close in \(C^2\)-topology to the linear map: \(\gamma' = \frac{1}{2}(x' + y')\). Therefore we see that

\[
\left|\frac{d^2}{ds^2}\gamma_{pq}\right|^2 \leq \frac{1}{2}\left(\frac{d^2}{ds^2}\gamma_p\right)^2 + \frac{d^2}{ds^2}\gamma_q\right)^2
\]

\[
\leq \varepsilon\left(\frac{d^2}{ds^2}\gamma_p\right)^2 + \frac{d^2}{ds^2}\gamma_q\right)^2
\]

\[
\leq (1-\varepsilon)\left(\frac{d^2}{ds^2}\gamma_p\right)^2 + \varepsilon\left(\frac{d^2}{ds^2}\gamma_q\right)^2
\]

hence that

\[
\frac{1}{4}\left|\frac{d^2}{ds^2}\gamma_p\right|^2 \leq \varepsilon - (1-\varepsilon)\left(\frac{d^2}{ds^2}\gamma_p\right)^2 + \begin{pmatrix} \frac{1}{2} & + \varepsilon \end{pmatrix}\left(\frac{d^2}{ds^2}\gamma_q\right)^2
\]

\[
\leq \varepsilon - (1-\varepsilon)\left|D_{l_{pq}}\hat{\gamma}_{pq}\right|^2 + \begin{pmatrix} \frac{1}{2} & + \varepsilon \end{pmatrix}\left(\left|D_{l_{pq}}\hat{\gamma}_{pq}\right|^2 + \left|D_{l_{pq}}\hat{\gamma}_{pq}\right|^2\right).
\]
Integrating this in $s$, we get
\[
\frac{1}{4} \int \left| \frac{d^2}{ds^2} \gamma_\sigma - \frac{d^2}{ds^2} \gamma_\epsilon \right|^2 ds \leq \varepsilon - (1 - \varepsilon) E(\gamma_\sigma) + \left( \frac{1}{2} + \varepsilon \right) (E(\gamma_\sigma) + E(\gamma_\epsilon)),
\]
where the integration of the left hand side means a finite sum of integrations over small intervals of $s$.

Combining it with inequality (***) and the fact that $F(\gamma_\rho)$ converges into $F_0$, we see that
\[
\int \left| \frac{d^2}{ds^2} \gamma_\rho - \frac{d^2}{ds^2} \gamma_\epsilon \right|^2 ds \leq \varepsilon(1 + E_0),
\]
which implies that the sequence $\{\gamma_\rho\}$ converges in $H^2$-topology. Therefore the limiting curve $\gamma_\omega$ of $\{\gamma_\rho\}$ is a $H^2$-curve and $F(\gamma_\omega) = F_0$.

Now it is standard to check that $\gamma_\omega$ is in fact $C^\infty$ and is a classical solution of the Euler-Larange equation. We complete the proof.

References


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