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A Theorem with Respect to the Unique Continuation for a Parabolic Differential Equation

By Taira SHIROTA

1. Introduction.

In the present paper we study the unique continuation of solutions $u(t, x)$ defined in the convex domain G of Euclidean $n+1$-space R_{n+1} satisfying the parabolic equation

$$(1.1)\qquad L(u) = 0\,,$$

$$(1.2)\qquad \left(L-\frac{\partial}{\partial t}\right)u(t, x) = \sum_{i,j=1}^{n} a_{i,j}(t, x)\frac{\partial^2 u}{\partial x_i \partial x_j}(t, x) + \sum_{i=1}^{n} b_i(t, x)\frac{\partial u}{\partial x_i}(t, x)+c(t, x)u(t, x)\,,$$

where we assume that the coefficients satisfy the following conditions:

(1.3) there are two positive numbers α_1 and α_2 such that

$$\alpha_1|\xi|^2 \geqq a_{ij}(t, x)\xi_i\xi_j \geqq \alpha_2|\xi|^2$$

for any real vector $\xi=(\xi_1, \xi_2, \cdots, \xi_n)$ and for any $(t, x)\in G$,

$$(1.4)\qquad a_{ij}(t, x)\,,\ \frac{\partial a_{ij}}{\partial x_k}(t, x)\,,\ \frac{\partial a_{ij}}{\partial t}(t, x)\,,\ \frac{\partial^2 a_{ij}}{\partial x_k \partial x_l}(t, x)\,,\ \frac{\partial^2 a_{ij}}{\partial x_l \partial t}(t, x)\,,$$

b_i and c are all continuous in G $(i, j, k, l=1, 2, \cdots, n)$.

In the case where the solution satisfies some boundary conditions unique continuation theorems are considered by H. Yamabe, S. Ito[5] and the author[8] using the unique continuation theorem of elliptic operator established by N. Aronszajn[1], H. O. Cordes[3] and applying the abstract analyticity of solutions of parabolic equations which is investigated also by K. Yosida[12] in another point of view. On the other hand the uniqueness of the solutions for Cauchy problem of (1.1) with non characteristic initial surface is established by S. Mizohata[6], modifying the methods used by A. P. Calderón[2], whose result is recently strengthened by Li der-Yuan[11] using the idea of E. Heinz[4] and H. O. Cordes[3].

The purpose of the present paper is to prove the following

Theorem. *Let* $\mathfrak{C}$ *be a curve*: $\{(t, x_i(t)) \mid t \in [a, b]\}$ *with* $x_i(t) \in C^1([a, b])$. *If* u *is continuously differentiable with respect to* x_i *of second order and with respect to* t *of first order on the domain* G *and if* u *satisfies the following two conditions*: *there is a positive number* M *such that*

$$(1.5) \qquad |L(u)(t, x)|^2 \leq M\left\{\sum_{i=1}^n \left|\frac{\partial u}{\partial x_i}(t, x)\right|^2 + |u(t, x)|^2\right\}$$

for any $(t, x) \in G$, *and for any* $\alpha > 0$

$$(1.6) \quad \lim_{r\to 0} \max_{\substack{|x-x(t)|=r \\ t\in[a,b] \\ i,j=1,2,\cdots,n}} \left\{\left|u(t, x)\right|, \left|\frac{\partial u}{\partial x_i}(t, x)\right|, \left|\frac{\partial^2 u}{\partial x_i \partial x_j}(t, x)\right|\right\} |x-x(t)|^{-\alpha} = 0,$$

then u *vanishes identically in the horizontal component* $G \frown \{(t, x) \mid t \in [a, b]\}$.

