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ELASTICAE IN A RIEMANNIAN SUBMANIFOLD

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For a curve $\gamma(s)$ in a riemannian manifold M we define two quantities: the length $L(\gamma)$ and the total square curvature $E(\gamma)$. A curve γ is called an elastica if it is a critical point of the functional E restricted to the space of curves of a fixed lnegth L_0. The notion of elastica is quite old. But modern approaches to it in differential geometry are rather new. J. Langer and D.A. Singer classified all closed elasticae in the euclidean space ([1]), and showed that Palais-Smale's condition (C) holds for the space of curves in a riemannian manifold ([2]).

In this paper we consider *elasticae restricted in a submanifold.* For example, let M be a compact surface of the euclidean space and C the set of all closed curves of given length in the surface. Is there a closed curve in C which minimizes the elastic energy $\bar{E}$ (defined as curves of the euclidean space)?

We will affirmatively answer to the question in a more general situation.

Theorem. *Let $\bar{M}$ be a riemannian manifold, M a compact submanifold of $\bar{M}$ and L_0 a positive real number. Let C be the space of all closed regular curves of length L_0 in M. For each $\gamma \in C$, we measure its (exterior) elastic energy $\bar{E}(\gamma)$ as a curve in the manifold $\bar{M}$. Then the infimum $\bar{E}_0$ of the energy $\bar{E}$ on the set C is attained by a C^∞ curve in C.*

Let γ be a curve in M of unit speed. We denote by $\bar{\tau}$ and τ its curvature vector as a curve of $\bar{M}$ and M, respectively. The difference of these two curvature is measured by the second fundamental form α of M in $\bar{M}$. That is,

$$\bar{\tau} = \tau + \alpha(\dot{\gamma}, \dot{\gamma}),$$
$$|\bar{\tau}|^2 = |\tau|^2 + |\alpha(\dot{\gamma}, \dot{\gamma})|^2.$$

This formula leads us to a more general situation: elastic energy with "potential". That is what we treat in the following. Let M be a riemannian manifold and ϕ a C^∞-function defined on the unit tangent vector bundle over M. For a (closed) regular curve γ in M, we reparametrize it by the arc length, and define the energy density $f(\gamma)$ and the energy $F(\gamma)$ by

$$f(\gamma) = |\tau|^2 + \phi(\dot{\gamma}),$$
$$F(\gamma) = \int f(\gamma) ds.$$

Since the curvature vector τ is given by

$$\tau^i = \frac{d^2}{ds^2}\gamma^i + \Gamma^i_{jk}\frac{d}{ds}\gamma^j\frac{d}{ds}\gamma^k$$

for a unit speed curve γ, where Γ is the Christoffel's symbol, the functional F is defined on the space of all regular H^2-curves. We will prove the following proposition, which implies our theorem as a corollary.

Proposition. *Let M be a compact riemannian manifold and L_0 a positive real number. Let C be the space of all closed C^∞ regular curves of length L_0, and C_0 a C^1 connected component of C. Then the infimum F_0 of the energy F on the set C_0 is attained by a C^∞ curve.*

Remark. The following proof also applies to equivariant situations. I.e., if a compact Lie group G acts on the manifold M as isometries and if it invariantly preserves the function ϕ, then we will get a G-invariant minimizer of F.

To prove this, we take a minimizing sequence $\{\gamma_p\}$ in C_0 of the functional F. We may assume that every curve γ_p has unit speed. Since the manifold M is compact, all curves of unit speed and of bounded length are bounded in C^1-topology. Moreover, since the function ϕ is bounded, the boundedness of values $F(\gamma_p)$ implies the boundedness of the family $\{\gamma_p\}$ in H^2-gopology. Therefore a subsequence converges into some C^1 curve γ_∞ in C^1-topology. Thus we may assume that the sequence $\{\gamma_p\}$ itself converges into γ_∞ in C^1-topology.

First, we consider the case that γ_∞ is a geodesic of M. Then the energy $F(\gamma_p)$ is greater or equal to the potential term $\int \phi(\dot{\gamma}_p)ds$, which converges to $F(\gamma_\infty)=\int \phi(\dot{\gamma}_\infty)ds$. Thus, we see that $F_0 \geq F(\gamma_\infty)$, hence γ_∞ is the desired curve.

Next, suppose that γ_∞ is not a goedesic. Note that even when γ_∞ has a self-intersection, we can take a covering of a neighbourhood of the image of γ_∞, and may assume that γ_∞ has no self-intersections. Thus we can take a C^∞ vector field V on a neighbourhood of the image of γ_∞ so that

$$\frac{d}{dt}\Big|_{t=0} L(\exp tV \circ \gamma_\infty) = 1\,,$$

because γ_∞ is not a critical point of the functional L.

