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ON STABLE JAMES NUMBERS OF STUNTED COMPLEX OR QUATERNIONIC PROJECTIVE SPACES

HIDEAKI ŌISHIMA

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Following James [7] we denote the stunted complex \( F=C \) or quaternionic \( F=H \) projective spaces by \( FP_{n+k} \) (or \( P_{n+k} \)) for positive integers \( n \) and \( k \), that is

\[
FP_{n+k} = FP_{n+k-1}/FP^{*+1}.
\]

Let \( d \) be the dimension of \( F \) over the real number field. Let \( i: S^{*d}=FP_{n+1} \rightarrow FP_{n+k} \) be the inclusion. By stable James number \( F\{n, k\} \) we mean the order of the cokernel of

\[
\deg = i^*: \{FP_{n+k}, S^{*d}\} \rightarrow \{S^{*d}, S^{*d}\} = Z
\]

where \( \{X, Y\} \) denotes the group of stable maps from a pointed space \( X \) to an other pointed space \( Y \). In the previous papers [5, 8, 9, 10] we used the notations \( k\{FP_{n+k-1}, S^{*d}\} \) instead of \( F\{n, k\} \) and estimated \( F\{1, k\} \).

The first purpose of this note is to determine \( F\{n, k\} \) for small \( k \), that is, we shall determine \( H\{n, k\} \) for \( k\leq 4 \), estimate them for \( k=5 \), determine \( C\{n, k\} \) for \( k\leq 8 \) and estimate them for \( k=9 \) and 10. These shall be done in §2 and §3. The second purpose is to show that \( F\{n, k\} \) can be identified with the James numbers defined by James in [6]. This shall be done in §4.

An application of this note to \( F \)-projective stable stems shall be given in [11].

In this note we work in the stable category of pointed spaces and stable maps between them, and we use Toda's notations of stable stems and Toda brackets in [14] freely.

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1. Preliminaries

In what follows we shall be working with both real \( K \)-cohomology theory \( KO^* \) and complex \( K \)-cohomology theory \( K^* \). We use the following notations. \( KO^* \) and \( K^* \) denote both the \( K \)-functors and the coefficient rings. By the same letter \( \xi=\xi \) we denote the canonical \( F \)-line bundle over \( FP_{n} \),
the underlying complex or real vector bundle of it. Put  \( z = \xi - \frac{d}{2} \in K(FP_n) \) and  \( t = (-1)^{1+d/2} c_d(z) \in H^d(FP_n; Z) \), where  \( c_d(z) \) denotes the  \( m \)-th Chern class of  \( \xi \). Put also
\[
\tilde{\xi} = \xi - \frac{d}{2} \in K(\tilde{S}^p(FP_n)) = \widetilde{KO}^*(FP_n). 
\]
The formal power series  \( \phi_F(x) \) are defined to be  \( \exp(x) - 1 \) for  \( F = \mathbb{C} \) or  \( \exp(\sqrt{x}) + \exp(-\sqrt{x}) - 2 \) for  \( F = H \). The rational numbers  \( \alpha_F(n, j) \) are defined by  \( (\phi_F(x)|x|^n)^j = \sum_{i=0}^n \alpha_F(n, j)x^i. \)

\( \text{ch}: K(\_ \to H^* \_ ; Q) \) denotes the Chern character. Then the followings are well known.

**Proposition 1.1.**
(i)  \( K(FP_n) = \mathbb{Z}[z]/(z^n) \).
(ii)  \( KO^*(HP_n) = KO^*[\tilde{\xi}]/(\tilde{\xi}^n) \) and  \( \tilde{\xi}_{|HP_{n-1}} = \tilde{\xi}_{n-1} \).
(iii)  \( H^*(FP_n; Z) = Z[t]/(t^n) \).
(iv)  \( \text{ch}(z) = \phi_F(t) \).

Let  \( i = i_0: FP_{n+k,k} \subset FP_{n+k+1,k+1} \) be the inclusion for  \( l > 0 \),  \( q = q_m: FP_{n+k,k} \to FP_{n+k,k-m} \) the canonical quotient map for  \( 0 \leq m < k \),  \( p = p_k: S^d \to FP_n \) the Hopf bundle projection, and  \( p_{n+k,k}: S^{(n+k)d-1} \to FP_{n+k,k} \) the composition of  \( p_{n+k} \) and  \( q_{n-1}: FP_{n+k} \to FP_{n+k,n+k-1} \). Let  \( G_k \) denote the  \( k \)-stem of the stable groups of spheres. Let  \( e_C: G_k \to \mathbb{Q}/\mathbb{Z} \) or  \( e_R: G_{8k+3} \to \mathbb{Q}/\mathbb{Z} \) be the Adams' complex or real  \( e \)-invariant respectively [1]. Then we have

**Proposition 1.2 (Adams[1]).**  \( e_C: G_1 \to Z_2, e_R: G_3 \to Z_{24}, e_C: G_7 \to Z_{240} \) and  \( e'_R: G_{11} \to Z_{504} \) are isomorphisms, while there is a split exact sequence
\[
0 \to Z_2 \{7k\} \to G_{15} \to Z_{480} \to 0.
\]

In [10] we obtained the following.

**Proposition 1.3.**  For  \( f \in \{FP_{n+k,k}, S^d \} \) we have
\[
e_C(f \circ p_{n+k,k}) = -\deg(f)\alpha_F(n, k).
\]

Since  \( e_R = 2e'_R \) on  \( (8k+3) \)-stems [1],  \( e_R \) gives more precise informations about 2-primary components, so we compute  \( e_R(f \circ p_{n+k,k}) \) for the case of  \( F = H \) and  \( k \equiv 1 \text{ mod}(2) \) or  \( F = C \) and  \( k \equiv 2 \text{ mod}(4) \).

We use the following notations. Let  \( g_C \in \tilde{K}(S^2) \) and  \( g_S \in \tilde{KO}(S^p) \) denote the Bott generators.  \( \psi^k \) denotes the Adams operation. Let  \( c: KO^* \to K^* \) be the complexification and  \( r: K^* \to KO^* \) the real restriction. Put  \( z_0 = r(z) \in \tilde{KO}(CP_n) \) and  \( z_j = r(g_C^jz) \in \tilde{KO}^{2j}(CP_n) \). Put also  \( y_{2k} = \tilde{g}^{2k}_C \in KO^{2k} \) and  \( y_{2k+1} \in KO^{2k+4} \) the generator satisfying  \( c(y_{2k+1}) = 2g_C^{2k-2} \) for integer  \( k \). For  \( f \in \{X, Y\}, C(f) \) denotes the mapping cone of  \( f \).

We consider the case of  \( F = H \) and  \( k \equiv 1 \text{ mod}(2) \) or  \( F = C \) and  \( k \equiv 2 \text{ mod}(4) \).
Given \( f \in \{ FP_{n+k,b}, S^{nd} \} \), we have the commutative diagram

\[
\begin{array}{ccc}
S^{(n+k)d-1} & \xrightarrow{p_{n+k,k}} & FP_{n+k,k} \\
\downarrow & & \downarrow f \\
S^{(n+k)d-1} & \xrightarrow{f \circ p_{n+k,k}} & S^{nd} \\
\end{array}
\]

Apply \( KO^{nd} \) and \( K^{nd} \) to this diagram; since \( KO^{nd}(S^{(n+k)d-1}) = K^{nd}(S^{(n+k)d-1}) \) is a finite group, we have the following commutative diagram in which the horizontal sequences are exact.

We can choose generators \( a, b \) of the free subgroup of \( KO^{nd}(C(f \circ p_{n+k,k})) \) such that \( a' = c(a), 2b' = c(b) \), \( j^*(a') \) generates \( K^{nd}(S^{nd}) \). Hence we can put

\[
f'^*(a') = g_{c^{nd/2}} z^{n+i}
\]

for some integers \( a_i \). Then by the proof of (1.1) of [10] we have

\[
\alpha_i = \deg(f) \alpha_F(n, i) \quad \text{for} \quad 0 \leq i \leq k-1,
\]

(1.4)

And we have

**Proposition 1.5.** In case of \( F=H \) and \( k \equiv 1 \mod(2) \) or \( F=C \) and \( k \equiv 2 \mod(4) \) we have

(i) \( \epsilon^f_k(f \circ p_{n+k,k}) = \frac{1}{2} a_k - \frac{1}{2} \deg(f) \alpha_F(n, k) \),

(ii) if \( F=H \), \( a_k \equiv 0 \mod(2) \),

(iii) if \( F=C \), \( n \equiv 1 \mod(2) \) and \( \deg(f) \) is known, \( a_k \mod(2) \) is computable.
Proof. First consider the case of $F=H$ and $n \equiv 0 \mod(2)$. By Bott periodicity we can use $\widetilde{KO}$ and $\tilde{K}$ instead of $\widetilde{KO}^n$ and $\tilde{K}^n$. Then we have
\[
\psi^2(a) = 4^*a + \lambda b
\]
for some integer $\lambda$, and
\[
e_k(f \circ p_{n+k,k}) = \lambda/(4^*(4^*-1)).
\]
We have
\[
\begin{align*}
\psi^2(a') &= c(\psi^2(a)) = 4^*a' + 2\lambda b', \\
\psi^2(f^*(a')) &= \psi^2(\sum_{i=0}^{k} a_i z^{n+i}) = \sum_{i=0}^{k} a_i (z^2 + 4z)^{n+i} \\
&= \sum_{j=0}^{k} \sum_{i=0}^{k} a_i (x^{-1}) 4^*4^{n+2i} z^{n+i}, \\
\psi^2(f^*(a')) &= f^*(\psi^2(a')) = f^*(4^*a' + 2\lambda b') \\
&= 4^* \sum_{i=0}^{k} a_i z^{n+i} + 2\lambda z^{n+k}.
\end{align*}
\]
Comparing the coefficients of $z^{n+k}$, we have
\[
2\lambda = 4^*(4^*-1)a_k + \sum_{i=0}^{k-1} a_i (x^{-1}) 4^*4^{n+2i-k}.
\]
Then by (1.4) we have
\[
e_k(f \circ p_{n+k,k}) = \frac{1}{2} a_k - \frac{1}{2} \deg(f) \alpha_B(n, k)
\]
as desired. Next we show (ii). Put $f^*(a) = \sum d_i y_i^* z^{n+i}$. Then
\[
c(f^*(a)) = \sum_{i=0}^{k} d_i c(y_i^* z^{n+i}) (c(\xi))^n = \sum_{i=0}^{k} d_i \xi_i z^{-2(n+i)} (\xi_2 z)^{n+i} \\
= \sum_{i=0}^{k} d_i \xi_i z^{n+i},
\]
where $\xi_i = 1$ (if $i$ is even) or $2$ (if $i$ is odd). We have also
\[
c(f^*(a)) = f^*(c(a)) = \sum_{i=0}^{k} a_i z^{n+i}.
\]
Therefore $a_k = d_k \xi_k = 2d_k$.