The following proof is the completion of my previous one, where I made errors in calculation, improved by some modifications. It is also based on the methods used by Heinz and Cordes. The main distinctive feature from other authors is that I am concerned with a strong unique continuation while they does only with the uniqueness of solutions of Cauchy problem. Therefore I must consider convex lens-shaped region and some damping factors in our estimates. Furthermore my conditions with respect to $a_{ij}(t, x)$, (1.4), is stronger than that used by Li Der-Yuan in his last lemma, that is, I add the assumption more with respect to $\frac{\partial^2}{\partial x_k \partial t} a_{ij}$, which seems to be necessary, so far as we do not employ other methods of consideration..(See p. 386)

2. Basic inequalities.

Before stating our results in this section we will describe some notations. We shall use the following convention:

$$u_{|t} = \frac{\partial u}{\partial t}, \quad u_{|x_i} = \frac{\partial u}{\partial x_i}, \quad u_{|\sigma} = \frac{\partial u}{\partial \varphi_\sigma}, \quad \text{etc.}$$

Furthermore for a domain D of R_{n+1} denote by $C_x^m(D)$ and $C_t^m(D)$ the sets of all functions v defined in D such that the derivatives of v of order m with respect to x_i $(i=1, 2, \cdots, n)$ and with respect to t are continuous on D respectively.

We assume here that the parabolic operator (1.2) is reduced to the following form:

$$(2.1) \qquad \begin{aligned} \bar{L}(v) &= a_{ij} u_{|x_i|x_j} + b_i u_{|x_i} - q u_{|t} \\ &= p\left(u_{|r|r} + \frac{n-1}{r} u_{|r} + \frac{1}{r^2} Nu\right) - q u_{|t}. \end{aligned}$$

Here the coefficients are functions defined in $D=\{t \mid t\in[-\varepsilon, 1+\varepsilon]\} \times\{x \mid |x|<R\}$ for some positive R and ε and satisfy the following conditions:

(i) for a finite number of systems of polar coordinates (r, φ_σ) covering the unit sphere of R_n the Laplace-Beltrami operator N is represented in the form

$$N=\frac{1}{\lambda(x)}\frac{\partial}{\partial\varphi_\sigma}\lambda(x)\bar{a}_{\sigma\tau}(t, x)\frac{\partial}{\partial\varphi_\tau}, \tag{2.2}$$

$$\lambda(x)=\frac{\partial O_1}{\partial\varphi_1\partial\varphi_2\cdots\partial\varphi_{n-1}},$$

where ∂O_1 is the usual surface element of the unit sphere,

(ii) $\quad p(t, x)\in C^1_{t,x}(D)$, $q(r)\in C^1_r([0, R])$, $b_i(t, x)\in L^\infty(D)$, $a_{ij}(t, x)\in L^\infty(D)$, and $\bar{a}_{\sigma,\tau}(t, x)\in C^1_{t,r}(D)$,

(iii) there are positive numbers β and γ $(\beta>\gamma)$ such that

$$\beta^{-1}(p\bar{a}_{\sigma,\tau})_{|r}(t, x)\eta_\sigma\eta_\tau\geqq\bar{a}_{\sigma,\tau}(t, x)\eta_\sigma\eta_\tau\geqq\gamma|\eta|^2, \tag{2.3}$$

$$-p_{|r}\geqq\beta \quad\text{and}\quad p, p^{-1}, q, q^{-1}\geqq\gamma \tag{2.4}$$

for any $(t, x)\in D$ and for any real vector $\eta=\{\eta_1, \eta_2, \cdots, \eta_{n-1}\}$, where β is sufficiently large.

Furthermore for $r_0<R$ let D_{r_0,K_0} be the domain:

$$\{(t, x) \mid 0<t<1,\ |x|<r_0\wedge K_0^{-1}t\wedge K_0^{-1}(1-t)\}.$$

Denote by $\mathfrak{K}$ the class of functions v such that i) $v\in C^2_x(D_{r_0,K_0})\cap C^1_t(D_{r_0,K_0})$ ii) the carrier of $v\subset D_{r_0,K_0}\cup\{(0, 0), (0, 1)\}$ and iii) v vanishes at $x=0$ as follows: for any $\alpha>0$

$$\lim_{r\to0}\max_{\substack{|x|=r\\ t\in[0,1]\\ i,j=1,2,\cdots,n}}\{|u(t, x)|, |u_{|x_i}(t, x)|, |u_{|x_i|x_j}(t, x)|, |u_{|t}(t, x)|\}r^{-\alpha}=0. \tag{2.5}$$

