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Osaka University
Potential Theory and its Applications, I.

By Zenjiro Kuramochi

Preface

The Riemann surface device as an instrument to study multiform functions of a complex variable in the $z$-plane, since it was defined strictly, has been one of the main subject in the study of function theory. The structure of the abstract Riemann surface has been approached chiefly from two standpoints, i.e. the topological and the metrical. But the latter is more complicated and it follows from this that the Riemann surface can be classified as zero or positive-boundary Riemann surface and so on.

From the theory of automorphic function, it is well known that there exists a one-valued meromorphic function on any given Riemann surface. It is quite natural to generalize the theorems obtained in the case when the domain of the function is the $z$-plane on an abstract Riemann surface.

In Chapter I, we discuss the topology of an abstract Riemann surface as done by Stoilow and the conformal mapping of the Riemann surface onto the unit-circle. In Chapter II, we study the behaviour of a harmonic function or meromorphic function in the neighbourhood of ideal boundary points of harmonic measure zero. Chapter III is concerned with Green function, especially, with Green function with its pole at an ideal boundary point. In Chapter VI the potential theory on the Riemann surface is discussed. The theorem of G. C. Evans in the potential theory is the most useful in the theory of the function of the $z$-plane. We generalize this theorem for an abstract Riemann surface under certain hypothesis, which means that the ideal boundary point is simple in a sense. The remainder of this paper is concerned with applications of G. C. Evans' theorem to the theory of function.

Chapter I.

Abstract Riemann surface.

1. Riemann surface. When a two-dimensional and orientable Hausdorff space satisfies the following conditions, it will be called a

---

1) The number indicates the reference at the end of this paper.
Riemann surface $F$.

1° $F$ is covered by at most enumerable number of discs $K_\nu$, which is mapped conformally and in a one-to-one manner on a circle of the $z$-plane by means of a local parameter.

2° When two discs $K_\mu$ and $K_\nu$ have the common part which is cut from $K_\mu$ and $K_\nu$ by analytic curves $\alpha_\mu$, $\alpha_\nu$, ending in two intersection points of peripheries of $K_\mu$ and $K_\nu$, these two part can be mapped conformally and directly each other.

3° For any point of $K_\mu$ there is another disc $K_\nu$ containing the point and between them correlating relation of 2° is defined.

4° Any disc has common points with only finite number of neighbourhood discs.

5° Any two discs of $F$ can be connected through a chain of a finite number of discs with the correlating of 2°.

Whenever any set of an infinite sequence of points in $F$ has at least a limit point, then $F$ is called a compact surface or closed surface, in other words if and only if $F$ is covered by a finite number of discs.

2. Exhausion. $F_t$ is a part of Riemann surface composed of a finite number of discs such that

$$F_0 \subset F_1 \subset F_2 \ldots \lim_{i \to \infty} F_i = F.$$  

The sequence of $F_i$ is an exhaustion. The boundary of $F_0$ will be denoted by $\Gamma_0$, and $F_t$ has a boundary composed of a finite number of closed analytic curves denoted by $\sum \Gamma_i = \Gamma'$.

We indicate the bounded harmonic function with the boundary values 1 on $\Gamma_i$ and 0 on $\Gamma_0$ defined in $F_i - F_0$, by $\omega_i(x, F_i - F_0); x \in F_i - F_0$. This $\omega_i$ is monotonously decreasing with $i$.

If $\lim \omega_i(x, F_i - F_0) = 0$, then $F$ is called a zero-boundary Riemann surface, otherwise $F$ is called a positive-boundary, this classification does not depend on the choice of the exhaustion of $F$.

More generally, let $F'$ be a part of Riemann surface with a relative boundary $\Gamma'$ and $F'_i (i = 1, 2, \ldots)$ be an exhaustion of $F'$. Let us denote by $\omega_i(x, F'_i, \Gamma')$ the bounded harmonic function in $F'_i$ with the boundary values 1 on $\Gamma_i$ and 0 on $\Gamma'$. If $\lim \omega_i(x, F'_i, \Gamma') = 0$, we call that $F'$ has a relative zero-boundary.

3. Boundary of Riemann surface. The boundary of $F_t$ is made up of a finite number of analytic curves $\Gamma_{n_1}, n_2, \ldots n_i : n_i \leq n_i : i = 1, 2, \ldots$, which are closed and have no common point each other. When a curve $\Gamma_{n_1}, n_2, \ldots n_i$ cuts $F$ completely into two parts, then $\Gamma_{n_1}, n_2, \ldots n_i$ will be named a proper cut. We assume that all of $\Gamma_{n_1}, n_2, \ldots n_i$ are proper cuts.
Ideal boundary. One of the two parts cut by \( V_{n_1}, n_2, \ldots, n_i \), not containing \( F_0 \) is denoted by \( V_{n_1}, n_2, \ldots, n_i \), if \( V_{n_1}, n_2, \ldots, n_i \supset V_{n_1}, n_2, \ldots, n_{n_i+1} \), if \( \bigcap_{i=n}^\infty V_{n_1}, n_2, \ldots, n_i \) has no common point in \( F \), then we say that the sequence of \( V_{n_1}, n_2, \ldots, n_i \) determines an ideal boundary point \( \alpha \), and the sequence is called a determining sequence of \( \alpha \), and \( V_{n_1}, n_2, \ldots, n_i \) are called a system of neighbourhood of \( \alpha \).

\[
\alpha = \{ V_{n_1}, V_{n_1, n_2}, \ldots \} = \{ n_1, n_2, \ldots \}
\]

Two determining sequences \( V_{n_1}, V_{n_1, n_2}, \ldots \) and \( V_{n_1}, V_{n_1, n_2}, \ldots \) are equivalent if and only if there exists for any \( V_{n_1}, n_2, \ldots, n_i \), a certain \( V_{n_1}, n_2, \ldots, n_j \) such as \( V_{n_1}, n_2, \ldots, n_i \subset V_{n_1}, n_2, \ldots, n_j \), and vice versa.

We say that two equivalent sequences determine the same ideal boundary point. In the sequel we use for simplicity \( V_{n_1}, n_2, \ldots, n_i \) defined by the exhaustion as the neighbourhood system \( V_{n_1}, n_2, \ldots, n_i \).

Let \( a_1, a_2, \ldots, a_i \in F \) and \( \alpha = (n_1, n_2, \ldots) \) be a sequence of points of \( F \) and an ideal boundary point. If for any given \( i \) there is a number \( j_0 \) such that

\[
a_j \in V_{n_1, \ldots, n_i}, \quad (j \geq j_0),
\]
then we say that the sequence of \( a_i \) converges to \( \alpha \). It is clear that any subsequence of \( a_i \) converges also to \( \alpha \).

We say that all boundary points constitute a boundary point set \( R \). From the definition, if infinitely many points \( a_i \) of \( F \) have no limit point in \( F \), then \( a_i \) converges to \( R \), and there exists at least a point \( \alpha = (n_1, n_2, \ldots) \) and the subsequence such as \( \lim a_{n_i} = \alpha \).

Limit of the ideal boundary points: Let

\[
\alpha^p = (V_{n_1}^p, V_{n_1, n_2}^p, \ldots)
\]

\[
\alpha = (V_{n_1}, V_{n_1, n_2}, \ldots)
\]

be ideal boundary points, if for any given \( i \), there exist \( p_0(i) \) such as for \( p > p_0(i) \) there holds \( V_{n_1}^p, \ldots, n_s \subset V_{n_1, n_2}^s \); where \( s \) depends on \( p \); \( s = s(p) \), then we say that \( \lim \alpha^p = \alpha \).

3. Theorem 1. In this topology the boundary set \( R \) is compact and closed.

Proof. Let

\[
\alpha_1 = (V_{n_1}^1, V_{n_1, n_2}^1, \ldots)
\]

\[
\alpha_2 = (V_{n_1}^2, V_{n_1, n_2}^2, \ldots)
\]

\[
\alpha_i = (V_{n_1}^i, V_{n_1, n_2}^i, \ldots)
\]
be an infinite number of ideal boundary points. Since there is only a finite number of $V_{n_1} : 0 \leq n_1 \leq n_0$, then there exists at least one $V^*_{n_1}$ such as $V^*_{n_1}$ contains an infinite subsequence $\{\alpha_1^2\}$ of $\{\alpha^2_1\}$, and there exists at least one $V^*_{n_1, n_2}$ such as $V^*_{n_1, n_2}$ contains an infinite subsequence $\{\alpha_1^1\}$ of $\{\alpha^1_1\}$. Thus we have

$$\alpha_1^1, \alpha_1^2 \quad \ldots \quad \in V^*_{n_1}$$

$$\alpha_2^1, \alpha_2^2 \quad \ldots \quad \in V^*_{n_1, n_2}$$

Put

$$\alpha^*_i = (V^*_{n_1}, V^*_{n_1, n_2}, \ldots)$$

It is clear that $\lim \alpha^*_i = \alpha^*_k$ and $\alpha^*_k \in \mathbb{R}$.

4. Let $F$ be a relative zero-boundary Riemann surface with a relative boundary $\Gamma$, and $\Gamma_{n_1, n_2}, \ldots n_i$ be the proper cuts defined in 3. $G_i = V_{n_1, n_i} - V_{n_1, n_i+1}$ is a tube with an open boundary curve $\Gamma_{n_1, \ldots n_i}$ and $k$ boundary curves $\Gamma_{n_1, \ldots n_i+1}; 1 \leq k < \infty$ contained in $\Gamma_{i+1}$ and has genus $g_i$.

In $G_i$ we make the conjugate loopcuts $\gamma_2, \gamma_2', \ldots, \gamma_{i'}, \gamma_{i'2}$ corresponding to $g_i$ and $\gamma_i$. Let us cut $G_i$ along $\gamma_2, \gamma_2' \ldots \gamma_{i'}, \gamma_{i2}$, then $G_i$ becomes a $\kappa + 2$ multiply connected domain $G_i'$, and take a point $p$ on $\gamma_1'$ and $q_j'; j = 1, 2 \ldots k$ on $\Gamma_{n_1, \ldots n_i}$ and connect $q_j'$ and $p$ with an analytic curve $q_jp$ for every $j$. After cutting $G_i'$ along them, $G_i'$ becomes a simply connected domain $G_i''$. In all $G_i$, if we construct a system of cuts, then $F$ becomes a simply connected domain denoted by $\bar{F}$ and $V_{n_1, \ldots n_i}$ becomes a simply connected domain $\bar{V}_{n_1, \ldots n_i}$ and further every boundary curve $\Gamma_{n_1, \ldots n_i}$ has only one intersecting point with the system of cuts.

We map $\bar{F}$ on to the unit-circle, making use of the universal covering surface $F^\infty$ of $F$. In this mapping the boundary $\Gamma$ of $F$ corresponds on the system of arcs with linear measure $2\pi$ on the periphery $|z| = 1$, and the ideal boundary point set will be transformed to the linear measure zero set on $|z| = 1$, and $F$ is mapped on to a system of equivalent fundamental domains. Let us take one of them which is the fundamental domain containing $z = z_0$, enclosed by the images of loopcuts and the images of cuts and denote it by $D_0$. In this mapping $\Gamma_{n_1, \ldots n_i}$ is transformed to the curve connecting two equivalent points. In the sequel we use the same notation with under-line for the image in $|z| < 1$ mapped conformally of the figure (point, curve, domain) in $F$.

In the fundamental domain, ideal boundary point $\alpha$ corresponds in one-to-one manner on the set $R$ on $|z| = 1$.

1) The image of curve $l$ passing $a_1, a_2, \ldots, \lim a_i = \alpha, a_i \in V_{n_1, \ldots n_i}$
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$-V_{n_1 \ldots n_t} F$ does not oscillate, otherwise, the image of $l$ converges on an arc $A$ of $|z| = 1$, then there is an arc $\Gamma^*$ of the image of $\Gamma$ in $A$, then there must be another fundamental domain $D^*$ with $\Gamma^*$, this contradicts that $a_i$ are contained in the former fundamental domain $D_0$. Thus $l$ converges to a certain point on $|z| = 1$. $\Gamma_{n_1} \ldots n_t$ cuts the fundamental domain into two parts, one of them containing $z = z_0$ and the other corresponds to $V_{n_1} \ldots n_t$ of $\alpha$.

2) If $\alpha_1 = \alpha_2$, then $\alpha_1 = \alpha_2$. Since $\alpha_1 = \alpha_2$, there is a pair of neighbourhoods such as $V'(\alpha_1) \cap V'(\alpha_2) = 0$ on $F$, then there is a Jordan curve $\alpha_1 \alpha_2$ connecting $\alpha_1$ and $\alpha_2$ on the boundary of $D_0$, accordingly there is at least an analytic curve to which a substitution $s$ corresponds, which must put $\Gamma$ of $D_0$ on an image which is one of equivalent system of $\Gamma$ on $|z| = 1$ between $\alpha_1$ and $\alpha_2$, this follows that $\alpha_1 = \alpha_2$. Therefore the set of ideal boundary points of Riemann surface corresponds in one-to-one manner on the set $R$ on the periphery of $|z| = 1$ of a certain fundamental domain.

We denote by $C$ the periphery $|z| = 1$, and by $\overline{D_0}$, the closure of $D_0$ then.

**Theorem 2.** $R = \overline{D_0} \setminus C - \Gamma$.

Proof. $R \subseteq \overline{D_0} \setminus C - \Gamma$ is clear, if $\overline{D_0} \setminus C - \Gamma \ni p$, then there exists a sequence of $a_i$ such as $\lim a_i = p$ ; $a_i \in D_0$ in the $z$-plane topology. In thinking the image $\alpha_i$ of $\alpha_i$ in $F$, which has no limit point in $F$, then they have a subsequence $a_{n_i}$ such as $\lim a_{n_i} = q \in R$. Let $z(q) = Q$, then $Q \in R$, $\lim a_{n_i} = Q \in R$ but $\lim a_{n_i} = p = \lim a_i$, therefore, we have $p = Q, \quad Q \in R$.

\[ \overline{D_0} \cap C - \Gamma = \overline{R} \]

Thus $R$ is closed in the $z$-plane topology and every point of $R$ is accessible in $F$, then all boundary of $D_0$ are accessible in $F$, then all boundary of $D_0$ are accessible and the set of ideal point $R$ is represented homeomorphically on $\overline{D_0} \setminus C - \Gamma$ and the system of neighbourhood of $\alpha$ can be defined on $D_0$ as their image.

$R$ lies on the periphery $|z| = 1$, and moreover the number of all fundamental domains is enumerable, then we have

**Corollary.** Let $F$ be a relative zero-boundary Riemann surface and if we map $F$ conformally onto $|z| < 1$, then the set of image of the ideal boundary point set is $F_0$ on $|z| = 1$. 
5. Smoothing process.

Theorem 3. (L. Sario.\textsuperscript{8}) If zero-boundary Riemann surface is divided into $F_0$ and other domains $F_1, F_2, \ldots, F_n$, where $F_i \cap F_j = 0, i \neq j$, and $F_0$ and $F_i$ have common boundary $\gamma_i$, and in each $F_i$ a harmonic function $u_i$ is defined and if
\[ \sum_{i=1}^{n} \int_{\gamma_i} \frac{\partial u_i}{\partial n} \, ds = 0 \]
is satisfied, then there exists a uniform harmonic function $f$ except at the singularity of $u_i$ in $F_i$ such that
\[ D_{F_i}(f-u_i) < \infty. \]
The proof of this theorem is shown in C. R. Paris (1949), p 229.

Chapter II.

6. The Behaviour of the harmonic function and analytic function in the neighbourhood of the harmonic measure zero-boundary.

Theorem 4. (R. Nevanlinna\textsuperscript{9}).

Let $F$ be a Riemann surface having the relative boundary $\Gamma$ and an ideal boundary set $R$ of harmonic measure zero. Let us denote by $dw = du + idv$ an uniform differential on $F$ with finite Dirichlet Integral over $F$. Then there exists a sequence of curves $\gamma_i = \sum_{j=1}^{\infty} \gamma_i(j); i = 1, 2, \ldots$ enclosing $R$ on which
\[ \lim_{i \to \infty} \int_{\gamma_i} |dw| = \lim_{i \to \infty} \varepsilon_i = 0. \]

i) if $u(x)$ is uniform and if we denote by $F_{\gamma_i}$ the non compact domain bounded by $\gamma_i$, then
\[ \min_{x \in \gamma_i} u(x) = \lim_{x \to F_{\gamma_i}} u(x) \leq \lim_{x \to F_{\gamma_i}} u(x) \leq \max_{x \in \gamma_i} u(x). \]

ii) if $u(x)$ is uniform and $\lim_{x \to F_{\gamma_i}} u(x) < \infty$, then $Du(x) < \infty$.

iii) if $\lim_{x \to F_{\gamma_i}} |u(x)| < \infty$ and uniform, then for the first kind of Stoiilow's ideal boundary point $p$ of $R$
\[ \lim_{x \to p} u(x) \]
exists, but if $p$ is of the second kind, then $\lim_{x \to p} u(x)$ does not necessarily exist.

Proof. If $p$ is of the first kind,\textsuperscript{3} then there exists a neighbourhood $V(p)$ of $p$ which is planer, therefore every $\bar{\gamma}_i$ which is the part of $\gamma_i$ of
Theorem 4 contained in $V(p)$ is made of only a proper cut $i_o \geq i(V(p))$.

If we denote by $V'(p)$ the neighbourhood of $p$ cut by $\gamma_i$ and contained in $V(p)$, then

$$\min_{x \in \gamma_i} u(x) \leq \lim_{x \to V'(p)} u(x) \leq \lim_{x \to V(p)} u(x) \geq \max_{x \in \gamma_i} u(x)$$

but $|\max_{x \in \gamma_i} u(x) - \min_{x \in \gamma_i} u(x)| \leq \int |d\nu| = \varepsilon_i : \lim_{x \to V(p)} \varepsilon_i = 0$

hence, $\lim_{x \to V(p)} u(x)$ exists.

