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ON STABLE JAMES NUMBERS OF STUNTED COMPLEX OR QUATERNIONIC PROJECTIVE SPACES

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Following James [7] we denote the stunted complex ($F=C$) or quaternionic ($F=H$) projective spaces by $FP_{n+k,k}$ (or $P_{n+k,k}$) for positive integers n and k, that is

$$FP_{n+k,k}=FP_{n+k}/FP_n=FP^{n+k-1}/FP^{n-1}\,.$$

Let d be the dimension of F over the real number field. Let i: $S^{nd}=FP_{n+1,1}\rightarrow FP_{n+k,k}$ be the inclusion. By *stable James number* $F\{n,k\}$ we mean the order of the cokernel of

$$\deg=i^*:\ \{FP_{n+k,k},S^{nd}\}\rightarrow\{S^{nd},S^{nd}\}=Z$$

where $\{X, Y\}$ denotes the group of stable maps from a pointed space X to an other pointed space Y. In the previous papers [5, 8, 9, 10] we used the notations $k_s(FP_n^{n+k-1}, S^{nd})$ instead of $F\{n,k\}$ and estimated $F\{1,k\}$.

The first purpose of this note is to determine $F\{n,k\}$ for small k, that is, we shall determine $H\{n,k\}$ for $k\leqq 4$, estimate them for $k=5$, determine $C\{n,k\}$ for $k\leqq 8$ and estimate them for $k=9$ and 10. These shall be done in §2 and §3. The second purpose is to show that $F\{n,k\}$ can be identified with the James numbers defined by James in [6]. This shall be done in §4.

An application of this note to F-projective stable stems shall be given in [11].

In this note we work in the stable category of pointed spaces and stable maps between them, and we use Toda's notations of stable stems and Toda brackets in [14] freely.

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1. Preliminaries

In what follows we shall be working with both real K-cohomology theory KO^* and complex K-cohomology theory K^*. We use the following notations. KO^* and K^* denote both the K-functors and the coefficient rings. By the same letter $\xi=\xi_n$ we denote the canonical F-line bundle over FP_n, the underlying complex or real vector bundle of it. Put $z=\xi-d/2\in\tilde{K}(FP_n)$ and $t=(-1)^{1+d/2}c_{d/2}(\xi)\in H^d(FP_n;Z)$, where $c_m(\xi)$ denotes the m-th Chern class of ξ. Put also $\tilde{\xi}=\tilde{\xi}_n=\xi_n-1\in\widetilde{KSp}^0(HP_n)=\widetilde{KO}^{-4}(HP_n)$. The formal power series $\phi_F(x)$ are defined to be $\exp(x)-1$ for $F=C$ or $\exp(\sqrt{x})+\exp(-\sqrt{x})-2$ for $F=H$. The rational numbers $\alpha_F(n,j)$ are defined by $(\phi_F^{-1}(x)/x)^n=\sum_{i=0}^{\infty}\alpha_F(n,j)x^j$. ch: $K(\quad)\to H^*(\quad;Q)$ denotes the Chern character. Then the followings are well known.

Proposition 1.1. (i) $K(FP_n)=Z[z]/(z^n)$.
(ii) $KO^*(HP_n)=KO^*[\tilde{\xi}_n]/(\tilde{\xi}_n^n)$ *and* $\tilde{\xi}_n|_{HP_{n-1}}=\tilde{\xi}_{n-1}$.
(iii) $H^*(FP_n;Z)=Z[t]/(t^n)$.
(iv) $ch(z)=\phi_F(t)$.

Let $i=i_l$: $FP_{n+k,k}\subset FP_{n+k+l,k+l}$ be the inclusion for $l>0$, $q=q_m$: $FP_{n+k,k}\to FP_{n+k,k-m}$ the canonical quotient map for $0\leqq m<k$, $p_n=p_n^F$: $S^{nd-1}\to FP_n$ the Hopf bundle projection, and $p_{n+k,k}$: $S^{(n+k)d-1}\to FP_{n+k,k}$ the composition of p_{n+k} and q_{n-1}: $FP_{n+k}=FP_{n+k,n+k-1}\to FP_{n+k,k}$. Let G_k denote the k-stem of the stable groups of spheres. Let e_C: $G_k\to Q/Z$ or e'_R: $G_{8k+3}\to Q/Z$ be the Adams' complex or real e-invariant respectively [1]. Then we have

Proposition 1.2 (Adams[1]). *e_C: $G_1\to Z_2$, e'_R: $G_3\to Z_{24}$, e_C: $G_7\to Z_{240}$ and e'_R: $G_{11}\to Z_{504}$ are isomorphisms, while there is a split exact sequence*

$$0\to Z_2\{\eta\kappa\}\to G_{15}\xrightarrow{e_C} Z_{480}\to 0\,.$$

In [10] we obtained the following.

Proposition 1.3. *For $f\in\{FP_{n+k,k}, S^{nd}\}$ we have*

$$e_C(f\circ p_{n+k,k}) = -\deg(f)\alpha_F(n,k)\,.$$

Since $e_C=2e'_R$ on $(8k+3)$-stems [1], e'_R gives more precise informations about 2-primary components, so we compute $e'_R(f\circ p_{n+k,k})$ for the case of $F=H$ and $k\equiv 1$ mod(2) or $F=C$ and $k\equiv 2$ mod(4).

We use the following notations. Let $g_C\in\tilde{K}(S^2)$ and $g_R\in\widetilde{KO}(S^8)$ denote the Bott generators. ψ^k denotes the Adams operation. Let c: $KO^*\to K^*$ be the complexification and r: $K^*\to KO^*$ the real restriction. Put $z_0=r(z)\in\widetilde{KO}(CP_n)$ and $z_j=r(g_C^jz)\in\widetilde{KO}^{-2j}(CP_n)$. Put also $y_{2k}=g_R^{-k}\in KO^{8k}$ and $y_{2k+1}\in KO^{8k+4}$ the generator satisfying $c(y_{2k+1})=2g_C^{-4k-2}$ for integer k. For $f\in\{X, Y\}$,$C(f)$ denotes the mapping cone of f.

We consider the case of $F=H$ and $k\equiv 1$ mod(2) or $F=C$ and $k\equiv 2$ mod(4).

Given $f\in\{FP_{n+k,k}, S^{nd}\}$, we have the commutative diagram

$$\begin{tikzcd}
S^{(n+k)d-1} \arrow[r, "p_{n+k,k}"] \arrow[d, "="] & FP_{n+k,k} \arrow[r] \arrow[d, "f"] & FP_{n+k+1,k+1} \arrow[d, "f'"] \\
S^{(n+k)d-1} \arrow[r, "f\circ p_{n+k,k}"] & S^{nd} \arrow[r] & C(f\circ p_{n+k,k}) .
\end{tikzcd}$$

Apply $\widetilde{KO}^{nd}$ and $\tilde{K}^{nd}$ to this diagram; since $\widetilde{KO}^{nd}(S^{(n+k)d-1})=\tilde{K}^{nd}(S^{(n+k)d-1})=\tilde{K}^{nd-1}(S^{nd})=0$ and $\widetilde{KO}^{nd-1}(FP_{n+k,k})$, $\tilde{K}^{nd-1}(FP_{n+k,k})$ and $\widetilde{KO}^{nd-1}(S^{nd})$ are finite groups, we have the following commutative diagram in which the horizontal sequences are exact.

$$\begin{tikzcd}
& 0 & \widetilde{KO}^{nd}(FP_{n+k,k}) \arrow[l] \arrow[dl, "c"'] & \widetilde{KO}^{nd}(FP_{n+k+1,k+1}) \arrow[l] \arrow[dl, "c"'] & \widetilde{KO}^{nd}(S^{(n+k)d}) \arrow[l] \arrow[dl, "c"'] & 0 \arrow[l] \\
0 & \tilde{K}^{nd}(FP_{n+k,k}) \arrow[l] & \tilde{K}^{nd}(FP_{n+k+1,k+1}) \arrow[l] & \tilde{K}^{nd}(S^{(n+k)d}) \arrow[l] & 0 \arrow[l] & \\
& 0 & \widetilde{KO}^{nd}(S^{nd}) \arrow[l] \arrow[dl, "c"'] \arrow[uu, "f^*"'] & \widetilde{KO}^{nd}(C(f\circ p_{n+k,k})) \arrow[l] \arrow[dl, "c"'] \arrow[uu, "f'^*"'] & \widetilde{KO}^{nd}(S^{(n+k)d}) \arrow[l] \arrow[dl, "c"'] \arrow[uu, "="'] & 0 \arrow[l] \\
0 & \tilde{K}^{nd}(S^{nd}) \arrow[l] \arrow[uu, "f^*"] & \tilde{K}^{nd}(C(f\circ p_{n+k,k})) \arrow[l, "j^*"'] \arrow[uu, "f'^*"] & \tilde{K}^{nd}(S^{(n+k)d}) \arrow[l] \arrow[uu, "="] & 0 \arrow[l] &
\end{tikzcd}$$

We can choose generators $a, b\in \widetilde{KO}^{nd}(C(f\circ p_{n+k,k}))$ and $a', b'\in\tilde{K}^{nd}(C(f\circ p_{n+k,k}))$ such that $a'=c(a)$, $2b'=c(b)$, $j^*(a')$ generates $\tilde{K}^{nd}(S^{nd})\cong Z$ and $f'^*(b')=g_C{}^{-nd/2}z^{n+k}$. Here we identify $\tilde{K}^{nd}(FP_{n+k+1,k+1})$ with the free subgroup of $\tilde{K}^{nd}(FP_{n+k+1})$ generated by $g_C{}^{-nd/2}z^n, g_C{}^{-nd/2}z^{n+1}, \cdots, g_C{}^{-nd/2}z^{n+k}$. Hence we can put

$$f'^*(a') = g_C{}^{-nd/2}\sum_{i=0}^{k} a_i z^{n+i}$$

for some integers a_i. Then by the proof of (1.1) of [10] we have

$$\text{(1.4)}\qquad \begin{aligned} a_i &= \deg(f)\alpha_F(n, i) \quad \text{for } 0\leqq i\leqq k-1 , \\ \sum_{i=1}^{k-1} \alpha_F(n, i) \tbinom{n+i}{k-i} d^{n+2i-k} &= d^n(1-d^k)\alpha_F(n, k) . \end{aligned}$$

And we have

Proposition 1.5. *In case of $F=H$ and $k\equiv 1$ mod(2) or $F=C$ and $k\equiv 2$ mod(4) we have*

(i) $e'_R(f\circ p_{n+k,k})=\frac{1}{2}a_k-\frac{1}{2}\deg(f)\alpha_F(n, k)$,

(ii) *if $F=H$, $a_k\equiv 0$ mod(2),*

(iii) *if $F=C$, $n\equiv 1$ mod(2) and $\deg(f)$ is known, a_k mod(2) is computable.*