In case of $F=H$ and $n \equiv 1 \mod(2)$, (i) and (ii) can be proved by the quite parallel arguments to the above. We omit the details.

For $F=C$ (i) can be proved by the same methods as the above. We only prove (iii). First we consider the case of $n \equiv 3 \mod(4)$. Put $n = 4m + 3$ and $k = 4l + 2$. By Bott periodicity we can use $\widetilde{KO}^{-2}$ and $\tilde{K}^{-2}$ instead of $\widetilde{KO}^n$ and $\tilde{K}^n$. By Theorem 2 of Fujii [4], it is easily seen that $\widetilde{KO}^{-2}(CP_{4m+4l+6,4l+3})$ can
be identified with the free subgroup of $\tilde{KO}^{-2}(CP_{4m+4l+6})$ generated by $z_1 z_0^{2m+1}$, $z_1 z_0^{2m+2}$, ..., $z_1 z_0^{2m+2l+2}$. So we can put $f^*(a) = \sum_{i=0}^{2l+1} d_i z_0^{2m+1+i}$ for some integers $d_i$. Then

$$c(f^*(a)) = \sum_{i=0}^{2l+1} d_i (c(z_1)(c(z_0))^{2m+1+i} = g_c \sum_{i=0}^{2l+1} d_i (z - \bar{z}) (z + \bar{z})^{2m+1+i}$$

where $\bar{z} = -z + z^2 - z^3 + \cdots$. We have also

$$c(f^*(a)) = f^*(c(a)) = g_c \sum_{i=0}^{4l+2} a_i z_0^{4m+3+i}.$$  

So we have

$$\sum_{i=0}^{4l+2} a_i z_0^{4m+3+i} = \sum_{i=0}^{2l+1} d_i (2z - z^2 + z^3 - \cdots) (z^2 - z^3 + \cdots)^{2m+1+i}.$$  

Calculating this equation over the mod 2 integers, we have

$$\sum_{i=0}^{4l+2} a_i z_0^{4m+3+i} \equiv \sum_{i=0}^{2l+1} d_i (z^2 + z^3 + \cdots)^{2m+2+i} \mod(2, z_0^{4m+4l+6})$$

$$\equiv \sum_{j=0}^{4l+2} \sum_{i=0}^{2l+1} d_i (z_0^{2m+1+i-j-i}) z_0^{4m+4+j} \mod(2),$$

since $(x^2 + x^3 + \cdots)^n = \sum_{j=0}^{n-1} \binom{n-1}{j} x^j$. Then

(1.6) $a_i \equiv \sum_{j=0}^{2l+1} d_j (z_0^{2m+1+i-j}) \mod(2)$ for $1 \leq i \leq 4l+2$.

By (1.4) and (1.6) for $1 \leq i \leq 4l+1$, $d_j \mod(2)$ is determined for $0 \leq j \leq 2l$, so the equation

(1.6)$'$

$$a_{4l+2} \equiv \sum_{j=0}^{2l+1} d_j (z_0^{2m+4l+2-j}) \mod(2)$$

$$\equiv \sum_{j=0}^{2l+1} d_j (z_0^{2m+4l+1-2j}) \mod(2)$$

determines $a_{4l+2} \mod(2)$, here we use the fact $(z_0^{2l}) \equiv 0 \mod(2)$ for any $i$ and $j$.

Next we consider the case of $n \equiv 1 \mod(4)$. Put $n = 4m+1$. We use $\widetilde{KO}^{-6}$ and $\widetilde{K}^{-6}$ instead of $KO^{-2n}$ and $K^{-2n}$. Then we can put $f^*(a) = \sum_{i=0}^{2l+1} d_i z_0^{2m+1+i}$ for some integers $d_i$. By the same arguments as the above we have

(1.7) $a_i \equiv \sum_{j=0}^{2l+1} d_j (z_0^{2m+1-i-j}) \mod(2)$ for $1 \leq i \leq 4l+2$

and in particular

(1.7)$'$

$$a_{4l+2} \equiv \sum_{i=0}^{2l+1} d_j (z_0^{2m+4l+2-i}) \mod(2).$$

These and (1.4) determine $a_{4l+2} \mod(2)$. This completes the proof.
To compute $F\{n, k\}$ by inductive step on $k$ we prepare the followings.

**Proposition 1.8.** $F\{n, k\}$ is a divisor of $F\{n, k+1\}$.

Proof. It is trivial by definition.

**Proposition 1.9.** For $f \in \{FP_{n+k,k}, S^{nd}\}$ with $\deg(f) = F\{n, k\}$ we have

$$F\{n, k\} \# e_c(f \circ p_{n+k,k}) | F\{n, k+1\} | F\{n, k\} \# (f \circ p_{n+k,k})$$

where $\#g$ denotes the order of $g$ and $a \mid b$ implies that $a$ is a divisor of $b$.

Proof. Choose $f' \in \{FP_{n+k+1,k+1}, S^{nd}\}$ with $\deg(f') = F\{n, k+1\}$. Since $i_1 \circ p_{n+k,k} = 0$, we have

$$0 = e_c(f' \circ i_1 \circ p_{n+k,k}) = -\deg(f' \circ i_1) \alpha_f(n, k) = -F\{n, k+1\} \alpha_f(n, k) = -F\{n, k\} \alpha_f(n, k) F\{n, k+1\} / F\{n, k\}$$

$$= -e_c(f \circ p_{n+k,k}) F\{n, k+1\} / F\{n, k\}.$$

Hence the first part of the conclusion is obtained. Since $(\#(f \circ p_{n+k,k})) / F\{n, k\} = 0$, there exists $h \in \{FP_{n+k+1,k+1}, S^{nd}\}$ with $h \circ i_1 = (\#(f \circ p_{n+k,k})) f$. Then $\deg(h) = \deg(f) \#(f \circ p_{n+k,k}) = F\{n, k\} \#(f \circ p_{n+k,k})$. Since $\deg(h)$ is a multiple of $F\{n, k+1\}$, the second part of the conclusion follows.

**Proposition 1.10.** For $f \in \{FP_{n+k,k}, S^{nd}\}$ with $\deg(f) = F\{n, k\}$ there exists $h \in \{FP_{n+k,k-1}, S^{nd}\}$ with $h \circ i_1 = (F\{n, k+1\} / F\{n, k\}) f \circ p_{n+k,k} = h \circ i_1 \circ p_{n+k,k}$.

Proof. Consider the exact sequence

$$\cdots \to \{FP_{n+k,k-1}, S^{nd}\} \xrightarrow{q_1^*} \{FP_{n+k,k}, S^{nd}\} \xrightarrow{\text{deg}} \{FP_{n+1,1}, S^{nd}\} \to \cdots$$

Take $f' \in \{FP_{n+k+1,k+1}, S^{nd}\}$ with $\deg(f') = F\{n, k+1\}$. Then $\deg((F\{n, k+1\} / F\{n, k\}) f - f' \circ i_1) = 0$. So there exists $h \in \{FP_{n+k,k-1}, S^{nd}\}$ with $q_1^*(h) = (F\{n, k+1\} / F\{n, k\}) f - f' \circ i_1$ by exactness. Then $h \circ i_1 \circ p_{n+k,k} = ((F\{n, k+1\} / F\{n, k\}) f - f' \circ i_1) \circ p_{n+k,k}$ as desired.

**Proposition 1.11.** $C\{2n, 2k\}$ is a divisor of $H\{n, k\}$.

Proof. Consider the commutative diagram

$$\begin{array}{ccc}
CP_{2n+2k,2k} & \supset & CP_{2n+1,1} = S^{4n} \\
S^{4n+4k-1} \downarrow \pi & & \downarrow \pi' \\
HP_{n+k,k} & \supset & HP_{n+1,1} = S^{4n}
\end{array}$$

in which all maps are the canonical ones. For our purpose it suffices to show that $\pi'$ is a homotopy equivalence. Indeed this holds because in the following
commutative diagram $\pi^*$ is an isomorphism.

\[
\begin{array}{ccc}
H^{n}(CP_{2n+2k};Z) & \xrightarrow{q^*} & H^{n}(CP_{2n+2k};Z) \\
\pi^* & \cong & \pi^* \\
H^{n}(HP_{n+k};Z) & \xrightarrow{q^*} & H^{n}(HP_{n+k};Z) \\
\end{array}
\]

Next we compute $\epsilon$-invariants of some elements.

**Lemma 1.12.** Suppose that there is a commutative diagram

\[
\begin{array}{cccc}
S^{(n+k)d-1} & \xrightarrow{\bar{p}} & FP_{n+k,k} & \subset FP_{n+k+1,k+1} \\
\downarrow & & \downarrow L & \downarrow L' \\
S^{(n+k)d-1} & \xrightarrow{\bar{p}} & FP_{n+k,k} & \rightarrow C(\bar{p}) \\
\uparrow i & \uparrow \cup i & \uparrow i' \\
S^{(n+k)d-1} & \xrightarrow{s} & FP_{n+1,1} & \rightarrow C(s) \\
\end{array}
\]

in which $L$ denotes the multiplication by non-zero integer $L$. Then

\[
\epsilon_C(s) = L \left( \sum_{j=1}^{k} \binom{n}{j} d^{n+j} C_j \right) / d^k (d^k - 1)
\]

where $C_j = C_j(n,k)$ is the coefficient of $x^{n+j}$ in $(\phi_F(x))^{n+j}$.