8. Example 1.

$$y^2 = \prod_{n=1}^{\infty} \left(1 - \frac{2^n}{2^{n+1}} \right) \left(1 - \frac{x}{2^{n+1}} \right) ; \quad n \geq 1$$

$I_n$ is defined afterward: $I_n = \text{real}$.

The Riemann surface $F^*$ of $y$ spread on the $x$-plane, composed of two sheets and have first order branch points $a_n = 2^n, b_n = 2^n + I_n$ on the real axis, and $x = \infty$ is the only singular point of the second kind ideal boundary point. We connect cross-wise the upper and lower sheets at the intervals $S_n = [b_n, a_n]$.

We denote by $F'$ the Riemann surface obtained after cutting two discs $|x| \leq 1$, from $F^*$, then $F'$ has two boundaries, $C_1, C_2$ on $|x| = 1$, and zero boundary. Denoting by $U(x)$ the bounded harmonic function with the boundary values 1 on $C_1$ and 0 on $C_2$. $F'$ has Green function (next chapter) denoted by $g(x, x'; c), g(x, x'; c)$, where $x', i = 1, 0$ means upper and lower sheets, and $x_j$ means the projection of $x$, then by Green's formula

$$u(x'^n) = \int_{c_1} \frac{\partial g}{\partial n}(x, x'^n) ds,$$

if $x \in S_n$, then

$$U(x'^1) = \frac{1}{2\pi} \int_{c_1} \frac{\partial}{\partial n} g(x, x'^1) ds = \frac{1}{2\pi} \int_{c_1} \frac{\partial}{\partial n} g(x, x'^1) ds = U(x'^0) = U(x).$$

$$\lim_{x \to S_n} U(x'^1) = \lim_{x \to S_n} U(x'^0).$$

$U(x'^1)$ is harmonic bounded in the $x$-plane out of $C_1$ and $\Sigma S_n$ with the boundary value 1 on $C_1$, $U(x)$ on $\Sigma S_n$, $U^2$ is bounded harmonic in the $x$-plane out of $C_2$ and with the boundary value 0 on $C_2$, $U(x)$ on $\Sigma S_n$.

$$\bar{U}(x) = U(x) - U(x')$$

is bounded harmonic in the $x$-plane except $C$ and $\Sigma S_n$ with the boundary value 1 on $C$, 0 on $\Sigma S_n$. 
By \( z = \frac{1}{\bar{w}} \), we inverse, then \( \bar{U}(z) = \bar{U}(z) \), \( \bar{U}(z) \) is harmonic bounded in \( |z| \leq 1 \).

\[
U(z) = 1: \quad |z| = 1 \\
U(z) = 0: \quad z \in S'_n,
\]

where \( \frac{1}{2^n} = \frac{1}{a_n} = a'_n, \frac{1}{b_n} = b'_n; S'_n = [a'_n, b'_n] \); where \( b'_n \) is defined by the next equation, where \( C \) is a constant such as \( 0 < C < \frac{1}{2} \).

\[
(1 + a'_n b'_n) - \sqrt{(1 - a'^2_n)(1 - b'^2_n)} \quad \frac{a'_n + b'_n}{\sqrt{b'_n/1 - a'_n + a'_n/1 - b'_n}}
\]

Let \( V(z) = 1 - \bar{U}(z) \), then \( 0 \leq V(z) \leq 1 \).

\[
V(z) = 0: \quad |z| = 1 \quad V(z) = 1: z \in \sum S'_n.
\]

We shall show that \( \lim_{x \to 0} V(0) < 1 \).

We denote by \( \omega_n(z) \) the harmonic function, \( 0 \leq \omega_n(z) \leq 1 \), in \( |z| < 1 \) 

\(- \sum S'_n \) such that

\[
\omega_n(z) = 0: z \in C \\
\omega_n(z) = 1: z \in S'_n,
\]

then \( V(z) \leq \sum \omega_n(z), V(0) \leq \sum \omega_n(0) \). We map \( |z| < 1 \) on to \( |w| < 1 \) by

\[
\frac{2 - p_n}{1 - p_n z} = w,
\]

where \( p_n = \frac{1 + a'_n b'_n - \sqrt{(1 - a'^2_n)(1 - b'^2_n)}}{a'_n + b'_n} \)

then

\[
b'_n \to \beta_n: \quad -1 < \beta_n = \frac{\sqrt{1 - b'^2_n} - \sqrt{1 - a'^2_n}}{a'_n + b'_n} < 0 \\
a'_n \to \alpha_n: \quad 0 < \alpha_n = \frac{\sqrt{1 - a'^2_n} - \sqrt{1 - b'^2_n}}{b'_n/1 - a'_n + a'_n/1 - b'_n} < 1 \\
\to 0' = 0 = \frac{-1 - a'_n b'_n + \sqrt{(1 - a'^2_n)(1 - b'^2_n)}}{a'_n + b'_n}.
\]

\( S'_n = [b'_n, a'_n] \to T_n = [\beta_n, \alpha_n] \) on real axis. Denoting by \( \omega_n(w) \) the function which is harmonic and bounded in \( |w| < 1 \). \( 0 \leq \omega_n \leq 1; \omega_n(w) = 1: w \in \) on the circle of which diameter is \( |\alpha_n| = |\beta_n| \), and \( \omega_n(w) = 0; |w| = 1 \), then

\[
\omega_n(w) = \frac{-\log |w|}{\log m_n}: m_n = |\alpha_n| = |\beta_n|.
\]

\( \omega_n(w) > \omega_n(w), \) then \( C(\frac{1}{2^n}) = \omega_n(0') > \omega_n(0'), \)

\( \sum \omega_n(0') > \sum \omega_n(0) > V(0), \)
but \( \omega_n(p') = C \left( \frac{1}{2} \right)^n \) : \( C < \frac{1}{2} \), therefore \( V(0) < 2C < 1 \), finally \( 1 - V(0) = \overline{U}(\infty) > 0 \), then \( \lim_{x \to \infty} |U(x) - U(x^2)| > 0 \). Accordingly \( U(x) \) has no limit when \( x \) converges to \( \infty \) on \( F - \sum S_n \). In reality \( x = \infty \) is irregular for Dirichlet problem, and

\[
I_n = \frac{2 \times 2^n \left( \frac{1}{2} \right)^n}{1 - 3K_n} = \frac{C}{2}, \quad \text{where} \quad K_n = \left( \frac{1}{2} \right)^n C.
\]

9. Theorem 6. Generalization of the identity theorem. Let \( F \) be a Riemann surface with relative boundary \( \Gamma \) and an ideal boundary \( R \) of a relative harmonic measure zero. If \( f(x) \) is in \( F \) a regular bounded, and non constant function, then for every ideal boundary point \( p \in R \)

1. \( \lim_{x \to p} f(x) \) exists.

2. \( f(x) \) is continuous in \( F + R \).

3. For every constant \( C \), the number of roots of the equation \( f(x) = C \) in \( F + R \) is uniformly bounded.

Proof. Let us denote by front \( A, \) int \( A \) and \( \bar{A}, \) the boundary, interior point and the closure of set \( A. \)

Lemma 1. If \( F \) is the Riemann surface satisfying the conditions of Theorem 6, then front \( (f(F')) - f(\Gamma') \) is a set of the logarithmic capacity zero.

Since \( f(F'): x \in F \) is continuous and bounded, then

\[
\text{front } f(F') + \text{int } f(F') = f(F') \subseteq f(\bar{F}) = f(\bar{F} - R - \Gamma) + f(R \cap \bar{F}) + f(\Gamma).
\]

Since \( f(x) \) is regular, if \( p \in F, \) then \( f(p) \in \text{int } f(F), \) therefore,

\[
\text{front } f(F') \subseteq f(R \cap \bar{F}) + f(\Gamma') \subseteq f(\bar{F}).
\]

Let \( E = \text{front } f(F') - f(\Gamma') \) then, \( E \cap f(F') = 0, E \cap f(\Gamma') = 0, \) and \( f(\Gamma') \) is closed.

We suppose that \( \text{Cap } E > 0, \) we denote by \( E_m \) the set of \( E \) having distance larger than \( \frac{1}{m} \) from \( f(\Gamma') \), then

\[
E = \sum_{n=1}^{\infty} E'_m,
\]

therefore there is a certain \( m_0 \) such as \( \text{Cap } (E_m) > 0 \), then there is at least one point \( w_0 \) of \( E_{m_0} \), such as for any small disc \( K \) of which is centre is \( w_0, \) \( \text{Cap } (E_{m_0} \cap K) > 0. \) therefore there exists a closed subset \( E'_{m_0} \) of \( E_m \) having no common point with the periphery of \( K \) and \( \text{Cap } (E_{m_0} \cap K) > 0, \) and \( \text{dia } K < \frac{1}{2m_0} \).

Hence \( E \in f(F') - f(\Gamma'), \) we take a connected piece on \( K \) which is
denoted by $F'$, and hence $\text{Cap}(E_{m_0} \setminus K) > 0$, there exists a bounded harmonic function such as $0 \leq U(w) \leq 1$.

$$\lim_{w \to m_0} U(w) = 1 \quad w \in (E_{m_0} \setminus K); \quad U(w) = 0 : w \in \text{boundary of } K$$

To $E_{m_0}$, a part of Riemann surface $F_{m_0}$ corresponds which have positive distance from $\Gamma$.

$$U(x) = U(f^{-1}(w)) ; U(x) \text{ is harmonic in } F_{m_0}.$$ 

We denote by $\omega_n(x)$ the harmonic function such as $0 \leq \omega_n(x) \leq 1$:

$$\omega_n(x) = 0 : x \in \Gamma, \quad \omega_n(x) = 1 : x \in \Gamma^n \text{ boundary of } F_n \text{ of exhaustion},$$

then $\omega_n(x) \geq U(x)$. But $R$ is harmonic measure zero set therefore

$$0 = \lim_{n \to \infty} \omega_n(x) \geq U(x) = 0,$$

this is absurd.

10. **Corollary.** Let us denote by $v(R)$ the non compact domain containing $R$ in its interior and bounded by the relative boundary $\gamma; \gamma = \sum_{i=1}^{n} \gamma_i$; in the $w$-plane denote by $D_{f}$ the maximal compact domain bounded by $f(\gamma)$. Then

$$\text{front } f(v) - f(\gamma) = E_1 \subset D_{f}.$$ 

We suppose that $E_1$ has at least one point in the exterior of $D_{f}$, it will be denoted by $p$, as $f(\gamma)$ is closed,

$$\text{dist. } (p, f(\gamma)) \geq \delta_0 > 0.$$ 

On the other hand there is at least an inner point $q$ of $f(V)$ such as $\text{dist } |p, q| < \frac{\delta_0}{4}$.

We can take a circular neighbourhood $v^*(q)$ of which the radius $\frac{\delta_0}{4}$ and composed of only inner point of $f(v)$, let us take a non compact and simply connected domain $G$ containing the point at infinity and $v^*(q)$ and denote its boundary by $\mathcal{B}$ satisfying $\text{dist. } (\mathcal{B}, f(\gamma)) \geq \frac{\delta_0}{8}$.

Hence $f(v)$ is compact, $\infty$ is exterior point of $f(v)$, dimension of $(G \cap f(v)) = 2$, but $(G \setminus \text{front } (f(v))) = \text{front } (G \cap f(v))$.

$$\dim \text{front } (G \setminus f(v)) = 1, \quad E_1 \ni (\text{front } (G \cap f(v)) - \mathcal{B})$$

but $\text{Cap } E_1 = 0$.

This is a contradiction.

Proof of 1. $D_{f}$ is a connected set, because $\gamma$ is connected by a curve in $v(R)$. Take a point $p \in R$, then there exists a sequence of curves $\gamma_n$ of theorem 4 on which

$$\lim_{n \to \infty} \int_{\gamma_n} |dw| = 0.$$ 

We denote by $V_n(p)$ the neighbourhood of $p$ determined by $\gamma_n$ and
denote its boundary by \( \bar{\gamma}_n \subseteq \gamma_n \).

If \( x \in V_{n+1}(p) - R \), then \( f(x) \) is regular, and \( D\nu_n \) is a connected set.

We have \( f(\bigcap V_{n+1}(p) - R) + E\nu_{n+1} \subseteq D\nu_n \).

\[ \text{diameter } D\nu_n \leq \frac{1}{2} \int |d\nu| = \frac{\varepsilon_n}{2} ; \lim_{n \to \infty} \varepsilon_n = 0 . \]

\( f(\bigcap V_n(x)) \subseteq \bigcap D\nu_n \) is only one point. Finally \( \lim_{x \to p} f(x) \) exists. It is clear that \( f(x) \) is continuous in \( F + R \).

11. Proof of 2. Take \( n \) points \( x_1, x_2, \ldots, x_n \) in the neighbourhood of \( \Gamma \) and denote by \( K_i \) the disc of which the centre is \( x_i \) and \( K_i \) contains \( x_{i-1}, x_{i+1} \) in its interior.

We connect \( x_i \) and \( x_{i+1} \) by a curve so that the image of the curve in the \( w \)-plane may be a straight line \( \lambda \), all straight lines \( \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n \) make up a polygon denoted by \( \pi \). Because, let \( K \), be a disc with centre \( p \) in \( F \) denote by \( K' \) the maximal disc contained in \( f(K) \), of which the centre is \( f(p) \). Then we can connect \( p \) and the other point of \( f^{-1}(K') \) with a curve of which the image in the \( w \)-plane is a straight line.

Let us denote by \( \Pi \) its outer polygon made of the outer side of \( \pi \), and the neighbourhood of \( R \) determined by \( f^{-1}(\pi) \) is denoted by \( V_{\pi_F} \) or \( F_{\pi_F}; \pi_F = f^{-1}(\pi) \).

12. Lemma 2. \( \Pi \setminus f(R \cap F_{\pi_F}) = 0 \).

Suppose \( \Pi \setminus f(R \cap F_{\pi_F}) \ni p \), we take a closed curves \( \pi'_F \) in the neighbourhood of \( \pi_F \). As \( F_{\pi_F} - V_{\pi'_F} \) is compact, therefore \( f^{-1}(p) \) is finite number of points \( x_1, \ldots, x_s \) in \( F_{\pi_F} - V_{\pi'_F} \), and take \( v_i(x_i) \) of neighbourhood in \( F - V_{\pi'_F} \), so that \( |f(x) - p| \geq \delta_0 > 0 \), if \( x \in \text{inte } (F_{\pi_F} - V_{\pi'_F}) - \sum v_i(x_i) \).

In \( F \) we construct a non compact domain containing \( R \) bounded by relative boundary \( \gamma^* \) contained in \( F_{\pi_F} - F_{\pi'_F} - \sum v_i(x_i) \) and denote by \( \Pi^* \) the maximal domain bounded by \( f(\gamma^*) \) in the \( w \)-plane, then dist \( (p, f(\gamma^*)) \) \( \geq \delta_0 ; \Pi \supset \Pi^* \), then \( p \in \text{exterior of } \Pi^* \), but \( p \in V_{\pi_F} \), this is a contradiction from corollary of Lemma 1.

We denote by \( \#(w) \) the number of times when \( w \) is covered by \( f(x) : x \in F_{\pi_F} \) and by \( D_n \) the set \( E[\#(w) \geq n] \), this is clearly open relative to \( \pi \).

Lemma 3. If \( E_n = \text{boundary of } (D_n - (\pi)) \) is not zero : \( n < \infty \),

then \( \text{Cap } E_n = 0 \).

Proof. Suppose that \( \text{Cap } E_n = 0 \), then the boundary \( E_n \) is the set of point which is covered by \( f(F_{\pi_F}) \) at most \( n-1 \) times.

We denote by \( S_i \) the set which is covered \( i \) times by \( f(F_{\pi_F}) \) exactly and denote by \( S_{im} \) the set of \( S_i \) which has a distance larger than \( \frac{1}{m} \) from \( \pi \).
therefore there exists at least one point $p_0$, and a number $m_0$ and $i_0$ such as $\text{Cap}(S_{i_0m_0} \cap K) > 0$ for any small disc $K$ of which the center is $p_0$, and the closed subset $S'_{i_0m_0}$ of $S_{i_0m_0}$ such as $(S'_{i_0m_0} \cap \text{Boundary of } K) = 0$, and $\text{Cap}(S_{i_0m_0} \cap K) > 0$.

Then there exist discs $K_1, K_2, ..., K_m$ on $p$, and there is another disc $K_0$ which does not cover a positive capacity set $S_{i_0m_0} \cap K = \text{far from } \pi$. Therefore there exists a non constant bounded harmonic function $1 \geq U(w) \geq 0$, such as

$$\lim U(w) = 1 : w \in S'_{i_0m_0} \cap K$$

$$U(w) = 0 : w \in \text{boundary of } K.$$  

This is a contradiction (see Lemma 1).

Since from Lemma, $f(R \cap F_\pi) \cap \Pi = 0$, and $f(R)$ is closed; accordingly $\text{dist}(f(R), \Pi) > \epsilon_0 > 0$. Take a point $q$ on $\Pi$, and denote by $V_{\frac{\epsilon_0}{2}}(q)$ a circular neighbourhood of $q$ of which radius is $\frac{\epsilon_0}{2}$.

If $(\text{inte } \Pi \cap V_{\frac{\epsilon_0}{2}}(q) - R) \ni w$, then $n(w) < \infty$.

Proof. Suppose $n(w_0) = \infty$, then there exists a sequenc of $x_1, x_2, ...$, such as $\lim x_i \in R$; $f(x) = w_0$, consequently we have, $f(R) \ni w_0 = \lim f(x_i) \in V_{\frac{\epsilon_0}{2}}(q)$. This is absurd.