Proof. First consider the case of $F=H$ and $n\equiv 0 \bmod(2)$. By Bott periodicity we can use $\widetilde{KO}$ and $\tilde{K}$ instead of $\widetilde{KO}^{4n}$ and $\tilde{K}^{4n}$. Then we have

$$\psi^2(a)=4^n a+\lambda b$$

for some integer λ, and

$$e'_R(f\circ p_{n+k,k})=\lambda/(4^n(4^k-1))\,.$$

We have

$$\psi^2(a')=c(\psi^2(a))=4^n a'+2\lambda b'\,,$$

$$\psi^2(f'^*(a'))=\psi^2(\sum_{i=0}^k a_i\, z^{n+i})=\sum_{i=0}^k a_i(z^2+4z)^{n+i}$$

$$=\sum_{j=0}^k\sum_{i=0}^k a_i\binom{n+i}{j-i}4^{n+2i-j}z^{n+j}\,,$$

$$\psi^2(f'^*(a'))=f'^*(\psi^2(a'))=f'^*(4^n a'+2\lambda b')$$

$$=4^n\sum_{i=0}^k a_i z^{n+i}+2\lambda z^{n+k}\,.$$

Comparing the coefficients of z^{n+k}, we have

$$2\lambda=4^n(4^k-1)a_k+\sum_{i=0}^{k-1}a_i\binom{n+i}{k-i}4^{n+2i-k}\,.$$

Then by (1.4) we have

$$e'_R(f\circ p_{n+k,k})=\frac{1}{2}a_k-\frac{1}{2}\deg(f)\alpha_H(n,k)$$

as desired. Next we show (ii). Put $f'^*(a)=\sum_{i=0}^k d_i y_{n+i}\xi^{n+i}$. Then

$$c(f'^*(a))=\sum_{i=0}^k d_i c(y_{n+i})\,(c(\xi))^{n+i}=\sum_{i=0}^k d_i\varepsilon_i g_C{}^{-2(n+i)}(g_C{}^2z)^{n+i}$$

$$=\sum_{i=0}^k d_i\varepsilon_i z^{n+i}\,,$$

where $\varepsilon_i=1$ (if i is even) or 2 (if i is odd). We have also

$$c(f'^*(a))=f'^*(c(a))=\sum_{i=0}^k a_i z^{n+i}\,.$$

Therefore $a_k=d_k\varepsilon_k=2d_k$.

In case of $F=H$ and $n\equiv 1 \bmod(2)$, (i) and (ii) can be proved by the quite parallel arguments to the above. We omit the details.

For $F=C$ (i) can be proved by the same methods as the above. We only prove (iii). First we consider the case of $n\equiv 3 \bmod(4)$. Put $n=4m+3$ and $k=4l+2$. By Bott periodicity we can use $\widetilde{KO}^{-2}$ and $\tilde{K}^{-2}$ instead of $\widetilde{KO}^{2n}$ and $\tilde{K}^{2n}$. By Theorem 2 of Fujii [4], it is easily seen that $\widetilde{KO}^{-2}(CP_{4m+4l+6,4l+3})$ can be identified with the free subgroup of $\widetilde{KO}^{-2}(CP_{4m+4l+6})$ generated by $z_1 z_0^{2m+1}$, $z_1 z_0^{2m+2}, \cdots, z_1 z_0^{2m+2l+2}$. So we can put $f'^*(a)=\sum_{i=0}^{2l+1} d_i z_1 z_0^{2m+1+i}$ for some integers d_i. Then

$$c(f'^*(a))=\sum_{i=0}^{2l+1} d_i c(z_1)\,(c(z_0))^{2m+1+i}=g_C\sum_{i=0}^{2l+1} d_i(z-\bar{z})\,(z+\bar{z})^{2m+1+i}$$

where $\bar{z}=-z+z^2-z^3+\cdots$. We have also

$$c(f'^*(a))=f'^*(c(a))=g_C\sum_{i=0}^{4l+2} a_i z^{4m+3+i}\,.$$

So we have

$$\sum_{i=0}^{4l+2} a_i z^{4m+3+i}=\sum_{i=0}^{2l+1} d_i(2z-z^2+z^3-\cdots)\,(z^2-z^3+\cdots)^{2m+1+i}\,.$$

Calculating this equation over the mod 2 integers, we have

$$\begin{aligned}\sum_{i=0}^{4l+2} a_i z^{4m+3+i} &\equiv \sum_{i=0}^{2l+1} d_i(z^2+z^3+\cdots)^{2m+2+i} \bmod(2,\, z^{4m+4l+6})\\ &\equiv \sum_{j=0}^{4l+1}\sum_{i=0}^{2l+1} d_i\binom{2m+1+j-i}{2m+1+i} z^{4m+4+j} \bmod(2)\,,\end{aligned}$$

since $(x^2+x^3+\cdots)^u=\sum_{j=2u}^{\infty}\binom{j-u-1}{u-1}x^j$. Then

$$a_i\equiv\sum_{j=0}^{2l+1} d_j\binom{2m+i-j}{2m+1+j} \bmod(2)\quad\text{for } 1\leqq i\leqq 4l+2\,. \tag{1.6}$$

By (1.4) and (1.6) for $1\leqq i\leqq 4l+1$, d_j mod(2) is determined for $0\leqq j\leqq 2l$, so the equation

$$\begin{aligned}a_{4l+2} &\equiv \sum_{j=0}^{2l+1} d_j\binom{2m+4l+2-j}{2m+1+j} \bmod(2)\\ &\equiv \sum_{j=0}^{l-1} d_{2j+1}\binom{2m+4l+1-2j}{2m+2j+2} \bmod(2)\end{aligned} \tag{1.6)$'$}$$

determines a_{4l+2} mod(2), here we use the fact $\binom{2i}{2j-1}\equiv 0$ mod(2) for any i and j.

Next we consider the case of $n\equiv 1$ mod(4). Put $n=4m+1$. We use $\widetilde{KO}^{-6}$ and $\tilde{K}^{-6}$ instead of $\widetilde{KO}^{2n}$ and $\tilde{K}^{2n}$. Then we can put $f'^*(a)=\sum_{i=0}^{2l+1} d_i z_3 z_0^{2m+i}$ for some integers d_i. By the same arguments as the above we have

$$a_i\equiv\sum_j d_j\binom{2m+i-j-1}{2m+j} \bmod(2)\quad\text{for } 1\leq i\leq 4l+2 \tag{1.7}$$

and in particular

$$a_{4l+2}\equiv\sum_{i=0}^{l} d_{2i}\binom{2m+4l-2i+1}{2m+2i} \bmod(2)\,. \tag{1.7)$'$}$$

These and (1.4) determine a_{4l+2} mod(2). This completes the proof.

To compute $F\{n, k\}$ by inductive step on k we prepare the followings.

Proposition 1.8. *$F\{n, k\}$ is a divisor of $F\{n, k+1\}$.*

Proof. It is trivial by definition.

Proposition 1.9. *For $f \in \{FP_{n+k,k}, S^{nd}\}$ with $\deg(f)=F\{n, k\}$ we have*

$$F\{n, k\}\#e_C(f \circ p_{n+k,k}) \mid F\{n, k+1\} \mid F\{n, k\}\#(f \circ p_{n+k,k})$$

where $\# g$ denotes the order of g and $a \mid b$ implies that a is a divisor of b.

Proof. Choose $f' \in \{FP_{n+k+1,k+1}, S^{nd}\}$ with $\deg(f')=F\{n, k+1\}$. Since $i_1 \circ p_{n+k,k}=0$, we have

$$\begin{aligned} 0 &= e_C(f' \circ i_1 \circ p_{n+k,k}) = -\deg(f' \circ i_1)\alpha_F(n, k) \\ &= -F\{n, k+1\}\alpha_F(n, k) = -F\{n, k\}\alpha_F(n, k)F\{n, k+1\}/F\{n, k\} \\ &= -e_C(f \circ p_{n+k,k})F\{n, k+1\}/F\{n, k\} . \end{aligned}$$

Hence the first part of the conclusion is obtained. Since $(\#(f \circ p_{n+k,k}))f \circ p_{n+k,k}=0$, there exists $h \in \{FP_{n+k+1,k+1}, S^{nd}\}$ with $h \circ i_1=(\#(f \circ p_{n+k,k}))f$. Then $\deg(h)= \deg(f)\#(f \circ p_{n+k,k})=F\{n,k\}\#(f \circ p_{n+k,k})$. Since $\deg(h)$ is a multiple of $F\{n, k+1\}$, the second part of the conclusion follows.

Proposition 1.10. *For $f \in \{FP_{n+k,k}, S^{nd}\}$ with $\deg(f)=F\{n, k\}$ there exists $h \in \{FP_{n+k,k-1}, S^{nd}\}$ with $(F\{n, k+1\}/F\{n, k\})f \circ p_{n+k,k}=h \circ q_1 \circ p_{n+k,k}$.*

Proof. Consider the exact sequence

$$\cdots \to \{FP_{n+k,k-1}, S^{nd}\} \xrightarrow{q_1^*} \{FP_{n+k,k}, S^{nd}\} \xrightarrow{\deg} \{FP_{n+1,1}, S^{nd}\} \to \cdots$$

Take $f' \in \{FP_{n+k+1,k+1}, S^{nd}\}$ with $\deg(f')=F\{n, k+1\}$. Then $\deg((F\{n, k+1\}/F\{n, k\})f-f' \circ i_1)=0$. So there exists $h \in \{FP_{n+k,k-1}, S^{nd}\}$ with $q_1^*(h)=(F\{n, k+1\}/F\{n, k\})f-f' \circ i_1$ by exactness. Then $h \circ q_1 \circ p_{n+k,k}=((F\{n, k+1\}/F\{n, k\})f-f' \circ i_1) \circ p_{n+k,k}=(F\{n, k+1\}/F\{n, k\})f \circ p_{n+k,k}$ as desired.

Proposition 1.11. *$C\{2n, 2k\}$ is a divisor of $H\{n, k\}$.*

Proof. Consider the commutative diagram

$$\begin{tikzcd} & CP_{2n+2k,2k} \arrow[d, "\pi"] & \supset & CP_{2n+1,1} = S^{4n} \arrow[d, "\pi'"] \\ S^{4n+4k-1} \arrow[ur] \arrow[dr] & & & \\ & HP_{n+k,k} & \supset & HP_{n+1,1} = S^{4n} \end{tikzcd}$$

in which all maps are the canonical ones. For our purpose it suffices to show that π' is a homotopy equivalence. Indeed this holds because in the following commutative diagram π'^* is an isomorphism.

$$\begin{array}{ccccc} H^{4n}(CP_{2n+2k};Z) & \xleftarrow[\cong]{q^*} & H^{4n}(CP_{2n+2k,2k};Z) & \xrightarrow[\cong]{} & H^{4n}(S^{4n};Z) \\ \uparrow{\scriptstyle \pi^*}\,\cong & & \uparrow{\scriptstyle \pi^*}\,\cong & & \uparrow{\scriptstyle \pi'^*} \\ H^{4n}(HP_{n+k};Z) & \xleftarrow[\cong]{q^*} & H^{4n}(HP_{n+k,k};Z) & \xrightarrow[\cong]{} & H^{4n}(S^{4n};Z). \end{array}$$

Next we compute e-invariants of some elements.