For two points x and y which are sufficiently close in M we denote by $\mu(x, y)$ their mid point on the distance geodesic joining x and y. Using it we define the mean curve γ_{pq} of two curves γ_p and γ_q which are sufficiently close to γ_∞ by

$$\gamma_{pq}(s) = \mu(\gamma_p(s), \gamma_q(s))\,.$$

Since the length $L(\gamma_{pq})$ may differ from L_0, we normalize it by a diffeomorphism

$\exp tV$ as follws: Note that the curve γ_{pq} is close to γ_∞ in C^1-topology. Therefore we can take a (small) real number t_{pq} so that

$$L((\exp t_{pq}V)\circ\gamma_{pq}) = L_0 .$$

The desired normalized curve $\tilde{\gamma}_{pq}$ is defined by

$$\tilde{\gamma}_{pq}(s) = (\exp t_{pq}V)(\gamma_{pq}(s)) .$$

Note that t_{pq} converges into 0 and that the sequences $\{\gamma_{pq}\}$ and $\{\tilde{\gamma}_{pq}\}$ converge into γ_∞ in C^1-topology.

To simplify the following calculation, we introduce the potential term $T(\gamma)$ and the modified energies $\tilde{E}(\gamma)$ and $\tilde{F}(\gamma)$ by

$$\begin{aligned} T(\gamma) &= \int \phi(|\dot{\gamma}|^{-1}\dot{\gamma})ds , \\ \tilde{E}(\gamma) &= \int |D_{\dot{\gamma}}\dot{\gamma}|^2 ds, \\ \tilde{F}(\gamma) &= \tilde{E}(\gamma)+T(\gamma) . \end{aligned}$$

Note that $\tilde{F}(\gamma)$ conicides with $F(\gamma)$ when γ is so of unit speed. We set $T_0 = T(\gamma_\infty)$ and $E_0 = F_0 - T_0$. We also denote by ε a positive quantity which converges into 0 when p (and q)$\to\infty$. For example, $|T(\gamma_p)-T(\gamma_q)|\leq\varepsilon$.

From the inequality

$$|\tau|^2 = |\dot{\gamma}|^{-4}|D_{\dot{\gamma}}\dot{\gamma}|^2 - |\dot{\gamma}|^{-6}(\dot{\gamma}, D_{\dot{\gamma}}\dot{\gamma})^2 \leq |\dot{\gamma}|^{-4}|D_{\dot{\gamma}}\dot{\gamma}|^2,$$

we see that

$$\tilde{E}(\gamma) = \int |D_{\dot{\gamma}}\dot{\gamma}|^2 ds \geq \int |\dot{\gamma}|^4|\tau|^2 ds = \int |\dot{\gamma}|^3|\tau|^2|\dot{\gamma}|ds .$$

Combining it with inequalities $||\dot{\tilde{\gamma}}_{pq}|-1|\leq\varepsilon$ and $E(\tilde{\gamma}_{pq})+T(\tilde{\gamma}_{pq})\geq F_0$, we get

$$\begin{aligned} \tilde{E}(\tilde{\gamma}_{pq}) &\geq \int |\dot{\tilde{\gamma}}_{pq}|^3|\tau(\tilde{\gamma}_{pq})|^2|\dot{\tilde{\gamma}}_{pq}|ds \\ &\geq (1-\varepsilon)\int |\tau(\tilde{\gamma}_{pq})|^2|\dot{\tilde{\gamma}}_{pq}|ds = (1-\varepsilon)E(\tilde{\gamma}_{pq}) \\ &\geq (1-\varepsilon)(F_0 - T(\tilde{\gamma}_{pq})) \qquad (*) \\ &= (1-\varepsilon)(F_0 - T_0) - (1-\varepsilon)(T(\tilde{\gamma}_{pq}) - T_0) \\ &\geq (1-\varepsilon)E_0 - \varepsilon . \end{aligned}$$

Now we take a sufficiently small geodesic coordinate system around each point $\gamma_\infty(s)$ so that the metric tensor g_{ij} and the Christoffel's symbol $\Gamma_i{}^k{}_j$ are sufficiently close to δ_{ij} and 0 respectively in C^0-topology. Also we denote by $|*|_e$ the euclidean norm defined by the coordinate functions. Then the expression

$$(D_{\dot\gamma}\dot\gamma)^i = \frac{d^2}{ds^2}\gamma^i + \Gamma_j{}^i{}_k \frac{d}{ds}\gamma^j \frac{d}{ds}\gamma^k$$

leads the inequality

$$(1-\varepsilon)\left|\frac{d^2}{ds^2}\gamma\right|_e - \varepsilon \leq |D_{\dot\gamma}\dot\gamma| \leq (1+\varepsilon)\left|\frac{d^2}{ds^2}\gamma\right|_e + \varepsilon\,,$$

for a curve γ which is sufficiently close to γ_∞ in C^1-topology. Moreover, $\tilde\gamma_{pq}$ differs from γ_{pq} only by a C^∞-diffeomorphism $\exp t_{pq}V$, where $|t_{pq}|\leq\varepsilon$. Therefore we get

