Komatsu, T.
Osaka J. Math.
25 (1988), 697–728

CONTINUITY ESTIMATES FOR SOLUTIONS OF PARABOLIC EQUATIONS ASSOCIATED WITH JUMP TYPE DIRICHLET FORMS

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(Received June 23, 1987)

0. Introduction

A priori estimates for solutions of linear equations generally play an important part in the study of non-linear equations. It is important that the estimates do not depend at all on the smoothness of coefficients. For uniformly parabolic equations of the form

$$\frac{\partial u}{\partial t} = \Sigma_{ij}\, \partial_i(a_{ij}(t, x)\partial_j u)\,, \tag{0.1}$$

the pioneering result of this kind is the Hölder continuity estimate due to Nash [14]. Generalizations of Nash's theorem were obtained by various methods in a number of works (see, for example, [8], [12]).

In the appendix to [14], Nash stated without detailed proof a Harnack type inequality for solutions of (0.1), while Moser [13] gave a proof of the Harnack type inequality different from that by Nash, and obtained the Hölder continuity estimate using the Harnack type inequality. For equation (0.1) with discontinuous coefficients, Aronson [1] proved the uniqueness of weak solutions of the Cauchy problem making use of Moser's result. The Harnack type inequality for solutions of quasi-linear parabolic equations was discussed in Aronson-Serrin [4] following the outline of Moser's proof. Applying theorems in [4], Aronson [2] showed that the fundamental solution of equation (0.1) is bounded below and above by functions of the form

$$Ct^{-d/2} \exp[-\beta|x|^2/t]\,,$$

where d is the dimension of the space, C and β denote positive constants. (cf. Ladyzhenskaja-Solonnikov-Uraltseva [11], Aronson [3]).

Parabolic equations of second order are closely related to Markov processes, so that Nash's methods have some probabilistic flavour. Above mentioned works were applied to various problems in stochastic analysis, especially to stochastic optimal control (cf. Bensoussan-Lions [5]) and homogenization (cf. Fukushima

[6]). Recently Osada [15] obtained a generalization of Aronson's results to discuss the propagation of chaos for Burger's equation. Equation (0.1) is associated with the Dirichlet form

$$\mathcal{E}_t(f, g) = \Sigma_{ij} \int a_{ij}(t, x)\partial_i f(x)\partial_j g(x)dx \tag{0.2}$$

and a diffusion process. In the theory of Dirichlet space, Dirichlet forms (1.1) of jump type are considered as well as Dirichlet forms (0.2) of diffusion type (cf. Fukushima [7]). Therefore it seems to be natural and important to consider a priori estimates similar to those due to Nash, Moser and Aronson for parabolic equations associated with jump type Dirichlet forms.

Markov processes whose generator are pseudo-differential operators are investigated in Komatsu [9] and [10], where the theory of singular integrals plays an essential part. Under a certain regularity condition, Markov processes associated with jump type Dirichlet forms are special ones of processes treated in [10]. Theorems in [10], however, can give little information to our problem which requires global theory. As we see later on, the situation is not the same as the case where Dirichlet forms are of type (0.2). It would be the main step to prove the Harnack type inequality, but this problem is still open. In the present paper, we shall obtain a priori estimates of modulus of continuity for fundamental solutions. These are weaker than the a priori Holder continuity estimates, but they can give a certain compactness to the class of possible solutions of parabolic equations. The outline of the proof is similar to that by Nash, for the idea comes from the probability theory.

1. Main results

Let $k(t, x, y)$ be a positive function on $\boldsymbol{R}_+ \times \boldsymbol{R}^d \times \boldsymbol{R}^d$ which is symmetric in x and y and $K(t, dx, dy) = (1/2)k(t, x, y)|x-y|^{-d-\alpha}dx\,dy$, where $0<\alpha<2$. We shall consider the Dirichlet form

$$\mathcal{E}_t(f, g) = \iint (f(x)-f(y))(g(x)-g(y))K(t, dx, dy) \tag{1.1}$$

of jump type defined for functions on $\boldsymbol{R}^d$. We assume that there are positive constants c_1 and c_2 such that

$$c_1 \le k(t, x, y) \le c_2 . \tag{1.2}$$

A function $u_t(x) = u(t, x)$ is said to be a (weak) solution of the parabolic equation associated with the Dirichlet form $\mathcal{E}_t(\cdot, \cdot)$ if