13. Since $D_n$ is compact, therefore the outer boundary $\Pi_n$ of $D_n$ is a continuum contained in $\pi$. Assuming that $\Pi_n \ni q$, we take a point $p$ in the neighbourhood of $q$ such as $V(q) \ni p \in f(R)$, then $n(p) = n_0 < \infty$, accordingly $D_{n+1}$ does not exist in the neighbourhood of $q$, because if it were not so, $q \in D_{n+1}$. As $q \in f(R)$, if we deform $\pi_x$ into $\pi_{x'}$ in adding a small disc of which the centre is $f^{-1}(q)$, so that $q \in \text{int } \Pi'$. Thus $n(q) = n(p) = n_0$.

The complement of $\pi$ in the w-plane is composed of a non compact domain and a finite number of compact domains which have no common point and their boundaries are made of $\pi$, we denote by $\mathcal{D}(p)$ the compact domain of the complement of $\pi$ devided by $\pi$ and containing $p$, the boundary of $\mathcal{D}(p)$ is a subset of $\pi$. Then $\mathcal{D}(p) \cap D_{n+1} = 0$. Take another compact domain next to $\mathcal{D}(p)$, and denote by $S$ the common boundary of $\mathcal{D}(p)$ and $\pi$, and take a point $q$ on $S$. Then for any point $t$ in the neighbourhood $v(q)$ such as $t \in v(q) \cap \mathcal{D}(p)$, we deform $F_\pi$ a little into $F_\pi'$ so that $q$ and $t$ may be contained in $\mathcal{D}'(p)$; where $\mathcal{D}'$ is $\mathcal{D}(p)$ corresponding to $\pi'$, then $n'(p) = n'(q)$, where $n'$ means the times when
\(p\), and \(q\) are covered by \(f(F_\pi)\), consequently the difference of \(n(q)\) and \(n(p)\) by \(f(\pi)\) is at most \(m\) which is the number of times when \(q\) is covered by \(\pi\).

\[
\text{then } \mathcal{D} \setminus D_{a_{0,k}} = 0 : \ k \geq m_0.
\]

But the number of domains is finite, therefore \(n(w) < M\); for every \(w \in F_\pi\).

In reality \(n(w)\) is equal to the order of \(w\) with respect to \(\pi\).

Let \(f(R) \ni p\), then \(p\) may be the image of only \(M\) different ideal boundary points.

**Proof.** Suppose that \(p = f(Q_1) = f(Q_2) \ldots f(Q_{M+1}) \ldots : Q_t \in R\).

Take neighbourhoods \(v(Q_i)\) of \(Q_i\), each of them has no common point and their boundaries may be denoted by \(\gamma_i\), and by \(D_{\gamma_i}\) the maximal compact domain bounded by \(f(\gamma_i)\), then the set of \(D_{\gamma_i}\) which is not covered by \(f(v(Q_i))\) is capacity zero set.

\[
\bigcap_{i=1}^{\infty} f(v(Q_i)) = D,
\]

\(D\) is covered at least \(M + 1\) times except at most capacity zero set of \(D\). This is a contradiction.

Consequently the number of roots of the equation \(f(x) = C\) is \(\leq 2M\) in \(\bar{F}_\pi\).

14. **Corollary.**

i. \(|f(x)| < +\infty, x \in F + R,\) if the number of root of \(f(x) = C\) is infinitely many, then \(f(x) \equiv C\).

ii. Let \(f(x)\) be non constant analytic function in \(F\) and the number of roots is infinite for at least a value, then \(f(x)\) is not bounded and further \(f(x): x \in F\) covers almost all point of the \(w\)-plane except at most a non dense set for any small neighbourhood \(V(p)\) of \(p \in R\).

15. **Definition.** Generalized local parameter of which the center is an ideal point \(p\). If \(|f(x)| < \infty : x \in F + R\), then we can take a small neighbourhood \(v(p)\) of \(p\) so that \(f^{-1}(f(p))\) may have only \(p\) in \(v(p)\), then \(f(v(p))\) covers at most finitely many times and \(f(v(p))\) is compact then we call \(v(p)\) a generalized disc of \(p\) and \(f(x)\) a generalized local parameter.

Chapter III.

16. **Green function on the relative zero-boundary Riemann surface \(F\) and its generalization.**

Let \(p\) be an inner point of \(F\) which has harmonic measure zero set ideal boundary \(R\) and has relative boundary \(\Gamma\).

**Definition.** If \(G(x, p)\) is harmonic positive in \(F\) except only \(p\) where \(G(x, p)\) has logarithmic singularity and zero on \(\Gamma\) and its Dirichlet integral on \(F - V(p)\) is finite, then \(G(x, p)\) is called the Green function
of $F$ with pole at $p$.

**Theorem 7.** If we denote by $x = m(z)$ the mapping function of $F$ onto $|z| < 1$, then,

$$G(x, p) = \sum g(z, s(p)), \quad x = m(z),$$

where the summation is taken over all the substitutions of Fuchsoid group, and $g(z, s(p))$ is the Green function of $|z| < 1$ with pole at $s(p)$.

**Proof.** From $\text{mes} \sum s_i(\Gamma) = 2\pi$,

i) $\sum g(z, s(p))$ is harmonic in $|z| < 1$ except at $\sum s_i(p)$ hence

$$\int \frac{\partial}{\partial n} \sum_{i=1}^{n} g(z, s_i(p))ds = \sum_{i=1}^{n} \int \frac{\partial}{\partial n} g(z, s_i(p))ds =$$

$$= \sum_{i=1}^{n} \int \frac{\partial}{\partial n} g(z, s_i(p))ds = 2\pi - \epsilon < \sum_{i=1}^{n} \int \frac{\partial}{\partial n} g(z, s_i(p))ds = 2\pi.$$

$m(\Gamma) = \Gamma$

ii) Let us denote by $z_0$ a certain point in the circle $|z| < 1$, except $\sum s_i(p)$, then there exists at least a fundamental domain $D_0$ which has $z_0$ at its inner point $D_0$ or at its boundary.

Case 1. $z_0$ is in the interior of a fundamental domain denoted by $D_0$, which has the pole $p$ and denote by $\alpha$ and $\beta$ the two ends of $\Gamma_0$, corresponding to $D_0$, and connect with curve $C$, $z$ and $a$ and with $C_2$, $z$ and $s(\alpha)$, where $a$ and $s(\alpha)$ are equivalent and situated on the two arcs of $D_0$ which are nearest to $\Gamma_0$, so that the simply connected subdomain of $D_0$, bounded by, $\alpha a$, $C_1$, $C_2$ and $s(\alpha)\beta$ does not contain $p$. Then $g(z, p) + g(z, s(p))$ is invariant with respect to the substitution $s$. In denoting by $G_n$ the sum $\sum g(z, s_i(p)) + g(z, ss_i(p))$, this is also invariant with respect to $s$ and is harmonic in the circle except the poles $\sum (s_i(p) + ss_i(p))$. All terms of this series are positive and zero on $\sum s_i(\Gamma) + ss_i(\Gamma)$. $m(C_1 + C_2)$ in $F$ enclose with $\Gamma$ a compact domain $F_{D_0}$ and indicate by $\omega(z)$ the harmonic function on $F_{D_0}$ having the boundary values 1 at $m(C_1) + m(C_2)$ and 0 on $\Gamma$, $\omega(z) = \omega(m(z)) = \omega(z)$ is automorphic in $|z| < 1$, therefore by Harnack's theorem, there exists a constant $q$ depending continuously on $C_1 + C_2$ such as

$$\frac{1}{q} G_n(z_0, p) \leq G_n(z, p) \leq q G_n(z_0, p) : z, z_0 \in J_1 + C_2,$$

$$G_n(z, p) = G_n(s(z), p). \quad \frac{\partial}{\partial n} G_n(z)ds = \frac{\partial}{\partial n} G_n(s(z))ds(s).$$
By Green's formula we have

$$\frac{\partial}{\partial n} \omega(z) ds = \omega(s(z)) ds(s),$$

where integration is taken on $\Gamma_Q + C_1 + C_2 + s_0(a) \beta + a \alpha$

but

$$\int G(z) \frac{\partial \omega}{\partial n} ds = 0, \quad \int \omega \frac{\partial G_n}{\partial n} ds = 0,$$

on the other hand

$$\int \frac{\partial \omega}{\partial n} ds \geq 0,$$

then we have

$$\frac{1}{q} \int \frac{2\pi - \varepsilon_n}{\int \frac{\partial \omega}{\partial n} ds} \leq G_n(z_0) \leq \frac{2\pi - \varepsilon_n}{\int \frac{\partial \omega}{\partial n} ds}.$$

the last inequality holds for every $n$, therefore $\sum_{i=1}^{\infty} g(z, s_i(a))$ is absolutely bounded except $\sum s(p) + p$.

Case 2. $z_0 \in \text{boundary of a certain fundamental domain } D_o$, let us denote by $\delta_0$ the side containing $z_0$, then there exists at least a fundamental domain $D_1$ which has $\delta_0$ in common with $D_o$, then we do in the same way in $D_o + D_1$ as in $D_o$, in taking $D_o + D_1$ in the place of $D_o$.

Therefore $\sum g(z, s(p))$ is convergent in $|z| < 1$ except at $\sum s(p)$.

iii) $G(z, p) = \sum g(z, s(p))$ is evidently automorphic

$$G(z, p) = \sum g(z, s(p)) = 0, \text{ if } z \in \sum s(\Gamma).$$

If $z \in \sum s(\Gamma)$, then there exists at least a $s(\Gamma) \ni z$, and $D^o \ni s(\Gamma)$ therefore $G_n(z, p)$ is regular in $D_o$ except only $s^o(p)$, and clearly

$$G_n(z, p) = 0 \text{ if } z \in s^o(\Gamma), \text{ so } 0 = \lim_n G_n(z, p) = G(z, p).$$

Finally we have that $G(x, p) = G(m(x), p)$ is harmonic positive in $F$ and $0$ on $\Gamma$, and has logarithmic pole at $p$.

17. **Lemma 1.** Let $u(x)$ be harmonic and positive on the noncompact part $F$ of zero-boundary Riemann surface with a relative boundary $\Gamma$. 
If\[
\int_{\Gamma} \frac{\partial u}{\partial n} \, ds = 0, \quad \text{then} \quad D_f(u) < +\infty.
\]

Proof. If \(D_f(u) = \infty\), by Nevanlinna’s theorem \(u\) is not bound in \(F\), therefore there exists a sequence of points \(p_1, p_2, \ldots\), \(\lim p_i = \text{ideal boundary point}\) such as \(u(p_i) = M_i; \lim M_i = \infty\).

Take \(M_0 \geq \max u(z): \ z \in \Gamma\), and trace a niveau curve \(C_{M_0}\) on which \(u(x) = M_0\). \(C_{M_0}\) divides \(F\) into two parts \(C_{\overline{M_0}}\) and \(C_{M_0}\) in which \(u(x)\) is \(\equiv M_0\) respectively.

Proof. Case 1. Every \(C_{M_i}\) encloses the compact part with \(\Gamma\) and does not intersect with \(\Gamma\), hence

\[
0 = \int_{\Gamma} \frac{\partial u}{\partial n} \, ds = \int_{C_{M_i}} \frac{\partial u}{\partial n} \, ds, \quad 0 \geq \frac{\partial u}{\partial n} \quad \text{on} \quad C_{M_i},
\]

then

\[
\lim D_{C_{M_i}}(u) = \int_{C_{M_i}} \frac{\partial u}{\partial n} \, ds - \int_{C_{\overline{M_i}}} \frac{\partial u}{\partial n} \, ds = \int_{C_{M_i}} \frac{\partial u}{\partial n} \, ds < \infty.
\]

Case 2. \(\Gamma\) and \(C_{M_i}\) enclose non compact part.

Let be \(\gamma_1, \gamma_2, \ldots\) a sequence of curves enclosing the ideal boundary point and denote by \(\tilde{\gamma}_i\) the part of \(\gamma_i\) lying in \(C_{M_i}\), then it is evident that \(\Gamma + \tilde{\gamma}_i + C_{M_i}\) enclose the compact part \(C_{M_i}\) of \(C_{\overline{M_i}}\).

Denoting \(\tilde{\omega}_i\) the harmonic function in \(C_{M_i}\) having the boundary values 0 and \(\Gamma + C_{M_i}\) and 1 on \(\tilde{\gamma}_i\), and denote by \(\omega_j\) being harmonic and in \(F - V(\gamma_J)\), having the boundary values 0 on \(\Gamma\) and 1 on \(\gamma_J\), we see directly \(0 < \tilde{\omega}_i < \omega_j\) for every \(i, j\).

Since \(F\) has zero-boundary,

\[
0 = \lim_{j \to \infty} \omega_j \geq \lim_{j \to \infty} \tilde{\omega}_i = 0.
\]

Denote by \(u_j\) the harmonic function in \(C_{M_i}\) such as

\[
\begin{align*}
 u_j &= u & \text{on} \quad \Gamma + C_{M_i}, \\
 u_j &= M_i & \text{on} \quad \tilde{\gamma}_i, \\
 0 \leq u_j - u & \leq \tilde{\omega}_i M_i, & \text{for every} \quad i,
\end{align*}
\]

therefore \(u_j\) converges uniformly in \(C_{M_i}\); \(\lim u_j = u\), but from

\[
\lim_{j \to \infty} \frac{\partial u_j}{\partial n} = \frac{\partial}{\partial n} \lim_{j \to \infty} u_j = \frac{\partial u}{\partial n},
\]
\[
\int_{\Gamma} u \frac{\partial u}{\partial n} ds = \lim_{j \to \infty} \int_{\Gamma_j} u_j \frac{\partial u_j}{\partial n} ds = \lim_{j \to \infty} D_{C_{\Gamma_j}} (u_j) = \lim D_{C_{\Gamma_j}} (u),
\]
this inequality holds for every \( M_i \), then
\[
D_{\Gamma}(u) = \int_{\Gamma} u \frac{\partial u}{\partial n} ds < +\infty.
\]

18. **Lemma 2.** Let \( C \) be a proper cut dividing \( F \) into two parts; if the part of \( F \) bounded by \( R + C \) has no pole, then \( \int_{C} \frac{\partial G}{\partial n} ds = 0 \).

**Proof.** Let \( C' \) be the image in the fundamental domain in \( |z| < 1 \) and is ending in two points \( a \) and \( b \) and \( s(a) = b \); where \( s \) is a substitution.

Case 1. \( s \) is parabolic.
\[
\bar{C}' = \sum_{n=1}^{\infty} s^n(C') + s^{-n}(C') + C' \text{ ends in the fixed point of } s \text{ on } |z| = 1
\]
making a closed curve \( C' \) not enclosing \( p \) by hypothesis, then we have
\[
\sum_{s} s(C') = \sum_{s} s(C),
\]
where the summation on the left is over all substitutions and the right is over all except \( s \). The right side is the sum of closed curves not containing \( p \) in their interior, finally
\[
\int_{C'} \frac{\partial G}{\partial n} ds = \sum_{s(C)} \int_{C} \frac{\partial g(z, p)}{\partial n} ds = 0.
\]

Case 2. \( s \) is hyperbolic.
\[
\sum_{n=1}^{\infty} s^n(C') + s^{-n}(C') + C' = \bar{C}',
\]
is a Jordan curve ending in the two fixed points \( a \) and \( b \). \( \bar{C}' \) is transformed by other substitution into \( s(C') \), every \( s(C') \) has on its outerside the image of \( \Gamma' \), but \( \text{mes} \sum_{s} s(\Gamma') = 2\pi \), therefore \( \sum_{s} s(C') = \sum_{s} s(C) \) are sum of closed Jordan curves not containing \( p \) in their interiors, then we have as in the case 1:
\[
\sum_{s} \frac{\partial G}{\partial n} ds = 0.
\]

From Lemma 1 and 2 \( D_{F-V(p)}(x, p) < +\infty \).

19.1. **Green function with its pole on an ideal boundary point.**

**Definition.** Generalized module. Let \( \gamma_i \) be a proper cut, we define
a harmonic function \( \omega \) in the surface bounded by \( \gamma, \gamma' \) and \( \Gamma \) and has the boundary value 0 on \( \Gamma \), and 1 on \( \gamma' + \gamma \), when \( \gamma' \) converges to ideal boundary set \( R, \omega \) is decreasing monotonously, therefore \( \omega \) converges to \( \omega \) being non constant harmonic function

\[
\lim_{z \to s} \int_{\gamma} \frac{\partial \omega}{\partial n} ds = \int_{\Gamma} \frac{\partial}{\partial n} \lim_{z \to \infty} \omega \ ds > 0.
\]

Then \( \infty > \int \frac{\partial \omega}{\partial n} ds > 0 \) is called generalized module of the surface \( F \) bounded by \( I' \) and \( \gamma' \),

\[
D_{F - V(\gamma)}(\omega) = \int_{\gamma} \frac{\partial \omega}{\partial n} ds = \int_{\Gamma} \frac{\partial \omega}{\partial n} ds = \int_{I} \frac{\partial \omega}{\partial n} ds,
\]

and if \( F \geq F' \), then \( \omega \geq \omega' \), in putting \( N = \frac{1}{\int \frac{\partial \omega}{\partial n} ds} \), we have \( N_i \leq N_{i'} \).

**Definition.** Regular ideal boundary point. From Harnack’s theorem, for positive harmonic function \( u(x) \), there exists a constant \( q \) depending on the curve \( C \) in the defining domain such that

\[
\frac{1}{q} u(x_0) \leq u(x) \leq qu(x_0) : x, x_0 \in C.
\]

Let us denote \( q \) by \( q(C) \).

If for an ideal boundary point \( \alpha \), there exists a sequence \( \gamma_i \) of proper cuts enclosing \( \alpha \), on which every positive and finite except \( \alpha \) harmonic function must satisfy \( q(\gamma_i) \leq q \), then \( \alpha \) is named a regular ideal boundary point.

19.2. **Theorem 7’.** If \( \alpha \) is an ideal boundary point, then we can define a Green function \( G(x, \alpha) \), and further if \( \alpha \) is a regular point, then \( G(x, \alpha) \) is uniquely determined.