Lemma 1.12. *Suppose that there is a commutative diagram*

$$\begin{array}{ccccc} S^{(n+k)d-1} & \xrightarrow{p_{n+k,k}} & FP_{n+k,k} & \subset & FP_{n+k+1,k+1} \\ \downarrow = & & \downarrow L & & \downarrow L' \\ S^{(n+k)d-1} & \xrightarrow{\tilde{p}} & FP_{n+k,k} & \longrightarrow & C(\tilde{p}) \\ \uparrow = & & \cup i & & \uparrow i' \\ S^{(n+k)d-1} & \xrightarrow{s} & FP_{n+1,1} & \longrightarrow & C(s) \end{array}$$

in which L denotes the multiplication by non-zero integer L. Then

$$e_C(s) = L\{\sum_{j=1}^{k-1}\binom{n}{j}d^{k-j}C_j+\binom{n}{k}\}/d^k(d^k-1)$$

where $C_j = C_j(n, k)$ is the coefficient of x^{n+k} in $(\phi_F(x))^{n+j}$.

Proof. Applying $\tilde{K}$ to the above diagram we have the following commutative diagram in which the horizontal sequences are exact.

$$\begin{array}{ccccccc} 0\leftarrow \tilde{K}(FP_{n+k,k}) & \longleftarrow & \tilde{K}(FP_{n+k+1,k+1}) & \longleftarrow & \tilde{K}(S^{(n+k)d}) & \leftarrow 0 \\ \uparrow L^* & & \uparrow L'^* & & \uparrow = & \\ 0\leftarrow \tilde{K}(FP_{n+k,k}) & \longleftarrow & \tilde{K}(C(\tilde{p})) & \longleftarrow & \tilde{K}(S^{(n+k)d}) & \leftarrow 0 \\ \downarrow i^* & & \downarrow i'^* & & \downarrow = & \\ 0\leftarrow \tilde{K}(S^{nd}) & \longleftarrow & \tilde{K}(C(s)) & \longleftarrow & \tilde{K}(S^{(n+k)d}) & \leftarrow 0. \end{array}$$

Choose $a_j\in\tilde{K}(C(\tilde{p}))$ for $0\leqq j\leqq k$ such that $L'^*(a_j)=Lz^{n+j}$ for $0\leqq j\leqq k-1$ and $L'^*(a_k)=z^{n+k}$. Then $i'^*(a_0)$ and $i'^*(a_k)$ generate $\tilde{K}(C(s))$. We have

$$\psi^2(i'^*(a_0)) = d^n i'^*(a_0)+\lambda i'^*(a_k)$$

for some $\lambda\in Z$ and

$$e_C(s) = \lambda/d^n(d^k-1)\,.$$

We compute λ. We have

$$\begin{aligned} L'^*(\psi^2(a_0)) &= \psi^2(L'^*(a_0)) = \psi^2(Lz^n) = L\,(z^2+dz)^n \\ &= L\sum_{j=0}^{k}\tbinom{n}{j}d^{n-j}z^{n+j} \\ &= \sum_{j=0}^{k-1}\tbinom{n}{j}d^{n-j}Lz^{n+j}+L\tbinom{n}{k}d^{n-k}z^{n+k} \\ &= L'^*\{\sum_{j=0}^{k-1}\tbinom{n}{j}d^{n-j}a_j+L\tbinom{n}{k}d^{n-k}a_k\}\,. \end{aligned}$$

Since L'^* is monomorphic, we have

$$\psi^2(a_0) = \sum_{j=0}^{k-1}\tbinom{n}{j}d^{n-j}a_j+L\tbinom{n}{k}d^{n-k}a_k\,.$$

Next consider the following commutative diagram

$$\begin{tikzcd} \tilde{K}(FP_{n+k+1,k+1}) \arrow[r, "ch"] & H^*(FP_{n+k+1,k+1};Q) \\ \tilde{K}(C(\tilde{p})) \arrow[u, "L'^*"] \arrow[r, "ch"] \arrow[d, "i'^*"] & H^*(C(\tilde{p});Q) \arrow[u, "L'^*"'] \arrow[d, "i'^*"'] \\ \tilde{K}(C(s)) \arrow[r, "ch"] & H^*(C(s);Q)\,. \end{tikzcd}$$

Choose the generators $x_{n+j}\in H^{(n+j)d}(C(\tilde{p});Z)$ for $0\leqq j\leqq k$ such that $L'^*(x_{n+j})=Lt^{n+j}$ for $0\leqq j\leqq k-1$ and $L'^*(x_{n+k})=t^{n+k}$. Then for $1\leqq j\leqq k-1$

$$\begin{aligned} L'^*(ch(a_j)) &= ch(L'^*(a_j)) = ch(Lz^{n+j}) = L(\phi_F(t))^{n+j} \\ &= L(t^{n+j}+\text{middle dim}+C_jt^{n+k}) \\ &= L'^*(x_{n+j}+\text{middle dim}+LC_jx_{n+k}) \end{aligned}$$

where the terms middle dim mean elements of middle dimensions. Since L'^* is monomorphic, we have

$$ch(a_j) = x_{n+j}+\text{middle dim}+LC_jx_{n+k} \text{ for } 1\leqq j\leqq k-1\,,$$

and so

$$ch(i'^*(a_j)) = i'^*(ch(a_j)) = LC_ji'^*(x_{n+k}) = ch(LC_ji'^*(a_k)) \quad \text{for } 1\leqq j\leqq k-1\,.$$

Since ch is monomorphic now, we have

$$i'^*(a_j) = LC_ji'^*(a_k) \text{ for } 1\leqq j\leqq k-1\,.$$

Then

$$\psi^2(i'^*(a_0)) = i'^*(\psi^2(a_0)) = i'^*\{\sum_{j=0}^{k-1}\binom{n}{j}d^{n-j}a_j+L\binom{n}{k}d^{n-k}a_k\}$$
$$= d^n i'^*(a_0)+\{\sum_{j=1}^{k-1}\binom{n}{j}d^{n-j}LC_j+L\binom{n}{k}d^{n-k}\}i'^*(a_k)$$
$$= d^n i'^*(a_0)+Ld^{n-k}\{\sum_{j=1}^{k-1}\binom{n}{j}d^{k-j}C_j+\binom{n}{k}\}i'^*(a_k)\,.$$

Therefore we have

$$\lambda = Ld^{n-k}\{\sum_{j=1}^{k-1}\binom{n}{j}d^{k-j}C_j+\binom{n}{k}\}$$

and

$$e_C(s) = L\{\sum_{j=1}^{k-1}\binom{n}{j}d^{k-j}C_j+\binom{n}{k}\}/d^k(d^k-1)\,.$$

This completes the proof.

As a corollary of the above lemma we have

Proposition 1.13. *In the same situation as* (1.12) *we have*

(i) *if* $(F, k)=(C, 1)$, $s=Ln\eta$ *and in particular* $p_{n+1,1}=n\eta$: $S^{2n+1}\to CP_{n+1,1}=S^{2n}$,

(ii) *if* $(F, k)=(H, 2)$, $e_C(s)=Ln(5n-1)/2^5\cdot 3^2\cdot 5$,

(iii) *if* $(F, k)=(C, 4)$, $e_C(s)=Ln(15n^3+30n^2+5n-2)/2^7\cdot 3^2\cdot 5$,

(iv) *if* $(F, k)=(C, 5)$, $e_C(s)=Ln(3n^4+10n^3+5n^2-2n+216)/2^8\cdot 3^2\cdot 5$.

Proof. Since

$$\phi_F(x) = \begin{cases} x+x^2/2!+x^3/3!+\cdots & \text{for } F=C \\ 2x/2!+2x^2/4!+2x^3/6!+\cdots & \text{for } F=H\,, \end{cases}$$

we can easily compute $e_C(s)$ for small k by elementary analysis, so we omit the details except (i). (i) follows from the fact that $e_C\colon G_1\to Z_2$ is an isomorphism and $e_C(s)=\dfrac{1}{2}Ln=e_C(Ln\eta)$.

REMARK. (i) is well known.

In case of $F=H$ and $k\equiv 1$ mod(2) or $F=C$ and $k\equiv 2$ mod(4) we have $e_C(s)=2e'_R(s)$ so the computation of $e'_R(s)$ may give more precise informations about the 2-primary components of the order of s. We do not require the whole computations but we only compute $e'_R(s)$ for the case of $(F, k)=(H, 1)$ or $(C, 2)$. Let $g_4=p_2$: $S^7\to S^4=HP_2$ be the Hopf map. Put $g_\infty=\{g_4\}\in G_3$. Then $e'_R(g_\infty)=1/24$ and

Proposition 1.14 (James [7]). $p_{n+1,1}=ng_\infty$: $S^{4n+3}\to HP_{n+1,1}=S^{4n}$

Proof. We have the short exact sequence

$$0 \leftarrow \widetilde{KO}^{-4n-8}(HP_{n+1,1}) \overset{i^*}{\leftarrow} \widetilde{KO}^{-4n-8}(HP_{n+2,2}) \overset{q^*}{\leftarrow} \widetilde{KO}^{-4n-8}(S^{4n+4}) \leftarrow 0\,.$$

It is easily seen by (1.1) that $\widetilde{KO}^{-4n-8}(HP_{n+1,1})=Z\{g_R\xi^n\}$, $\widetilde{KO}^{-4n-8}(HP_{n+2,2}) =Z\{g_R\xi^n, y_{-1}\xi^{n+1}\}$, $\widetilde{KO}^{-4n-8}(S^{4n+4})=Z\{e\}$, $i^*(g_R\xi^n)=g_R\xi^n$ and $q^*(e)=y_{-1}\xi^{n+1}$. We have

$$\psi^2(g_R\xi^n) = \psi^2(g_R)\psi^2(\xi^n) = 2^4 g_R\{2^{4n}\xi^n+n2^{4n-3}y_1\xi^{n+1}\}\,.$$

Then

$$e'_R(p_{n+1,1}) = 2^{4n+1}n/(2^{4n+6}-2^{4n+4}) = n/24 = e'_R(ng_\infty)\,.$$

This shows that $p_{n+1,1}=ng_\infty$, since $e'_R\colon G_3\to Z_{24}$ is an isomorphism by (1.2).