$$\sup_{s\le\tau\le t} (u_\tau, u_\tau)_{L^2} + \int_s^t \mathcal{E}_\tau(u_\tau, u_\tau)\,d\tau < \infty$$

and

$$(u_t, f)_{L^2}-(u_s, f)_{L^2}+\int_s^t \mathcal{E}_\tau(u_\tau, f)d\tau = 0 \tag{1.3}$$

for any $0<s<t$ and any test function $f(x)$ on $\boldsymbol{R}^d$, where $(\cdot,\cdot)_{L^2}$ denotes the usual inner product of the Hilbert space $L^2(\boldsymbol{R}^d, dx)$.

Let A_t be the operator defined by the relation

$$\mathcal{E}_t(f, g) = -(A_t f, g)_{L^2}\,. \tag{1.4}$$

Then equation (1.3) is equivalent to the parabolic equation $\partial u/\partial t=A_t u$. If $(\partial/\partial y_j)k(t, x, y)$, $1\leq j\leq d$, are bounded continuous in (t, x, y), then

$$A_t f(x) = \int (f(x+z)-f(x)-\Theta(z)\cdot\partial f(x))k(t, x, x)\,|z|^{-d-\alpha}dz$$
$$+\int (f(x+z)-f(x))(k(t, x, x+z)-k(t, x, x))\,|z|^{-d-\alpha}dz$$

for smooth bounded functions $f(x)$, where $\Theta(z)=z$ if $|z|<1$ and $\Theta(z)=0$ if $|z|>1$ and $\partial f(x)=(\partial f/\partial x_1, \cdots, \partial f/\partial x_d)$.

It can be proved that, if $(\partial/\partial x)^\mu(\partial/\partial y)^\nu k(t, x, y)$ are bounded and continuous for all $\mu, \nu\in\boldsymbol{Z}_+^d$ and if

$$\lim_{t\to s}\sup_{x,y}\,|k(t, x, y)-k(s, x, y)| = 0\,, \tag{1.5}$$

then there exists a fundamental solution $S(s, x; t, y)$ of the parabolic equation such that $(\partial/\partial x)^\mu(\partial/\partial y)^\nu S(s, x; t, y)$ are continuously differentiable on $\{(s, x, t, y);\ s<t \text{ and } x, y\in\boldsymbol{R}^d\}$ for all $\mu, \nu\in\boldsymbol{Z}_+^d$. (This fact will be shown in another paper by the author.) The purpose of this paper is to prove continuity estimates of the special solution $S(0, 0; t, x)=T(t, x)=T_t(x)$ depending only on α, d, c_1 and c_1. After the continuity estimates are obtained, a passage to the limit can remove the restrictions on the function $k(t, x, y)$ except (1.2) and (1.5) (see Section 5).

In Section 2, 3 and 4, we assume that $(\partial/\partial x)^\mu(\partial/\partial y)^\nu k(t, x, y)$ are bounded and continuous for all $\mu, \nu\in\boldsymbol{Z}_+^d$. Then we see that, for each $\nu\in\boldsymbol{Z}_+^d$, $(\partial/\partial x)^\nu T(t, x)$ is continuous on $(0, \infty)\times\boldsymbol{R}^d$ and

$$\int [T(t, x)^2+|\partial T(t, x)|^2]dx <\infty \qquad (t>0)\,, \tag{1.6}$$

moreover the equation $(\partial/\partial t)T(t, x)=A_t T(t, x)$ is satisfied in the strong sense. From the arguement for stochastic differential equations, it is easy to see that

$$\sup_{t\leq n}\int (1+|x|)^\beta T(t, x)\,dx <\infty \tag{1.7}$$

for any $\beta<\alpha$ and n.

We shall use the convention of letting c's stand for positive constants depending only on α, d, c_1 and c_2. Each c may denote a constant different from other c's. The next moment bound is essential in the proof of continuity estimates.