**Proof.** Take a sequence of point \( p_t \) in \( F \), such as \( p_t \in V_\alpha(\alpha) - V_{i+1}(\alpha) \): \( \lim_{i \to \infty} p_i = \alpha \) and sequence of Green function corresponding to \( p_i \)

\[
C(x, p_i), \ G(x, p_i) \ldots \ldots \ldots
\]

So long as \( p_i \) is contained in \( F \)

\[
G(x, p) = \sum g(x, s(p)).
\]

Therefore

\[
\int_{\Gamma} \frac{\partial G}{\partial n} ds = 2\pi
\]

and from Lemma 2, for every proper cut \( \gamma \) of which the domain bounded does not contain \( p \),
\[
\int_{\gamma} \frac{\partial G}{\partial n} \, ds = 0.
\]

If \( p \in V(\gamma) \) and \( p \in \sum V(\gamma_i) \), then \( G(x, p) \) is regular harmonic in the domain bounded by \( \Gamma + \gamma \) and \( \sum \gamma_i \), accordingly

\[
\int_{\Gamma + \gamma + \sum \gamma_i} \frac{\partial G}{\partial n} \, ds = 0.
\]

Finally

\[
\int_{\gamma} \frac{\partial G}{\partial n} \, ds = 2\pi.
\]

As \( G(x, p) \geq 0 \), there exists \( q \) for such that

\[
\frac{1}{q} G(x_0, p) \leq G(x, p) \leq q G(x_0, p); \quad x, x_0 \in \gamma,
\]

and if we denote by \( F_\gamma \) the non compact domain not containing \( p \) bounded by \( \gamma \), then

\[
\int_{\gamma} \frac{\partial G}{\partial n} \, ds = 0,
\]

accordingly

\[
D_{F - V(\gamma)} (G(x, p)) < \infty.
\]

Then

\[
\max_{x \in F} G(x, p) \geq \lim_{i \to \infty} G(x, p); \quad x \in F_\gamma,
\]

after all

\[
G(x, p_i) \leq M_i_0; \quad x \in F - V(\gamma_i_0), \quad V(\gamma') \ni p_i, i \geq i_0.
\]

We can extract a sequence of \( G(x, p_i) \) which converges uniformly in every compact domain contained in \( F \),

\[
\lim_{i \to \infty} G(x, p_i) = G(x, \alpha).
\]

Then the limit function \( G(x, \alpha) \) is clearly non constant and

\[
\int_{\gamma} \frac{\partial G(x, \alpha)}{\partial n} \, ds = 2\pi, \quad \text{because} \quad \lim_{i \to \infty} \frac{\partial G(x, p_i)}{\partial n} = \frac{\partial}{\partial n} \lim_{i \to \infty} G(x, p_i).
\]

We call \( G(x, \alpha) \) a Green function also.

20. The behaviour of \( G(x, \alpha) \) in the neighbourhood \( S \) of ideal boundary points.

Case 1. When \( x \) converges in the other boundary point \( \alpha' \), there exists a number \( j_0 \) a proper cut \( \gamma_j; j \geq j_0 \), such that the non compact part \( F_{\gamma_j} \) of \( F \) cut by \( \gamma_j \) do not contain \( p_i: i \geq i_0 \), hence by Lemma 1,
\[ \frac{\partial G}{\partial n} ds = 0, \text{ then} \]

\[ \max_{x \in \gamma_i} G(x, \alpha) \geq \lim_{x \in F \gamma_i} G(x, \alpha). \]

Case 2. We see directly that \( G(x, \alpha) \) is not bounded in \( V(\alpha) \), if \( \alpha \) is a regular ideal point there exist a sequence of \( p_i \) such as

\[ G(p_i, \alpha) = M_i, \quad p_i \in V(\alpha): M_i = \infty. \]

We make the curve \( C_{M_i} \) on which \( G(x, \alpha) = M_i \), these curves are composed of a finite number of closed curves or open curves tending to \( \alpha \) and each of them divides \( F \) into two parts in which \( G(x, \alpha) \leq M_i \) respectively. But in this case \( C_{M_i} \) does not tend to the ideal point \( \alpha \), if it were so then there exists a certain \( \gamma \) such as \( \gamma \) intersects the curve \( C_{M_i} \) where

\[ M_i < \frac{1}{q} N_{g,2\pi} \cdot \lim N_\gamma = \infty. \]

This is a contradiction.

Remark. When \( p \in F \), \( G(x, p) \) is expressed in a uniformly convergent series of Green functions \( g(z, s(p)) : |x| < 1 \). But when \( p \) converges to the ideal boundary set, this loses its meaning, because \( |s(p)| \to 1 \) as \( p \to R \) and all \( g(z, s(p)) \to 0 \), but \( G(x, p) \neq 0 \), that is, an ideal boundary is singular point with respect to this series.

21. For the regular boundary point \( \alpha \), there exists a sequence of \( \gamma_i \) on which

\[ \max_{x \in \gamma_i} G(x, \alpha) \leq q^2 : x \in \gamma_i. \]

If there were two \( G_1(x, \alpha) \) and \( G_2(x, \alpha) \), then \( \frac{G_1}{G_2} \) is non constant. Let us denote by \( k_i \)

\[ \min_{x \in \gamma_i} \frac{G_1(x, \alpha)}{G_2(x, \alpha)} = k_i: x \in \gamma_i, \]

then, \( k_i \) is a constant and \( G_1 - k_i G_2 \geq 0 \) in the domain bounded by \( \Gamma \) and \( \gamma_i \), because other boundary is harmonic measure zero set, then

\[ D_{F-V_1}(G_1 - k_i G_2) < + \infty. \]

From the maximum principle \( k_i \) is taken on \( \gamma_i \) and

\[ k_1 > k_2 > \ldots > k_n > \ldots > k > \frac{1}{q^2} \geq 0. \]

\( k \geq 0 \) follows from that
where $N_t$ means the generalized module of the domain bounded by $\gamma_t$ and $\Gamma$. Let $\varepsilon_n = k_n - k$, then $\lim_{n \to \infty} \varepsilon_n = 0$, and there exist $x_i$ on $\gamma_t$ such as

$$G_i(x_i) - kG_2(x_i) = \varepsilon_iG_2(x_i)$$

$$0 < G_i(x) - kG_2(x) < q\varepsilon_iG_2(x) : x \in \gamma_t.$$ is true for every $i$. That is on $\gamma_i$.

$$G_i(x) - kG_2(x) = o(G_2(x)) : n \to \infty .$$ (1)

But $G(x_i) - k_i G_2(x^*_i)$ for a certain $x^*_i$ on $\gamma_i$, for every $i$,

$$G_i(x^*_i) - k_i G_2(x^*_i) = \varepsilon_i G_2(x^*_i).$$ (2)

(1) and (2) contradict each other for $\varepsilon_1 \leq \frac{\varepsilon_n}{q}$, finally $G_i(x, \alpha) = G_2(x, \alpha)$.

For closed Riemann surface $\alpha$ is a regular ideal boundary point.

$G(x, \alpha)$ generally depends on the sequence of $p_i : \lim_{i \to \infty} p_i = \alpha$ and is not always uniquely determined.

22. Example 2. Let us consider the Riemann surface of Example 1.

Let us denote by $G(x, 1.5)$ the Green function with pole at 1.5 on the upper sheet, then $G(x, 1.5)$ has no limit. When $x$ converges to on the upper and lower sheets, it has no limit.

Take two sequences $p_i$, $q_i$ on the upper or the lower sheet such as:

$$\lim_{i \to \infty} p_i = \infty, \lim_{i \to \infty} q_i = \infty$$

$$\lim_{i \to \infty} G(p_i, 1.5) = A = B = \lim_{i \to \infty} G(q_i, 1.5)$$

$$G(1.5, p_i) = G(p_i, 1.5); G(1.5, q_i) = G(q_i, 1.5).$$

Then we have two Green functions

$$\lim_{i \to \infty} G(1.5, p_i) = G(1.5, \infty) + G_2(1.5, \infty) = \lim_{i \to \infty} G(1.5, q_i).$$

Property of Green function 1.

23.1. Let $F$ be a Riemann surface with relative boundary $\Gamma$ and a harmonic measure zero ideal boundary point set $R$. We denote by $G(x, p_i)$ the Green function of $F$ with pole $p_i$ : where $p_i \in F + R - \Gamma$, then $G(x, p_i)$ are not always uniquely determined, and denote by $G'_M$, $\overline{C}_M$, $C'_M$ the niveau curve of $G(x, p_i)$ on which $G(x, p_i) = M$, domain in which
\( G(x, p) = M, \) and domain in which \( g(x, p) < M, \) then

\[
\int_{C_M^*} (G(x, p_1) \frac{\partial G(x, p_2)}{\partial n}) \, ds = \int_{C_M^*} G(x, p_1) \frac{\partial G(x, p_1)}{\partial n} \, ds
\]

Proof. \( C_M \) may converge to the ideal point set, we denote them \( q_1 \ldots q_\infty \in C_M, \) then we take a system of neighbourhood of \( q_1 \ldots q_\infty, \) such as \( \bigcap_k (\sum V_k(q_i)) = R \setminus C_M \) and the boundary of \( V_k(q_i) \) will be denoted by \( \gamma_k(q_i), \) then \( C_M - \sum V_k(q_i) \) is a compact domain with the boundary \( \Gamma, \) the part \( \sum \gamma_k'(q_i) \) or \( \sum \gamma_k(q_i), \) and the part \( k C'_M \) of \( C_M. \)

Denoting by \( k_\omega M \) the harmonic measure of \( \sum \gamma_k \) with respect to \( (C_M - \sum V(q_i)), \) then we have

\[
k_\omega M = 1 : \quad x \in \sum \gamma_k \gamma_k'
k_\omega M = 0 : \quad x \in \Gamma + C'_M.
\]

Let \( k_\omega \) be the harmonic measure of \( \sum \gamma_k \) with respect to \( (F - \sum V_k), \) clearly

\[
k_\omega M < k_\omega.
\]

Hence harmonic measure of \( R \) is zero:

\[
\lim k_\omega M = 0,
\]

but by Green's formula, we have

\[
\int_{k_\omega M} \frac{\partial G(x, p_1)}{\partial n} \, ds = \int_{k_\omega M} G \frac{\partial G(x, p_1)}{\partial n} \, ds
\]

\[
\lim_{k_\omega M} \int \frac{\partial G(x, p_1)}{\partial n} \, ds = 0.
\]

From

\[
\frac{\partial G(x, p_1)}{\partial n} \geq 0 : x \in k C_M,
\]

\[
\int_{k C'_M + k C'_M} \frac{\partial}{\partial n} G(x, p) \, ds = 2\pi,
\]

then

\[
\lim_{k_\omega M} \int \frac{\partial}{\partial n} G(x, p_1) \, ds = \int_{C_M} \frac{\partial}{\partial n} G(x, p_1) = 2\pi.
\]

As \( G(x, p_1) \) is bounded in the neighbourhood of \( p_1, j = i, \) we have

\[
\lim_{k_\omega M} \int \frac{\partial G(x, p_1)}{\partial n} \, ds = \int_{C_M} \frac{\partial G(x, p_1)}{\partial n} \, ds.
\]
and by Green's formula we have the conclusion.

Since $G(x, p_i)$ is finite except for $p_i$,

$$\lim_{n \to \infty} \frac{C_n}{M} = p_i,$$

$$\lim_{x \to P_j} G(x, p_i) = G(p_j, p),$$

$$\lim_{n \to \infty} \int_{C_n} G(x, p_i) \frac{\partial G(x, p_i)}{\partial n} \, ds = 2\pi G(p_j, p_i).$$

Especially if $p_i \in F - R - \Gamma$,

$$G(p_1, p_2) = G(p_2, p_1).$$

23.2 The properties of Green function. If $G(x, \alpha)$ is a Green function with pole $\alpha : \lim_{i \to \infty} p_i = \alpha \in \mathbb{R}$, and $V_2(\alpha) \subset V_1(\alpha)$ are two neighbourhoods and their boundary curves are denoted by $C_2$, and $C_1$, then

$$m_1 = \min_{x \in C_1} G(x, \alpha) \leq \min_{x \in C_2} G(x, \alpha) = m_2.$$

Proof. If $m_1 > m_2$, let us take $\delta$ such as $m_1 - m_2 > \delta > 0$, then for any small number $\varepsilon < \frac{\delta}{4}$ there exists a number $i_0 = i_0(\varepsilon)$ such as

$$|\min_{x \in C_1} G(x, p) - m_1| < \varepsilon, \quad |\min_{x \in C_2} G(x, p) - m_2| < \varepsilon$$

for every $p_i : i \geq i_0$, hence $p_i \in F$, $\lim_{x \to P_i} G(x, p_i) = \infty$, taking a small neighbourhood $v(p_i)$ of $p_i,$ then

$$\min_{x \to P_i} D(G(x, p_i)) < +\infty$$

$$\min_{x \in C_1} G(x, p_i) \leq \lim_{x \in V_2 - v} G(x, p_i) \leq \lim_{x \in V_1 - v} G(x, p_i) = \min_{x \in C_2} G(x, p_i)$$

$$m_1 - \varepsilon < m_2 + \varepsilon, \quad \delta < m_2 - m_1 < 2\varepsilon.$$

This is absurd.

23.3 If $G^*(x, \alpha)$ is the function satisfying the following conditions

- $G^*(x, \alpha) \geq 0 : x \in F, \quad G^*(x, \alpha) = 0 : x \in \Gamma$
- $\min_{x \in C_1} G^*(x, \alpha) \leq \min_{x \in C_2} G^*(x, \alpha)$: if $V_2(\alpha) \subset V_1(\alpha)$
- $\int_{\Gamma} \frac{\partial G^*}{\partial n} \, ds = 2\pi, \quad G^*(x, \alpha) < +\infty : x \in F - R$

Then for any point $x_0 \in F$, we can choose a sequence of $p_i, \lim_{i \to \infty} p_i = \alpha$ such that

$$\lim_{i \to \infty} G(x_0, p_i) = G^*(x_0, \alpha).$$

Proof. Let us denote by $C_M$ the niveau curve on which $G^*(x, \alpha) = M$ and $C_M$ such as $G^*(x, \alpha) \geq M$, then $M_1 < M_2$ it follows that $\bar{C}_M \supset \bar{C}_M$. 
from \( c^\circ \),

\[
\int_{C_M} \frac{\partial G^*}{\partial n} ds = 2\pi \quad \text{on} \quad C_M.
\]

Since

\[
\lim_{M \to -\infty} C_M = \alpha : \lim M = \infty,
\]

\( \sigma^j \) is a niveau curve of \( G(x, x_0) \), \( \sigma^j \) is a niveau curve of \( G^* (x, \alpha) \), then

\[
G^*(x_0, \alpha) = \frac{1}{2\pi} \int_{\sigma^j} G^*(x, \alpha) \frac{\partial}{\partial n} G(x, x_0) ds = \frac{1}{2\pi} \int_{\sigma^j} G(x, x_0) \frac{\partial}{\partial n} G^*(x, \alpha) ds.
\]

therefore there exists a point \( p_i \) on \( \sigma^j \), such as \( G^*(x_0, \alpha) = G(p_i, x_0) \), but

\[
\lim_{t \to -\infty} G(p_i, x_0) = \lim_{t \to -\infty} G(x_0, p_i) = G^*(x_0, \alpha), = \lim_{t \to -\infty} G(x_0, p_i)
\]

and it is clear that

\[
\int_{C_M} \frac{\partial}{\partial n} G(x, \alpha) ds = 2\pi + 0.
\]

24. It is clear that \( G(x, \alpha) \) is not bounded in the neighbourhood of \( \alpha \), but not always

\[
\lim G(x, \alpha) = \infty,
\]

nevertheless we see directly that if \( \lim G(x, \alpha) < +\infty \), then from 23, there is a sequence of \( C_M \) which converges into \( \alpha \).

Definition. If \( \lim G(x, \alpha) = \infty \), we call it a regular Green function.

It is easily seen that there is only regular function on the regular ideal point or inner point.

When an ideal point \( \alpha \) has at least one regular Green function, we call \( \alpha \) a regular ideal point for Evans' problem. This notion is a clearly local property.

Theorem 8. It is necessary and sufficient for \( \alpha \) to be regular for Evans' Problem, that there is a certain neighbourhood \( V(\alpha) \) and a harmonic function \( U(x) \) satisfying the following conditions:

1. \( U(x) \) is lower bounded in \( \overline{V(\alpha)} \) (\( \overline{V} \) is \( V \)'s closure),
2. \( U(x) < +\infty \quad x \in \overline{V(\alpha)} - \alpha \),
3. \( U(x) \) is harmonic in \( \overline{V(\alpha)} - R \),
4. \( \lim_{x \to +\infty} U(x) = +\infty \).

The necessity is clear, if \( U(x) \) exists we can make a regular Green function. Let \( W(x) \equiv 0 \ ; \ x \in F - \overline{V(\alpha)} \), by the smoothing process we gain a harmonic function \( H(x) \) such as

\[
0 < d = \int_{\gamma} \frac{\partial H}{\partial n} ds = \int_{\gamma} \frac{\partial U}{\partial n} ds,
\]

where \( \gamma \) is the boundary of \( V(\alpha) \).
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\[ |H - U| < M : x \in \overline{V}(\alpha) , \quad H = 0 : x \in \Gamma , \]

accordingly \( \frac{2\pi H(x)}{d} \) is the Green function which we require.

25. **Theorem 9.** If there is a certain neighbourhood \( V(\alpha) \) of \( \alpha \) in which a one-valued bounded analytic function \( f(x) : x \in F - V \) exists, then all points of \( V(\alpha) \setminus R \) is regular for Evans' problem.

Proof. By Theorem 7 there is a certain neighbourhood \( V'(\alpha) \subseteq V(\alpha) \) in which \( f(x) = f(\alpha) \) has no root except only \( \alpha \). \( -\log |f(x) - f(\alpha)| \) is the function of theorem 9, accordingly \( \alpha \) is regular for Evans' Problem.