Finally let $\varphi(t)$ be the smooth function such that

$$\begin{aligned}\varphi(t)&=t \quad\text{for}\quad t\in\left[0, \frac{1}{5}\right]\\ &=1 \quad\text{for}\quad t\in\left[\frac{2}{5}, \frac{3}{5}\right]\\ &=1-t \quad\text{for}\quad t\in\left[\frac{4}{5}, 1\right], \quad\text{and}\end{aligned}$$

$$|\varphi(t)|\leqq1 \quad\text{for any}\quad t\in[0, 1].$$

We are now in a position to state the basic lemma of this section.

Lemma 1. *For sufficiently small* r_0 *and sufficiently large* K_0, *there are constants* α_0 *and* C *such that for any* $\alpha > \alpha_0$ *and* $v \in \mathfrak{K}$,

$$\iiint (\bar{L}v)^2 r^{3-2\alpha} \varphi(t)^{3\alpha} p^{-1} dr\, dO_1 dt \tag{2. 6}$$
$$\geqq C\alpha^3 \iiint v^2 r^{-2\alpha} \varphi(t)^{3\alpha} p^{-1} dr\, dO_1 dt\,,$$

where r_0, K_0 *and* C *depend only on the absolute value of derivatives of* $\bar{a}_{\sigma,\rho}$, *of* p *and of* q *with respect to* (t, r), (t, x) *and* r *of order* 1 *respectively, on the absolute value of* a_{ij} *and on the values of* β *and* γ. (In the following we denote such a constant also by C_i $(i=1, 2, \cdots, 8)$).

Proof. After substituting $z = v\gamma^{-\alpha}$ into (2. 1) the integral on the left-hand side of (2. 6) becomes, denoting the integral by A,

$$A \geqq \iiint \{(L^{(1)}z)^2 p^{-1} + (Lz^{(2)})^2 p^{-1} + 2Lz^{(1)} \cdot Lz^{(2)} p^{-1}\} r^{-1} \cdot \varphi(t)^{3\alpha} dr\, dO_1 dt\,,$$

where

$$Lz^{(1)} = \{\alpha(\alpha+n-2)z + Nz + r^2 z_{|r|r}\}\, p$$
$$Lz^{(2)} = (2\alpha+n-1) r z_{|r} \cdot p - qr^2 z_{|t}\,.$$

The right hand side of the above inequality is denoted by $\sum_{k=1}^{3} A_k$, where the A_k are defined in the obvious manner. We shall now reduce A_2, A_3 to simpler forms using integration by parts:

$$\begin{aligned} A_2 &= \iiint (Lz^{(2)})^2 r^{-1} p^{-1} \varphi^{3\alpha} dO_1 dr\, dt \\ &= (2\alpha+n-1)^2 \iiint r z_{|r}{}^2 p \varphi(t)^{3\alpha} dO_1 dr\, dt \\ &\quad + \iiint r^3 q^2 z_{|t}{}^2 p^{-1} \varphi^{3\alpha} dO_1 dr\, dt \\ &\quad - 2(2\alpha+n-1) \iiint r^2 z_{|r} q z_{|t} \varphi^{3\alpha} dO_1 dr\, dt\,. \end{aligned}$$

Furthermore

$$\begin{aligned} A_3 &= 2 \iiint Lz^{(1)} \cdot Lz^{(2)} p^{-1} r^{-1} \varphi^{3\alpha} dO_1 dr\, dt \\ &\geqq 2\alpha(\alpha+n-2)(2\alpha+n-1) \iiint z \cdot z_{|r} \cdot p\, dO_1 dr\, dt \\ &\quad + 2 \iiint \alpha(\alpha+n-2) z \cdot (-rqz_{|t}) \varphi^{3\alpha} dO_1 dr\, dt \\ &\quad + 2 \iiint (Nz + r^2 z_{|r|r})((2\alpha+n-1) z_{|r} p - rqz_{|t}) \varphi^{3\alpha} dO_1 dr\, dt \end{aligned}$$