Now consider the following commutative diagram in which the horizontal sequences are exact.

$$\begin{array}{ccccccc}
\cdots \longrightarrow & \{S^{2n+1}, S^{2n-1}\} & \xrightarrow{p_{n*}} & \{S^{2n+1}, CP_n\} & \xrightarrow{i_*} & \{S^{2n+1}, CP_{n+1}\} \\
 & \downarrow = & & \downarrow q_* & & \downarrow \\
\cdots \longrightarrow & \{S^{2n+1}, S^{2n-1}\} & \xrightarrow{p_{n,1*}} & \{S^{2n+1}, CP_{n,1}\} & \xrightarrow{j_*} & \{S^{2n+1}, CP_{n+1,2}\}
\end{array}$$

$$\begin{array}{ccccc}
\xrightarrow{q_*} & \{S^{2n+1}, S^{2n}\} & \longrightarrow \cdots & & \\
 & \downarrow = & & & \\
\longrightarrow & \{S^{2n+1}, S^{2n}\} & \xrightarrow{p_{n,1*}} & \{S^{2n+1}, S^{2n-1}\} & \longrightarrow \cdots
\end{array}$$

By (1.13) $q_*(p_{n+1})=n\eta$. Then we have

Proposition 1.15. *If* $Ln\equiv 0 \bmod(2)$

$$q_*(i_*)^{-1}(Lp_{n+1}) = \begin{cases} \frac{1}{2}L(n-1)g_\infty & \textit{for } n \textit{ odd} \\ \left\{\frac{1}{2}L(n+2)g_\infty,\ \left(\frac{1}{2}L(n+2)+12\right)g_\infty\right\} & \textit{for } n \textit{ even.} \end{cases}$$

Proof. The above diagram shows that $q_*(i_*)^{-1}(Lp_{n+1})=(j_*)^{-1}(Lp_{n+1,2})$. Since $\{S^{2n+1}, S^{2n-1}\}=Z_2\{\eta^2\}$ and $p_{n,1*}(\eta^2)=(n-1)\eta^3=12(n-1)g_\infty$, $(j_*)^{-1}(Lp_{n+1,2})$ is a coset of the subgroup of $\{S^{2n+1}, CP_{n,1}\}=G_3$ generated by $12(n-1)g_\infty$. This coset consists of a single element if n is odd or two elements if n is even. In case of n being odd we have the following commutative diagram by the proof of

(1.11), (i) of (1.13) and (1.14).

$$\begin{tikzcd}
 & CP_{n+1,2}=S^{2n-2}\vee S^{2n} \arrow[dd] & \supset CP_{n,1}=S^{2n-2} \arrow[ddl, "="] \\
S^{2n+1} \arrow[ur, "p_{n+1,2}"] \arrow[dr, "(1/2)(n-1)g_\infty"'] & & \\
 & HP_{(n+1)/2,1}=S^{2n-2} &
\end{tikzcd}$$

This diagram proves Proposition if n is odd. If n is even, we have the short exact sequence

$$0 \to \{S^{2n+1}, S^{2n-1}\} \to \{S^{2n+1}, S^{2n-2}\} \xrightarrow{j_*} \{S^{2n+1}, CP_{n+1,2}\} \to 0$$

since $p_{n,1}=(n-1)\eta$ by (i) of (1.13). For our purpose it suffices to show that

$$(j_*)^{-1}(p_{n+1,2}) = \{(n/2+1)g_\infty,(n/2+13)g_\infty\}\ .$$

For any $f\in(j_*)^{-1}(p_{n+1,2})$ the equation
(*) $\quad e'_R(f)=(n/2+1+12e)/24$ for some integer e
implies this, because $e'_R((n/2+1)g_\infty)=(n/2+1)/24$. We prove (*). We use $\widetilde{KO}^{-2}$ if $n\equiv 0 \bmod(4)$ or $\widetilde{KO}^{-6}$ if $n\equiv 2 \bmod(4)$. The methods are quite parallel, so we only prove (*) for the case of $n\equiv 0 \bmod(4)$. Put $n=4m$. There is the following commutative diagram in which the horizontal sequences are exact.

$$\begin{tikzcd}
0 & \widetilde{KO}^{-2}(CP_{4m+1,2}) \arrow[l] \arrow[d, "i^*"] & \widetilde{KO}^{-2}(CP_{4m+2,3}) \arrow[l] \arrow[d, "i'^*"] & \widetilde{KO}^{-2}(S^{8m+2}) \arrow[l] \arrow[d, "="] & 0 \arrow[l] \\
0 & \widetilde{KO}^{-2}(S^{8m-2}) \arrow[l] & \widetilde{KO}^{-2}(C(f)) \arrow[l, "u^*"'] & \widetilde{KO}^{-2}(S^{8m+2}) \arrow[l, "v^*"'] & 0 \arrow[l]
\end{tikzcd}$$

By Theorem 2 of Fujii [4] it is easy to see that $\widetilde{KO}^{-2}(CP_{4m+1,2})=Z\{z_1z_0^{2m-1}\}$, $\widetilde{KO}^{-2}(CP_{4m+2,3})=Z\{z_1z_0^{2m-1}, z_1z_0^{2m}\}$, $\widetilde{KO}^{-2}(CP_{4m,1})=Z\{w\}$ with $2w=z_1z_0^{2m-1}$ and $\widetilde{KO}^{-2}(CP_{4m+2,1})=Z\{z_1z_0^{2m}\}$. Take $a\in\widetilde{KO}^{-2}((C(f))$ with $u^*(a)=w$. Then a and $v^*(z_1z_0^{2m})=i'^*(z_1z_0^{2m})$ generate $\widetilde{KO}^{-2}(C(f))$. By definition $2a=i'^*(z_1z_0^{2m-1})+ei'^*(z_1z_0^{2m})$ for some integer e. We have $\psi^2(a)=2^{4m}a+\lambda i'^*(z_1z_0^{2m})$ for some integer λ, and $e'_R(f)=\lambda/2^{4m}\cdot 3$. We have also

$$\begin{aligned} c(2a) &= c(i'^*(z_1z_0^{2m-1})+ei'^*(z_1z_0^{2m})) \\ &= g_Ci'^*\{2z^{4m-1}-(4m-1)z^{4m}+(4m^2+2e)z^{4m+1}\} \end{aligned}$$

and

$$c(i'^*(z_1z_0^{2m})) = 2g_Ci'^*(z^{4m+1})$$

and then

$$
\begin{aligned}
c(\psi^2(2a)) &= c(2^{4m+1}a+2\lambda i'^*(z_1 z_0^{2m})) \\
&= g_C i'^*\{2^{4m+1}z^{4m-1}-2^{4m}(4m-1)z^{4m}+(2^{4m+2}m^2+2^{4m+1}e+4\lambda)z^{4m+1}\}\ .
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
c(\psi^2(2a)) &= \psi^2(c(2a)) = \psi^2[g_C i'^*\{2z^{4m-1}-(4m-1)z^{4m}+(4m^2+2e)z^{4m+2}\}] \\
&= 2g_C\psi^2[i'^*\{2z^{4m-1}-(4m-1)z^{4m}+(4m^2+2e)z^{4m+1}\}] \\
&= g_C i'^*\{2^{4m+1}z^{4m-1}-2^{4m}(4m-1)z^{4m}+2^{4m-1}(2^3m^2+2m+1+16e)z^{4m+1}\}\ .
\end{aligned}
$$

Comparing the coefficients of z^{4m+1}, we have

$$\lambda = 2^{4m-3}(2m+1+12e)$$

and so

$$e'_R(f) = (2m+1+12e)/24\ .$$

This completes the proof.

In the sequel we shall need the explicit form of $\alpha_F(n,k)$ for small k. Since the expansion of $\phi_F^{-1}(x)$ is known (see e.g. [10]), we can obtain the following by elementary calculations.

Lemma 1.16.

$$
\begin{aligned}
&\alpha_F(n,0) = 1, \\
&\alpha_H(n,1) = -n/2^2\cdot 3\ , \\
&\alpha_H(n,2) = n(5n+11)/2^5\cdot 3^2\cdot 5\ , \\
&\alpha_H(n,3) = -n(35n^2+231n+382)/2^7\cdot 3^4\cdot 5\cdot 7\ , \\
&\alpha_H(n,4) = n(175n^3+2310n^2+10181n+14982)/2^{11}\cdot 3^5\cdot 5^2\cdot 7\ , \\
&\alpha_H(n,5) = -n(385n^4+8470n^3+69971n^2+257246n+355128)/2^{13}\cdot 3^6\cdot 5^2\cdot 7\cdot 11, \\
&\alpha_C(n,1) = -n/2\ , \\
&\alpha_C(n,2) = n(3n+5)/2^3\cdot 3\ , \\
&\alpha_C(n,3) = -n(n+2)\,(n+3)/2^4\cdot 3\ , \\
&\alpha_C(n,4) = n(15n^3+150n^2+485n+502)/2^7\cdot 3^2\cdot 5\ , \\
&\alpha_C(n,5) = -n(3n^4-30n^3+785n^2-78n+1240)/2^8\cdot 3^2\cdot 5\ , \\
&\alpha_C(n,6) = n(63n^5+1575n^4+15435n^3+73801n^2+171150n+152696) \\
&\qquad\qquad /2^{10}\cdot 3^4\cdot 5\cdot 7\ , \\
&\alpha_C(n,7) = -n(9n^6+315n^5+4515n^4+33817n^3+139020n^2+295748n \\
&\qquad\qquad +252336)/2^{11}\cdot 3^4\cdot 5\cdot 7\ , \\
&\alpha_C(n,8) = n(135n^7+6300n^6+124110n^5+1334760n^4+8437975n^3 \\
&\qquad\qquad +74777100n^2-68303596n+138452016)/2^{15}\cdot 3^5\cdot 5^2\cdot 7\ ,
\end{aligned}
$$

$$\alpha_C(n,9) = -n(15n^8+900n^7+23310n^6+339752n^5-829745n^4+38354500n^3 +27449684n^2+112877136n+100476288)/2^{16}\cdot 3^5\cdot 5^2\cdot 7\,,$$
$$\alpha_C(n,10)= n(99n^9+7425n^8+244530n^7+4634322n^6+55598235n^5 +436886945n^4+2242194592n^3+7220722828n^2 +38722058672n-15239326848)/2^{18}\cdot 3^6\cdot 5^2\cdot 7\cdot 11\,.$$

2. $H\{n, k\}$ for $k\leqq 5$

The results of this section are summarized as follows.

Theorem 2.1. (i) $H\{n, 1\} = 1$,

(ii) $H\{n, 2\} = 24/(n, 24)$,

(iii) $H\{n, 3\} = H\{n, 2\}\,\mathrm{den}[H\{n, 2\}\alpha_H(n, 2)]$,

(iv) $H\{n, 4\} = H\{n, 3\}\,\mathrm{den}\left[\frac{1}{2}H\{n, 3\}\alpha_H(n, 3)\right]$,

(v) $H\{n, 5\}/(H\{n, 4\}\,\mathrm{den}[H\{n, 4\}\alpha_H(n, 4)])$
$$= \begin{cases} 1 \text{ or } 2 \text{ if } n\equiv 1 \bmod(2^5) \text{ or } 34 \bmod(2^6) \\ 1 \quad \text{otherwise,} \end{cases}$$

where den(a) *denotes the denominator of a rational number* a *when the fraction* a *is expressed in its lowest terms.*

Proof. (i) is trivial.

By (1.14), $\sharp p_{n+1,1}=24/(n,24)$, since $\sharp g_\infty=24$. Then $H\{n, 2\}\,|\,24/(n, 24)$ by (1.9). Choose $f\in\{HP_{n+2,2}, S^{4n}\}$ with $\deg(f)=H\{n,2\}$. Then

$$0 = f\circ i_1\circ p_{n+1,1} = \deg(f)p_{n+1,1} = H\{n, 2\}\,p_{n+1,1}\,.$$

Therefore $24/(n,24)\,|\,H\{n,2\}$. Hence (ii) follows.