Theorem 1. (i) *$T(t, x)\leq ct^{-d/\alpha}$.*
(ii) *Let $r(\sigma)=\sigma^{\alpha/2}(1+c(\log\sigma)^2)^{-1/2}$. Then*

$$\int r(t^{-1/\alpha}|x|)T(t, x)dx\leq c\,. \tag{1.8}$$

We do not know whether the function $r(\sigma)$ can be replaced by the simple function $\sigma^{\alpha/2}$ in moment bound (1.8) or not. This is the reason why we were unable to obtain the Hölder estimate so far. Define

$$\Phi(\sigma)=\begin{cases}\exp\left[-\dfrac{\log(1/\sigma)}{\log\log(1/\sigma)}\right] & (0<\sigma<e^{-e})\\ e^{-e} & (e^{-e}\leq\sigma)\,.\end{cases} \tag{1.9}$$

The $\Phi(\sigma)$ is increasing and $\Phi(+0)=0$. We see that $\sigma^{\varepsilon}\leq c\,\Phi(\sigma)$ for any $\varepsilon>0$ and $0<\sigma<1$, so that the following continuity estimates are weaker than the Hölder estimate.

Theorem 2. (i) *For any $t>s>0$ and $x, x'\in\boldsymbol{R}^d$,*

$$|T(t, x)-T(t, x')|\leq cs^{-d/\alpha}\Phi((t-s)^{-1/\alpha}|x-x'|)^c\,.$$

(ii) *For any $t'>t>s>0$ and $x\in\boldsymbol{R}^d$,*

$$|T(t, x)-T(t', x)|\leq cs^{-d/\alpha}\Phi((t-s)^{-1}(t'-t))^c\,.$$

From this theorem, fundamental solutions $S(s, x; t, y)$ of the parabolic equations associated with the Dirichlet forms satisfying condition (1.2) are equicontinuous on any compact domain in $\{(s, x, t, y);\ 0<s<t \text{ and } x, y\in\boldsymbol{R}^d\}$.

REMARK. Here we shall state the canonical change of scales. Let $\mathcal{E}_t(\lambda||f, g)$ be the Dirichlet form associated with the function $k(\lambda^\alpha t, \lambda x, \lambda y)$ in place of the function $k(t, x, y)$, where λ is a positive constant. Then the function

$$S(\lambda||s, x; t, y)=\lambda^d S(\lambda^\alpha s, \lambda x; \lambda^\alpha t, \lambda y)$$

is the fundamental solution of the parabolic equation

$$(\partial/\partial t)u=A_t(\lambda)u$$

associated with the Dirichlet form $\mathcal{E}_t(\lambda||\cdot, \cdot)$. Let

$$U(t, x) = T(t^{1/\alpha}\|1, x) = t^{d/\alpha} T(t, t^{1/\alpha} x) . \tag{1.10}$$

Then Theorem 1 is equivalent to say that $U(t, x) \leq c$ and

$$\int r(|x|) U(t, x) dx \leq c .$$

2. The moment bound

Let $\mathcal{F}$ denote the Fourier transform and $\mathcal{F}^{-1}$ the inverse Fourier transform for functions on $\boldsymbol{R}^d$:

$$\mathcal{F} f(\xi) = \int e^{-i\xi \cdot x} f(x) dx , \quad \mathcal{F}^{-1} \phi(x) = (2\pi)^{-d} \int e^{i\xi \cdot x} \phi(\xi) d\xi .$$

For any test function $f(x)$,

$$f(x+z) - f(x) = \mathcal{F}^{-1}[(e^{i\xi \cdot z} - 1) \mathcal{F} f(\xi)](x) .$$

From the elemental equality

$$\int (e^{i\xi \cdot z} - 1)(e^{-i\eta \cdot z} - 1) |z|^{-d-\alpha} dz$$
$$= (2\pi)^d a (|\xi|^\alpha + |\eta|^\alpha - |\xi - \eta|^\alpha) ,$$