**Corollary.** If \( f(x) : x \in V(\alpha) \setminus F \) remains a second Category set in the \( w \)-plane in which \( n(w) \leq +\infty \) then \( \alpha \) is regular for Evans' Problem.

If there is no generalized local parameter for any small \( V(\alpha) \) then by theorem 7, \( f(x) \) covers in \( V(\alpha) \) the \( w \)-plane except at most non-dense set, therefore

\[ V_1 \supset V_2 \ldots ; \cap V_i = \alpha , \quad f(x) \in V_i(\alpha) \setminus F \]

covers the \( w \)-plane infinitely many times except at most first category set. This is a contradiction.

**Corollary.** When the Riemann surface \( F \) is given as the covering surface of an other abstract Riemann surface \( F^* \), if \( F \) covers finitely many times and all points of \( F^* \) have generalized local parameter, then all points of \( F \) have generalized local parameter.

We denote by \( x^* \) the projection of \( x : x \in F \) on \( F^* \). We define \( f(x) = f(x^*) \), this is clearly the generalized local parameter. Especially if we take the \( w \)-plane as ground surface, if \( V(\alpha) \) has finitely many times covers the \( w \)-plane, then all points of \( V(\alpha) \in R \) is regular for Evans' problem.

Especially, let \( F^{**} \) be the covering surface, being finitely many sheeted on \( F \), if \( F \) has finite numbers of genus then \( F \) is representable conformally as a sub-Riemann surface of closed Riemann surface \( F_c \), since all point of \( F_c \) are regular for Evans's problem it follows that all point of \( F^{**} \) are regular, however infinitely many times sheeted covering surface on the \( w \)-plane \( F^* \) may be represented.

The problem whether all ideal harmonic measure zero points are regular for Evans’ Problem is quite difficult but it seems very true and admissible.

**Extension of Cauchy's integral formula.**

26. When a curve \( C \) on \( F \) converges to the ideal point set \( R \), we call \( C \) non compact. If \( F \) is bounded by a compact or non compact
curve \( \Gamma \), and ideal point set, we take a system of neighbourhood \( V_k \) such as
\[
\sum V_k \supset \overline{F} \cap R : \lim_{k \to \infty} \sum V_k = \overline{F} \cap R
\]
then \( F_k = F - \sum V_k \) is compact domain with the boundary \( \Gamma_k \) which is a part of \( \Gamma \) and \( \gamma_k \) which is a part of the boundary \( \sum V_k \), we denote by \( \omega_k(x) \) a positive bounded harmonic function in \( F \) with the boundary value 1 on \( \gamma_k \), 0 on \( \Gamma_k \), if \( \lim_{k \to \infty} \omega_k(x) = 0 \), then we say that \( F \) has zero ideal boundary point.

On the other hand we denote a neighbourhood systems
\[
V_k'(p) : p \in R \cap \Gamma', \text{ such as } \sum V'(p) \supset R \cap \Gamma', \text{ then } \sum V_k' = R \cap \Gamma',
\]
\( \Gamma_k' = \Gamma' - \sum V_k' \), \( F_k' = F' - \sum V_k' \); \( \gamma_k' = \text{boundary of } (\sum V_k' \cap F) \)
then \( F_k' \) satisfies the conditions of theorem 7 then \( F_k' \) has Green function \( G_k'(x, x_0) \).

**Theorem 10.** If \( f(x) \) is a one-valued bounded analytic function in \( F \), and
\[
\left| \frac{\partial f}{\partial n} \right| < \infty \text{ on } \Gamma - R,
\]
then
\[
f(x_0) = \lim_{k \to \infty} \int_{\Gamma_k'} f(x) \frac{\partial G_k(x, x_0)}{\partial n} ds : x_0 \in \Gamma
\]
Proof. Hence \( F_k' \) is compact then \( G_k' = 0; x \in \Gamma_k' + \gamma_k' \). In denoting by \( C_{x_0} \) the niveau curve of \( G_k'(x, x_0) \) then
\[
\int_{\Gamma_k' + \gamma_k'} \frac{\partial G_k'}{\partial n} ds = \int_{\Gamma_k' + \gamma_k'} \frac{\partial G_k'}{\partial n} ds = 2\pi \text{ (see Lemma 1 of Nr. 17)}
\]
\[
\lim_{k \to \infty} C_{x_0} = x_0, \text{ by theorem 7 } \lim_{x \to x_0} f(x) = f(x_0)
\]
and in using Green's formula, we have
\[
\int_{\Gamma_k' + \gamma_k' + C_{x_0}} f(x) \frac{\partial G_k'}{\partial n} ds = \int_{\Gamma_k' + \gamma_k' + C_{x_0}} G_k' \frac{\partial f}{\partial n} ds
\]
Let \( \omega'(x) \) be the bounded harmonic function in \( F_k' \) such as then we have
\[
f(x_0) = \frac{1}{2\pi} \int_{\Gamma_k' + \gamma_k'} f \frac{\partial G_k'}{\partial n} ds
\]
\[
0 \leq \omega'(x) \leq 1, \quad \omega'(x) = 0 : x \in \Gamma_k', \quad \omega_k' = 1 : x \in \gamma_k',
\]
then
\[
\omega_k' \leq \omega_k : \lim_{k \to \infty} \omega_k' = 0.
\]
\[
\int_{\gamma_k} G_k' \frac{\partial \omega_k}{\partial n} ds = \int_{\gamma_k} G_k' \frac{\partial \omega_k}{\partial n} ds : |G_k| \leq M; x \in V(x_0)
\]
\[
\lim_{k \to \infty} \int_{\Gamma_k'} \frac{\partial G_k'}{\partial n} \, ds = 0 \quad \text{but} \quad \frac{\partial G_k'}{\partial n} \geq 0 \quad \text{on} \quad \Gamma_k',
\]
then
\[
\lim_k \int_{\Gamma_k'} \frac{\partial G_k'}{\partial n} \, ds = \int_{\Gamma} \frac{\partial G_{\infty}}{\partial n} \, ds
\]
finally
\[
\frac{1}{2\pi} \lim_k \int_{\Gamma_k'} f \frac{\partial G_k'}{\partial n} \, ds \quad \text{exists and equal is to} \quad f(x_0).
\]

Remark. \( G_k(x, x_0) \) is not always uniquely determined.

Chapter IV.

Potential theory on the abstract zero-boundary Riemann surface.

27. If we would like to establish the potential theory on the zero-boundary Riemann surface, we must construct the function \( \chi(p, 0, q) \) which has the same role as \( \phi\left(\frac{1}{r}\right) \) in \( u(p) = \int \phi\left(\frac{1}{r_{pq}}\right) d(q) \) in the general potential theory, and study its fundamental properties which are very useful.

Distance function \( \chi \). Let \( H \) be the disc, that is simply connected, and compact domain of zero-boundary Riemann surface \( F \) which is mapped conformally on to the circle \(|z| < 1\) of the \( z \)-plane, and its centre is denoted by 0. In the preceding we recognized that Green function \( G_1(x, p) \) of \( F - H \); exists where \( p \) is an inner point of \( F - H \) or the boundary point \( R \).

And we make the Green function of \( H \) with its pole at 0, which is \(- \log |z - 0| : z \in H\)
\[
\int_{|z| = 1} \frac{\partial}{\partial n} G_1(x) ds = 2\pi, \quad \text{and} \quad G_2 = \log |x|.
\]
By smoothing process, we can construct the function \( \chi(x, 0, p) \) such as
\[
D_{F-H}(-G_1 + \chi) < \infty, \quad D_R(-G_2(x, 0) + \chi) < \infty,
\]
in this process we used an assistant curve \( \Gamma' \) which is on the outer side of the boundary of \( H \). Then, in the neighbourhood of \( x = 0 \)
\[
\chi(x, 0, p) = - \log |x - 0| + U(x),
\]
where \( U(x) \) is harmonic in the neighbourhood of 0, and determine an adequate constant so that
\[ \chi(x, 0, p) = |x| + \varepsilon(x): \lim_{x \to 0} \varepsilon(x) = 0. \]

**Theorem 11.** After the normalization in the above process, \( \chi(x, 0, p) \) is uniquely determined depending only on \( G_1(x, p) \) and \( G_2(x, 0) \), but not on the assistant curve \( \Gamma \).

**Proof.** If there exist two functions \( \chi_1 \) and \( \chi_2 \) which have the same singularity, then
\[
D_h(x_1 - G_2(x, 0), x_0) + D_{f-h}(x_1 - G_1(x, p)) < +\infty
\]
and
\[
D_h(x_2 - G_2(x, 0), x_0) + D_{f-h}(x_2 - G_1(x, p)) < +\infty
\]
which do not depend on the assistant curve \( \Gamma \).

On the other hand
\[
D(x_1 - G_1(x, p), x_2 - G_2(x, p)) \leq \sqrt{D(x_1 - G_1(x, p))D(x_2 - G_2(x, p))}
\]
finally
\[
D_h(x_1 - x_2) < +\infty,
\]
but \( x_1 - x_2 \) is uniform, then \( x_1 - x_2 = \text{const} \), but this constant is zero by the normalization at 0.

**Remark.** We can prove easily that
\[ \chi(x, 0, p) = \chi(p, 0, x): \quad x, p \in F \]

\[ \chi(x, 0, p) = \int_{\gamma} \chi(p, 0, x) \frac{\partial}{\partial n} \chi(x, 0, p) ds: \quad p \in R, \]

where the integration is on the curve \( \gamma \) which is the nimeau curve of \( \chi(p, 0, x) \).

**28.1. Property A.** Let us denote by \( V(x) \) and \( V(0) \) the neighbourhood of \( x \) and 0 respectively. Then \( \chi(x, 0, p) \leq M(x) \) wherever the parameter point \( p \) may be situated, including ideal boundary points, so long as \( p \in V(x) + V(0) \), where \( M \) depends on only \( x \) but not on \( p \).

**Case 1.** from \( \chi(x, 0, p) = \chi(p, 0, x): \quad p \in F, \ x \in F \)

\[ |\chi(x, 0, p)| \leq M(x) \quad \text{if} \quad p \in V(x) + V(0): \quad p \in F, \]

because \( \chi(x, 0, p) \) is harmonic in \( F - V(p) - V(0) \), \( |\chi(x, 0, p)| \leq \max |\chi(x, 0, p)| \) on the boundaries of \( V(x) \) and \( V(0) \), (see Nevanlinna’s theorem) Case 2. when \( p \) converges in the set of ideal boundary point \( \alpha \), let us fix \( x \) at present.

If \( p \in V(x): \alpha \in R \), then there exist two neighbourhoods \( V(x) \) and \( V(0) \) such as if \( p \in V(0) + V(x) \) then, \( D(x, 0, p) < +\infty \), it follows that
\[
\lim_{p \in F - V(0) - V(x)} \max |\chi(x, 0, p)| \quad \text{on the boundary of} \ V(0) \text{ and} \ V(x).
\]

**Property A’.** If \( x \in V(p) + V(0) \), then
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This can be proved in the same way.

28.2. Property B. In the neighbourhood of a positive pole \( p \),

\[
\chi(\lambda - p, 0, p) = \log \left( \frac{1}{|\lambda - p|} \right) + U_p(\lambda),
\]

where the distance is in the local parameter, and \( U_p(\lambda) \) is a harmonic function depending on \( p \), and \( U_p(\lambda) \) is continuous with respect to \( p \), that is to say for every positive number \( \epsilon \), there exists a circle \( pC_d \) with a diameter \( d \) and the centre at \( p \) and \( \delta \) such as if \( |p - p'| < \delta \), then \( |U_p(\lambda) - U_{p'}(\lambda)| < \epsilon \) : \( \lambda \in pC_d \cap \lambda', C_d : d \) is larger than \( \delta \).

Proof. It can be considered that \( \chi(x, 0, p) \) is made by smoothing process from two Green functions \( G_1 \) and \( G_2 \)

\[
G_1(x, 0, p) = \log \left( \frac{1}{|x - p|} \right) + \bar{U}_p(x) : x \in pC_d.
\]

First we prove that \( \bar{U}_p(x) \) is a continuous function with respect to \( p \). \( G(x, p) \) seems that which is made by smoothing process from

\[
U_0 = \log \left( \frac{1}{|x - p|} \right) \text{ in } pC_d, \quad \omega_0 = 0 \text{ in } F - H - pC_d \text{ with assistant curve } \Gamma \text{ traced in } pC_d.
\]

\( \omega_1 \) is a harmonic function in the domain bounded by \( \Gamma_0 \) and \( \Gamma' \), and \( \omega_1 = 0 \) on \( \Gamma_0 \), \( \omega_1 = U_0 \) on \( \Gamma' \); where \( \Gamma' \) is the boundary of \( H \).

\[
D_{pC_d}(U_1 - U_0) < +\infty, \quad U_1 = \omega_1 \text{ on } pC_d's \text{ periphery.}
\]

Let \( S_1 = U_1 - U_0 : S_1 \) being harmonic in \( pC_d \)

\[
................ S_n = U_n - U_{n-1} ; \quad \omega_n = U_n \text{ on } C's \text{ periphery} \quad ; \quad \omega_n = 0 \text{ on } \Gamma_0
\]

\[
T_n = \omega_n - \omega_{n-1}.
\]

If \( |p - p'| \) is so small that \( |U_0 - U_0'| < \varepsilon, \) on \( \Gamma' \), then \( |\omega_1 - \omega_1'| \leq \varepsilon \)

\[
|S_1 - S_1'| \leq L|\omega_1 - \omega_1'|
\]

\[
|\omega_2 - \omega_2'| \leq K|S_1 - S_1''| \leq KL|\omega_1 - \omega_1'|
\]

\[
|S_2 - S_2'| \leq L|\omega_2 - \omega_2'| \leq L^2K|\omega_1 - \omega_1'|
\]

in general

\[
|S_n - S_n'| \leq L^nK^{n-1}|\omega_1 - \omega_n'|, \quad |\omega_n - \omega_n'| \leq L^{n-1}K^{n-1}|\omega_1 - \omega_1'|,
\]

after all

\[
G(x, p) = U_0 + \sum S_n = U^\infty : \quad x \in pC_a,
\]

\[
G(x, p) = U_0' + \sum S_n' = U'^\infty : \quad x \in p'C_a,
\]

therefore

\[
\bar{U} = \sum S_n, \quad \bar{U}' = \sum S_n'.
\]
\[ \sum S_n - \sum S'_n \leq \sum |S_n - S'_n| \leq M\varepsilon : M = M(K, L) < +\infty \text{ in } pC_d \bigcap p' C_d \]
\[ |\bar{U} - \bar{U}'| \leq M\varepsilon \text{ in } pC_d \bigcap p' C_d , \]
in the same way we have \[ |pU(x) - p'U(x)| \leq M\varepsilon . \]

29. The function \( \chi(x, 0, p) \) is harmonic when \( x \in F \) except at 0, where it is negatively infinite and \( p \), where it is positively infinite \((p \in F)\) or a regular ideal point, but if \( p \) is not a regular boundary point the function has no limit but in all cases \( \frac{1}{2\pi} \int \frac{\partial \chi}{\partial n} ds = 1 \). This fact means that a positive mass one is distributed on \( p \) in the sense of the potential theory, then we can define ideal mass on the ideal boundary point.

**Definition. Mass distribution \( \mu \).** Mass distribution is so defined as in the general potential theory, \( \mu \) is defined for the set in the Riemann surface and its regular boundary (or all ideal boundary point set). The family of which \( \mu \) is defined must be additive class and \( \mu \) is completely additive. The corn of mass distribution is defined in the same manner (of course \( \mu \) is invariant with respect to conformal mapping).

Then the potential will be defined as the Lebesgue-Stieltjes-Radon integral
\[ u(x) = \int \chi(x, 0, p)d\mu(p) . \]
The value of the function \( \chi(x, 0, p) \) is not determined only by the distance \(|x-p|\) as \( \phi\left(\frac{1}{r}\right) \), but it depends on the location of \( x \), and further distance is not defined in the Riemanian surface in general except locally, so the potential defined with \( \chi \) is not homogenous.

We must verify to what extent the properties of general potential will hold. We see directly

1° at \( x = 0 \) \( u(x) = \int \chi(x, 0, p)d\mu(p) = \log|x| \int d\mu(p) \),

and from the properties A and A', \( u(p) \) is harmonic, continuous and finite wherever no mass is scattered, for instance at inner point of \( F \) or at the boundary, if only \( x \) is not situated in the corn of mass distribution.

2° Hence \( u(p) = \lim_{N \to \infty} \int \chi^N(p, 0, Q)d\mu(Q) \), \( u(p) \) is lower semi-continuous, where
\[ \chi^N \begin{cases} = \chi, & \text{if } \chi \leq N \\ = N, & \text{if } \chi > N . \end{cases} \]

3° From the definition of integral which expresses the potential, it is necessary and sufficient for the potential to be bounded and con-
continuous on the closed set $T$ not containing $x = 0$, that there exists a 
circles of radius $\delta$, so as the potential engendered by the mass con-
tained in this circle $C_\delta$ is $< \varepsilon$, for any positive number $\varepsilon$.

30. **Theorem 12.** (G. C. Evans, F. Vasilescu) Let $\mu$ is zero out side 
of the closed set $T$ not containing $x = 0$. If the potential is continuous 
as the function defined on $T$ where the mass is distributed, then it is 
also continuous in $F$.

This theorem will be proved by means of the property B.
Proof. In the sequel the distance is assumed that it is defined by 
the local parameter.