Proof. Applying $K$ to the above diagram we have the following commutative diagram in which the horizontal sequences are exact.

\[
\begin{array}{cccc}
0 & \xleftarrow{L^*} & K(FP_{n+k,k}) & \xleftarrow{L^*} K(FP_{n+k+1,k+1}) \\
0 & \xleftarrow{i^*} & K(C(\bar{p})) & \xleftarrow{i^*} K(S^{(n+k)d-1}) \\
0 & \xleftarrow{} & K(S^{nd}) & \xleftarrow{} K(C(s)) \xleftarrow{} K(S^{(n+k)d-0})
\end{array}
\]

Choose $a_j \in K(C(\bar{p}))$ for $0 \leq j \leq k$ such that $L'^*(a_j) = L x^{n+j}$ for $0 \leq j \leq k - 1$ and $L'^*(a_k) = x^{n+k}$. Then $i'^*(a_0)$ and $i'^*(a_k)$ generate $K(C(s))$. We have

\[
\psi^2(i'^*(a_0)) = d^* i'^*(a_0) + \lambda i'^*(a_k)
\]

for some $\lambda \in \mathbb{Z}$ and
\[ e_c(s) = \lambda / d^s (d^s - 1) \).

We compute \( \lambda \). We have

\[
L'^*(\psi^2(a_0)) = \psi^2(L'^*(a_0)) = \psi^2(Lz^n) = L(z^2 + dz)^n
\]

\[ = L \sum_{j=0}^{k-1} (\xi) d^{n-j} z^{n+j} \]

\[ = \sum_{j=0}^{k-1} (\xi) d^{n-j} z^{n+j} + L(\xi) d^{n-k} z^{n+k} \]

\[ = L'^* \left( \sum_{j=0}^{k-1} (\xi) d^{n-j} a_j + L(\xi) d^{n-k} a_k \right). \]

Since \( L'^* \) is monomorphic, we have

\[ \psi^2(a_0) = \sum_{j=0}^{k-1} (\xi) d^{n-j} a_j + L(\xi) d^{n-k} a_k. \]

Next consider the following commutative diagram

\[
\begin{array}{ccc}
\tilde{K}(FP_{n+k+1, k+1}) & \xrightarrow{ch} & H^*(FP_{n+k+1, k+1}; Q) \\
\uparrow L'^* & & \uparrow L'^* \\
\tilde{K}(C(\tilde{p})) & \xrightarrow{ch} & H^*(C(\tilde{p}); Q) \\
\downarrow i'^* & & \downarrow i'^* \\
\tilde{K}(C(s)) & \xrightarrow{ch} & H^*(C(s); Q). \\
\end{array}
\]

Choose the generators \( x_{n+j} \in H^{(n+j)d}(C(\tilde{p}); Z) \) for \( 0 \leq j \leq k \) such that \( L'^*(x_{n+j}) = L d^n \) for \( 0 \leq j \leq k-1 \) and \( L'^*(x_{n+k}) = \xi^n \). Then for \( 1 \leq j \leq k-1 \)

\[ L'^*(ch(a_j)) = ch(L'^*(a_j)) = ch(Lz^n) = L(\phi F(t)^n) \]

\[ = L(d^n + \text{middle dim } + C_{j, n+k}) \]

\[ = L'^*(x_{n+j} + \text{middle dim } + LC_{j, n+k}) \]

where the terms middle dim mean elements of middle dimensions. Since \( L'^* \) is monomorphic, we have

\[ ch(a_j) = x_{n+j} + \text{middle dim } + LC_{j, n+k} \text{ for } 1 \leq j \leq k-1, \]

and so

\[ ch(i'^*(a_j)) = i'^*(ch(a_j)) = LC_{j, i'^*(x_{n+k})} = ch(LC_{j, i'^*(a_k)}) \text{ for } 1 \leq j \leq k-1. \]

Since \( ch \) is monomorphic now, we have

\[ i'^*(a_j) = LC_{j, i'^*(a_k)} \text{ for } 1 \leq j \leq k-1. \]
Then
\[ \psi^2(i^*(a_0)) = i^*(\psi^2(a_0)) = i^*\left\{ \sum_{j=0}^{k-1} \binom{\ell}{j} d^{n-j}a_j + L(z) d^{n-k}a_k \right\} \]
\[ = d^* i^*(a_0) + \left\{ \sum_{j=1}^{k-1} \binom{\ell}{j} d^{n-j}LC_j + L(z) d^{n-k} \right\} i^*(a_k) \]
\[ = d^* i^*(a_0) + L d^{n-k} \left\{ \sum_{j=1}^{k-1} \binom{\ell}{j} d^{n-j}C_j + (z) \right\} i^*(a_k). \]

Therefore we have
\[ \lambda = L d^{n-k} \left\{ \sum_{j=1}^{k-1} \binom{\ell}{j} d^{n-j}C_j + (z) \right\} \]
and
\[ e_c(s) = L \left\{ \sum_{j=1}^{k-1} \binom{\ell}{j} d^{n-j}C_j + (z) \right\} / d^n(d^n-1). \]

This completes the proof.

As a corollary of the above lemma we have

**Proposition 1.13.** In the same situation as (1.12) we have

(i) if \((F, k) = (C, 1)\), \(s = Ln\eta\) and in particular \(p_{n+1,1} = n\eta: S^{2n+1} \to CP_{n+1,1} = S^{2n}\),

(ii) if \((F, k) = (H, 2)\), \(e_c(s) = L n(5n-1)/2^2 - 3^2 \cdot 5\),

(iii) if \((F, k) = (C, 4)\), \(e_c(s) = L n(15n^3 + 30n^2 + 5n - 2)/2^2 \cdot 3^2 \cdot 5\),

(iv) if \((F, k) = (C, 5)\), \(e_c(s) = L n(3n^4 + 10n^3 + 5n^2 - 2n + 216)/2^3 \cdot 3^2 \cdot 5\).

**Proof.** Since
\[ \phi_F(x) = \begin{cases} x + x^2/2! + x^3/3! + \cdots & \text{for } F = C \\ 2x/2! + 2x^2/4! + 2x^3/6! + \cdots & \text{for } F = H, \end{cases} \]
we can easily compute \(e_c(s)\) for small \(k\) by elementary analysis, so we omit the details except (i). (i) follows from the fact that \(e_c: G_1 \to \mathbb{Z}_2\) is an isomorphism and \(e_c(s) = \frac{1}{2}Ln = e_c(Ln\eta)\).

**Remark.** (i) is well known.

In case of \(F = H\) and \(k = 1 \mod(2)\) or \(F = C\) and \(k = 2 \mod(4)\) we have \(e_c(s) = 2e_k(s)\) so the computation of \(e_k(s)\) may give more precise informations about the 2-primary components of the order of \(s\). We do not require the whole computations but we only compute \(e_k(s)\) for the case of \((F, k) = (H, 1)\) or \((C, 2)\). Let \(g_s = p_2: S^3 \to S^3 = HP_2\) be the Hopf map. Put \(g_\eta = \{g_s\} \in G_3\). Then \(e_k(s) = 1/2\) and

**Proposition 1.14** (James [7]). \(p_{n+1,1} = n\eta: S^{4n+3} \to HP_{n+1,1} = S^{4n}\).
Proof. We have the short exact sequence
\[ 0 \leftarrow \overline{KO}^{-4n-8}(HP_{n+1,1}) \leftarrow \overline{KO}^{-4n-8}(HP_{n+2,2}) \leftarrow \overline{KO}^{-4n-8}(S^{4n+4}) \leftarrow 0. \]

It is easily seen by (1.1) that \( \overline{KO}^{-4n-8}(HP_{n+1,1}) = \mathbb{Z}\{g_{S^5}^n, y_{-1}S^{n+1}\} \), \( \overline{KO}^{-4n-8}(HP_{n+2,2}) = \mathbb{Z}\{g_{S^5}^n, y_{-1}S^{n+1}\} \), \( \overline{KO}^{-4n-8}(S^{4n+4}) = \mathbb{Z}\{e\} \), \( \iota^*(g_{S^5}^n) = g_{S^5}^n \), and \( q^*(e) = y_{-1}S^{n+1} \). We have
\[ \psi^2(g_{S^5}^n) = \psi^2(g_{S^5}) = 2^4 g_{S^5} \{2^{4n}S^{n+3} + n2^{4n+3}S^{n+1}\}. \]

Then
\[ e^*_R(p_{n+1,1}) = 2^{n+1}n/(2^{n+6} - 2^{n+4}) = n/24 = e^*_R(n). \]

This shows that \( p_{n+1,1} = ng_{\infty} \), since \( e^*_R: G_3 \rightarrow Z_{24} \) is an isomorphism by (1.2).