where a is a constant, we have

$$\iint |f(x+z) - f(x)|^2 |z|^{-d-\alpha} dx dz$$
$$= (2\pi)^{-2d} \iint \{ \iint (e^{i\xi \cdot z} - 1)(e^{-i\eta \cdot z} - 1)$$
$$\times e^{i(\xi-\eta)\cdot x} \mathcal{F} f(\xi) \overline{\mathcal{F} f(\eta)} d\xi d\eta \} |z|^{-d-\alpha} dx dz$$
$$= (2\pi)^{-d} a \int \{ \iint (|\xi|^\alpha + |\eta|^\alpha - |\xi - \eta|^\alpha)$$
$$\times e^{i(\xi-\eta)\cdot x} \mathcal{F} f(\xi) \overline{\mathcal{F} f(\eta)} d\xi d\eta \} dx$$
$$= a \iint (|\xi|^\alpha + |\eta|^\alpha - |\xi - \eta|^\alpha) \delta(\xi - \eta) \mathcal{F} f(\xi) \overline{\mathcal{F} f(\eta)} d\xi d\eta$$
$$= 2a \int |\xi|^\alpha |\mathcal{F} f(\xi)|^2 d\xi .$$

From (1.6) we see that

$$\int |\xi|^\alpha |\mathcal{F} T_t(\xi)|^2 d\xi < \int (1 + |\xi|^2) |\mathcal{F} T_t(\xi)|^2 d\xi < \infty ,$$

therefore a passage to the limit gives

$$\iint |T_t(x)-T_t(y)|^2|x-y|^{-d-\alpha}dx\,dy = 2a\int|\xi|^\alpha|\mathcal{F}T_t(\xi)|^2 d\xi\,. \tag{2.1}$$

Let $\rho(x)$ be a smooth function such that $0\le\rho(x)\le 1$, $\rho(x)=1$ for $|x|\le 1$ and $\rho(x)=0$ for $|x|\ge 2$, and set $\rho_n(x)=\rho(x/n)$. Define $E(t)$ by

$$E(t)=\int T_t(x)^2dx\,. \tag{2.2}$$

For $0<s<t$, we have

$$\begin{aligned}\int T_t^2\rho_n dx-\int T_s^2\rho_n dx &= 2\int_s^t\int T_\tau((\partial/\partial\tau)T_\tau)\rho_n d\tau\,dx\\ &= 2\int_s^t (A_\tau T_\tau,\ T_\tau\rho_n)_{L^2}d\tau = -2\int_s^t \mathcal{E}_\tau(T_\tau,\ T_\tau\rho_n)d\tau\,,\end{aligned}$$

whence

$$E(t)-E(s) = -2\int_s^t \mathcal{E}_\tau(T_\tau,\ T_\tau)d\tau\,,$$

or $(d/dt)E(t)=-2\mathcal{E}_t(T_t,\ T_t)$.

Lemma 2.1. *There exists a positive constant c depending only on α, d, c_1 and c_2 such that $T(t,\ x)\le ct^{-d/\alpha}$.*

Proof. From (2.1) and

$$\mathcal{E}_t(T_t,\ T_t)\ge(c_1/2)\iint(T_t(x)-T_t(y))^2|x-y|^{-d-\alpha}dx\,dy\,,$$

we have

$$-(d/dt)E(t)\ge c\int|\xi|^\alpha|\mathcal{F}T_t(\xi)|^2d\xi\,.$$

On the other hand, for any $r>0$,

$$\begin{aligned}E(t) &= (2\pi)^{-d}\int|\mathcal{F}T_t(\xi)|^2d\xi\\ &\le c\sup_\xi|\mathcal{F}T_t(\xi)|^2\int_{|\xi|\le r}d\xi+c\int_{|\xi|>r}|\mathcal{F}T_t(\xi)|^2d\xi\\ &\le c\,r^d+c\,r^{-\alpha}\int|\xi|^\alpha|\mathcal{F}T_t(\xi)|^2d\xi\,.\end{aligned}$$

Choosing the best constant r, we obtain

$$\int|\xi|^\alpha|\mathcal{F}T_t(\xi)|^2d\xi\ge cE(t)^{1+\alpha/d}\,.$$

From these inequalities,

$$-(d/dt)E(t)\ge cE(t)^{1+\alpha/d}\quad\text{or}\quad(d/dt)(E(t)^{-\alpha/d})\ge c\,.$$

Since $E(+0)=\infty$, we have $E(t)^{-\alpha/d}\geq ct$. From this estimate and the semigroup property,

$$T(2t, x) = \int T(t, y)S(t, y; 2t, x)dy$$
$$\leq \{E(t)\int S(t, y; 2t, x)^2dy\}^{1/2}$$
$$\leq \{(ct^{-d/\alpha})(ct^{-d/\alpha})\}^{1/2} = ct^{-d/\alpha},$$

hence $T(t, x)\leq ct^{-d/\alpha}$. q.e.d.