From B $\chi(x, 0, p) = \log \frac{1}{|x - p|} + U_p(x)$ : $x \in C_\delta$, where $C_\delta$ is the 
center at $p$ of radius $\delta$ and moreover $U_p(x)$ is continuous with respect 
to $p$ in this circle. We denote by $u_\delta(p)$ the potential engendered by the 
mass in the circle with its center at $p$ and radius $\delta$. As $u(p)$ is con-
tinous as the function on the corn of mass distribution, we can choose $2\delta$ and $d$ for any positive number $\varepsilon$ so that the following condition may 
be satisfied, at any point $Q$ of the corn.

i. $|U_p(x) - U_p(x)| \leq \varepsilon < \frac{1}{2} \log 2 :$ if $x \in C_\delta(Q) : d > 2\delta ; |p - Q| \leq 2\delta$

ii. $|U_p(Q + h) - U_p(Q)| \leq \varepsilon :$ $0 \leq h \leq 2\delta$

iii. $\log \frac{1}{2\delta} \geq 2U_p(Q)$ : if $|U_p(R)| > 5$

$\geq 10$ : if $|U_p(Q)| \leq 5$

iv. $|U_{2\delta}(Q)| \leq \frac{\varepsilon}{3 \log 2}$ : if $|U_p(Q)| > 5$

$1 + \frac{3 \log 2}{U_p(Q) - 1 \log 2}$

$|U_{2\delta}(Q)| \leq \frac{\varepsilon}{1 + \frac{3 \log 2}{2}}$ : if $|U_p(Q)| \leq 5$

Take a point $p$ of $F$, and denote by $u_\delta$ the potential engendered 
by the mass in the circle with the centre at $p$ of radius $\delta$.

If the distance between $p$ and the set of corn $K$ of the mass 
distribution is larger than $\delta$, then $u_\delta = 0$, and if this distance is smaller 
than $\delta$, then $C_\delta(p)$ is contained completely in a circle $C_{2\delta}(Q) : Q \in K$.

Let $Q$ be a certain point of $K$ which is nearest to $p$, then
\[ |u_\delta(p)| \leq \int_{K \cap C(p)} X(p, 0, M)d\mu(M) \leq \int_{K \cap C(p)} |X(p, 0, M)|d\mu(M) \]

\[ = \int \frac{|X(p, 0, M)||X(Q, 0, M)|}{|X(Q, 0, M)|} d\mu(M). \]

\[ X(p, 0, M) = \frac{\log \frac{1}{pM} + U_\delta(M)}{\log \frac{1}{QM} + U_\delta(M)} = 1 + \frac{\log \frac{QM - U_\delta(M) - U_\delta(M)}{pM}}{\log \frac{QM}{pM} + U_\delta(M)} \]

\[ m \begin{cases} \frac{3}{2} \log 2 & \text{if } U_\delta(Q) > 5, \\ U_\delta(Q) - \frac{1}{2} \log 2 & \text{if } U_\delta(Q) \leq 5, \end{cases} \]

therefore \[ |u_\delta(p)| \leq m \int |X(Q, 0, M)||d\mu(M)| \leq \varepsilon, \] consequently \( u(p) \) is is continuous.

Remark. \[ \frac{M}{MP} < 2, \] because their proportion is greatest when \( M \) is situated at the point where \( C_\delta \) and the extension of \( QP \) intersect each other.

From the property of \( X(x, 0, p) \), \( u(p) \) is sub-harmonic except at 0, and satisfies all conditions of logarithmic potential, accordingly energy integral, problem of equilibrium, sweeping out process, capacity and the transfinite diameter will be defined in the same manner and all theorems of general potential theory will hold so long as we consider only the set (point, curve, domain) of inner point of \( F \) being different from 0.

But it is necessary and interesting to consider the problem regarding the ideal boundary point set \( R \), where \( X(x, 0, p) \) is not always determined uniquely.

It is well known that the equilibrium problem becomes easy in the case of logarithmic potential by making use of Green function, so called Robin’s problem.

31. Robin’s problem. Let \( D \) be a domain compact or not (i.e. bounded by ideal boundary point set) composed of a finite number of domains \( D_i \) satisfying the following conditions: 1°. The boundary of \( D_i \) are analytic curves \( \Gamma_i \) or more generally regular curves for Dirichlet problem, 2°. Every \( D_i \) does not contain the point 0. 3°. Every curve \( \Gamma_i \) does never converge to any ideal boundary point.

Theorem 13. The Equilibrium Problem is soluble with respect to \( D \).
Let us denote by $g_{F-D}(\zeta,0) : \zeta \in F$ the Green function of $F-D$ with pole at 0 and call $\gamma_D$ Robin's constant of $D$. Defined by the next formula

$$\gamma_D = \lim_{x \to 0} g_{F-D}(x,0) - \log \frac{1}{|x-0|},$$

where $g_{F-D}(\zeta,0) = \text{Green function of } F-D \text{ with pole at } 0$. Hence

$$u(\zeta,0,x) = -\chi(\zeta,0,x) + g_{F-D}(\zeta,x) - g_{F-D}(\zeta,0)$$

is regular harmonic in $F-D$ and finite in the neighbourhood of ideal boundary point contained in $F-D$, and $u = \chi$ on $\Sigma = \Gamma$.

By Green's formula

$$u(\zeta,0,x) = -\frac{1}{2\pi} \int_{\Gamma} \chi(x,0,\xi^*) \frac{\partial}{\partial n} g_{F-D}(\xi^*,\zeta) \, ds,$$

and hence

$$\lim_{\zeta \to 0} (g_{F-D}(\zeta,x) + \chi(\zeta,0,x)) = \gamma_D.$$

Case 1. If $x \in F-D - \gamma_D + g_{F-D}(\zeta,0) = -\frac{1}{2\pi} \int_{\Gamma} \chi(\zeta,0,\xi^*) \frac{\partial}{\partial n} g_{F-D}(\xi^*,0) \, ds$.

Case 2. if $x \in \Gamma$, then $g_{F-D}(\zeta,x) = 0$

$$\gamma_D = \frac{1}{2\pi} \int_{\Gamma} \chi(\zeta,0,\xi^*) \frac{\partial}{\partial n} g_{F-D}(\xi^*,0) \, ds.$$

Case 3. If $x \in D$, $g_{F-D}(\zeta,x)$ is cancelled.

$$\gamma_D = \frac{1}{2\pi} \int_{\Gamma} \chi(\zeta,0,\xi^*) \frac{\partial}{\partial n} g_{F-D}(\xi^*,0) \, ds.$$

Here the potential $u(x)$ engendered by the positive mass distribution $\frac{\partial}{\partial n} g_{F-D}(\xi^*,0)$ is continuous ($= \gamma_D$) on $\Gamma$, where the mass is distributed, on account of the theorem 14, $u(x)$ is continuous in $F$ except 0.

The behaviour of $u(x)$ in the neighbourhood of the ideal boundary. We see directly that $u(x)$ is bounded in absolute value depending on $D$, on account of property A, especially in side of $C$ with respect to $D$, by Nevanlinna's theorem $u(x) = \gamma_D$, because $u(x)$ is harmonic except 0 and $\Gamma$ and $R$ where $u(x)$ is finite, therefore $u(x)$ is the potential being the solution of Robin's problem.

Or more precisely we take $\Gamma'_i$ near $\Gamma'_i$ surrounding $E$ which is the part of $R$ contained in $D$, since $x \in V(0)$, $u(x) \geq M(D'_i) > -\infty$ in $D$ and $\int \frac{\partial U}{\partial n} \, ds = 0$, then by Lemma 1 of Nr. 7, $D_{D'_i}(u) \leq +\infty$, so we have $\Sigma V'_i$. 

\[ \]
\[ u(x) = \gamma_D : x \in D. \]

From Case 1, we have
\[ \min_{-\chi(x,0)} \leq g_{F-D}(0,x) - \gamma_D \leq \max_{-\chi(x,0)} : \zeta \in \Gamma \]

this follows that the theorems about the z-plane set in R. Nevanlinna's (pp. 111-129) will be proved in the zero-boundary Riemannian surface.

**Green's function of** \( F \), \( H < F_1 < F_2 \ldots \lim F_i = F \) is the exhaustion of \( F \), and denote by \( g_n \) the Green function of \( F_n \) with pole at 0, then
\[ g_1 < g_2 < g \ldots \]

If \( F \) is a zero boundary Riemannian surface, then \( \lim g_n = \infty \). This equivalency with \( \lim_{\omega_n} = 0 \), is proved by P. J. Myrberg^4. Let us denote by \( \gamma_n \) the Robin's constant of \( F - F_n \), then capacity of \( F - F_n \) is defined by \( C = e^{-\gamma_n} \) where \( F - F_n \) is non compact.

Clearly \( F_i < F_j \) follows \( g_i < g_j \) and \( \text{Cap}(F - F_i) > \text{Cap}(F - F_j) \).

Finally \( \lim \text{Cap}(F - F_n) = 0 \) is equivalent with \( \lim_{\omega_n} = 0 \) and \( \lim g_n = \infty \).

32. Let \( F \) be the Riemann surface with relative boundary \( \Gamma_0 \) and relative harmonic measure zero ideal boundary point set \( R \).

Denote by \( G(x, p) \) the Green function of \( F \) with pole at \( p \), then we can discuss the potential defined with \( G(x, p) \) as in the case of \( \chi(x,0,p) \)
\[ u(x) = \frac{1}{2\pi} \int G(x, p) d\mu(p), \]

\( u(x) \) has property \( A, A', B \) and its lower semi-continuity and theorem of Evans-Vasilesco is valid.

If the mass distribution \( \mu \) is zero on \( \Gamma_0 \) and on \( R \), then we call
\[ I(\mu) = \frac{1}{2\pi} \int G(p,q) d\mu(p) d\mu(q), \]

the energy integral^10 with respect to \( \mu \).

33.1. Let \( D \) be a compact domain of \( F \) not containing \( \Gamma_0 \) on \( \bar{D}, F \), and the boundary of \( D \) is regular for Dirichlet Problem, then there exists a positive mass distribution \( \mu_D \) on \( D, \) of which total is 1 and the energy integral is minimum, so called the equilibrium distribution, in this case \( \mu \) is zero out of the boundary of \( D \) and the potential engendered by this distribution is constant on \( D \) which is equal to \( I_D(\mu_D) \).

This is proved by using the following properties in the same way as in general potential theory^6:
1. If $\mu D = \lim \mu_n D$, then $I_D(\mu) \leq \lim I_D(\mu_n)$.

2. $U(x)$ is harmonic in $F$ except the corn of mass distribution $\mu$.

3. $U(x)$ is lower semi-continuous in $F-R$.

If $D$ is not compact, denote by $\bar{D} \cap R$ the subset of $R$ which is contained in $\bar{D}$. Let be $G_1, G_2, ...$ a sequence of closed domains enclosing $\bar{D} \cap R$, such as

$$\bigcap_i G_i = \bar{D} \cap R,$$

and every boundary of $G_i$ never converges to $R$, then we define

$$I_D(\mu^*) = \lim I_{D-G_i}(\mu^*)$$

for $I_D(\mu^*)$ decreasing function of set.

33.2. Lemma 1.

$$J_XO = \lim I_D \quad G < \mu^* \quad = I_D - G \quad G \quad (\mu^*) = I_D - G \quad (\mu^*)$$

If we denote by $\Gamma_D$ the outer boundary of $D$, then

$$U_1(x) = \int G(x, p) \mu^*_{D-G_i}(p) = I_{D-G_i}(\mu^*),$$

$U_1(x)$ is harmonic in $F-D$, but $\int_{\partial_D} \partial U_1 d s = 1$.

Since

$$D_{p-p}(U_1(x) - U_2(x)) = \left(I_{D-G_i}(\mu^*) - I_{D-G_i}(\mu^*)\right) \int_{\partial_D} (U_1(x) - U_2(x)) d s = 0,$$

therefore

$$I_{D-G_i}(\mu^*) = \lim I_{D-G_i}(\mu^*).$$

33.3. Transfinite diameter$^3$.

We denote by $D_n$, the transfinite diameter of order $n$ of non-compact set $D$.

$$\frac{1}{D_n} = \frac{1}{D} \lim \frac{1}{G_2} \left(\min_{p_n} \sum_{i=1}^{n} G(p_n \cdot p_i)\right).$$

For $D-G_j D_n$ is monotonously decreasing with respect to $j$.

Lemma 2.

$$\frac{1}{D_n} = \frac{1}{D-G_1 D_n} = \frac{1}{D-G_2 D_n} = \cdots$$
If we denote by $C$ and $C_i$ the boundary of $D$ and $G_i$ respectively then all $p_i (i = 1, 2, 3, \ldots)$ lie on $C$.

If it were not so, there is at least one point $p_0$ on $C_j$ such as

$$\frac{1}{D - G_i D_n} = \frac{1}{2\pi n} \left( \sum_{s = t} G(p_0 \cdot p_s) + \sum_{s < t} G(p_s \cdot p_t) \right)$$

is minimum.

Hence $\lim_{p \to p_s} G(p \cdot p_s)$, and $U(p_0) = \sum_i G(p_0 \cdot p_s) < +\infty$, $p_0 \in F - \sum \nu(p_s)$, where $\nu(p_s)$ is a neighbourhood of $p_s$, and $R$ is harmonic measure zero therefore $U(p_0)$ takes its minimum on $C$, this is a contradiction, accordingly we have the conclusion.

$p D_n$ is monotonously decreasing with $n$.

Then $\lim_{n} D_n = D$ is called the transfinite diameter of $D$.

Lemma 3. From general potential theory

$$\frac{1}{D - G_i D} = I_{D - G_i (\mu^*)} : \text{ for each } i,$$

then

$$\frac{1}{\nu D} = I_{D (\mu^*)}.$$

Lemma 4. We denote by $\omega_D(x)$ the bounded harmonic function of $F - D$; such that $0 \leq \omega_D(x) \leq 1$, $\omega_D(x) = 0 : x \in \Gamma_D$, $\omega_D(x) = 1 : x \in C$, and let

$$w_D(x) = \frac{\omega_D(x)}{\int_0^1 \frac{\partial \omega_D(x)}{\partial n} ds} , \quad W_D = \frac{1}{\int_0^1 \frac{\partial \omega_D(x)}{\partial n} ds}.$$

Then $w_D(x)$ is constant for $x \in C$ and $\int_0^1 \frac{\partial \omega_D(x)}{\partial n} ds = 1$.

We easily see that

$$I_{D (\mu^*)} = W_D.$$

Let

$$V_n(M) = \frac{\sum_{i=1}^n G(M \cdot p_i)}{2\pi n} : M \in C.$$  

Since $C$ is closed $V_n(M)$ is lower semi-continuous, $V(M)$ attains its minimum $M_n$ on $C$ which is denoted by $V(p_1, p_2, \ldots, p_n)$. We take $p_1, p_2, \ldots, p_n$ on $C$ so that $V(p_1, \ldots, p_n)$ may be its upper bound $p V_n$ and $p_i; i = 1, 2, \ldots, n$ and $M$ converge

$$p_i \to p_i^* ; \quad i = 1, 2, \ldots, n.$$

$$M \to M^*.$$
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\[ V^* = \frac{1}{2\pi n} \left( \sum_{i=1}^{n} G(M \cdot p_i^*) \right), \]
then

\[ V^*(M) \geq_d V_n. \]

Lemma 5.

\[ \frac{1}{\mu D_{n+1}} \leq \frac{1}{2\pi n+1} \min \left( \sum_{p \in C} G(p \cdot p_i) \right) \]

\[ = \frac{1}{4\pi n+1} \min \left( \sum_{p \in C} \sum_{i=1}^{n+1} G(p \cdot p_i) \right), \]

then \[ \frac{1}{\mu D_{n+1}} \leq \frac{1}{2\pi n+1} \min \left( \sum_{p \in C} G(p \cdot p_i) \right). \]

\( p \) in the right hand term is \( p_1 \), otherwise \( \frac{1}{\mu D_{n+1}} \) cannot be minimum, because \( G(p_1 \cdot p_i) = G(p \cdot p_i). \)

We have \( \lim V^*(M) \geq \min M \geq V_n \geq \frac{1}{\mu D_{n+1}}. \)

Let \( A \) be closed set contained in \( F+R-\Gamma_0 \), and denoted by \( D_i \) domains such as

\[ \bigcap_{t} D_t = A. \]

We define \( D_n, I_A(M^k), \) and \( W_A \) in the following manner:

\[ A D_n = \lim_{t} D_t D_n, \]
\[ I_A = \lim_{t} I_D_t, \]
\[ W_A = \lim_{t} W_D_t. \]

If \( A \) is harmonic measure zero, then \( W_A = \infty. \)

Theorem. If \( A \) is harmonic measure zero set, then

\[ I_A = \infty, D_A = 0, W_A = \infty \] and vice versa.

34. Theorem 14. (G. C. Evans)\(^6\)

Let \( A \) be a closed and a set of relative harmonic measure zero of \( F+R \) with respect to \( F \), and let every point of \( A \) be regular for Evans Problem. Then there exists a positive harmonic function satisfying the following conditions,

1°. \( U(x) \geq 0 : x \in F. \)
2°. \( U(x) = 0 : x \in \Gamma_0. \)
3°. \( \int_{\Gamma_0} \frac{\partial U}{\partial n} ds = 1. \)
4°. \( \lim_{x \to A} U(x) = \infty. \)
5°. \( \lim_{x \to F-A} U(x) = +\infty. \)

Proof.
\[
\frac{1}{D_2} = \frac{1}{2\pi} \lim \lim \lim \min_{p_s, p_t \in D_j} \sum_{i < t} G(p_s, p_t) \cdot C_2
\]

\[
= \lim j I_D(\mu^*) = \lim j W_{D_j} = \infty, \quad \lim j D_2 = \infty.
\]

Therefore, for every number \( N \), there exists \( j_0(N) \) such as

\[
\frac{1}{D_j D_j} = \lim n \min_{p_s, p_t} \sum_{i = 1}^{n} G(p_s, p_t) \geq 2N: \quad j \geq j_0(N)
\]

therefore there exists \( n_0(D_j) = n_0(N) \) such as

\[
\frac{1}{n+1} = \min_{p_s, p_t \in \text{boundary of } D_j} \sum_{i = 1}^{n+1} G(p_s, p_t) \geq N: \quad n \geq n_0(N).
\]

But since \( D' \supset D'' \), it follows that \( \frac{1}{D_n D_{n+1}} \leq \frac{1}{D'' D_{n+1}} \) for every \( n+1 \).