Now consider the following commutative diagram in which the horizontal sequences are exact.

\[ \cdots \rightarrow \{S^{2n+1}, S^{2n-1}\} \xrightarrow{p_{2n}} \{S^{2n+1}, CP_{n}\} \xrightarrow{\iota_*} \{S^{2n+1}, CP_{n+1}\} \xrightarrow{q_*} \{S^{2n+1}, S^{2n}\} \xrightarrow{q_*} \{S^{2n+1}, S^{2n-1}\} \rightarrow \cdots \]

By (1.13) \( q_*(p_{n+1}) = m\). Then we have

**Proposition 1.15.** If \( Ln \equiv 0 \bmod(2) \)

\[ q_*(\iota_*)^{-1}(Lp_{n+1}) = \begin{cases} \frac{1}{2} L(n-1)g_{\infty} & \text{for } n \text{ odd} \\ \left\{ \frac{1}{2} L(n+2)g_{\infty}, \left( \frac{1}{2} L(n+2) + 12 \right)g_{\infty} \right\} & \text{for } n \text{ even.} \end{cases} \]

Proof. The above diagram shows that \( q_*(\iota_*)^{-1}(Lp_{n+1}) = (j_*)^{-1}(Lp_{n+1,2}). \) Since \( \{S^{2n+1}, S^{2n-1}\} = Z_2\{\gamma^2\} \) and \( p_{n+1}(\gamma^2) = (n-1)\gamma^3 = 12(n-1)g_{\infty}, (j_*)^{-1}(Lp_{n+1,2}) \) is a coset of the subgroup of \( \{S^{2n+1}, CP_{n,1}\} \) generated by \( 12(n-1)g_{\infty} \). This coset consists of a single element if \( n \) is odd or two elements if \( n \) is even. In case of \( n \) being odd we have the following commutative diagram by the proof of
This diagram proves Proposition if \( n \) is odd. If \( n \) is even, we have the short exact sequence

\[
0 \rightarrow \{S^{2n+1}, S^{2n-1}\} \rightarrow \{S^{2n+1}, S^{2n-3}\} \xrightarrow{j_*} \{S^{2n+1}, CP_{n+1,2}\} \rightarrow 0
\]

since \( p_{n,1} = (n-1)\eta \) by (i) of (1.13). For our purpose it suffices to show that

\[
(j_*)^{-1}(p_{n+1,2}) = \{(n/2+1)g_{\infty},(n/2+13)g_{\infty}\}.
\]

For any \( f \in (j_*)^{-1}(p_{n+1,2}) \) the equation

\[
(*) \quad e'(f) = (n/2 + 1 + 12 \epsilon)/24
\]

implies this, because \( e'_b((n/2+1)g_{\infty}) = (n/2+1)/24 \). We prove \((*)\). We use \( KO^{-2} \) if \( n \equiv 0 \mod(4) \) or \( KO^{-8} \) if \( n \equiv 2 \mod(4) \). The methods are quite parallel, so we only prove \((*)\) for the case of \( n \equiv 0 \mod(4) \). Put \( n = 4m \). There is the following commutative diagram in which the horizontal sequences are exact.

\[
\begin{array}{ccccccc}
0 & \rightarrow & KO^{-2}(CP_{4m+1,2}) & \rightarrow & KO^{-2}(CP_{4m+2,3}) & \rightarrow & KO^{-2}(S^{8m+2}) & \rightarrow & 0 \\
\downarrow i^* & & \downarrow i'^* & & \downarrow v^* & & \downarrow & \\
0 & \rightarrow & KO^{-2}(S^{8m+2}) & \rightarrow & KO^{-2}(C(f)) & \rightarrow & KO^{-2}(S^{8m+2}) & \rightarrow & 0
\end{array}
\]

By Theorem 2 of Fujii [4] it is easy to see that \( KO^{-2}(CP_{4m+1,2}) = \mathbb{Z}\{z_1z_2^{2m-1}\} \), \( KO^{-2}(CP_{4m+2,3}) = \mathbb{Z}\{z_1z_2^{2m-1}, z_1z_2^{2m}\} \), \( KO^{-2}(CP_{4m,1}) = \mathbb{Z}\{w\} \) with \( 2w = z_1z_2^{2m-1} \) and \( KO^{-2}(CP_{4m+2,1}) = \mathbb{Z}\{z_1z_2^{2m}\} \). Take \( a \in KO^{-2}((C(f)) \) with \( u^*(a) = w \). Then \( a \) and \( v^*(z_1z_2^{2m}) = i'^*(z_1z_2^{2m}) \) generate \( KO^{-2}(C(f)) \). By definition \( 2a = i'^*(z_1z_2^{2m-1}) + ei'^*(z_1z_2^{2m}) \) for some integer \( e \). We have \( \psi^2(a) = 2a^m + \lambda i'^*(z_1z_2^{2m}) \) for some integer \( \lambda \), and \( e'_b(f) = \lambda/2^{4m-3} \). We have also

\[
c(2a) = c(i'^*(z_1z_2^{2m-1}) + ei'^*(z_1z_2^{2m})) \\
= g_0c i'^*\left\{2z_1z_2^{4m-1} - (4m-1)z_1z_2^{4m} + (4m^2 + 2e)z_1z_2^{4m+1}\right\}
\]

and then

\[
c(i'^*(z_1z_2^{2m})) = 2g_0c i'^*(z_1z_2^{4m+1})
\]
\[ c(\psi^2(2a)) = c(2^{4m+1}a + 2\lambda i^*(z_1z_0^{2m})) = g_{c'}(2^{4m+1}a - 2^{4m}(4m-1)z^{4m} + (2^{4m+1}m^2 + 2^{4m+1}e + 4\lambda)z^{4m+1}). \]

On the other hand
\[ c(\psi^2(2a)) = \psi^2(c(2a)) = g_{c'}(2^{4m-1} - (4m-1)z^{4m} + (4m^2 + 2e)z^{4m+2}). \]
\[ = g_{c'}(2^{4m-1} - (4m-1)z^{4m} + (4m^2 + 2e)z^{4m+1}). \]
Compared to the coefficients of \( z^{4m+1}, \) we have
\[ \lambda = 2^{4m-3}(2m+1+12e), \]
and so
\[ e_k(f) = (2m+1+12e)/24. \]
This completes the proof.

In the sequel we shall need the explicit form of \( \alpha_f(n,k) \) for small \( k. \) Since the expansion of \( \phi_f^{-1}(x) \) is known (see e.g. [10]), we can obtain the following by elementary calculations.

**Lemma 1.16.**
\[ \alpha_f(n,0) = 1, \]
\[ \alpha_f(n,1) = -n/2^3, \]
\[ \alpha_f(n,2) = n(5n+11)/2^{7} \cdot 3^5 \cdot 5, \]
\[ \alpha_f(n,3) = -n(35n^2 + 231n + 382)/2^7 \cdot 3^4 \cdot 5 \cdot 7, \]
\[ \alpha_f(n,4) = n(175n^3 + 2310n^2 + 10181n + 14982)/2^{11} \cdot 3^5 \cdot 5 \cdot 7, \]
\[ \alpha_f(n,5) = -n(385n^4 + 8470n^3 + 69971n^2 + 257246n + 355128)/2^{13} \cdot 3^5 \cdot 5 \cdot 7 \cdot 11, \]
\[ \alpha_c(n,1) = -n/2, \]
\[ \alpha_c(n,2) = n(3n+5)/2^3 \cdot 3, \]
\[ \alpha_c(n,3) = -n(n+2)(n+3)/2^4 \cdot 3, \]
\[ \alpha_c(n,4) = n(15n^3 + 150n^2 + 485n + 502)/2^7 \cdot 3^5 \cdot 5, \]
\[ \alpha_c(n,5) = -n(3n^4 - 30n^3 + 785n^2 - 78n + 1240)/2^8 \cdot 3^2 \cdot 5, \]
\[ \alpha_c(n,6) = n(63n^5 + 1575n^4 + 15435n^3 + 73801n^2 + 171150n + 152696)/2^{10} \cdot 3^4 \cdot 5 \cdot 7, \]
\[ \alpha_c(n,7) = -n(9n^6 + 315n^5 + 4515n^4 + 33817n^3 + 139020n^2 + 295748n + 252356)/2^{11} \cdot 3^4 \cdot 5 \cdot 7, \]
\[ \alpha_c(n,8) = n(135n^7 + 6300n^6 + 124110n^5 + 1334760n^4 + 8437975n^3 + 74777100n^2 - 68303596n + 138452016)/2^{15} \cdot 3^5 \cdot 5 \cdot 7, \]
\[ \alpha_c(n, 9) = -n(15n^6 + 900n^7 + 23310n^6 + 339752n^5 - 829745n^4 + 3835450n^3 + 27449684n^2 + 112877136n + 100476288)/2^{18} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11. \]

\[ \alpha_c(n, 10) = n(99n^6 + 7425n^7 + 244530n^6 + 4634322n^5 + 55598235n^4 + 436886945n^3 + 2242194592n^2 + 7220722828n + 38722058672 - 15239326848)/2^{18} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11. \]

2. \( H\{n, k\} \) for \( k \leq 5 \)

The results of this section are summarized as follows.

**Theorem 2.1.**

(i) \( H\{n, 1\} = 1 \),

(ii) \( H\{n, 2\} = 24/(n, 24) \),

(iii) \( H\{n, 3\} = H\{n, 2\} \text{den}[H\{n, 2\} \alpha_{n}(n, 2)] \),

(iv) \( H\{n, 4\} = H\{n, 3\} \text{den} \left[ \frac{1}{2} H\{n, 3\} \alpha_{n}(n, 3) \right] \),

(v) \( H\{n, 5\}/(H\{n, 4\} \text{den}[H\{n, 4\} \alpha_{n}(n, 4)]) = \begin{cases} 1 & \text{or 2 if } n \equiv 1 \text{ mod}(2^5) \text{ or } 34 \text{ mod}(2^6) \\ 1 & \text{otherwise,} \end{cases} \)

where \( \text{den}(a) \) denotes the denominator of a rational number \( a \) when the fraction \( a \) is expressed in its lowest terms.

**Proof.**

(i) is trivial.

By (1.14), \#\( p_{n+1, 1} = 24/(n, 24) \), since \( g_{n} = 24 \). Then \( H\{n, 2\} = 24/(n, 24) \) by (1.9). Choose \( f \in \{H_{p_{22+2}}, S^w\} \) with \( \deg(f) = H\{n, 2\} \). Then 

\[ 0 = f \circ i_1 \circ p_{n+1, 1} = \deg(f) p_{n+1, 1} = H\{n, 2\} p_{n+1, 1}. \]

Therefore \( 24/(n, 24) \) \( \mid H\{n, 2\} \). Hence (ii) follows.

Take \( f \in \{H_{p_{n+2}}, S^w\} \) with \( \deg(f) = H\{n, 2\} \). We have \#\( e_{c}(f \circ p_{n+2}) = \#(f \circ p_{n+2, 2}) \), since \( e_{c}: G_{7} \rightarrow Z_{960} \) is an isomorphism by (1.2). They by (1.9) \( H\{n, 3\} = H\{n, 2\} \text{den}[f \circ p_{n+2} \circ H\{n, 2\} \alpha_{n}(n, 2)] \). Hence (iii) is obtained.