Fix a positive constant γ satisfying $2\sqrt{\gamma}\leq 1-|\alpha-1|=\alpha\wedge(2-\alpha)$, and let $r(0)=0$ and

$$r(\sigma) = \sigma^{\alpha/2}(1+\gamma(\log\sigma)^2)^{-1/2} \qquad (\sigma>0)\,. \tag{2.3}$$

Set $s=\log\sigma$. Since

$$2r'/r = e^{-s}\{\alpha-2\gamma s/(1+\gamma s^2)\}\geq e^{-s}\{\alpha-\sqrt{\gamma}\}\geq 0\,,$$

$r(\sigma)$ is an increasing function on $\boldsymbol{R}_+$. Moreover

$$\begin{aligned}4r''/r &= (2r'/r)^2+2(2r'/r)'\\ &= \{e^{-s}(\alpha-2\gamma s/(1+\gamma s^2)\}^2+2e^{-s}(d/ds)\{e^{-s}(\alpha-2\gamma s/(1+\gamma s^2)\}\\ &= e^{-2s}\{-1+(\alpha-1-2\gamma s/(1+2\gamma s^2))^2-4\gamma(1-\gamma s^2)/(1+\gamma s^2)^2\}\\ &\leq e^{-2s}(-1+(\alpha-1)^2-2(\alpha-1)\Gamma+2\Gamma^2)\,,\end{aligned}$$

where $\Gamma=2\gamma s/(1+\gamma s^2)$. Since $|\Gamma|\leq\sqrt{\gamma}$ and

$$-1+(\alpha-1)^2+2|\alpha-1|\sqrt{\gamma}+2\gamma\leq 0\,,$$

we have $r''(\sigma)\leq 0$, hence $r(\sigma)$ is a concave function.

Let

$$M(t) = \int r(|x|)T(t, x)dx\,, \tag{2.4}$$

which takes finite values by (1.7), and let $Q(t)$ be an entropy

$$Q(t) = -\int T_t(x)\log T_t(x)\,dx \qquad (t>0)\,. \tag{2.5}$$

Since $-\log T_t\geq 1-T_t$, we see that $Q(t)\geq 1-E(t)>-\infty$. The following lemma shows that $Q(t)<\infty$.

Lemma 2.2. *Let* $0<\beta<\alpha/2<\beta'<1$. *Then*

$$M(t)\geq c\exp[(\beta/d)Q(t)]\wedge\exp[(\beta'/d)Q(t)]\,. \tag{2.6}$$

Proof. Observe that $\tau\log\tau+\lambda\tau\geq -e^{-\lambda-1}$ for $\tau, \lambda>0$. Let $\tau=T(t, x)$,

$\lambda = ar(|x|)+b$, where $a>0$ and b are constants, and integrate over $\boldsymbol{R}^d$, obtaining

$$
\begin{aligned}
-Q+aM+b &= \int T(\log T+ar(|x|)+b)\,dx \\
&\geq -\int e^{-1-b-ar(|x|)}dx = -ce^{-b}\int_0^\infty e^{-ar(\sigma)}\sigma^{d-1}d\sigma \\
&\geq -ce^{-b}\int_0^\infty (e^{-a\sigma^\beta}+e^{-a\sigma^{\beta'}})\sigma^{d-1}d\sigma \\
&\geq -ce^{-b}(a^{-d/\beta}+a^{-d/\beta'})\,.
\end{aligned}
$$

Suppose $M\geq 1$, and set $a=M^{-1}$ and $e^b=M^{d/\beta}$. Then

$$-Q+1+(d/\beta)\log M \geq -cM^{-d/\beta}(M^{d/\beta}+M^{d/\beta'})\geq -c\,,$$

or $M\geq c\exp[(d/\beta)Q]$. Next suppose $M<1$, and set $a=M^{-1}$ and $e^b=M^{d/\beta'}$. Then we have $M\geq c\exp[(d/\beta')Q]$. Combining these estimates, we see that

$$M\geq c\exp[(d/\beta)Q]\wedge\exp[(d/\beta')Q]\,. \qquad \text{q.e.d.}$$

Formally differentiating (2.5), we have

$$(d/dt)Q(t) = -((\partial/\partial t)T_t, \log T_t)_{L^2} = \mathcal{E}_t(T_t, \log T_t)\,.$$