We can choose adequately \( n+1 \) points \( p_1', ..., p_{n+1}' \) on the boundary \( D_j : j \geq j_0(N) \) so that

\[
\xi'(x) = \frac{1}{2n} \sum_{i = 1}^{n} G(x, p_i') \geq N: \quad x \in D_j,
\]

because harmonic measure of \( D_j \cap R \) is zero.

Since \( D_j, D_{j+1}, ..., \lim D_k = A \), and since \( A \) is closed,

we can choose from the sequence of systems \( ((p_1', p_2', ..., p_n'), j = 1, 2, 3, ..., \) a subsequence of \( i > i_0 \)

such as

\[
\lim_{j} p_n^{(i)} = p_k : \quad p_k \in A: \quad k = 1, 2, ..., n.
\]

Since from the hypothesis regular Green functions exist at \( p_k (k = 1, 2, ..., n) \), we denote them by \( G^*(x, p_k) \).

Let

\[
\eta_n(x) = \frac{1}{2n} \sum_{k = 1}^{n} G^*(x, p_k),
\]

Since \( \lim x \to p_k \eta_n(x) = \infty \), there exists a system of neighbourhoods \( v(p_k) \)

whose boundary is \( \gamma_k \) curve in \( F \), satisfying the following condition:

\[
\eta_n(x) \geq N; \quad \text{if} \quad x \in \sum_{k = 1}^{n} v(p_k).
\]

We choose a subsequence \( \{z_n'(x)\} \) from \( \{z_n(x)\} \) such as

\[
p_k' \in v(p_k) : \quad k = 1, 2, ..., n.
\]
Then \( \xi_n'(x) \) has the properties,

1. \( \int_{\Gamma_0} \frac{\partial \xi_n'}{\partial n} \, ds = \frac{1}{2} \cdot \xi_n'(x) \leq 0 \cdot \xi_n'(x) = 0 : x \in \Gamma_0 \).

2. \( \xi_n''(x) \leq m(x) \) if \( x \in \sum(p, \rho) + (R) \cap A \), for every \( n' \).

3. \( \lim_{x \to \sum(p, \rho)} \xi_n''(x) \geq N \).

From 2° \( \xi_n''(x) \) constitute a normal family in \( F \), we can choose a subsequence which converges uniformly in \( F \) to the limit function \( \xi_n(x) \)

\[
\lim_{n \to \infty} \xi_n''(x) = \xi_n(x) .
\]

The boundary of \( (D_j - \sum v(p, \rho)) = C_j + \sum \gamma_k \)

Since \( D_j - \sum v(p, \rho) \) has no mass for \( \xi_n''(x) \), then

\[
\lim_{x \to A \cap D_j - \sum v(p, \rho)} |\xi_n''(x) - \xi_n(x)| = \max_{x \in C_j + \sum \gamma_k} |\xi_n''(x) - \xi_n(x)| ,
\]

therefore \( \xi_n''(x) \) uniformly converges to \( \xi_n(x) \) in \( D_j - \sum v(p, \rho) \).

Then, \( \lim_{x \to A \cap D_j - \sum v(p, \rho)} \xi_n(x) = \lim_{j \to \infty} \lim_{x \to A \cap D_j - \sum v(p, \rho)} \xi_n''(x) \geq N \).

Let \( \xi_n(x) = \xi(x) + \eta_n(x) \), then \( \xi_n(x) \) has next properties,

1. harmonic positive when \( x \in F - \sum p_k \).

2. \( \int_{\Gamma_0} \frac{\partial \xi}{\partial n} \, ds = 1 \cdot \xi_n(x) = 0 : x \in \Gamma_0 \).

3. \( \lim_{x \to A - \delta} \xi_n(x) < +\infty \).

4. \( \lim_{x \to A - \delta} \xi_n(x) \geq N \).

We denote this function by \( \xi_n(x) \).

Take \( N \geq 3, N^1, N^2, \ldots, N^n : \lim N^n = \infty \), and corresponding to \( N^i \),

\[
\frac{1}{2^i} \xi_N(x), \frac{1}{2^{i+1}} \xi_N^2(x), \ldots, \frac{1}{2^n} \xi_N^N(x) ,
\]

\[
U^N(x) = \sum \frac{\xi_N^N(x)}{2^i}, \quad U(x) = \lim_n U^n(x) .
\]

\( U(x) \) has the properties mentioned in the theorem.

1° and 2° are clear, because if \( F - p \), \( U(x) \) is harmonic and
If \( p \in R \), \( \rho \in A \), there exists \( j_0 \) such as \( p \in D_j \); \( j \geq j_0 \) therefore there is a neighbourhood \( v(p) \), \( v(p) \cap D_j = 0 \), but \( v(p) \) has no mass, then
\[
\lim_{x \in \text{boundary of } v(p)} U(x) \leq \max_{x \in \text{boundary of } v(p)} U(x) < +\infty.
\]

\[
\lim_{x \to A} U(x) \geq U^n(x), \text{ for every } n.
\]

\[
\lim_{x \to A} U(x) \geq \lim_{x \to A} U^n(x) \geq \lim_{x \to A} \sum_{l=1}^{n} \frac{1}{2^l} \xi_{N^l}(x) \geq \lim_{n \to \infty} \left( \frac{N}{2} \right)^n = \infty.
\]

If \( F \) is a zero boundary Riemann surface, let us denote by \( 0 \) an inner point of \( F \) and take a disc of centre \( 0 \) then by the smoothing process and normalization of constant we easily have the harmonic function \( U(x) \) satisfying the next conditions:

1°. \( U(x) = \log |x| \) in the neighbourhood of \( 0 \) and \( x \) is the local parameter.

2°. \( \lim_{x \to A} U(x) = \infty. \)

3°. \( \lim_{x \to R - A} U(x) < \infty, \int_{\gamma} \frac{\partial U}{\partial n} ds = 1: \gamma \) curve enclosing 0 or \( A. \)

We can discuss with \( \chi \) in the same manner as \( G. \)

35. Let \( F \) be a Riemann surface with the relative boundary \( \Gamma \) and the ideal boundary point set \( R \). If there exists a harmonic function \( U(x) \) such that

\[
U(x) \geq 0, \quad \lim_{x \to R} U(x) = \infty.
\]

Then \( R \) is of a set of relative harmonic measure zero.

Proof. Let us denote by \( C_M \) the niveau curve on which \( U(x) = M \), and by \( C_M^N \) the domain in which \( U(x) \geq M \) respectively. If \( \infty > M \geq \max_{x \in \Gamma} U(x) \), then \( C_M \) is compact curve and surrounds \( R \), and \( \lim_{N \to \infty} C_M = R. \)

On the other hand we dedote by \( \omega_M(x) \) the bounded positive harmonic function such as

\[
\omega_M(x) = 1 : x \in C_M, \quad \omega_M(x) = 0 : x \in \Gamma,
\]

then \( \frac{\partial \omega_M}{\partial n} \leq 0 \) on \( C_M \), where normal derivative is inner direction with respect to \( C_M \),

\[
0 \leq \frac{\partial \omega_{M+\delta}}{\partial n} \leq \frac{\partial \omega_M}{\partial n} : x \in \Gamma; \text{ if } \delta \geq 0.
\]

By Green’s formula,

\[
\int_{\Gamma \cap C_M} \frac{\partial U}{\partial n} \omega_M ds = \int_{\Gamma \cap C_M} U \frac{\partial \omega_M}{\partial n} ds
\]
then
\[ \int \frac{\partial U}{\partial n} \, ds = \int \frac{\partial U}{\partial n} \, ds = \int U \frac{\partial \omega_\mu}{\partial n} \, ds = M \frac{\partial \omega_\mu}{\partial n} \, ds , \]
but
\[ \left| \int \frac{\partial U}{\partial n} \, ds - \int M \frac{\partial \omega_\mu}{\partial n} \, ds \right| \]
is bounded for every \( M \), on the other hand \( M \to \infty \) and \( \frac{\partial \omega_\mu}{\partial n} \int ds \geq 0 \) then
\[ \lim_{M \to \infty} \int \frac{\partial \omega_\mu}{\partial n} \, ds = 0, \]
this follows that \( \lim_{M \to \infty} \int \frac{\partial \omega_\mu}{\partial n} \, ds = 0 \),
then
\[ \lim \omega_\mu = 0 . \]

Thus \( R \) is a set of relative harmonic measure zero.

**Corollary.** Let \( F^* \) be a Riemann surface of which every ideal boundary point is regular for Evans's problem of zero boundary, and let \( F \) be a covering surface of \( F^* \). We denote by \( n(p) \) the number of times when \( p \) is covered by \( F \) and by \( D_n(F^*) \) the set
\[ D_n = E[n(p) \geq n] . \]
If \( \sup_{p \in F^*} n(p) \leq N \), then it is necessary and sufficient for \( F \) to be of zero boundary, that \( D_n(F^*) \) is a zero boundary Riemann surface.

Proof. The necessity is clear. We denote by \( F' \) the sub-Riemann surface which has its projection on \( D_n(F^*) \), and denote by \( 0 \) an inner point of \( D_n(F^*) \). We can construct a harmonic function \( U(x^*): x \in F^* \) such as negatively infinite at \( 0 \) and positively infinite at every point of the boundary of \( D_n(F^*) \) and let
\[ U(x) = U(x^*) ; x \in F', \ x \in D_n(F^*) . \]
Then from 32, \( F' \) has zero boundary, accordingly, \( F(F \supseteq F') \) is of zero boundary Riemann surface.

**Chapter IV.**

**Function theory on an abstract Riemman surface.**

36. The function theory of the \( z \)-plane has made much progress but in the Riemann surface, it is in infancy, this owing to the fact that the \( z \)-plane has very adequate metric when \( z = \infty \) is the only essential singularity and even when the set of singularity is not one point but of capacity zero set, the same metric in a sense can be constructed by the benevolance of Evans' theorem of the potential theory. On the contrary in the Riemann surface, there is no adequate metric except conformal
distance from the automorphic function theory, that is an invariant metric with respect to Fuchsoid group named hyperbolic metric.

It is clear that the metric defined by the Evans’ theorem is the best to study the function theory of Riemann surface, as in the $z$-plane.

We must begin with the notion of regularity of the function at $x: x \in F$, we call that $f(x)$ is regular at $x = x_0$, when $f(x)$ is regular with respect to the local parameter defined in the neighbourhood of $x_0$, then the notion of regularity will be defined at every inner point of $F$, nevertheless it loses its meaning at an ideal boundary point $\alpha$, and when we can prolong the Riemann surface $F$ so that $F$ may be contained as an inner point in the prolonged surface, we can define regularity as the preceding, for instance, if $F$ has a finite number of genus then $F$ is contained in a closed Riemann surface, in this case the essential difference about the notion of the regularity or the singularity of the function between the $z$-plane and in the Riemann surface does not occur.

But the fatal distinction between the $z$-plane and the Riemann surface is that there can exist the genuine ideal boundary point, which cannot be inner point in the other surface by no means as the second kind boundary point of Stoilow, at this point the notion of regularity or the singularity loses its meaning completely. Therefore in the theory of function on the Riemann surface, there is two cases when $f(x)$ is not regular, one of them is the case when $f(x)$ is not regular with the local parameter defined in the neighbourhood of $x = x_0$, and the other case is when $f(x)$ has no local parameter. When there is no local parameter at $x = x_0$, we call that $f(x)$ has a genuine singular point at $x = x_0$.

The behaviour of the function in the neighbourhood of a genuine singular point is most complicated, it can take any value without condition perfectly. But the theorem 6 shows the behaviour of the function to some extent. Thus even at a genuine singular point, we can define the regularity of $f(x)$ in the following manner. Let us denote by $V(p)$ a neighbourhood of $p$, 1°. We say that $f(x)$ is regular at $p$ in extended meaning, if $f(x)$ is regular in $V(p) \cap F$ and finite in $V(p)$, and $f(x)$ is meromorphic in $V(p)$ extended meaning in the case when $f(x)$ is a rational function of regular function 2°. If $f(x)$ is not meromorphic at $p$, then we say that $f(x)$ has an essential singular point in extended meaning. We see directly that $f(x)$ is meromorphic at $p$ then the number of roots of the equation $f(x) C$ is finite and if $f(x)$ is essential singular, then $f(x)$ covers almost all the $w$-plane except at most a non dense set infinitely many times for any small neighbourhood of $p$. 
Hypothesis

In the sequel we presuppose that all boundary ideal point are regular for Evans' problem.

The First fundamental theorem of Nevanlinna.

37. Let \( f(x) \) be one-valued and meromorphic function on the zero-boundary Riemann surface and denote by \( E \) the set of genuine or essential singular point set, then we see directly that \( E \subseteq F + R \) and closed. If \( F - E \) is zero-boundary Riemann surface, we say that \( E \) is capacity zero set, If \( E \) is capacity zero set, and all point of \( E \) is regular for Evans' problem, take an inner point denoted by 0 of \( F - E \) and call it the origin.

By Theorem 14 there exists a harmonic function \( U(x) \) which is negatively infinite at 0 and positively infinite at \( E \) and only there, take a conjugate function \( h(x) \) of \( U(x) \)

\[
z = e^{U(x) + i\theta(x)} = re^{i\theta}, \quad 0 \leq r < \infty.
\]

This parameter corresponds to \( z \), \( 0 < |z| < \infty \), in the \( z \)-plane. Let \( C_r \) be the niveau curve \( r(x) = \text{const} \), then \( C_r \) consists of a finite number of Jordan curves surrounding \( E \), we remark that

\[
\oint_{C_r} d\theta(x) = \int_{C_r} \frac{\partial U}{\partial n} ds = 2\pi,
\]

where \( ds \) is the arc length on \( C_r \) and \( n \) is the inner normal of \( C_r \), we use the same notation in R. Nevanlinna's book.

As \( w(x) \) is meromorphic, it is expressed in a power series with respect to the local parameter \( t \) in the neighbourhood of \( x = x_0 \) and the function \( z \) is one valued in the neighbourhood.

\[
w(x) = c_k t^{-k} + c_{k+1} t^{-k+1} + \ldots, \quad z(x) = d_0 + d_{\lambda} t^\lambda + d_{\lambda+1} t^{\lambda+1} + \ldots
\]

\( w(x) \) can have a finite number of negative power terms but \( z(x) \) is finite in \( F - E \) accordingly has no negative power terms exact at 0.

The differential \( \frac{dw}{dt}, \frac{dz}{dt} \) have the next transformation in the change of localparameter from \( t \) to \( \tau \).

\[
\frac{dw}{d\tau} = \frac{dw}{dt} \frac{dt}{d\tau}, \quad \frac{dz}{d\tau} = \frac{dz}{dt} \frac{dt}{d\tau}.
\]
then \( \frac{dw}{dt} = \frac{dw}{dz} \) is one-valued and meromorphic function we denote it by \( |w'(x)| \). In the neighbourhood of 0,
\[
x = e^{\frac{t}{\gamma-x}} = e^{\log t^n + \varepsilon} = t^* + \varepsilon,
\]
where \( t^* \) is the local parameter used in the normalization of constant of \( U(x) \) or \( \chi(x, 0, p) \) in the neighbourhood of 0,
then
\[
\frac{dz}{dt} = 1 \quad \frac{dw}{dt} = \frac{dw}{dz} \quad \text{at} \quad x = 0.
\]

**Theorem 1.** Let the domain bounded by \( C_r \) be denoted by \( \Delta_r \),
\[
n(r, a) = \text{the number of zero point of } w-a \text{ in } \Delta_r.
\]
\[
m(r, a) = \frac{1}{2\pi} \int_{C_r} \log \frac{1}{|w-a|} \, d\theta(x).
\]
\[
N(r, a) = \int_0^r \frac{n(r, a)}{r} \, dr.
\]
Then
\[
m(r, a) + N(r, a) = T(r) + \varphi(r),
\]
where \( \varphi(r) \leq \log |a| + \log |c| + \log 2 \) and \( c \) is the first non-vanishing coefficient of the Taylor's expansion of \( w-a \) in the neighbourhood of 0 with respect to \( t^* \).

Proof. \( U(x) \) is is one-valued, \( C_r \) does never intersect other \( C_{r'} \), and \( C_r \) is composed of a finite number of analytic, compact and closed curve. They enclose a compact domain which is denoted by \( \Delta_r \), therefore \( \Delta_r \) has only finite number of zero or pole of \( w(x) \), we denote by \( a, b \), and by \( k_v, h_v \) their multiplicity, we assume that \( C_r \) has no \( a, b \) on it.

If \( g_r(x, b) \) is the Green function of \( \Delta_r \) with pole at \( b \), then \( \log |w(x)| - \sum h_v g_r(x, b_v) - \sum k_v g_r(x, a_v) \) is regular harmonic in \( \Delta_r \) and \( \log |w(x)| \) on \( C_r \), accordingly by Green's formula
\[
\log |w(x)| = \int_{C_r} \log |w(x)| \frac{\partial g_r(x, x_v)}{\partial n} ds + \sum h_v g_r(x, b_v) - \sum k_v g_r(x, a_v).
\]

We put \( x = 0 \),
\[
\log |w(0)| = \int_{C_r} \log |w(x)| d\theta + \sum h_v g_r(0, b_v) - \sum k_v g_r(0, a_v).
\]
If \( w(0) = 0 \) or \( \infty \),
\[
w(x) = c_1 t^{k_1} + c_{k+1} t^{k+1} \ldots \text{ in the neighbourhood of 0. Since}\]
\[
\log |w(x)| - \lambda \log |z| = \lambda \log |t^*| + \log c_\lambda - \lambda \log |t^*| + \varepsilon, \text{ and } |\log z| \text{ is log r on } C_r,
\]
\[ \log |c_\lambda| = \frac{1}{2\pi} \int_{\mathcal{C}_r} \log |w(x)|d\theta + \sum h\nu g(x, b) - \sum k\nu g(x, a) + \lambda \log r. \]

On the other hand, \( g(0, a) = g(a, 0) = -U(a) + \log r = -\log r + \log r, \) where \( U(a) = \log r \) then we have the theorem.