Take $f\in\{HP_{n+2,2}, S^{4n}\}$ with $\deg(f)=H\{n,2\}$. We have $\sharp e_C(f\circ p_{n+2,2})=\sharp(f\circ p_{n+2,2})$, since $e_C\colon G_7\to Z_{240}$ is an isomorphism by (1.2). They by (1.9) $H\{n,3\}=H\{n, 2\}\cdot\sharp e_C(f\circ p_{n+2,2})$. By (1.3) $e_C(f\circ p_{n+2,2})=-H\{n,2\}\alpha_H(n,2)$. Hence (iii) is obtained.

For any $h\in\{HP_{n+3,2}, S^{4n}\}$ we have

$$e'_R(h\circ q_1\circ p_{n+3,3}) = -\frac{1}{2}\deg(h\circ q_1)\alpha_H(n, 3) = 0$$

by (1.5). Since $e'_R\colon G_{11}\to Z_{504}$ is an isomorphism by (1.2), $h\circ q_1\circ p_{n+3,3}=0$.

Then by (1.10), for $f\in\{HP_{n+3,3}, S^{4n}\}$ with $\deg(f)=H\{n,3\}$, $\sharp(f\circ p_{n+3,3})$ is a divisor of $H\{n,4\}/H\{n,3\}$. Conversely (1.9) implies that $\sharp(f\circ p_{n+3,3})$ is a multiple of $H\{n,4\}/H\{n,3\}$. Hence $\sharp(f\circ p_{n+3,3})=H\{n,4\}/H\{n,3\}$. On the other hand $e'_R(f\circ p_{n+3,3})=-\frac{1}{2}H\{n,3\}\alpha_H(n,3)$ by (1.5). Hence $\sharp(f\circ p_{n+3,3})=\mathrm{den}\left[\frac{1}{2}H\{n,3\}\alpha_H(n,3)\right]$. Therefore

$$H\{n, 4\}/H\{n, 3\} = \mathrm{den}\left[\frac{1}{2}H\{n, 3\}\alpha_H(n, 3)\right]$$

and this implies (iv).

For the proof of (v) we prepare a lemma.

Lemma 2.2. *If* $n\equiv 0$ *or* 3 mod(4), *the image of* $p_{n+4,2}{}^*\colon \{HP_{n+4,2}, S^{4n}\}\to \{S^{4n+15}, S^{4n}\}$ *contains the element* $\eta\kappa\in G_{15}$.

The proof of (2.2): Since all Toda brackets which appear in the proof have zero indeterminacies, we have

$$\eta\kappa = \langle\varepsilon, 2\iota, \nu^2\rangle = \langle\varepsilon, 2\nu, \nu\rangle = \langle\varepsilon, 2g_\infty, g_\infty\rangle\,.$$

Consider the diagram

$$\begin{tikzcd}
S^{4n+14} \arrow[d, "(n+3)g_\infty"] & & S^{4n+15} \arrow[d, "p_{n+4,2}"] \arrow[dr, "p_{n+4,1}"] & \\
S^{4n+11} \arrow[r, "p_{n+3,1}"] & HP_{n+3,1}=S^{4n+8} \arrow[r, hook, "i"] \arrow[dr, "\varepsilon"'] & HP_{n+4,2} \arrow[r] & S^{4n+12} \\
 & & S^{4n} &
\end{tikzcd}$$

By (1.14) $p_{n+3,1}=(n+2)g_\infty$ and $p_{n+4,1}=(n+3)g_\infty$. So $p_{n+3,1}\circ(n+3)g_\infty=\varepsilon\circ p_{n+3,1}=0$, since $2g_\infty^2=\varepsilon g_\infty=0$. Then there exists $f\in\{HP_{n+4,2}, S^{4n}\}$ with $f\circ i=\varepsilon$, and by definition of Toda bracket

$$f\circ p_{n+4,2}\in\langle\varepsilon, (n+2)g_\infty, (n+3)g_\infty\rangle$$

and

$$\begin{aligned}\langle\varepsilon, (n+2)g_\infty, (n+3)g_\infty\rangle &= \frac{1}{2}(n+2)(n+3)\langle\varepsilon, 2g_\infty, g_\infty\rangle \\ &= \frac{1}{2}(n+2)(n+3)\eta\kappa\,.\end{aligned}$$

Thus $f\circ p_{n+4,2}=\frac{1}{2}(n+2)(n+3)\eta\kappa$. Since the order of $\eta\kappa$ is 2, the conclusion follows.

Now we prove (v). Take $f\in\{HP_{n+4,4}, S^{4n}\}$ with $\deg(f)=H\{n,4\}$. Then $e_C(f\circ p_{n+4,4})=-H\{n,4\}\alpha_H(n,4)$ by (1.3), and $\#(f\circ p_{n+4,4})/\# e_C(f\circ p_{n+4,4})=1$ or 2 by (1.2). From (1.9) $H\{n,5\}/(H\{n,4\}\mathrm{den}[H\{n,4\}\alpha_H(n,4)])=1$ or 2. And by (1.2), if $\nu_2(H\{n,4\}\alpha_H(n,4))\leqq -1$, we have $\#(f\circ p_{n+4,4})=\# e_C(f\circ p_{n+4,4})=\mathrm{den}[H\{n,4\}\alpha_H(n,4)]$ and

$$H\{n, 5\} = H\{n, 4\}\mathrm{den}[H\{n, 4\}\alpha_H(n, 4)]\,,$$

where $\nu_p(n/m)=\nu_p(n)-\nu_p(m)$ for a prime number p and integers m and n. (1.16), (ii), (iii), (iv) and elementary analysis show that $\nu_2(H\{n,4\}\alpha_H(n,4))\geqq 0$ if and only if $n\equiv 3 \bmod(2^3)$, $1 \bmod(2^5)$, $34 \bmod(2^6)$ or 0 and (2^{10}). Consider the case of $n\equiv 3 \bmod(2^3)$ or $0 \bmod(2^{10})$. By (2.2) there exists $h\in\{HP_{n+4,2}, S^{4n}\}$ with $h\circ p_{n+4,2}=\eta\kappa$. Then f or $f+h\circ q_2$, say f', satisfies the conditions $\#e_C(f'\circ p_{n+4,4})=\#(f'\circ p_{n+4,4})$ and $\deg(f')=H\{n,4\}$. Then by (1.3) $\#e_C(f'\circ p_{n+4\ 4})=\mathrm{den}[H\{n,4\}\alpha_H(n,4)]$ and the conclusion (v) follows from (1.9).

3. $C\{n,k\}$ for $k\leqq 10$

In this section we determine inductively $C\{n,k\}$ for $k\leqq 8$ and estimate them for $k=9$ and 10. The results are as follows.

Theorem 3.1. (i) $C\{n,1\}=1$,

(ii) $C\{n,2\}=2/(n,2)$,

(iii) $C\{n,4\}=C\{n,3\}=\begin{cases}24/(n,24) & \textit{if } n\not\equiv 1 \bmod(4)\\ 12/(n,3) & \textit{if } n\equiv 1 \bmod(8)\\ 6/(n,3) & \textit{if } n\equiv 5 \bmod(8),\end{cases}$

(iv) $C\{n,5\}=C\{n,4\}\mathrm{den}[C\{n,4\}\alpha_C(n,4)]$,

(v) $C\{n,6\}=C\{n,5\}\mathrm{den}[C\{n,5\}\alpha_C(n,5)]$

$=\begin{cases} C\{n,5\} & \textit{if } n\equiv 0 \bmod(2),\ 1,\ 11 \textit{ or } 27 \bmod(32)\\ 2C\{n,5\} & \textit{otherwise,}\end{cases}$

(vi) $C\{n,7\}=\begin{cases} C\{n,6\}\mathrm{den}[C\{n,6\}\alpha_C(n,6)] & \textit{if } n\equiv 0 \bmod(2) \textit{ or } 19 \bmod(32)\\ 2C\{n,6\}\mathrm{den}[C\{n,6\}\alpha_C(n,6)] & \textit{otherwise}\end{cases}$

(vii) $C\{n,8\}=C\{n,7\}$,

(viii) $C\{n,9\}/(C\{n,8\}\mathrm{den}[C\{n,8\}\alpha_C(n,8)])$

$=\begin{cases}1 \text{ or } 2 & \textit{if } n\equiv 3 \bmod(2^7) \textit{ or } 1 \bmod(2^9)\\ 1 & \textit{otherwise,}\end{cases}$

(ix) $C\{n,10\}/C\{n,9\}=\begin{cases}1 & \textit{if } n\equiv 0,\ 6 \bmod(2^3),\ 10,\ 12 \bmod(2^4),\\ & \quad 18,\ 20 \bmod(2^5),\ 34,\ 36 \bmod(2^6) \textit{ or } 4 \bmod(2^7)\\ 1 \textit{ or } 2 & \textit{otherwise.}\end{cases}$

Proof. (i) is trivial. (ii) is proved by the same methods as the proof of (ii) of (2.1).

The proof of (iii): The first equality is a consequence of (1.9) and the fact $G_5=0$. We prove the second equality. Choose $f\in\{CP_{n+2,2}, S^{2n}\}$ with $\deg(f)=C\{n,2\}$. Then $C\{n,3\}/C\{n,2\}$ is a divisor of $\#(f\circ p_{n+2,2})$ from (1.9), there exists $h\in\{CP_{n+2,1}, S^{2n}\}$ with $(C\{n,3\}/C\{n,2\})f\circ p_{n+2,2}=h\circ q_1\circ p_{n+2,2}$ from (1.10), while $q_1\circ p_{n+2,2}=(n+1)\eta$ from (i) of (1.13), so $C\{n,3\}/C\{n,2\}$ is a multiple of $\sharp(f\circ p_{n+2,2})$ if n is odd, and therefore $C\{n,3\}/C\{n,2\}=\sharp(f\circ p_{n+2,2})$ if n is odd. From (1.5), $e'_R(f\circ p_{n+2,2})=\frac{1}{2}a_2-\frac{1}{2}C\{n,2\}\alpha_C(n,2)$ for some integer a_2. If $n\equiv 3$ mod(4), say $n=4m+3$, $a_2\equiv 0$ mod(2) by (1.6)$'$, then $e'_R(f\circ p_{n+2,2})=-(4m+3)(6m+7)/12$ by (1.16) and (ii), hence $\sharp(f\circ p_{n+2,2})=\mathrm{den}[(4m+3)/12]=12/(n,24)$ by (1.2), and therefore the conclusion follows in this case since $C\{n,2\}=2$. If $n\equiv 1$ mod(4), say $n=4m+1$, $a_2\equiv 1$ mod(2) by (1.4), (1.7), (1.7)$'$ and (ii), then $e'_R(f\circ p_{n+2,2})=-(12m-1)(m+1)/6$ by (1.16) and (ii), hence $\sharp(f\circ p_{n+2,2})=\mathrm{den}[(m+1)/6]$ and the conclusion follows easily in this case also.