For any \( h \in \{H_{p_{n+3, 2}}, S^w\} \) we have 

\[ e_{c}(h \circ q_{1} \circ p_{n+3, 3}) = -\frac{1}{2} \deg(h \circ q_{1}) \alpha_{H}(n, 3) = 0 \]

by (1.5). Since \( e_{k}: G_{7} \rightarrow Z_{960} \) is an isomorphism by (1.2), \( h \circ q_{1} \circ p_{n+3, 3} = 0 \).

Then by (1.10), for \( f \in \{H_{p_{n+3}}, S^w\} \) with \( \deg(f) = H\{n, 3\} \), \#(f \circ p_{n+3}) is a divisor of \( H\{n, 4\}/H\{n, 3\} \). Conversely (1.9) implies that \#(f \circ p_{n+3}) is a multiple of \( H\{n, 4\}/H\{n, 3\} \). Hence \#(f \circ p_{n+3}) \( = H\{n, 4\}/H\{n, 3\} \). On the other hand 

\[ e_{c}(f \circ p_{n+3}) = -\frac{1}{2} H\{n, 3\} \alpha_{H}(n, 3) \] by (1.5). Hence \#(f \circ p_{n+3}) \( = \text{den} \left[ \frac{1}{2} H\{n, 3\} \alpha_{H}(n, 3) \right] \). Therefore
and this implies (iv).

For the proof of (v) we prepare a lemma.

**Lemma 2.2.** If \( n \equiv 0 \) or \( 3 \mod(4) \), the image of \( p_{n+4,i}^* : \{H_{n+4,2}, S^{4n}\} \rightarrow \{S^{4n+15}, S^{4n}\} \) contains the element \( \eta \kappa \in G_{15} \).

The proof of (2.2): Since all Toda brackets which appear in the proof have zero indeterminacies, we have

\[
\eta \kappa = \langle \varepsilon, 2t, \nu^2 \rangle = \langle \varepsilon, 2t, \nu \rangle = \langle \varepsilon, 2g_\infty, g_\infty \rangle.
\]

Consider the diagram

\[
\begin{array}{ccc}
S^{4n+14} & \xrightarrow{(n+3)g_\infty} & S^{4n+11} \\
\downarrow p_{n+3,1} & & \downarrow p_{n+4,2} \\
HP_{n+3,1} = S^{4n+8} & \subset & HP_{n+4,2} = S^{4n+12} \\
\downarrow i & & \downarrow \varepsilon \\
S^{4n} & & \end{array}
\]

By (1.14) \( p_{n+3,1} = (n+2)g_\infty \) and \( p_{n+4,1} = (n+3)g_\infty \). So \( p_{n+3,1} \circ (n+3)g_\infty = \varepsilon \circ p_{n+3,1} = 0 \), since \( 2g_\infty^2 = \varepsilon g_\infty = 0 \). Then there exists \( f \in \{H_{n+4,2}, S^{4n}\} \) with \( f \circ i = \varepsilon \), and by definition of Toda bracket

\[
f \circ p_{n+4,2} \in \langle \varepsilon, (n+2)g_\infty, (n+3)g_\infty \rangle
\]

and

\[
\langle \varepsilon, (n+2)g_\infty, (n+3)g_\infty \rangle = \frac{1}{2} (n+2)(n+3) \langle \varepsilon, 2g_\infty, g_\infty \rangle = \frac{1}{2} (n+2)(n+3) \eta \kappa.
\]

Thus \( f \circ p_{n+4,2} = \frac{1}{2} (n+2)(n+3) \eta \kappa \). Since the order of \( \eta \kappa \) is 2, the conclusion follows.

Now we prove (v). Take \( f \in \{H_{n+4,2}, S^{4n}\} \) with \( \deg(f) = H\{n, 4\} \). Then \( e_c(f \circ p_{n+4,4}) = -H\{n, 4\} \alpha_H(n, 4) \) by (1.3), and \( \#(f \circ p_{n+4,4})\#e_c(f \circ p_{n+4,4}) = 1 \) or 2 by (1.2). From (1.9) \( H\{n, 5\}/H\{n, 4\} \text{den}[H\{n, 4\} \alpha_H(n, 4)] = 1 \) or 2. And by (1.2), if \( \nu_2(H\{n, 4\} \alpha_H(n, 4)) = -1 \), we have \( \#(f \circ p_{n+4,4}) = \#e_c(f \circ p_{n+4,4}) = \text{den}[H\{n, 4\} \alpha_H(n, 4)] \) and

\[
H\{n, 5\} = H\{n, 4\} \text{den}[H\{n, 4\} \alpha_H(n, 4)]
\]
where $v_p(n/m) = v_p(n) - v_p(m)$ for a prime number $p$ and integers $m$ and $n$. (1.16), (ii), (iii), (iv) and elementary analysis show that $v_p[H\{n,4\} \alpha_H(n,4)] \geq 0$ if and only if $n \equiv 3 \mod(2^3)$, $1 \mod(2^3)$, $34 \mod(2^6)$ or $0 \mod(2^6)$. Consider the case of $n \equiv 3 \mod(2^3)$ or $0 \mod(2^6)$. By (2.2) there exists $h \in \{HP_{n+4,2}, S^{\infty}\}$ with $h \circ p_{n+4,2} = \eta \kappa$. Then $f \circ f \circ q_2$, say $f'$, satisfies the conditions $\#(f' \circ p_{n+4,2}) = \#(f' \circ p_{n+4,2})$ and $\deg(f') = H\{n,4\}$. Then by (1.3) $\#(f' \circ p_{n+4,2}) = \text{den}[H\{n,4\} \alpha_H(n,4)]$ and the conclusion (v) follows from (1.9).

3. $C(n,k)$ for $k \leq 10$

In this section we determine inductively $C(n,k)$ for $k \leq 8$ and estimate them for $k = 9$ and $10$. The results are as follows.

**Theorem 3.1.**

(i) $C(n,1) = 1$,

(ii) $C(n,2) = 2(n,2)$,

$$\frac{24}{(n,24)} \text{ if } n \equiv 1 \mod(4)$$

(iii) $C(n,4) = C(n,3)$

$$\begin{cases} 12/(n,3) & \text{if } n \equiv 1 \mod(8) \\ 6/(n,3) & \text{if } n \equiv 5 \mod(8) \end{cases}$$

(iv) $C(n,5) = C(n,4)\text{den}[C(n,4)\alpha_c(n,4)]$,

(v) $C(n,6) = C(n,5)\text{den}[C(n,5)\alpha_c(n,5)]$

$$\begin{cases} C(n,5) & \text{if } n \equiv 0 \mod(2), 1, 11 \text{ or } 27 \mod(32) \\ 2C(n,5) & \text{otherwise} \end{cases}$$

(vi) $C(n,7) = C(n,6)\text{den}[C(n,6)\alpha_c(n,6)]$

$$\begin{cases} 2C(n,6) & \text{if } n \equiv 0 \mod(2) \text{ or } 19 \mod(32) \\ 2C(n,6) & \text{otherwise} \end{cases}$$

(vii) $C(n,8) = C(n,7)$,

(viii) $C(n,9)/(C(n,8)\text{den}[C(n,8)\alpha_c(n,8)])$

$$\begin{cases} 1 \text{ or } 2 & \text{if } n \equiv 3 \mod(2^7) \text{ or } 1 \mod(2^7) \\ 1 & \text{otherwise} \end{cases}$$

$$\begin{cases} 1 & \text{if } n \equiv 0, 6 \mod(2^7), 10, 12 \mod(2^7), \\ 18, 20 \mod(2^9), 34, 36 \mod(2^9) \text{ or } 4 \mod(2^7) \\ 1 \text{ or } 2 & \text{otherwise} \end{cases}$$

(x) $C(n,10)/C(n,9)$

Proof. (i) is trivial. (ii) is proved by the same methods as the proof of (ii) of (2.1).

The proof of (iii): The first equality is a consequence of (1.9) and the fact $G_5 = 0$. We prove the second equality. Choose $f \in \{CP_{n+2,2}, S^{\infty}\}$ with $\deg(f) = C(n,2)$. Then $C(n,3)/C(n,2)$ is a divisor of $\#(f \circ p_{n+2,2})$ from (1.9), there exists $h \in \{CP_{n+2,2}, S^{\infty}\}$ with $(C(n,3)/C(n,2))f \circ p_{n+2,2} = h \circ q_1 \circ p_{n+2,2}$ from (1.10), while $q_1 \circ p_{n+2,2} = (n+1)\eta$ from (i) of (1.13), so $C(n,3)/C(n,2)$ is a multiple
of \( \#(f \circ p_{n+2,2}) \) if \( n \) is odd, and therefore \( C \{n,3\}/C \{n,2\} = \#(f \circ p_{n+2,2}) \) if \( n \) is odd. From (1.5), \( e'_K(f \circ p_{n+2,2}) = \frac{1}{2} a_2 \alpha_c(n,2) \) for some integer \( a_2 \). If \( n \equiv 3 \mod(4) \), say \( n=4m+3 \), \( a_2 \equiv 0 \mod(2) \) by (1.6)', then \( e'_K(f \circ p_{n+2,2}) = -(4m+3) \), \( 6m+7)/12 \) by (1.16) and (ii), hence \( \#(f \circ p_{n+2,2}) = \text{den}[(4m+3)/12] = 12/(n,24) \) by (1.2), and therefore the conclusion follows in this case since \( C \{n,2\} = 2 \). If \( n \equiv 1 \mod(4) \), say \( n=4m+1 \), \( a_2 \equiv 1 \mod(2) \) by (1.4), (1.7) and (ii), then \( e'_K(f \circ p_{n+2,2}) = -(12m-1)/(m+1)/6 \) by (1.16) and (ii), hence \( \#(f \circ p_{n+2,2}) = \text{den}[(m+1)/6] \) and the conclusion follows easily in this case also.