We denote by \( K \) the Riemann of diameter 1 contacting the \( w \)-plane at \( w = 0 \), we put the ordinal distance,

\[ [a, b] = \frac{|a-b|}{\sqrt{1+|a|^2} \sqrt{1+|b|^2}}. \]

If we denote by \( A(r) \) the area on \( K \), which is covered by \( w(x) \) when \( x \) varies in \( \Delta_r \) then

\[ m^*(r, a) + N(r, a) = \frac{1}{2\pi} \int_{\mathcal{C}_r} \log \frac{1}{[w, a]} d\theta. \]

If we denote by \( |d\sigma| \) the line element on the \( w \)-sphere

\[ |d\sigma| = \frac{|dw|}{1+|w|^2}. \]

\[ U(w) = \log \frac{|dw|}{|d\sigma|} = \log |1+|w|^2| = 2 \log |w| + \varepsilon \left( \frac{1}{w} \right): \lim \varepsilon \left( \frac{1}{w} \right) = 0. \]

\[ \Delta z v(x) = \Delta z U(w(x)) = \Delta w \frac{dw}{dz} \] in an invariant,

\[ ds \frac{\partial v}{\partial r} = \frac{\partial v}{\partial t} \frac{\partial s}{\partial r} \] is an invariant also,

\[ df_z = \text{area element with respect to } z. \]

In the same manner as R. Nevanlinna,

\[ r \int_{\mathcal{C}_r} V(z)d\theta + 4\pi n(r, \infty) = 4 \int_{\Delta_r} \frac{|w'(x)|^2}{(1+|w|^2)^2} df_z \]

where \( A(r) = \int_{\Delta_r} \frac{|w'(x)|^2}{(1+|w|^2)^2} df_z \) is the area on \( K \) when \( x \) varies in \( \Delta_r \), we integrate between \( r_0 \) and \( r(0 < r_0 < r < \infty) \) and make \( r_0 \) converge to 0.

Then we have

\[ \log \frac{1}{2\pi} \int_{\mathcal{C}_r} \log \sqrt{1+|w|^2} (re^{i\theta})^2 d\theta + N(r, \infty) = \frac{1}{\pi} \int_{\mathcal{C}_r} A(t) dt + \log \sqrt{1+|w(0)|^2} \]

If we transform \( w \) into \( w_1 \), by \( w_1 = \frac{1+aw}{w-a} \), which this is the rotation of the Riemann sphere, then
If \( w(0) = a \) the right hand term is infinite but we cannot now replace other term as in the case of the \( z \)-plane because \( z \) is not uniform generally and has infinite number of periods corresponding to genus. Accordingly we assume that \( w(0) = \infty \), which is always possible. We denote \( T(r, \infty) = \int_{C_r} [w(re^{i\theta})a] + N(r, a) = \frac{1}{\pi} \int_0^r A(t) \frac{dt}{t} \log \frac{1}{[w(0), a]} \).

\[
\frac{1}{2\pi} \int_{C_r} \log \frac{1}{[w(re^{i\theta}), a]} + N(r, a) = \frac{1}{\pi} \int_0^r A(t) \frac{dt}{t} + \log \frac{1}{[w(0), a]}
\]

Theorem 2. \( T^*(r) \) is a monotone and convex function of \( \log r \).

Theorem 3. \( \lim_{r \to \infty} \frac{T(r)}{\log r} > 0 \).

If \( \lim_{r \to \infty} T(r) = 0 \), \( w(x) \) must reduce to a constant.

Proof.

\[
\int_{r_0}^r \frac{A(t)}{t} \frac{dt}{t} > \int_{r_0}^r \frac{A(t)}{t} \frac{dt}{t} > A(r_0) \log \frac{r}{r_0}.
\]

Theorem 3'. If \( f(x) \) has an essential singularity (classical or extended meaning), then

\[
\lim_{r \to \infty} \frac{T(r)}{\log r} = \infty.
\]

Let \( p \) be an essential singularity and \( V_1 \supset V_2 \supset \ldots \supset V_n \supset \ldots \rightarrow p \) be a sequence of neighbourhoods covering to \( p \) and \( e_n \) be the set of values ommitted by \( f(x) \) in \( V_n \), then \( e_n \) is non dense, so that

\[
e = \sum e_n \text{ is of first category.}
\]

Hence there exists a point \( w_0 \), which does not belong to \( e \), then \( w_0 \) is converyed by \( f(x) \) infinitely many times about \( p \) so that

\[
\lim_{r \to \infty} \frac{T(r)}{\log r} = \infty.
\]

38. **Meromorphic functions defined in a compact domain in the zero-boundary Riemann surface.** Let \( D \) be a compact domain in the zero boundary surface bouded by Jordan curve \( C \), then by theorem 11 we can construct the domain function \( U(x) \) which is negatively infinite at an inner point 0 of \( D \). Accordingly we can discuss the function theory as in the case \( 0 \leq |z| \leq 1 \) in the \( z \)-plane.
39. **Meromorphic functions in a neighbourhood of a closed harmonic measure zero ideal boundary point**.

Let $D$ be a domain in the Riemann surface (positive or zero boundary) bounded by Jordan curve $C$ and closed set $E$ of $F+R$ of relative harmonic measure zero, with respect to domain $D$.

We easily have all theorems studies about the behaviour of function in the $z$-plane. Since harmonic measure $E = 0$, by theorem 15 we can construct a harmonic function $U(x)$ satisfying all conditions of $U(x)$.

Then we write results without proof because it is the same in the $z$-plane.

**Theorem 1'**. First fundamental theorem of Nevanlinna.

$$ T(r, a) = T(r) + 0(\log r), \text{ where } T(r) = \frac{1}{\pi} \int_{r_0}^{r} \frac{A(t)}{t} \, dt $$

**Theorem 4'**. Second fundamental theorem of Nevanlinna.

Let $e$ be a bounded closed set of positive capacity on $K$. Then we can distribute a positive mass $d\nu(a)$ on $e$, such that

$$ \int_{e} \log \left( \frac{1}{w(a)} \right) \, d\nu(a) : \int_{e} d\nu(a) = 1, $$

is bounded on $K$, hence by Theorem 1',

$$ T(r) = \int N(r, a) d\nu(a) + 0(\log r), $$

and the order is defined by the formula

$$ \lim_{r \to \infty} \frac{\log T(r)}{\log r} = \rho, $$

**Theorem 3'**.

$$ \lim_{r \to \infty} \frac{T(r)}{\log r} > 0. $$

$E$ is singular point set, but in the Riemann surface, $E$ can consist of only genuine singular points where $w(x)$ may have its behaviour as if it were regular point therefore we cannot expect that $w(x)$ is not bounded in the neighbourhood of $E$. But further if we suppose that $E$ has at least an essential singular (classical or extended meaning) point, we can conclude that

$$ \lim_{r \to \infty} \frac{T(r)}{\log r} = \infty, $$

as in the same way used in Theorem 3.

40. **Some consequences of Fundamental theorems**.

**Theorem**. Let $D$ be a part domain which is bounded by Jordan curves $C$ and by a closed set of $F+R$ of relative harmonic measure zero...
lying inside of \( C \), and let \( w(x) \) be one-valued and meromorphic in \( D \) and 
\[
\lim_{r \to \infty} \frac{T(r)}{\log r} = \infty \quad (\text{or } E \text{ has at least one essential singular \( (\text{classical or extended meaning) point). Then,})
\]

1°, \( w(x) \) takes any value infinitely many times, except a set of 
capacity zero. More precisely, 
\[
\lim_{r \to \infty} \frac{N(r, a)}{T(r, a)} = 1 \quad \text{except values of capacity zero.}
\]

2°. If further \( w(x) \) is of finite order \( \rho > 0 \) and \( x_n = x(a_n) \) is the zero 
point of \( w - a \) and \( r_n(a) = r(z_n) \) then 
\[
\sum_{n=1}^{\infty} \left[ \frac{1}{r_n(a)^{\rho+\epsilon}} \right] < \infty : \quad \text{for all } a.
\]

\[
\sum_{n=1}^{\infty} \left[ \frac{1}{r_n(a)^{\rho-\epsilon}} \right] = \infty,
\]
except values of capacity zero, where \( \epsilon \) is any positive number.

41. Theorem 6. (W. Gross)\(^7\). Let \( w = w(x) \) be one valued and meromorphic in \( F - E \) have at least one essential singular point of \( E \), and let 
\( x = x(w) \) be its inverse function.

i. If \( x(w) \) is regular at \( w \), then we can continue \( x(w) \) analytically 
on half lines; \( w = w_0 + r \theta (0 \leq r < \infty) \) indefinitely, except for values or 
measure zero.

ii. If \( w(x) \) is regular on a segment; \( w = w_0 + r^\theta (\leq r_0 < r < r_1) \), then 
starting from \( w = w_0 + r \), we can continue \( x(w) \) analytically along 
circles; \( w = w_0 + r^\theta (-\infty < \theta < \infty) \) indefinitely, except for \( r \) values of measure 
zero.

42. Theorem 7. (Cartwright-Noshiro)\(^12\). From Theorem 6 under the 
same condition as Theorem 5, without \( \lim_{r \to \infty} \frac{T(r)}{\log r} = \infty \). Let \( x = x(w) \) be 
the inverse function of \( w = w(x) \) and \( F \) be its Riemann surface spread 
over the \( w \)-plane and \( (w_0) \) be its boundary point, whose projection on 
the \( w \)-plane is \( w \). Then \( (w_0) \) is an accessible boundary point and \( w_0 \) is 
asymptotic values of \( w(x) \), i.e., there exists a curve \( L \) in \( D \) ending at a 
point on \( E \), such that \( w(x) \to w_0 \) when \( x \to x_0 \) along \( L \).

Theorem 8. Under the same condition as 5′ and if \( E \) has further 
at least one essential singularity, and if \( f(x) = a \) in \( D \), then there exists 
a curve \( L \) in \( D \) ending at a point \( x \) on \( E \) such that \( w(x) \to a \), when \( x \to x_0 \) along \( L \).

43. Direct transcendental singularity.

Let \( z(w) \) be defined on a Riemann surface \( F_w \) spread over the \( w \)-plane 
and \( (w_0) \) be a boundary point of \( F_w \), whose projection on the \( w \)-plane
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is w. F. Iversen called \((w_0)\) a direct transcendental singularity of \(w(x)\) if \(w_0\) is lacunary for a connected piece \(F_0\) of \(F\) which has \((w_0)\) as its boundary and lies above a disc \(K; |w-w_0| \leq \rho\) about \(w_0\).

Theorem 9. Under the same condition as Theorem 5', the set of projection of direct transcendental singularity of the inverse function \(x(w)\) of \(w(x)\) on the \(w\)-plane is of capacity zero.\(^{13}\)

44. Behaviour of the inverse function \(x(w)\) of \(w(x)\) at its transcendental singularity.

Let \(w = w(x)\) satisfy the same condition as theorem 5' and \(x = x(w)\) be its inverse function and \(F\) be its Riemannian surface spread over the \(w\)-plane. Let \((w_0)\) be its boundary point, whose projection on the \(w\)-plane is \(w_0\).

A \(\delta\)-neighbourhood \(U\) of \((w_0)\) is defined by a connected piece of \(F\) which lies above a disc \((w-w_0) \leq \delta\) and has \((w_0)\) as its boundary point, let \(U\) correspond to a domain \(\Delta\) on the Riemannian surface, then \([w, w_0] \leq \delta\) in \(\Delta\) and \([w(x)-w_0] = \delta\) on the boundary \(\partial\) of \(\Delta\), except the point on \(E\). Since \((w_0)\) is an accessible boundary point of \(F\), there exists a curve on \(F\) ending at \((w_0)\), which corresponds to a curve \(L\) in \(\Delta\) ending at a point \(x\), on \(E\). Let \(z = r(x)e^{i\theta(x)}\) be defined as theorem 14 and the part of \(\Delta\), such that \(r(x) \leq r\) and \(r(x) = r\) be denoted by \(\Delta_r\) and \(\theta_r\) respectively, let \(K\) be the Riemannian sphere of diameter 1, which touches the \(w\)-plane at \(w = 0\) we put \(n(r, a)\) the number of zero point of \(w(x)-a\) in \(\Delta\), where \([a-w_0] \leq \delta\),

\[
N(r, a, \Delta) = \int_{r_0}^{r} \frac{n(r, a; \Delta)}{r} \, dr ,
\]

\[
m(r, a, \Delta) = \frac{1}{2\pi} \int_{0}^{2\pi} \log \left( \frac{1}{w(x), a} \right) d\theta(x) ,
\]

\[
T(r, a; \Delta) = m(r, a; \Delta) + N(r, a; \Delta) ,
\]

\(A(r; \Delta), S(r; \Delta)\) are the same for \(\Delta\).

\(L(r)\) the sum of length of the curves on \(K\), which corresponds to Theorem 1'.

Theorem 1'.'

\[
T(r, a; \Delta) = T(r, \Delta) + 0(\int_{r_0}^{r} \frac{L(r)}{r} \, dr) + 0(\log r) ,
\]

\[
T(r, \Delta) = \int_{r_0}^{r} \frac{S(r; \Delta)}{r} \, dr ,
\]

where
\[ L(r) = O(\sqrt{T(2r, \Delta) \log r}) \quad : \quad \text{for all } r \geq r_0. \]
\[ L(r) = O(\sqrt{T(r ; \Delta) \log T(r ; \Delta)}), \]

except certain intervals \( I_n \) such that

\[ \sum_{n=1}^{\infty} d \int_{I_n} \log \log r < +\infty, \]

the order being

\[ \rho = \lim_{r \to \infty} \frac{T(r ; \Delta)}{\log r}. \]

**Theorem 3'.** If

\[ \lim_{r \to \infty} \frac{T(r ; \Delta)}{\log r} < 0, \]

and if \( \Delta \) is bounded by \( E \) which containing at last one essential singular (classical or extended meaning) point, then

\[ \lim_{r \to \infty} \frac{T(r ; \Delta)}{\log r} = \infty. \]

More generally, if

\[ \lim_{r \to \infty} \frac{T(r ; \Delta)}{\log r} = \infty, \]

then

i. \( w(x) \) takes only values in \([w, w_0] < \delta\) infinitely many times in \( \Delta \), except a set of values in \([w, w_0] < \delta\) of capacity zero.

ii. If further \( w(x) \) is of finite order \( \rho(>0) \) in \( \Delta \), then

\[ \sum \left[ \frac{r_i(a)}{x^{\delta - \delta}} \right] < \infty, \quad : \quad \text{for all } a \in [w, w_0] < \delta. \]
\[ \sum \left[ \frac{r_i(a)}{x^{\delta - \delta}} \right] = \infty, \]

except values in \([w - w_0] < \delta\) of capacity zero, where \( \delta \) is any small positive number and \( r(a_n) = r(x_n), x_n \) being the zero point of \( w - a \) in \( \Delta \).

45. **Applications to the theory of the cluster set.**

Let \( F \) be an abstract Riemnn surface with a relative boundary \( \Gamma_0 \).

In the sense of Stoilow, we call an ideal point \( \alpha \) defined by the system of the neighbourhood of \( \alpha \) such as \( \bigcap_{\delta} V_{\delta}(\alpha) = \alpha. \)

\( F - V_{\delta}(\alpha) \) has another set of boundary point \( R' \) defined by the system of the neighbourhood \( W_{\delta}(R') \) such as \( \bigcap_{\delta} W_{\delta}(R') = R'. \)

Let us denote by \( \omega_{\delta}(x) \) the positive harmonic function in \( F - V_{\delta}(\alpha) - W_{\delta}(R') \) with the boundary values 1 on the boundary of \( V_{\delta}(\alpha) \) and 0 on \( \Gamma_0 \) and the boundary of \( W_{\delta}(R') \).

If \( \lim_{\delta \to 0} \lim_{r \to \infty} \omega_{\delta}(x) = 0 \), then we call \( \alpha \) a **point-wise** boundary point.

Let \( D \) be an arbitrary connected domain of Riemnn surface \( F \) and
$C$ be its boundary point set of $D$ included in $F + R$, and $E$ be closed relative harmonic measure zero boundary point defined as the preceding, being contained in $C$ and further suppose that $\alpha$ is a point-wise and not isolated from $C$.

$f(x)$ is one valued meromorphic function in $D$. We denote by $\overline{\{f(x)\}}_{x \in \Lambda}$ the closure of the set attained by $f(x)$ in $\Lambda$.

Let us associate two cluster sets,

$$S_a^{(D)} = \bigcap_{t} \overline{\{f(x)\},} \quad S_a^{(E)} = \bigcap_{p \in \partial \Omega \cap \mu \neq \eta} \bigcup_{p \in \Omega} S_{\mu \eta}^{(D)},$$

then we easily have the following theorems as in the case of $D$ being planer,

Theorem\textsuperscript{11).} (F. Iversen, A. Beuring, K. Kunugi, M. Tsuji)

$B(S_a^{(D)}) \subset S_a^{(E)}$, that is $\Omega = S_a^{(D)} - S_a^{(E)}$ is an open set, where $B(S_a^{(E)})$ is the boundary of $S_a^{(D)}$.

Theorem\textsuperscript{12).} (S. Kametani, M. Tsujii).

Let $F$ have a boundary point set $R$ of at most relative harmonic measure zero and let all points of $E$ are regular for Evans’ problem.

If $\Omega = S_a^{(D)} - S_a^{(E)}$ is not empty, then $f(x)$ takes any value, except a set of at most capacity zero, belonging to $\Omega$ infinitely many times in any neighbourhood of $\alpha$.

Reference


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