$$
\begin{aligned}
&= \alpha(\alpha+n-2)(2\alpha+n-1)\iiint z^2(-p_{|r})\,dO_1\,dr\,dt \\
&\quad +\alpha(\alpha+n-2)\iiint r\cdot 3\alpha\varphi_{|t}\varphi^{-1}qz^2\varphi^{3\alpha}\,dO_1\,dr\,dt \\
&\quad +2\iiint Nz\{(2\alpha+n-1)z_{|r}p-rqz_{|t})\}\varphi^{2\alpha}\,dO_1\,dr\,dt \\
&\quad +2\iiint r^2 z_{|r|r}(2\alpha+n-1)z_{|r}p\varphi^{3\alpha}\,dO_1\,dr\,dt \\
&\quad -2\iiint r^2 z_{|r|r}rqz_{|t}\varphi^{3\alpha}\,dO_1\,dr\,dt \\
&= \textstyle\sum_{k=1}^{5}\bar{A}_{3,k}\,,
\end{aligned}
$$

where the $\bar{A}_{3,k}$ are also defined in the obvious manner. The terms $\bar{A}_{3,4}$ and $\bar{A}_{3,5}$ are also calculated by using integration by parts:

$$
\bar{A}_{3,4} = -(2\alpha+n-1)\{2\iiint rz_{|r}{}^2 p\varphi^{3\alpha}\,dO_1\,dr\,dt+\iiint r^2 z_{|r}{}^2 p_{|r}\varphi^{3\alpha}\,dO_1\,dr\,dt\}\,,
$$

and

$$
\begin{aligned}
\bar{A}_{3,5} &= -2\iiint r^3 z_{|r|r}qz_{|t}\varphi^{3\alpha}\,dO_1\,dr\,dt \\
&= 2\iiint r^3 z_{|r}z_{|r|t}q\varphi^{3\alpha}\,dO_1\,dr\,dt+6\iiint r^2 z_{|t}z_{|r}q\varphi^{3\alpha}\,dO_1\,dr\,dt \\
&\quad +2\iiint r^3 z_{|r}z_{|t}q_{|r}\varphi^{3\alpha}\,dO_1\,dr\,dt \\
&= -2\iiint r^3 z_{|r}{}^2 3\alpha q\varphi_{|t}\varphi^{-1}\varphi^{3\alpha}\,dO_1\,dr\,dt-2\iiint r^3 z_{|r|t}z_{|r}q\varphi^{3\alpha}\,dO_1\,dr\,dt \\
&\quad +6\iiint r^2 z_{|t}z_{|r}q\varphi^{3\alpha}\,dO_1\,dr\,dt+2\iiint r^3 z_{|r}z_{|t}q_{|r}\varphi^{3\alpha}\,dO_1\,dr\,dt \\
&= -2\iiint r^3 z_{|r}{}^2 3\alpha q\varphi_{|t}\varphi^{-1}\varphi^{3\alpha}\,dO_1\,dr\,dt+2\iiint r^3 z_{|t}z_{|r|r}q\varphi^{3\alpha}\,dO_1\,dr\,dt \\
&\quad +12\iiint r^2 z_{|t}z_{|r}q\varphi^{3\alpha}\,dO_1\,dr\,dt+4\iiint r^3 z_{|r}z_{|t}q_{|r}\varphi^{3\alpha}\,dO_1\,dr\,dt\,,
\end{aligned}
$$

therefore

$$
\begin{aligned}
\bar{A}_{3,5} &= 6\iiint r^2 z_{|t}z_{|r}q\varphi^{3\alpha}\,dO_1\,dr\,dt-3\iiint \alpha r^3 z_{|r}q^2\varphi_{|t}\varphi^{-1}\varphi^{3\alpha}\,dO_1\,dr\,dt \\
&\quad +2\iiint r^3 z_{|r}z_{|t}q_{|r}\varphi^{3\alpha}\,dO_1\,dr\,dt\,.
\end{aligned}
$$