Next we consider the case of \( n \) being even. Take \( f \in \{ CP_{n+3,3}, S^{2n} \} \) with \( \deg(f) = C \{n,3\} \). First we show that \( C \{n,3\} \) is a multiple of \( 24/(n,24) \). Since arguments are quite parallel we only consider the case of \( n \equiv 0 \mod(4) \). Let \( n=4m \) and consider the commutative diagram

\[
\begin{array}{ccc}
\widetilde{K}(CP_{4m+3,3}) & \xrightarrow{c} & \tilde{K}(CP_{4m+3,3}) \\
\downarrow f^* & & \downarrow f^* \\
\widetilde{K}(S^{2m}) & \xrightarrow{c} & \tilde{K}(S^{2m}) \\
\end{array}
\]

We can put \( f^*(g_{2m}^n) = b_0 z_0^{2m} + b_1 z_0^{2m+1} \) for some integers \( b_0 \) and \( b_1 \). We have

\[
c(f^*(g_{2m}^n)) = b_0 (x+z)^{2m} + b_1 (x+z)^{2m+1} \\
= b_0 z^{2m} - 2d_2 z^{4m+1} + ((2m^2+m)d_0 + d_1) z^{4m+2},
\]

\[
c(f^*(g_{2m}^n)) = f^*(c(g_{2m}^n)) = a_0 z^{4m} + a_1 z^{4m+1} + a_2 z^{4m+2}
\]
for some integers \( a_0 \), \( a_1 \) and \( a_2 \). Comparing the coefficients of the powers of \( z \), by (1.4) we have

\[
d_0 = a_0 = C \{n,3\},
\]

\[
(2m^2+m)d_0 + d_1 = a_2 = C \{4m,3\} \alpha_c(4m,2) = C \{4m,3\} m(12m+5)/6,
\]

and so \( d_1 = -C \{4m,3\} m/6 \). Thus \( C \{4m,3\} \) is a multiple of \( \text{den}(m/6) = 24/(4m,24) \) as desired. Second we show that \( C \{n,3\} \) is a divisor of \( 24/(n,24) \). We define \( h: CP_{n+2,2} = S^{2n} \setminus S^{2n+2} \rightarrow S^{2n} \) by \( h|_{S^{2n}} = 24/(n,24) \) and

\[
h|_{S^{2n+2}} = \begin{cases} 
0 & \text{if } n \equiv 0 \mod(16) \\
\eta^2 & \text{for other even } n.
\end{cases}
\]

Since \( p_{n+2,2} = \frac{1}{2} ng_0 \setminus \eta \), \( h \circ p_{n+2,2} = (12n/(n,24)) g_\eta + h|_{S^{2n+2}} \circ \eta = 0 \). Hence there exists \( f' \in \{ CP_{n+3,3}, S^{2n} \} \) with \( f'|_{CP_{n+2,2}} = h \). Clearly \( \deg(f') = 24/(n,24) \), so \( C \{n,3\} \) is a divisor of \( 24/(n,24) \). Thus \( C \{n,3\} = 24/(n,24) \) if \( n \) is even. This completes the proof of (iii).

The proof of (iv): By (1.3), \( e_c(h \circ q_1 \circ p_{n+4,4}) = 0 \) for any \( h \in \{ CP_{n+4,3}, S^{2n} \} \) and then \( h \circ q_1 \circ p_{n+4,4} = 0 \) by (1.2). So by (1.3), (1.9) and (1.10)
The proof of (v): First consider the case of $n \equiv 1 \mod(2)$. Choose $f \in \{CP_{*+5,5}, S^{2n}\}$ with $\deg(f) = C\{n, 5\}$. Recall that $G = Z_2 \{\eta\} \oplus Z_2 \{\eta\} \oplus Z_2 \{\mu\}$ and the kernel of $e_c: G \rightarrow \mathbb{Z}/2\mathbb{Z}$ is $Z_2 \{\eta\} \oplus Z_2 \{\eta\}$. Hence, if $e_c(f \circ p_{*+5,5}) = 0$, we can choose $h \in \{CP_{*+5,5}, S^{2n}\} = G_8$ with $(f \circ p_{*+5,5}) = 0$, because $q_c \circ p_{*+5,5} = p_{*+5,1} = \eta$ by (i) of (1.13). Since $\deg(f \circ p_{*+5,5}) = \deg(f) = C\{n, 5\}$, by (1.9) we have

$$C\{n, 6\} = C\{n, 5\} = C\{n, 5\} \# e_c(f \circ p_{*+5,5}).$$

If $e_c(f \circ p_{*+5,5}) \neq 0$, (1.9) implies

$$C\{n, 6\} = 2C\{n, 5\} = C\{n, 5\} \# e_c(f \circ p_{*+5,5}).$$

Since $C\{n, 5\}$ and $\alpha_c(n, 5)$ are known, we can easily compute $\den[C\{n, 5\} \alpha_c(n, 5)]$ by elementary analysis. Indeed

$$\# e_c(f \circ p_{*+5,5}) = \den[C\{n, 5\} \alpha_c(n, 5)]$$

$$= \begin{cases} 1 & \text{if } n \equiv 1, 11 \text{ or } 27 \mod(32) \\ 2 & \text{for other odd } n. \end{cases}$$

This completes the proof of (v) if $n$ is odd.

Suppose that $n$ is even. It is easy to see that $\den[C\{n, 5\} \alpha_c(n, 5)] = 1$. From (1.8) and (1.11)

$$C\{n, 5\} \mid C\{n, 6\} \mid H\{n/2, 3\}.$$  

By the previous calculations $C\{n, 5\}$ and $H\{n/2, 3\}$ are coincide if $n \equiv 0 \mod(4), 6, 10$ or $14 \mod(16)$, so $C\{n, 5\} = C\{n, 6\}$ in this case, while if $n \equiv 2 \mod(16)$ the odd components are coincide but

$$2 = \nu_2(C\{n, 5\}) \leq \nu_2(C\{n, 6\}) \leq \nu_2(H\{n/2, 3\}) = 3.$$

Put $n = 16m + 2$. We construct a commutative diagram in which $\deg(f) = C\{16m + 2, 5\}$.  

$$C\{16m + 2, 5\} \mid C\{16m + 2, 4\}.$$
By (i) of (1.13), \( q_{16m+5} \circ p_{16m+7} = p_{16m+7,1} = 0 \) and so by (1.15) we have

\[
q_{16m+7} \circ (i_{16m+7})^{-1} (p_{16m+7}) = \{(8m+4)g_{\infty}, (8m+16)g_{\infty}\}.
\]

Take \( s_1 \in (i_{16m+7})^{-1}(p_{16m+7}) \subset \{S_{32m+13}^{2}, CP_{16m+6}^{2}\} \) with \( q_{16m+4} \circ s_1 = (8m+16)g_{\infty} \). Put \( s_1 = q_{16m+1} \circ s_1' \). Then

\[
q_{3} \circ 3s_1 = q_{16m+4} \circ 3s_1' = 3(8m+16)g_{\infty} = 0.
\]

Hence there exists \( s_2 \in \{S_{32m+13}^{2}, CP_{16m+5,3}^{2}\} \) with \( i_1 \circ s_2 = s_1 \). Since \( q_2 \circ s_2 \in G_3 = 0 \), there exists \( s_3 \in \{S_{32m+13}^{2}, CP_{16m+4,3}^{2}\} \) with \( i_1 \circ s_3 = s_2 \). Next we define \( h \) by \( h|_{S_{32m+4}}^{2} = C\{16m+2,4\} \) and \( h|_{S_{32m+6}}^{2} = \eta^{2} \). Since \( p_{16m+4,3} = (8m+1)g_{\infty} \vee \eta \) by the proof of (1.11), (1.14) and (i) of (1.13), we have

\[
h \circ p_{16m+4,2} = C\{16m+2,4\} \frac{(8m+1)g_{\infty} + \eta^{3}}{(16m+2,24)g_{\infty}} = 0.
\]

So there exists \( h' \in \{CP_{16m+5,3}^{2}, S_{32m+4}^{2}\} \) with \( h' \circ i = h \). Since \( h' \circ p_{16m+5,3} \in G_3 = 0 \), there exists \( h'' \in \{CP_{16m+6,4}^{2}, S_{32m+4}^{2}\} \) with \( h'' \circ i = h' \). By (1.2), (1.3) and (iv) we have

\[
\#(h'' \circ p_{16m+6,4}) = \#e_c(h'' \circ p_{16m+6,4}) = \operatorname{deg}(h'')c(16m+24)]
\]

\[
= C\{16m+2,5\}/C\{16m+2,4\}.
\]

Hence there exists \( f \in \{CP_{16m+7,5}^{2}, S_{32m+4}^{2}\} \) with \( (C\{16m+2,5\}/C\{16m+2,4\})h'' = f \circ i \) and \( \operatorname{deg}(f) = \operatorname{deg}(h'')c\{16m+2,5\}/C\{16m+2,4\} = C\{16m+2,5\} \). This completes the construction of the above diagram.

Now we proceed to the proof of (v). We may write \( s_3 = s'_3 \setminus q_1 \circ s_3 \) for some \( s'_3 \in \{S_{32m+13}^{2}, S_{32m+4}^{2}\} \). By (iii) of (1.13)

\[
e_c(q_3 \circ s_3) = (16m+3)(3840m^3 + 2640m^2 + 590m + 43)/2^3 \cdot 3^5
\]

so by (1.2) \( q_1 \circ s_3 \) is divisible by 2. Then

\[
f \circ p_{16m+7,5} = f \circ 3p_{16m+7,5}, \text{ since } 2G_3 = 0,
\]

\[
= (C\{16m+2,5\}/C\{16m+2,4\})h \circ s_3
\]

\[
= (C\{16m+2,5\}/C\{16m+2,4\}) (C\{16m+2,4\}) s'_3 + \eta^3 \circ q_1 \circ s_3
\]

\[
= (C\{16m+2,5\}/C\{16m+2,4\}) (0+0), \text{ since } C\{16m+2,4\} \equiv 0 \mod(2)
\]

\[
\text{and } 2\eta = 0
\]

\[
= 0.
\]

Thus by (1.9), \( C\{16m+2,6\} = C\{16m+2,5\} \). This completes the proof of (v).
The proof of (vi): First consider the case of \( n \) being odd. For any \( h \in \{CP_{n+6,6}, S^{2n}\} \), by (i) of (1.5) we have

\[
e'_k(h \circ q \circ p_{n+6,6}) = \frac{1}{2} a
\]

for some integer \( a \). By (1.6) and (1.7) \( a \) is even. Then \( h \circ q \circ p_{n+6,6} = 0 \) by (1.2). Thus (1.9) and (1.10) imply

\[
C\{n, 7\} = C\{n, 6\} \#(f \circ p_{n+6,6})
\]

for \( f \in \{CP_{n+6,6}, S^{2n}\} \) with \( \deg(f) = C\{n, 6\} \). Again by (i) of (1.5)

\[
e'_k(f \circ p_{n+6,6}) = \frac{1}{2} a_{6} - \frac{1}{2} C\{n, 6\} \alpha_{c}(n, 6)
\]

for some integer \( a_{6} \), and by the proof of (iii) of (1.5) we have

\[
a_{6} \equiv 0 \mod(2) \text{ if } n \equiv 3 \mod(4) \text{ or } 33 \mod(64)
\]

Then since \( \#(f \circ p_{n+6,6}) \) is equal to \( \#e'_k(f \circ p_{n+6,6}) = \text{den} \left[ \frac{1}{2} a_{6} - \frac{1}{2} C\{n, 6\} \alpha_{c}(n, 6) \right] \) by (1.2), elementary analysis draws the conclusion for odd \( n \) by (iii), (iv), (v) and (1.16).