Next we consider the case of n being even. Take $f\in\{CP_{n+3,3}, S^{2n}\}$ with $\deg(f)=C\{n,3\}$. First we show that $C\{n,3\}$ is a multiple of $24/(n,24)$. Since arguments are quite parallel we only consider the case of $n\equiv 0$ mod(4). Put $n=4m$ and consider the commutative diagram

$$\begin{array}{ccc}\widetilde{KO}(CP_{4m+3,3}) & \xrightarrow{c} & \tilde{K}(CP_{4m+3,3}) \\ \uparrow f^* & & \uparrow f^* \\ \widetilde{KO}(S^{8m}) & \underset{\cong}{\xrightarrow{c}} & \tilde{K}(S^{8m})\,. \end{array}$$

We can put $f^*(g_R^m)=d_0z_0^{2m}+d_1z_0^{2m+1}$ for some integers d_0 and d_1. We have

$$\begin{aligned} c(f^*(g_R^m)) &= d_0(z+\bar{z})^{2m}+d_1(z+\bar{z})^{2m+1} \\ &= d_0z^{4m}-2d_0mz^{4m+1}+((2m^2+m)d_0+d_1)z^{4m+2}\,, \\ c(f^*(g_R^m)) &= f^*(c(g_R^m)) = a_0z^{4m}+a_1z^{4m+1}+a_2z^{4m+2} \end{aligned}$$

for some integers a_0, a_1 and a_2. Comparing the coefficients of the powers of z, by (1.4) we have

$$d_0=a_0=C\{n,3\}\,,$$
$$(2m^2+m)d_0+d_1=a_2=C\{4m,3\}\alpha_C(4m,2)=C\{4m,3\}m(12m+5)/6$$

and so $d_1=-C\{4m,3\}m/6$. Thus $C\{4m,3\}$ is a multiple of $\mathrm{den}(m/6)=24/(4m,24)$ as desired. Second we show that $C\{n,3\}$ is a divisor of $24/(n,24)$. We define $h\colon CP_{n+2,2}=S^{2n}\vee S^{2n+2}\to S^{2n}$ by $h|_{S^{2n}}=24/(n,24)$ and

$$h|_{S^{2n+2}}=\begin{cases}0 & \text{if } n\equiv 0 \text{ mod}(16)\\ \eta^2 & \text{for other even } n.\end{cases}$$

Since $p_{n+2,2}=\frac{1}{2}ng_\infty\vee\eta$, $h\circ p_{n+2,2}=(12n/(n,24))g_\infty+h|_{S^{2n+2}}\circ\eta=0$. Hence there exists $f'\in\{CP_{n+3,3}, S^{2n}\}$ with $f'|_{CP_{n+2,2}}=h$. Clearly $\deg(f')=24/(n,24)$, so $C\{n,3\}$ is a divisor of $24/(n,24)$. Thus $C\{n,3\}=24/(n,24)$ if n is even. This completes the proof of (iii).

The proof of (iv): By (1.3), $e_C(h\circ q_1\circ p_{n+4,4})=0$ for any $h\in\{CP_{n+4,3}, S^{2n}\}$ and then $h\circ q_1\circ p_{n+4,4}=0$ by (1.2). So by (1.3), (1.9) and (1.10)

$$C\{n, 5\}/C\{n, 4\} = \#(f\circ p_{n+4,4}) = \mathrm{den}[C\{n, 4\}\alpha_C(n, 4)]\,.$$

The proof of (v): First consider the case of $n\equiv 1 \bmod(2)$. Choose $f\in \{CP_{n+5,5}, S^{2n}\}$ with $\deg(f)=C\{n, 5\}$. Recall that $G_9=Z_2\{\eta\bar{\nu}\}\oplus Z_2\{\eta\varepsilon\}\oplus Z_2\{\mu\}$ and the kernel of e_C: $G_9\to Q/Z$ is $Z_2\{\eta\bar{\nu}\}\oplus Z_2\{\eta\varepsilon\}$. Hence, if $e_C(f\circ p_{n+5,5})=0$, we can choose $h\in\{CP_{n+5,1}, S^{2n}\}=G_8$ with $(f+h\circ q_4)p_{n+5,5}=0$, because $q_4\circ p_{n+5,5}=p_{n+5,1}=\eta$ by (i) of (1.13). Since $\deg(f+h\circ q_4)=\deg(f)=C\{n,5\}$, by (1.9) we have

$$C\{n, 6\} = C\{n, 5\} = C\{n, 5\}\#e_C(f\circ p_{n+5,5})\,.$$

If $e_C(f\circ p_{n+5,5})\neq 0$, (1.9) implies

$$C\{n, 6\} = 2C\{n, 5\} = C\{n, 5\}\#e_C(f\circ p_{n+5,5})\,.$$

Since $C\{n,5\}$ and $\alpha_C(n,5)$ are known, we can easily compute $\mathrm{den}[C\{n,5\}\alpha_C(n,5)]$ by elementary analysis. Indeed

$$\begin{aligned}\#e_C(f\circ p_{n+5,5}) &= \mathrm{den}[C\{n, 5\}\alpha_C(n, 5)]\\ &= \begin{cases}1 \text{ if } n\equiv 1,\ 11 \text{ or } 27 \bmod(32)\\ 2 \text{ for other odd } n.\end{cases}\end{aligned}$$

This completes the proof of (v) if n is odd.

Suppose that n is even. It is easy to see that $\mathrm{den}[C\{n,5\}\alpha_C(n,5)]=1$. From (1.8) and (1.11)

$$C\{n, 5\}\,|\,C\{n, 6\}\,|\,H\{n/2, 3\}\,.$$

By the previous calculations $C\{n, 5\}$ and $H\{n/2, 3\}$ are coinside if $n\equiv 0 \bmod(4)$, 6, 10 or 14 mod(16), so $C\{n, 5\}=C\{n, 6\}$ in this case, while if $n\equiv 2 \bmod(16)$ the odd components are coinside but

$$2 = \nu_2(C\{n, 5\})\leqq \nu_2(C\{n, 6\})\leqq \nu_2(H\{n/2, 3\}) = 3\,.$$

Put $n=16m+2$. We construct a commutative diagram in which $\deg(f)=C\{16m+2, 5\}$.

$$\begin{tikzcd}
 & & & S^{32m+13} \arrow[dlll, "s_3"', bend right=0] \arrow[ddll, "s_2"'] \arrow[dl, "s_1"'] \arrow[d, "p_{16m+7,5}"] \arrow[ddlll, "s_3"'] \\
 & & CP_{16m+6,4} \arrow[r, phantom, "\subset"] \arrow[d, "3"] & CP_{16m+7,5} \arrow[d, "3"] \\
CP_{16m+4,2}=S^{32m+4}\vee S^{32m+6} \arrow[r, phantom, "\overset{i}{\subset}"] \arrow[d, "h"] & CP_{16m+5,3} \arrow[r, phantom, "\overset{i}{\subset}"] & CP_{16m+6,4} \arrow[r, phantom, "\overset{i}{\subset}"] & CP_{16m+7,5} \arrow[d, "f"] \\
S^{32m+4} \arrow[rrr, "C\{16m+2,\, 5\}/C\{16m+2,\, 4\}"'] & & & S^{32m+4}
\end{tikzcd}$$

By (i) of (1.13), $q_{16m+5}\circ p_{16m+7}=p_{16m+7,1}=0$ and so by (1.15) we have

$$q_{16m+4*}(i_{1*})^{-1}(p_{16m+7}) = \{(8m+4)g_\infty, (8m+16)g_\infty\} .$$

Take $s_1'\in(i_{1*})^{-1}(p_{16m+7})\subset\{S^{32m+13}, CP_{16m+6}\}$ with $q_{16m+4}\circ s_1'=(8m+16)g_\infty$. Put $s_1=q_{16m+1}\circ s_1'$. Then

$$q_3\circ 3s_1 = q_{16m+4}\circ 3s_1' = 3(8m+16)g_\infty = 0 .$$

Hence there exists $s_2\in\{S^{32m+13}, CP_{16m+5,3}\}$ with $i_1\circ s_2=3s_1$. Since $q_2\circ s_2\in G_5=0$, there exists $s_3\in\{S^{32m+13}, CP_{16m+4,2}\}$ with $i_1\circ s_3=s_2$. Next we define h by $h|_{S^{32m+4}}=C\{16m+2,4\}$ and $h|_{S^{32m+6}}=\eta^2$. Since $p_{16m+4,2}=(8m+1)g_\infty\vee\eta$ by the proof of (1.11), (1.14) and (i) of (1.13), we have

$$\begin{aligned} h\circ p_{16m+4,2} &= C\{16m+2,4\}\,(8m+1)g_\infty+\eta^3 \\ &= \frac{24(8m+1)}{(16m+2,24)}g_\infty+12g_\infty \\ &= 0 . \end{aligned}$$

So there exists $h'\in\{CP_{16m+5,3}, S^{32m+4}\}$ with $h'\circ i=h$. Since $h'\circ p_{16m+5,3}\in G_5=0$, there exists $h''\in\{CP_{16m+6,4}, S^{32m+4}\}$ with $h''\circ i=h'$. By (1.2), (1.3) and (iv) we have

$$\begin{aligned} \#(h''\circ p_{16m+6,4}) &= \# e_C(h''\circ p_{16m+6,4}) \\ &= \mathrm{den}[\deg(h'')\alpha_C(16m+2,4)] \\ &= C\{16m+2,5\}/C\{16m+2,4\} . \end{aligned}$$

Hence there exists $f\in\{CP_{16m+7,5}, S^{32m+4}\}$ with $(C\{16m+2,5\}/C\{16m+2,4\})h''=f\circ i$ and $\deg(f)=\deg(h'')C\{16m+2,5\}/C\{16m+2,4\}=C\{16m+2,5\}$. This completes the construction of the above diagram.

Now we proceed to the proof of (v). We may write $s_3=s_3'\vee q_1\circ s_3$ for some $s_3'\in\{S^{32m+13}, S^{32m+4}\}$. By (iii) of (1.13)

$$e_C(q_3\circ s_3) = (16m+3)\,(3840m^3+2640m^2+590m+43)/2^3\cdot 3\cdot 5$$

so by (1.2) $q_1\circ s_3$ is divisible by 2. Then

$$\begin{aligned} f\circ p_{16m+7,5} &= f\circ 3p_{16m+7,5}, \text{ since } 2G_9=0\,, \\ &= (C\{16m+2,5\}/C\{16m+2,4\})h\circ s_3 \\ &= (C\{16m+2,5\}/C\{16m+2,4\})\,(C\{16m+2,4\}s_3'+\eta^2\circ q_1\circ s_3) \\ &= (C\{16m+2,5\}/C\{16m+2,4\})\,(0+0), \text{ since } C\{16m+2,4\}\equiv 0 \bmod(2) \\ &\qquad\qquad \text{and } 2\eta=0 \\ &= 0 . \end{aligned}$$

Thus by (1.9), $C\{16m+2,6\}=C\{16m+2,5\}$. This completes the proof of (v).