By combining the previous equalities and inequalities we see that

$$(2.7)\quad A \geqq \iiint \{\alpha((\alpha+n-2)(2\alpha+n-1)(-p_{|r})+3\alpha^2(\alpha+n-2)rq\varphi_{|t}\varphi^{-1}\} z^2\varphi^{3\alpha}dO_1\,dr\,dt$$
$$+\iiint \{(2\alpha+n-1)^2rp-2(2\alpha+n-1)rp+r^2p_{|r}-3\alpha r^3q\varphi_{|t}\varphi^{-1}\} z_{|r}{}^2\varphi^{3\alpha}dO_1\,dr\,dt$$
$$+\iiint r^3q^2z_{|t}{}^2p^{-1}\varphi^{3\alpha}dO_1\,dr\,dt$$
$$+2\iiint (3r^2-(2\alpha+n-1)r^2)+r^3q_{|r}q^{-1}\}\,qz_{|t}\cdot z_{|r}\varphi(t)^{3\alpha}dO_1\,dr\,dt$$
$$+\iiint [2Nz\cdot\{(2\alpha+n-1)z_{|r}p-rqz_{|t}\}]\varphi^{3\alpha}dO_1\,dr\,dt$$

Here we remark that since $r<r_0\wedge K_0^{-1}t\wedge K_0^{-1}(1-t)$,

$$(2.8)\qquad |r\varphi_{|t}\varphi^{-1}|\leqq (K_0^{-1}\vee r_0)k \qquad \text{in } D_{r_0,K_0},$$

where k depends only on $\varphi(t)$. Then from (2.4) and (2.8) we obtain the estimate of the first term of the right hand side of (2.7): the first term, which we denote by I, is

$$(2.9)\qquad I\geqq \alpha^3C_1\iiint z^2\cdot p^{-1}\varphi^{3\alpha}dO_1\,dr\,dt,$$

for sufficiently large α and K_0 and for sufficiently small r_0. 'In order to estimate the last term we remark that

$$2\iint Nz\cdot z_{|r}p\,dO_1\,dr = -2\iint z_{|\rho}\cdot\bar{a}_{\rho,\sigma}(z_{|r}p)_{|\sigma}dO_1\,dr$$
$$=\iint z_{|\rho}(p\bar{a}_{\rho,\sigma})_{|r}z_{|\sigma}p\,dO_1\,dr$$
$$-2\iint z_{|\rho}\bar{a}_{\rho,\sigma}\cdot z_{|r}p_{|\sigma}dO_1\,dr.$$

Hence we obtain that the fifth term, which we denote by V is,

$$V\geqq (2\alpha+n-1)\iiint z_{|\rho}(p\bar{a}_{\rho,\sigma})_{|r}\cdot z_{|\sigma}\varphi^{3\alpha}dO_1\,dr\,dt$$
$$+\iiint z_{|\rho}\bar{a}_{\rho,\sigma}\{-2(\alpha+n-1)z_{|r}p_{|\sigma}-rz_{|\sigma}\cdot 3\alpha\cdot q\varphi_{|t}\varphi^{-1}\}\,\varphi^{3\alpha}dO_1\,dr\,dt$$
$$-\iiint rz_{|\rho}\cdot\bar{a}_{\rho,\sigma|t}z_{|\sigma}\cdot q\varphi^{3\alpha}dO_1\,dr\,dt,$$

hence

$$(2.10)\qquad V\geqq \{(2\alpha+n-1)(\beta-\gamma)-3\alpha q(r_0\vee k_0^{-1})-r_0qC_2\}\,.$$
$$\cdot\iiint z_{|\rho}\bar{a}_{\rho,\sigma}z_{|\sigma}\varphi^{3\alpha}dO_1\,dr\,dt$$
$$-\alpha C_3\iiint r^2z_{|r}{}^2\varphi^{3\alpha}dO_1\,dr\,dt,$$

since $|\bar{a}_{\rho,\sigma}p_{|\rho}p_{|\sigma}| \leqq C_4 r$, where we use (2.3), (2.4), (2.8) and take α, K_0 sufficiently large and r_0 sufficiently small.