Next suppose that \( n \) is even. Choose \( f \in \{CP_{n+6,6}, S^{2n}\} \) with \( \deg(f) = C\{n, 6\} \). (1.2) says that \( e_{c} = 2e'_{k} : G_{11} \rightarrow Q/Z \) is monomorphic on the odd component, so (vi) is true about the odd components by (1.3) and (1.9). So we only see the 2-primary part. Recall that \( G_{11} = Z_{8} \{ \xi \} \oplus Z_{63} \). By (1.3), (1.16) and elementary analysis show that

\[
\nu_{2}(\#e_{c}(f \circ p_{n+6,6})) \leq 2.
\]

If \( \nu_{2}(\#e_{c}(f \circ p_{n+6,6})) = 0 \), \( \nu_{2}(\#(f \circ p_{n+6,6})) \leq 1 \) by (1.2) and (1.5). If \( \nu_{2}(\#(f \circ p_{n+6,6})) = 0 \), the result follows by (1.9). If \( \nu_{2}(\#(f \circ p_{n+6,6})) = 1 \), we have

\[
f \circ p_{n+6,6} \equiv 4z \mod(odd \text{ components}).
\]

Since \( 4z = \mu \eta^{2} \) and \( p_{n+6,1} = q_{6} \circ p_{n+6,6} = \eta \),

\[
(f + \mu \eta q_{6})p_{n+6,6} \equiv 0 \mod(odd \text{ components}).
\]

Clearly \( \deg(f + \mu \eta q_{6}) = \deg(f) = C\{n, 6\} \), so the result follows again by (1.9). If \( \nu_{2}(\#e_{c}(f \circ p_{n+6,6})) = u = 1 \) or 2,

\[
\nu_{2}(C\{n, 6\}) + u \leq \nu_{2}(C\{n, 7\})
\]

by (1.9), and

\[
\nu_{2}(\#(f \circ p_{n+6,6})) = u + 1
\]
by (1.2) and (1.5), so
\[ f \circ p_{n+6,6} \equiv 2^{n-3} \mod (2^{n-5}, \text{odd components}) \]
and then
\[ (2^n f + \mu \eta q_6) \circ p_{n+6,6} \equiv 0 \mod (\text{odd components}). \]
Put \( \#((2^n f + \mu \eta q_6) \circ p_{n+6,6}) = 2m + 1 \). Then there exists \( h \in \{ CP_{n+7,7}, S^{2a} \} \) with \( h \mid p_{n+6,6} = (2m+1)(2^n f + \mu \eta q_6) \). Clearly \( \deg(h) = 2^n(2m+1)\deg(f) = 2^n(2m+1) \cdot C \{n,6\} \). Since \( \deg(h) \) is a multiple of \( C \{n,7\} \), we have
\[ \nu_2(C \{n,7\}) \leq \nu_2(C \{n,6\}) + u \]
and hence
\[ \nu_2(C \{n,7\}) = \nu_2(C \{n,6\}) + u = \nu_2(C \{n,6\}) + \nu_2(\#(f \circ p_{n+6,6})) = \nu_2(C \{n,6\}) \ den(C \{n,6\} \ \alpha_c(n,6)) \]
as desired. This completes the proof of (vi).

The proof of (vii): Since \( G_{13} = Z_3 \{ \alpha_1 \beta_1 \} \), \( C \{n,8\}/C \{n,7\} = 1 \) or 3 by (1.9).
In case of \( n \) being even, the relations
\[ C \{n,7\}/C \{n,8\}/H \{n/2,4\} \]
and the previous calculations show that the 3-components of the first and the third are equal so that the 3-components of these three are equal. Thus \( C \{n,8\} = C \{n,7\} \) if \( n \) is even.
Choose \( h \in \{ CP_{n+7,2}, S^{2n+10} \} \) with \( \deg(h) = C \{n+5,2\} \). Then
\[ e_c(h \circ q_5 \circ p_{n+7,7}) = -C \{n+5,2\} \alpha_c(n+5,2) = -(n+5)(3n+20)/(12(n+5,2)) \]
so by (1.2)
\[ \#(h \circ q_5 \circ p_{n+7,7}) \equiv 0 \mod (3) \]
if and only if \( n \equiv 1 \mod (3) \).
Therefore if \( n \not\equiv 1 \mod (3) \), the image of
\[ p_{n+7,7}^* = (q_5 \circ p_{n+7,7})^*: \{ CP_{n+7,2}, S^{2n+10} \} \rightarrow \{ S^{2n+13}, S^{2n+10} \} = G_3 \]
contains \( Z_3 \{ \alpha_1 \} \).
Take \( f \in \{ CP_{n+7,7}, S^{2n} \} \) with \( \deg(f) = C \{n,7\} \). Suppose that \( n \not\equiv 1 \mod (3) \).
If \( f \circ p_{n+7,7} = 0, C \{n,8\} = C \{n,7\} \) by (1.9). If \( f \circ p_{n+7,7} \not= 0 \), that is \( f \circ p_{n+7,7} = \pm \beta_1 \alpha_1 \),
the above implies that there exists \( h' \in \{ CP_{n+7,2}, S^{2n+10} \} \) with \( h' \circ q_5 \circ p_{n+7,7} = \mp \alpha_1 \), and we have
\[(f + \beta_1 \circ h' \circ \eta_3) \circ p_{n+7,7} = 0,\]
\[\deg(f + \beta_1 \circ h' \circ \eta_3) = \deg(f) = C\{n, 7\}\]

and so by (1.9)
\[C\{n, 8\} = C\{n, 7\}.\]

Therefore \(C\{n, 8\} = C\{n, 7\}\) if \(n \equiv 1 \mod(3)\).

We must prove (vii) for the case of \(n \equiv 1 \mod(6)\). Put \(n = 6m+1\). Take \(f \in \{CP_{6m+8,7}, S^{12m+2}\}\) with \(\deg(f) = C\{6m+1,7\}\). By the same methods as the proof of (v) we can construct a commutative diagram

\[
\begin{array}{c}
S^{12m+15} \\
\downarrow p_{6m+8,7} \\
CP_{6m+8,7} \\
\downarrow 2 \\
CP_{6m+7,6} \\
\downarrow 4 \\
CP_{6m+6,5} \\
\downarrow 4 \\
CP_{6m+5,4} \\
i \\
\end{array}
\]

Take \(a \in \{CP_{6m+5,4}, S^{12m+2}\}\) with \(\deg(a) = C\{6m+1,4\}\) and \(b \in \{CP_{6m+3,2}, S^{12m+2}\}\) with \(\deg(b) = C\{6m+1,2\} = 2\). Consider the diagram

\[
\begin{array}{c}
\{S^{12m+6}, S^{12m+2}\} = 0 \\
\downarrow \\
\{S^{12m+8}, S^{12m+2}\} \approx \{CP_{6m+5,4} \times S^{12m+2}\} \rightarrow \{CP_{6m+4,3}, S^{12m+2}\} \rightarrow \{S^{12m+7}, S^{12m+2}\} = 0 \\
\downarrow \\
\{S^{12m+3}, S^{12m+2}\} \approx \{S^{12m+4}, S^{12m+2}\} \rightarrow \{CP_{6m+3,2}, S^{12m+2}\} \rightarrow \{S^{12m+2}, S^{12m+2}\}
\end{array}
\]

in which the horizontals and the vertical are the parts of suitable Puppe exact sequences. Then \(a\) generates a free part of \(\{CP_{6m+5,4}, S^{12m+2}\}\) which is of rank 1, and so

\[f \circ \iota \circ \iota \circ i = (\deg(f) / \deg(a)) a + q^*(e)\]

for some \(e \in \{S^{12m+8}, S^{12m+2}\} = G_6\). Then
\[ 2f \circ p_{6m+8,7} = 8f \circ p_{6m+8,7}, \text{ since } G_{13} = Z_3 = f \circ q \circ s_3, \]
\[ = (C\{6m+1,7\}/C\{6m+1,4\})a \circ s_3 + e \circ q \circ s_3, \]
\[ = (C\{6m+1,7\}/C\{6m+1,4\})a \circ s_3, \text{ since } G_6 \circ G_7 = 0. \]

By the previous calculations and elementary analysis it follows that

\[ \nu_3(C\{6m+1,7\}) = \begin{cases} 3 & \text{if } m \equiv 1 \text{ or } 2 \text{ mod}(3) \\ 2 & \text{if } m \equiv 3 \text{ or } 6 \text{ mod}(9) \\ 1 & \text{if } m \equiv 0 \text{ mod}(9), \end{cases} \]

so if \( m \equiv 0 \text{ mod}(9) \) we have

\[ C\{6m+1,7\}/C\{6m+1,4\} \equiv 0 \text{ mod}(3) \]

and so

\[ f \circ p_{6m+8,7} = 0 \]

and then by (1.9)

\[ C\{6m+1,8\} = C\{6m+1,7\} \text{ if } m \equiv 0 \text{ mod}(9). \]

Next suppose that \( m \equiv 0 \text{ mod}(9) \). By (iii) of (1.13) we can easily see that

\[ \nu_3(#c(q_3 \circ s_3)) = 0. \]

So by (1.13) and the same methods as the proof of (v), we can construct a commutative diagram.