$$\begin{aligned}|D_{\dot\gamma_{pq}}\dot\gamma_{pq}| &\geq (1-\varepsilon)\left|\frac{d^2}{ds^2}\gamma_{pq}\right|_e - \varepsilon \\ &\geq (1-\varepsilon)\left((1-\varepsilon)\left|\frac{d^2}{ds^2}\tilde\gamma_{pq}\right|_e - \varepsilon\right) - \varepsilon \geq (1-\varepsilon)\left|\frac{d^2}{ds^2}\tilde\gamma_{pq}\right|_e - \varepsilon \\ &\geq (1-\varepsilon)\left((1-\varepsilon)|D_{\dot{\tilde\gamma}_{pq}}\dot{\tilde\gamma}_{pq}| - \varepsilon\right) - \varepsilon \geq (1-\varepsilon)|D_{\dot{\tilde\gamma}_{pq}}\dot{\tilde\gamma}_{pq}| - \varepsilon\,.\end{aligned}$$

Combining it with (*), we see that

$$(**)\qquad \tilde E(\gamma_{pq}) \geq (1-\varepsilon)\tilde E(\tilde\gamma_{pq}) - \varepsilon \geq (1-\varepsilon)E_0 - \varepsilon\,.$$

The relation $z=\mu(x, y)$ is expressed by a goedisic $\eta(\sigma)$:

$$\frac{d^2}{d\sigma^2}\eta^i + \Gamma_j{}^i{}_k \frac{d}{d\sigma}\eta^j \frac{d}{d\sigma}\eta^k = 0\,,$$
$$x=\eta(-1),\quad y=\eta(1),\quad z=\eta(0).$$

Regarding η to depend on a parameter t, we take second derivative with respect to t at the origin: $x^i=y^i=0$, $\Gamma_j{}^i{}_k=0$. Then we get

$$\frac{d}{d\sigma}\frac{d}{dt}\eta^i = \frac{d^2}{d\sigma^2}\frac{d^2}{dt^2}\eta^j = 0\,,$$

which implies that μ is sufficiently close in C^2-topology to the linear map: $z^j=\frac{1}{2}(x^i+y^i)$. Therefore we see that

$$\begin{aligned}&\left|\frac{d^2}{ds^2}\gamma_{pq}\right|_e^2 - \frac{1}{2}\left(\left|\frac{d^2}{ds^2}\gamma_p\right|_e^2 + \left|\frac{d^2}{ds^2}\gamma_q\right|_e^2\right) \\ &\leq \varepsilon\left(1+\left|\frac{d^2}{ds^2}\gamma_p\right|_e^2 + \left|\frac{d^2}{ds^2}\gamma_q\right|_e^2\right) - \frac{1}{4}\left|\frac{d^2}{ds^2}\gamma_p - \frac{d^2}{ds^2}\gamma_q\right|_e^2\,,\end{aligned}$$

hence that

$$\begin{aligned}\frac{1}{4}\left|\frac{d^2}{ds^2}\gamma_p - \frac{d^2}{ds^2}\gamma_q\right|_e^2 &\leq \varepsilon - (1-\varepsilon)\left|\frac{d^2}{ds^2}\gamma_{pq}\right|_e^2 + \left(\frac{1}{2}+\varepsilon\right)\left(\left|\frac{d^2}{ds^2}\gamma_q\right|_e^2 + \left|\frac{d^2}{ds^2}\gamma_q\right|_e^2\right) \\ &\leq \varepsilon - (1-\varepsilon)|D_{\dot\gamma_{pq}}\dot\gamma_{pq}|^2 + \left(\frac{1}{2}+\varepsilon\right)\left(|D_{\dot\gamma_p}\dot\gamma_p|^2 + |D_{\dot\gamma_p}\dot\gamma_p|^2\right).\end{aligned}$$

Integrating this in s, we get

$$\frac{1}{4}\int |\frac{d^2}{ds^2}\gamma_p-\frac{d^2}{ds^2}\gamma_q|_e^2 ds \leq \varepsilon-(1-\varepsilon)\tilde{E}(\gamma_{pq})+(\frac{1}{2}+\varepsilon)(\tilde{E}(\gamma_p)+\tilde{E}(\gamma_q)),$$

where the integration of the left hand side maens a finite sum of integrations over small intervals of s.

Combining it with inequelity (**) and the fact that $F(\gamma_p)$ converges into F_0, we see that

$$\int |\frac{d^2}{ds^2}\gamma_p-\frac{d^2}{ds^2}\gamma_q|_e^2 ds \leq \varepsilon(1+E_0),$$

which implies that the seqeunce $\{\gamma_p\}$ converges in H^2-topology. Therefore the limiting curve γ_∞ of $\{\gamma_p\}$ is a H^2-curve and $F(\gamma_\infty)=F_0$.

Now it is standard to check that γ_∞ is in fact C^∞ and is a classical solution of the Euler-Larange equation. We complete the proof.

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