Thus from (2.7), (2.8), (2.9) and (2.10) we see that if the coefficient of the second term of (2.7), i.e. of the term containing $z_{|r}{}^2$ is larger than $[\{(2\alpha+n-1)-3-rq_{|r}q^{-1}\}^2 rp+\alpha C_3 r^2]$, then $A \geqq \alpha^3 \iiint z^2 p^{-1} \varphi^{3\alpha} dO_1 dr\, dt$. But this condition follows also from conditions ii) and (2.8). (Q.E.D.)

From Lemma 1 we obtain the following basic inequality.

Lemma 2. *Under the same assumption of Lemma* 1,

$$\iint (\bar{L}v)^2 r^{4-2\alpha-n} \varphi(t)^{3\alpha} dx\, dt$$
$$\geqq C \iint \left(\frac{\alpha^3}{r_0^3}|v|^2 + \frac{\alpha}{r_0} \textstyle\sum_{i=1}^n |v_{|x_i}|^2\right) r^{4-2\alpha-n} \varphi(t)^{3\alpha} dx\, dt\,.$$

Proof. From the relation

$$\bar{L}(v^2) = 2v\bar{L}(v) + 2a_{ij} v_{|x_i} v_{|x_j}\,, \qquad p > \gamma$$

and the positive definiteness of the matrices $((a_{ij}))$, we have

$$\begin{aligned} B &= C_5 \iint \textstyle\sum_{i=1}^n v_{|x_i}{}^2 r^{-2\alpha-n+4} p^{-1} \varphi(t)^{3\alpha} dx\, dt \\ &\leqq -\iint v\bar{L}(v) r^{-2\alpha-n+4} p^{-1} \varphi^{3\alpha} dx\, dt + \frac{1}{2} \iint \bar{L}(v) r^{-2\alpha-n+4} p^{-1} \varphi^{3\alpha} dx\, dt \\ &\leqq \{\iint v^2 r^{-2\alpha-n+4} p^{-1} \varphi^{3\alpha} dx\, dt\}^{1/2} \{\iint \bar{L}(v) r^{-2\alpha-n+4} p^{-1} \varphi^{3\alpha} dx\, dt\}^{1/2} \\ &\quad + 2^{-1}(2\alpha+2n-5)(2\alpha+3n-5) \iint v^2 r^{-2\alpha-n+2} \varphi^{3\alpha} dx\, dt \\ &\quad + 2^{-1} \iint v^2 r^{-2\alpha-n+4} \{3p^{-1}\varphi_{|t}\varphi + (p^{-1})_{|t}\}\, q\varphi^{3\alpha} dx\, dt\,, \end{aligned}$$

since

$$\begin{aligned} \bar{L}^*(p^{-1} r^{-2\alpha-n+4} \varphi^{3\alpha}) &= (2\alpha+2n-5)(2\alpha+3n-5) r^{-2\alpha-n+2} \varphi^{3\alpha} \\ &\quad + q(p^{-1}\varphi^{3\alpha})_{|t} r^{-2\alpha-n+4}\,. \end{aligned}$$

Therefore from ii) and iii) using Lemma 1 repeatedly we see that

$$\begin{aligned} B &\leqq \left(\frac{r_0^3}{C\alpha^3}\right)^{1/2} \iint \bar{L}(v)^2 r^{-2\alpha-n+4} \varphi(t)^{3\alpha} dx\, dt \\ &\quad + \{2^{-1}(2\alpha+2n-5)(2\alpha+3n-5) + C_6 \alpha r_0 (r_0 \vee k_0^{-1}) + C_7 r_0^2\} \cdot \\ &\quad \cdot \iint v^2 r^{-2\alpha-n+2} \varphi^{3\alpha} dx\, dt \\ &\leqq \left[\left(\frac{r_0^3}{C\alpha^3}\right)^{1/2} + \frac{r_0 C_8 \{\alpha^2 + \alpha r_0 (r_0 \vee k_0^{-1}) + r_0^2\}}{C\alpha^3}\right] \iint \bar{L}(v)^2 r^{-2\alpha-n+4} \varphi(t)^{3\alpha} dx\, dt\,, \end{aligned}$$

from which we have the desired inequality.