Then

\[ f \circ p_{6m+8,7} = 640f \circ p_{6m+8,7} \]
\[ C\{6m+1,8\} = C\{6m+1,7\} \quad \text{if } m \equiv 0 \mod(9). \]

This completes the proof of (vii).

The proof of (viii): Take \( f \in \{CP_{n+8,8}, S^n\} \) with \( \deg(f) = C\{n,8\} \). First consider the case of \( n \) being even. By (i) of (1.13) \( p_{n+8,1} = q_7 \circ p_{n+8,8} = \eta \). Then \( f \) or \( f + \kappa q_n \), say \( f' \), satisfies

\[
\#(f' \circ p_{n+8,8}) = \#e_C(f' \circ p_{n+8,8}) = \text{den}[C\{n,8\}\alpha_C(n,8)], \quad \deg(f') = \deg(f) = C\{n,8\}
\]

by (1.2), and so the conclusion follows from (1.9). Next suppose that \( n \) is odd. By (1.2)

\[
\#(f \circ p_{n+8,8})/\#e_C(f \circ p_{n+8,8}) = 1 \text{ or } 2.
\]

By the previous calculations and elementary analysis we have

\[
\nu_2(\text{den}[C\{n,8\}\alpha_C(n,8)]) = 0 \quad \text{if and only if } n \equiv 3 \mod(2^7) \text{ or } 1 \mod(2^9).
\]

Therefore if \( n \equiv 3 \mod(2^7) \) and \( 1 \mod(2^9) \), by (1.2) we have

\[
\#(f \circ p_{n+8,8}) = \#e_C(f \circ p_{n+8,8}) = \text{den}[C\{n,8\}\alpha_C(n,8)]
\]

and so the conclusion follows.

The proof of (ix): Since \( 2G_{17} = 0 \), by (1.9) we have

\[
C\{n,10\}/C\{n,9\} = 1 \text{ or } 2.
\]

In case of \( n \) being even, by the following relations and an elementary analysis conclusion follows if \( n \equiv 0 \mod(2^3) \), 10, 12, 14 mod(2^4), 18, 20, 22 mod(2^5), 34, 36 mod(2^6) or 4 mod(2^7)

\[
C\{n,9\}/C\{n,10\}/H\{n/2,5\}.
\]

If \( n \equiv 0 \mod(2^5) \), the conclusion follows from the same methods as the proof of (vii).

4. Relations with other James numbers

In this section we use the notations and terminologies of James [6,7] freely.
Consider the fibration of Stiefel manifolds

\[ O_{n-k-1} \to O_{n,k} \overset{p}{\to} O_{n,1} = S^{n-1} \]

and the cofibration of quasi-projective spaces

\[ Q_{n-k-1} \to Q_{n,k} \overset{q}{\to} Q_{n,1} = S^{n-1} \]

where \( n > k > 0 \). Following James [6] we define non-negative integers \( O\{n,k\} \), \( O^t\{n,k\} \), \( Q\{n,k\} \) and \( Q^t\{n,k\} \) by the equations

\[
\begin{align*}
\pi_0^t(O_{n,k}) &= O\{n,k\} \\
\pi_1^t(O_{n,k}) &= O^t\{n,k\} \\
\pi_0^t(Q_{n,k}) &= Q\{n,k\} \\
\pi_1^t(Q_{n,k}) &= Q^t\{n,k\}
\end{align*}
\]

here \( \pi_n^t(X) = \{S^n, X\} \) for a pointed space \( X \). We have

**Lemma 4.1.** \( O\{n,k\} \mid Q\{n,k\} \), \( O^t\{n,k\} \mid O\{n,k\} \) and \( Q^t\{n,k\} \mid Q\{n,k\} \).

**Proof.** The first conclusion follows from the commutative diagram

\[
\begin{array}{ccc}
Q_{n,k} & \overset{q}{\to} & Q_{n,1} \\
\cap & \downarrow & = \\
O_{n,k} & \overset{p}{\to} & O_{n,1}
\end{array}
\]

and the others follow immediately by definition.

Let \( M_k(F) \) be the order of the canonical \( F \)-line bundle over \( FP_k \) in the \( J \)-group \( J(FP_k) \) [3] which was determined by Adams-Walker [2] and Sigrist-Suter [13]. We have

**Lemma 4.2.** \( Q^t\{n,k\} = O^t\{n,k\} \).

**Proof.** For any \( m \) with \( m \equiv 0 \mod(M_k(F)) \) there exists \( S^0 \)-section \( \omega: Q_{m,1} \to Q_{m,k} \), that is, \( q \circ \omega = 1 \). By James [7] we have the diagram

\[
\begin{array}{c}
Q_{m,1} \times Q_{n,k} \xrightarrow{1 \times i} Q_{m,1} \times O_{n,k} \xrightarrow{\omega \times 1} Q_{m,k} \times O_{n,k} \xrightarrow{g'} Q_{m+n,k} \\
\downarrow 1 \times q \quad \quad \downarrow 1 \times p \quad \quad \downarrow q \times p \quad \quad \downarrow q \\
Q_{m,1} \times Q_{n,1} = Q_{m,1} \times O_{n,1} = Q_{m,1} \times O_{n,1} \xrightarrow{g' \circ (\omega \times 1) \circ (1 \times i)} Q_{m+n,1}
\end{array}
\]

in which \( g' \circ (\omega \times 1) \circ (1 \times i) \) is a homotopy equivalence by (7.3) of [7], the first
square is commutative, the second is homotopy commutative and the third is homotopy commutative up to sign from quasi-projective case of (5.2) of [7]. Applying $\pi_{(m+n)d-1}$ to this diagram we have the following diagram

\[
\begin{array}{c}
\pi_{(m+n)d-1}(Q_{n,k}) \xrightarrow{i_*} \pi_{(m+n)d-1}(O_{n,k}) \xrightarrow{g_*} \pi_{(m+n)d-1}(Q_{m+k}\ast O_{n,k}) \\
\downarrow q_* \quad \downarrow p_* \quad \downarrow (q\ast p)_* \\
\pi_{(m+n)d-1}(Q_{n,k}) \xrightarrow{\cong} \pi_{(m+n)d-1}(Q_{m+k}\ast O_{n,k}) \xrightarrow{\cong} \pi_{(m+n)d-1}(S^{(m+n)d-1})
\end{array}
\]

in which the first and second squares are commutative and the third is commutative up to sign. Hence $Q^i\{m-n, k\} \mid O^i\{n, k\} \mid Q^i\{n, k\}$. Since $Q^i\{m+n, k\} = Q^i\{n, k\}$, the conclusion follows.

We have also

**Lemma 4.3.** If $n \geq 2(k-1)+2/d$, then

$Q^i\{n, k\} = O^i\{n, k\} = O\{n, k\} = Q\{n, k\}$.

**Proof.** Since $Q_{n,k}$ and $O_{n,k}$ are $(n-k+1)d-2$ connected, the canonical homomorphisms $\pi_{(m+n)d-1}(Q_{n,k}) \rightarrow \pi_{(m+n)d-1}(Q_{m+k})$ and $\pi_{(m+n)d-1}(O_{n,k}) \rightarrow \pi_{(m+n)d-1}(O_{m+k})$ are epimorphisms if $n \geq 2(k-1)+2/d$. Then $Q^i\{n, k\} = O\{n, k\}$ and $Q^i\{n, k\} = O\{n, k\}$ in this case, and the conclusion follows from (4.2).

Atiyah [3] proved that $Q_{n,k}$ and $P_{k-n,k}$ are $S$-duals. His proof gives the following precise theorem.

**Theorem 4.4.** For any $j$ with $jM(F) \geq n$, there exists a $(djM(F)-1)$-duality $u \in \{Q_{jM(F)-n+k,k} \wedge P_{n,k}, S^{djM(F)-1}\}$.

Consider the cofibrations

\[
\begin{align*}
S^{(n-k)d} & \xleftarrow{i} P_{n,k} \rightarrow P_{n,k-1} \rightarrow S^{(n-k)d+1} \\
S^{md-2} & \rightarrow Q_{m-1,l-1} \xrightarrow{q} Q_{m,l} \rightarrow S^{md-1}
\end{align*}
\]

We have

**Proposition 4.5.** If $jM_{n}(F) \geq n$, $(djM_{n}(F)-1)$-dual of $i$: $S^{(n-k)d} \rightarrow P_{n,k}$ is $q: Q_{jM_{n}(F)-n+k,k} \rightarrow S^{(jM_{n}(F)-n+k)d-1}$, and hence $F\{n-k, k\} = Q^i\{jM_{n}(F)-n+k,k\}$.

**Proof.** By Puppe exact sequences associated with the above cofibrations it is easily seen that $\{S^{(n-k)d}, P_{n,k}\}$ and $\{Q_{jM_{n}(F)-n+k,k}, S^{(jM_{n}(F)-n+k)d-1}\}$ are infinite cyclic groups with generators $i$ and $q$ respectively. Then the conclusion follows from (4.4).
As a corollary of (4.3) and (4.5) we have

**Theorem 4.6.** \( F\{n,k\} \) is equal to \( O\{jM_4(F) - n, k\} \) if \( jM_4(F) \geq n + 2k - 2 + 2/d \).

In case of \( F=C \), Sigrist [12, Théorème I] proved that a prime number \( p \) is a factor of \( O\{m,l\} \) if and only if \( p \) is a factor of \( M_4(C)/(m,M_4(C)) \). His proof is valid for the case of \( F=H \), since \( M_4(H) \) is known [13]. Then by (4.6) we have

**Proposition 4.7.** A prime number \( p \) is a factor of \( F\{n,k\} \) if and only if \( p \) is a factor of \( M_4(F)/(n,M_4(F)) \).

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**References**