The proof of (vi): First consider the case of n being odd. For any $h \in \{CP_{n+6,5}, S^{2n}\}$, by (i) of (1.5) we have

$$e'_R(h \circ q_1 \circ p_{n+6,6}) = \frac{1}{2} a$$

for some integer a. By (1.6) and (1.7) a is even. Then $h \circ q_1 \circ p_{n+6,6} = 0$ by (1.2). Thus (1.9) and (1.10) imply

$$C\{n, 7\} = C\{n, 6\} \#(f \circ p_{n+6,6})$$

for $f \in \{CP_{n+6,6}, S^{2n}\}$ with $\deg(f) = C\{n, 6\}$. Again by (i) of (1.5)

$$e'_R(f \circ p_{n+6,6}) = \frac{1}{2} a_6 - \frac{1}{2} C\{n, 6\} \alpha_C(n, 6)$$

for some integer a_6, and by the proof of (iii) of (1.5) we have

$$a_6 \equiv \begin{cases} 0 \bmod(2) & \text{if } n \equiv 3 \bmod(4) \text{ or } 33 \bmod(64) \\ 1 \bmod(2) & \text{for other odd } n. \end{cases}$$

Then since $\#(f \circ p_{n+6,6})$ is equal to $\# e'_R(f \circ p_{n+6,6}) = \mathrm{den}\left[\frac{1}{2} a_6 - \frac{1}{2} C\{n, 6\} \alpha_C(n, 6)\right]$ by (1.2), elementary analysis draws the conclusion for odd n by (iii), (iv), (v) and (1.16).

Next suppose that n is even. Choose $f \in \{CP_{n+6,6}, S^{2n}\}$ with $\deg(f) = C\{n, 6\}$. (1.2) says that $e_C = 2e'_R : G_{11} \to Q/Z$ is monomorphic on the odd component, so (vi) is true about the odd components by (1.3) and (1.9). So we only see the 2-primary part. Recall that $G_{11} = Z_8\{\zeta\} \oplus Z_{63}$. By (1.3), (1.16) and elementary analysis show that

$$\nu_2(\# e_C(f \circ p_{n+6,6})) \leqq 2 .$$

If $\nu_2(\# e_C(f \circ p_{n+6,6})) = 0$, $\nu_2(\#(f \circ p_{n+6,6})) \leqq 1$ by (1.2) and (1.5). If $\nu_2(\#(f \circ p_{n+6,6})) = 0$, the result follows by (1.9). If $\nu_2(\#(f \circ p_{n+6,6})) = 1$, we have

$$f \circ p_{n+6,6} \equiv 4\zeta \text{ mod (odd components)}.$$

Since $4\zeta = \mu\eta^2$ and $p_{n+6,1} = q_5 \circ p_{n+6,6} = \eta$,

$$(f + \mu\eta q_5) p_{n+6,6} \equiv 0 \text{ mod(odd components)}.$$

Clearly $\deg(f + \mu\eta q_5) = \deg(f) = C\{n, 6\}$, so the result follows again by (1.9). If $\nu_2(\# e_C(f \circ p_{n+6,6})) = u = 1$ or 2,

$$\nu_2(C\{n, 6\}) + u \leqq \nu_2(C\{n, 7\})$$

by (1.9), and

$$\nu_2(\#(f \circ p_{n+6,6})) = u + 1$$

by (1.2) and (1.5), so

$$f\circ p_{n+6,6}\equiv 2^{2-u}\zeta \bmod(2^{3-u}\zeta, \text{ odd components})$$

and then

$$(2^u f+\mu\eta q_5)\circ p_{n+6,6}\equiv 0 \bmod(\text{odd components}).$$

Put $\#((2^u f+\mu\eta q_5)\circ p_{n+6,6})=2m+1$. Then there exists $h\in\{CP_{n+7,7}, S^{2n}\}$ with $h|_{CP_{n+6,6}}=(2m+1)(2^u f+\mu\eta q_5)$. Clearly $\deg(h)=2^u(2m+1)\deg(f)=2^u(2m+1)\cdot C\{n,6\}$. Since $\deg(h)$ is a multiple of $C\{n,7\}$, we have

$$\nu_2(C\{n, 7\})\leqq\nu_2(C\{n, 6\})+u$$

and hence

$$\begin{aligned}\nu_2(C\{n, 7\}) &= \nu_2(C\{n, 6\})+u\\ &= \nu_2(C\{n, 6\})+\nu_2(\# e_C(f\circ p_{n+6,6}))\\ &= \nu_2(C\{n, 6\}\,\mathrm{den}[C\{n, 6\}\alpha_C(n, 6)])\end{aligned}$$

as desired. This completes the proof of (vi).

The proof of (vii): Since $G_{13}=Z_3\{\alpha_1\beta_1\}$, $C\{n,8\}/C\{n,7\}=1$ or 3 by (1.9). In case of n being even, the relations

$$C\{n, 7\}\,|\,C\{n, 8\}\,|\,H\{n/2,4\}$$

and the previous calculations show that the 3-components of the first and the third are equal so that the 3-components of these three are equal. Thus $C\{n,8\}=C\{n,7\}$ if n is even.

Choose $h\in\{CP_{n+7,2}, S^{2n+10}\}$ with $\deg(h)=C\{n+5,2\}$. Then

$$\begin{aligned}e_C(h\circ q_5\circ p_{n+7,7}) &= -C\{n+5,2\}\alpha_C(n+5,2)\\ &= -(n+5)(3n+20)/(12(n+5,2))\end{aligned}$$

so by (1.2)

$$\#(h\circ q_5\circ p_{n+7,7})\equiv 0 \bmod(3) \text{ if and only if } n\not\equiv 1 \bmod(3).$$

Therefore if $n\not\equiv 1 \bmod(3)$, the image of

$$p_{n+7,2}{}^* = (q_5\circ p_{n+7,7})^*\colon \{CP_{n+7,2}, S^{2n+10}\}\to\{S^{2n+13}, S^{2n+10}\} = G_3$$

contains $Z_3\{\alpha_1\}$.

Take $f\in\{CP_{n+7,7}, S^{2n}\}$ with $\deg(f)=C\{n,7\}$. Suppose that $n\not\equiv 1 \bmod(3)$. If $f\circ p_{n+7,7}=0$, $C\{n,8\}=C\{n,7\}$ by (1.9). If $f\circ p_{n+7,7}\neq 0$, that is $f\circ p_{n+7,7}=\pm\beta_1\alpha_1$, the above implies that there exists $h'\in\{CP_{n+7,2}, S^{2n+10}\}$ with $h'\circ q_5\circ p_{n+7,7}=\mp\alpha_1$, and we have

$$(f+\beta_1\circ h'\circ q_5)\circ p_{n+7,7}=0\,,$$
$$\deg(f+\beta_1\circ h'\circ q_5)=\deg(f)=C\{n,7\}$$

and so by (1.9)

$$C\{n,8\}=C\{n,7\}\,.$$

Therefore $C\{n,8\}=C\{n,7\}$ if $n\not\equiv 1 \bmod(3)$.

We must prove (vii) for the case of $n\equiv 1 \bmod(6)$. Put $n=6m+1$. Take $f\in\{CP_{6m+8,7}, S^{12m+2}\}$ with $\deg(f)=C\{6m+1,7\}$. By the same methods as the proof of (v) we can construct a commutative diagram

$$\begin{tikzcd}
 & & & S^{12m+15} \arrow[d, "p_{6m+8,7}"] \arrow[ddlll, bend right, "s_3"'] \arrow[ddll, "s_2"'] \arrow[dl, "s_1"'] \\
 & & & CP_{6m+8,7} \arrow[d, "2"] \\
 & & CP_{6m+7,6} \arrow[r, phantom, "\subset"] \arrow[d, "4"] & CP_{6m+8,7} \arrow[d, "4"] \\
CP_{6m+5,4} \arrow[r, phantom, "\overset{i}{\subset}"] & CP_{6m+6,5} \arrow[r, phantom, "\overset{i}{\subset}"] & CP_{6m+7,6} \arrow[r, phantom, "\overset{i}{\subset}"] & CP_{6m+8,7} \arrow[d, "f"] \\
 & & & S^{12m+2}
\end{tikzcd}$$

Take $a\in\{CP_{6m+5,4}, S^{12m+2}\}$ with $\deg(a)=C\{6m+1,4\}$ and $b\in\{CP_{6m+3,2}, S^{12m+2}\}$ with $\deg(b)=C\{6m+1,2\}=2$. Consider the diagram

$$\begin{tikzcd}
 & & \{S^{12m+6}, S^{12m+2}\}=0 \arrow[d] & \\
\{S^{12m+8}, S^{12m+2}\} \arrow[r, "q^*"] & \{CP_{6m+5,4}, S^{12m+2}\} \arrow[r] & \{CP_{6m+4,3}, S^{12m+2}\} \arrow[r] \arrow[d] & \{S^{12m+7}, S^{12m+2}\}=0 \\
\{S^{12m+3}, S^{12m+2}\} \arrow[r, "\eta^*", "\cong"'] & \{S^{12m+4}, S^{12m+2}\} \arrow[r] & \{CP_{6m+3,2}, S^{12m+2}\} \arrow[r] & \{S^{12m+2}, S^{12m+2}\}
\end{tikzcd}$$

in which the horizontals and the vertical are the parts of suitable Puppe exact sequences. Then a generates a free part of $\{CP_{6m+5,4}, S^{12m+2}\}$ which is of rank 1, and so

$$\begin{aligned} f\circ i\circ i\circ i &= (\deg(f)/\deg(a))a+q^*(e) \\ &= (C\{6m+1,7\}/C\{6m+1,4\})a+q^*(e) \end{aligned}$$

for some $e\in\{S^{12m+8}, S^{12m+2}\}=G_6$. Then

$$\begin{aligned}
2f\circ p_{6m+8,7} &= 8f\circ p_{6m+8,7}, \text{ since } G_{13}=Z_3\\
&= f\circ i\circ i\circ i\circ s_3\\
&= (C\{6m+1,7\}/C\{6m+1,4\})a\circ s_3 + e\circ q\circ s_3\\
&= (C\{6m+1,7\}/C\{6m+1,4\})a\circ s_3, \text{ since } G_6\circ G_7 = 0\,.
\end{aligned}$$

By the previous calculations and elementary analysis it follows that

$$\nu_3(C\{6m+1,7\}) = \begin{cases} 3 \text{ if } m\equiv 1 \text{ or } 2 \bmod(3)\\ 2 \text{ if } m\equiv 3 \text{ or } 6 \bmod(9)\\ 1 \text{ if } m\equiv 0 \bmod(9)\,,\end{cases}$$

$$\nu_3(C\{6m+1,4\}) = 1$$

so if $m\not\equiv 0 \bmod(9)$ we have

$$C\{6m+1,7\}/C\{6m+1,4\}\equiv 0 \bmod(3)$$

and so

$$f\circ p_{6m+8,7} = 0$$

and then by (1.9)

$$C\{6m+1,8\} = C\{6m+1,7\} \quad \text{if } m\not\equiv 0 \bmod(9).$$

Next suppose that $m\equiv 0 \bmod(9)$. By (iii) of (1.13) we can easily see that

$$\nu_3(\# e_C(q_3\circ s_3)) = 0\,.$$

So by (1.13) and the same methods as the proof of (v), we can construct a commutative diagram

$$\begin{tikzcd}
 & & & S^{12m+15} \arrow[dl, "s_3"'] \arrow[ddll, bend right, "s_4"'] \arrow[dddlll, bend right, "s_5"'] \arrow[d, "8p_{6m+8,7}"] \\
 & & CP_{6m+5,4} \arrow[r, phantom, "\subset"] \arrow[d, "40"] & CP_{6m+8,7} \arrow[d, "40"] \\
 & CP_{6m+4,3} \arrow[r, phantom, "\subset"] \arrow[d, "2"] & CP_{6m+5,4} \arrow[r, phantom, "\subset"] & CP_{6m+8,7} \arrow[d, "2"] \\
CP_{6m+3,2} \arrow[r, phantom, "\subset"] & CP_{6m+4,3} \arrow[rr, phantom, "\subset"] & & CP_{6m+8,7} \arrow[d, "f"] \\
 & & & S^{12m+2}
\end{tikzcd}$$

Then

$$f\circ p_{6m+8,7} = 640f\circ p_{6m+8,7}$$

$$\begin{aligned} &= f\,|_{CP_{6m+3,2}} \circ s_5 \\ &= (\deg(f)/\deg(b))b \circ s_5 \\ &= (C\{6m+1,7\}/2)b \circ s_5 \\ &= 0\,, \ \text{ since } C\{6m+1,7\} \equiv 0 \bmod(6) \end{aligned}$$

so by (1.9)

$$C\{6m+1,8\} = C\{6m+1,7\} \quad \text{if } m \equiv 0 \bmod(9).$$

This completes the proof of (vii).