3. The proof of Theorem.

By certain coordinate transformations of R_{n+1} we can reduce the general operator (1. 2) to (2. 1) with the conditions i), ii) and iii) (in particular the boundedness of $\bar{a}_{\sigma,\rho|t}$). But in order to obtain such a coordinate transformation, the existence of the derivatives of a_{ij} of order 3 such as $a_{ij|x_k|x_l|x_m}$, and $a_{ij|x_k|x_l|t}$ will be required, if we use the geodesic differential equation. To avoid this we use the transformations considered by Cordes[3)], where the constants r_0, K_0, C in §2 depend only on the absolute values of functions of (1. 4) on a certain small domains and on the numbers α_1, α_2 in §1. (See p. 386) Then it is not difficult to show that our theorem is reduced to the following Lemma 3. For from Lemma 3 we see that under the condition of Theorem u vanishes identically in a (small) lens-shaped region with axis $\mathfrak{C}$ and such regions cover the horizontal component mentioned in Theorem.

Lemma 3. *Let $\bar{L}$ be the operator with the conditions* i) ii) iii) *in* §2 *on the domain* $D=\{(t, x)\,|\,t\in[-\varepsilon, 1+\varepsilon],\ |x|\leqq R\}$. *If* $u(\in C_x^2(D)\cap C_t^1(D))$ *satisfies conditions* (1. 5) *and* (1. 6) *with respect to* $\mathfrak{C}=[-\varepsilon, 1+\varepsilon]\times\{0\}$, *then u vanishes identically in the small domain* $\{(t, x)\,|\,t\in\left[\frac{2}{5}, \frac{3}{5}\right],$ $|x|\leqq r_0<R\}$ *for a sufficiently small* r_0.

Proof. Let $\rho(r)$ and $\sigma(t)$ be the smooth functions such that $\rho(r)=1$ for any $r\in\left[0, \frac{3}{4}\right]$, $=0$ for $r\in\left[\frac{4}{5}, \infty\right]$ and $|\rho(r)|\leqq 1$ for any $r\geqq 0$, and such that $|\sigma(t)-\sigma_0(t)|\leqq\delta\sigma_0(t)$ for $t\in[0, 1]$ and for a sufficiently small δ, where $\sigma_0(t)$ is the function such that

$$\begin{aligned}\sigma_0(t) &= K_0^{-1}t && \text{for } t\in[0, K_0r_0],\\ &= r_0 && \text{for } t\in[K_0r_0, 1-K_0r_0],\\ &= K_0^{-1}(1-t) && \text{for } t\in[1-K_0r_0, 1].\end{aligned}$$

Now let $v=u(t, x)\cdot\rho(r\cdot\sigma(t)^{-1})$. Then from (1. 5) and (1. 6) with $\mathfrak{C}=[-\varepsilon, 1+\varepsilon]\times\{0\}$, we see that $v\in\mathfrak{K}$. Therefore from Lemma 2 we have

$$\begin{aligned}&\alpha C\iint_{D_{K_0, r_0}}\{|v|^2+\sum|v_{|x_i}|^2\}r^{-2\alpha-n+4}\varphi^{3\alpha}\,dx\,dt\\ &\leqq\iint_{D_{K_0, r_0}}\bar{L}(v)^2r^{-2\alpha-n+4}\varphi^{3\alpha}\,dx\,dt\\ &\leqq\iint_{D_{2K_0, r_0/2}}\bar{L}(u)^2r^{-2\alpha-n+4}\varphi^{3\alpha}\,dx\,dt+\iint_{D_{K_0, r_0}-D_{2K_0, r_0/2}}\bar{L}(v)^2r^{-2\alpha-n+4}\varphi^{3\alpha}\,dx\,dt\,.\end{aligned}$$

Accordingly from (1. 5) we see that for sufficiently large

$$\alpha > \frac{2M}{K_0} \vee \alpha_0 \vee (n-4) \quad \text{and} \quad K_0 r_0 \leqq \frac{1}{5},$$

$$\alpha \frac{C}{2}\left(\frac{r_0}{3}\right)^{-2\alpha-n+4} \iint_{|x| \leq r_0/3,\, t \in [2/5,\, 3/5]} |u|^2 dx\, dt$$

$$\leqq \alpha \frac{C}{2} \iint_{D_{2K_0,\, r_0/2}} |u|^2 r^{-2\alpha-n+4} \varphi^{3\alpha} dx\, dt$$

$$\leqq \left(\frac{r_0}{2}\right)^{-2\alpha-n+4} \iint_{D_{K_0,\, r_0} - D_{2K_0,\, r_0/2},\, [1/5,\, 4/5]} \bar{L}(v)^2 dx\, dt$$

$$+ \left(\frac{1}{K_0}\right)^{-2\alpha-n+4} \iint_{D_{K_0,\, r_0} - D_{2K_0,\, r_0/2},\, [0,\, 1/5] \cup [4/5,\, 1]} \bar{L}(v)^2 dx\, dt\,.$$

Then letting $\alpha \to \infty$, we see that $u \equiv 0$ for (t, x); $|x| \leqq r_0$, $t \in \left[\frac{2}{5}, \frac{3}{5}\right]$. Here we remark that the number 5 in the definition $\varphi(t)$, Lemma 1, 2 and 3 is chosen only for the convenience of descriptions, but r_0 depends on this number, therefore we obtain that $u \equiv 0$ for a small lens-shaped region surrounding the curve $\mathfrak{C}$. (Q.E.D.).

REMARK. If one is interested only in proving the uniqueness of solutions for Cauchy problem of (1. 1) with non characteristic initial surface S, we have only to replace first S by a strictly convex surface[7)] by the use of a smooth transformation with the t-coordinate fixed and then to apply the fact that the integral inequality of Lemma 2 with the integral domain $D = \{(t, x) \mid |t| \leqq 1,\ |x| \leqq r_0\}$ and with $\varphi(t) = 1$ is valid for any v with the condition (1. 6), vanishing on the boundary of D. I have learned this method of consideration from Prof. M. Nagumo early in my investigation.

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Added in proof. We see in proof that in our theorem the conditions with respect to $a_{ij|x_l|t}$ in (1. 4) can be removed. In fact, let $a_{ij}(t, 0) = \delta_{ij}$, then without using the coordinate transformation of the unit sphere, applying only the transformation with respect to the distance from the origin, we obtain by the same method used above the inequality (2. 6) and therefore the inequality in Lemma 2. To prove (2. 6) we first replace $L^{(1)}z$ and $L^{(2)}z$ as follows: setting

$$b_\sigma = a_{ij}(t, x)x_i \frac{\partial\theta_\sigma}{\partial x_j} \Big/ a_{ij}(t, x)\frac{x_i x_j}{r^2}$$

$$L^{(1)}z = \{\alpha(\alpha+n-2)z+Nz+r(rz_{|r})_{|r}+r(b_\sigma z_{|\sigma})_{|r}+\lambda^{-1}(\lambda b_\sigma rz_{|r})_{|\sigma}\}\cdot p$$
$$L^{(2)}z = (2\alpha+n-2)(rz_{|r}+b_\sigma z_{|\sigma})\cdot p-q\cdot r^2 z_{|t}\,,$$

and then calculate as above replacing $(rz_{|r})$ in $L^{(2)}z(A_2)$ by $(rz_{|r}+b_\sigma z_{|\sigma})$ and considering the following conditions: for a fixed polar coordinate system

$$|b_\sigma|,\quad |b_{\sigma|\rho}|\leqq\gamma r\,,\quad |b_{\sigma|r}|,\ |b_{\sigma|t}|\leqq\gamma \qquad \text{in} \quad D_{r_0, K_0}\,.$$

Finally we remark that for calculations it is convenient to consider the transformation $r=e^{-s}(s\to\infty)$.