The proof of (viii): Take $f \in \{CP_{n+8,8}, S^{2n}\}$ with $\deg(f) = C\{n, 8\}$. First consider the case of n being even. By (i) of (1.13) $p_{n+8,1} = q_7 \circ p_{n+8,8} = \eta$. Then f or $f + \kappa q_7$, say f', satisfies

$$\begin{aligned} &\#(f' \circ p_{n+8,8}) = \#e_C(f' \circ p_{n+8,8}) = \operatorname{den}[C\{n,8\}\alpha_C(n,8)]\,, \\ &\deg(f') = \deg(f) = C\{n,8\} \end{aligned}$$

by (1.2), and so the conclusion follows from (1.9). Next suppose that n is odd. By (1.2)

$$\#(f \circ p_{n+8,8})/\#e_C(f \circ p_{n+8,8}) = 1 \text{ or } 2\,.$$

By the previous calculations and elementary analysis we have
$\nu_2(\operatorname{den}[C\{n,8\}\alpha_C(n,8)]) = 0$ if and only if $n \equiv 3 \bmod(2^7)$ or $1 \bmod(2^9)$.
Therefore if $n \not\equiv 3 \bmod(2^7)$ and $1 \bmod(2^9)$, by (1.2) we have

$$\#(f \circ p_{n+8,8}) = \#e_C(f \circ p_{n+8,8}) = \operatorname{den}[C\{n, 8\}\alpha_C(n, 8)]$$

and so the conclusion follows.

The proof of (ix): Since $2G_{17} = 0$, by (1.9) we have

$$C\{n, 10\}/C\{n, 9\} = 1 \text{ or } 2\,.$$

In case of n being even, by the following relations and an elementary analysis conclusion follows if $n \equiv 0 \bmod(2^3)$, 10, 12, 14 $\bmod(2^4)$, 18, 20, 22 $\bmod(2^5)$, 34, 36 $\bmod(2^6)$ or 4 $\bmod(2^7)$

$$C\{n, 9\} \mid C\{n, 10\} \mid H\{n/2, 5\}\,.$$

If $n \equiv 6 \bmod(2^5)$, the conclusion follows from the same methods as the proof of (vii).

4. Relations with other James numbers

In this section we use the notations and terminologies of James [6,7] freely.

Consider the fibration of Stiefel manifolds

$$O_{n-1,k-1} \to O_{n,k} \xrightarrow{p} O_{n,1} = S^{nd-1}$$

and the cofibration of quasi-projective spaces

$$Q_{n-1,k-1} \to Q_{n,k} \xrightarrow{q} Q_{n,1} = S^{nd-1}$$

where $n>k>0$. Following James [6] we define non-negative integers $O\{n,k\}$, $O^s\{n,k\}$, $Q\{n,k\}$ and $Q^s\{n,k\}$ by the equations

$$\begin{aligned} p_*\pi_{nd-1}(O_{n,k}) &= O\{n,k\}\pi_{nd-1}(S^{nd-1}), \\ p_*\pi^s_{nd-1}(O_{n,k}) &= O^s\{n,k\}\pi^s_{nd-1}(S^{nd-1}), \\ q_*\pi_{nd-1}(Q_{n,k}) &= Q\{n,k\}\pi_{nd-1}(S^{nd-1}), \\ q_*\pi^s_{nd-1}(Q_{n,k}) &= Q^s\{n,k\}\pi^s_{nd-1}(S^{nd-1}) \end{aligned}$$

here $\pi^s_m(X)=\{S^m, X\}$ for a pointed space X. We have

Lemma 4.1. *$O\{n,k\} \mid Q\{n,k\}$, $O^s\{n,k\} \mid O\{n,k\}$ and $Q^s\{n,k\} \mid Q\{n,k\}$.*

Proof. The first conclusion follows from the commutative diagram

$$\begin{tikzcd} Q_{n,k} \arrow[r, "q"] \arrow[d, phantom, "\cap"] & Q_{n,1} \arrow[d, "="] \\ O_{n,k} \arrow[r, "p"] & O_{n,1} \end{tikzcd}$$

and the others follow immediately by definition.

Let $M_k(F)$ be the order of the canonical F-line bundle over FP_k in the J-group $J(FP_k)$ [3] which was determined by Adams-Walker [2] and Sigrist-Suter [13]. We have

Lemma 4.2. *$Q^s\{n,k\}=O^s\{n,k\}$.*

Proof. For any m with $m\equiv 0 \bmod(M_k(F))$ there exists S^0-section $w\colon Q_{m,1} \to Q_{m,k}$, that is, $q\circ w\simeq 1$. By James [7] we have the diagram

$$\begin{tikzcd} Q_{m,1}*Q_{n,k} \arrow[r, "1*i"] \arrow[d, "1*q"] & Q_{m,1}*O_{n,k} \arrow[r, "w*1"] \arrow[d, "1*p"] & Q_{m,k}*O_{n,k} \arrow[r, "g'"] \arrow[d, "q*p"] & Q_{m+n,k} \arrow[d, "q"] \\ Q_{m,1}*Q_{n,1} \arrow[r, phantom, "="] & Q_{m,1}*O_{n,1} \arrow[r, phantom, "="] & Q_{m,1}*O_{n,1} \arrow[r, "\simeq"'] & Q_{m+n,1} \end{tikzcd}$$

in which $g'\circ(w*1)\circ(1*i)$ is a homotopy equivalence by (7.3) of [7], the first square is commutative, the second is homotopy commutative and the third is homotopy commutative up to sign from quasi-projective case of (5.2) of [7]. Applying $\pi^s_{(m+n)d-1}$ to this diagram we have the following diagram

$$\begin{tikzcd}
\pi^s_{nd-1}(Q_{n,k}) \arrow[r, "i*"] \arrow[d, "q*"] & \pi^s_{nd-1}(O_{n,k}) \arrow[r] \arrow[d, "p*"] & \pi^s_{(m+n)d-1}(Q_{m,k}*O_{n,k}) \arrow[r, "g'_*"] \arrow[d, "(q*p)*"] & \pi^s_{(m+n)d-1}(Q_{m+n,k}) \arrow[d, "q*"] \\
\pi^s_{nd-1}(Q_{n,1}) \arrow[r, equal] & \pi^s_{nd-1}(O_{n,1}) \arrow[r, "\cong"'] & \pi^s_{(m+n)d-1}(Q_{m,1}*O_{n,1}) \arrow[r, "\cong"'] & \pi^s_{(m+n)d-1}(S^{(m+n)d-1})
\end{tikzcd}$$

in which the first and second squares are commutative and the third is commutative up to sign. Hence $Q^s\{m+n,k\} \mid O^s\{n,k\} \mid Q^s\{n,k\}$. Since $Q^s\{m+n,k\} = Q^s\{n,k\}$, the conclusion follows.

We have also

Lemma 4.3. *If $n \geqq 2(k-1)+2/d$, then*

$$Q^s\{n,k\} = O^s\{n,k\} = O\{n,k\} = Q\{n,k\} .$$

Proof. Since $Q_{n,k}$ and $O_{n,k}$ are $(n-k+1)d-2$ connected, the canonical homomorphisms $\pi_{nd-1}(Q_{n,k}) \to \pi^s_{nd-1}(Q_{n,k})$ and $\pi_{nd-1}(O_{n,k}) \to \pi^s_{nd-1}(O_{n,k})$ are epimorphisms if $n \geqq 2(k-1)+2/d$. Then $Q^s\{n,k\} = Q\{n,k\}$ and $O^s\{n,k\} = O\{n,k\}$ in this case, and the conclusion follows from (4.2).

Atiyah [3] proved that $Q_{n,k}$ and $P_{k-n,k}$ are S-duals. His proof gives the following precise theorem.

Theorem 4.4. *For any j with $jM_k(F) \geqq n$, there exists a $(djM_k(F)-1)$-duality $u \in \{Q_{jM_k(F)-n+k,k} \wedge P_{n,k}, S^{djM_k(F)-1}\}$.*

Consider the cofibrations

$$S^{(n-k)d} \overset{i}{\subset} P_{n,k} \to P_{n,k-1} \to S^{(n-k)d+1}$$

$$S^{md-2} \to Q_{m-1,l-1} \subset Q_{m,l} \xrightarrow{q} S^{md-1}$$

We have

Proposition 4.5. *If $jM_k(F) \geqq n$, $(djM_k(F)-1)$-dual of i: $S^{(n-k)d} \to P_{n,k}$ is q: $Q_{jM_k(F)-n+k,k} \to S^{(jM_k(F)-n+k)d-1}$, and hence $F\{n-k,k\} = Q^s\{jM_k(F)-n+k,k\}$.*

Proof. By Puppe exact sequences associated with the above cofibrations it is easily seen that $\{S^{(n-k)d}, P_{n,k}\}$ and $\{Q_{jM_k(F)-n+k,k}, S^{(jM_k(F)-n+k)d-1}\}$ are infinite cyclic groups with generators i and q respectively. Then the conclusion follows from (4.4).

As a corollary of (4.3) and (4.5) we have

Theorem 4.6. *$F\{n,k\}$ is equal to $O\{jM_k(F)-n,k\}$ if $jM_k(F)\geqq n+2k-2+2/d$.*

In case of $F=C$, Sigrist [12, Théorème I] proved that a prime number p is a factor of $O\{m,l\}$ if and only if p is a factor of $M_l(C)/(m,M_l(C))$. His proof is valid for the case of $F=H$, since $M_l(H)$ is known [13]. Then by (4.6) we have

Proposition 4.7. *A prime number p is a factor of $F\{n,k\}$ if and only if p is a factor of $M_k(F)/(n,M_k(F))$.*

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