This can be justified in the case $0<\alpha<1$ but not in the case $1\leq\alpha<2$. Fortunately it is sufficient for our purpose to show the inequality

$$Q(t)-Q(s)\geq\int_s^t \mathcal{E}_\tau(T_\tau, \log T_\tau)d\tau \qquad (0<s<t)\,. \tag{2.7}$$

Let $\rho_n(x)=\rho(x/n)$ be the same function as before. For $\delta>0$, we have

$$
\begin{aligned}
&-\int \rho_n T_t \log(T_t+\delta)dx+\int \rho_n T_s\log(T_s+\delta)dx \\
&= -\int_s^t\Big\{\int \rho_n((\partial/\partial\tau)T_\tau)(\log(T_\tau+\delta)+T_\tau/(T_\tau+\delta))dx\Big\}d\tau \\
&= \int_s^t\{\mathcal{E}_\tau(\rho_n\log(T_\tau+\delta), T_\tau)+\mathcal{E}_\tau(\rho_n T_\tau/(T_\tau+\delta), T_\tau)\}d\tau\,.
\end{aligned}
$$

Since $\rho_n\to 1$ as $n\to\infty$, we have

$$
\begin{aligned}
&-\int T_t\log(T_t+\delta)dx+\int T_s\log(T_s+\delta)dx \\
&= \int_s^t\{\mathcal{E}_\tau(\log(T_\tau+\delta), T_\tau)+\mathcal{E}_\tau(T_\tau/(T_\tau+\delta), T_\tau)\}d\tau \\
&\geq \int_s^t \mathcal{E}_\tau(\log(T_\tau+\delta), T_\tau)d\tau\,.
\end{aligned}
$$

Since $0 \leq \mathcal{E}_\tau(\log(T_\tau+\delta), T_\tau) \uparrow \mathcal{E}_\tau(\log T_\tau, T_\tau)$ as $\delta \downarrow 0$, we obtain inequality (2.7). q.e.d.

Lemma 2.3. *$M(t)$ is absolutely continuous and*

$$(d/dt)M(t) \leq c\sqrt{\mathcal{E}_t(\log T_t, T_t)} < \infty \qquad (t>0)\,. \tag{2.8}$$

Proof. Observe that

$$\begin{aligned}&\sup_{n,x} r(|x|)^2 \int (\rho_n(x+z)-\rho_n(x))^2 |z|^{-d-\alpha} dz \\ &\leq \sup_{n,x} |x|^\alpha \int (\rho_n(x+z)-\rho_n(x))^2 |z|^{-d-\alpha} dz \\ &= \sup_x |x|^\alpha \int (\rho(x+z)-\rho(x))^2 |z|^{-d-\alpha} dz < \infty\,.\end{aligned}$$

Since $r''(\sigma) \leq 0$, we have

$$\begin{aligned}&\sup_x \int (r(|x+z|)-r(|x|))^2 |z|^{-d-\alpha} dz \\ &= \int r(|z|)^2 |z|^{-d-\alpha} dz = \int (1+\gamma(\log|z|)^2)^{-1} |z|^{-d} dz \\ &= c\int_{-\infty}^{+\infty} (1+\gamma s^2)^{-1} ds < \infty\,.\end{aligned}$$

Let $R_n(x) = r(|x|)\rho_n(x)$. Then

$$(R_n(x+z)-R_n(x))^2 \leq 2(r(|x+z|)-r(|x|))^2 + 2r(|x|)^2(\rho_n(x+z)-\rho_n(x))^2\,,$$

so that

$$\int (R_n(x+z)-R_n(x))^2 |z|^{-d-\alpha} dz \leq c\,.$$

For $0<s<t$, we have

$$\begin{aligned}M(t)-M(s) &= \lim_{n\to\infty} \{\int R_n(x)T(t,x)dx - \int R_n(x)T(s,x)dx\} \\ &= -\lim_{n\to\infty} \int_s^t \mathcal{E}_\tau(R_n, T_\tau)d\tau\,.\end{aligned}$$

Let $[A]_+$ denote the positive part of A. Then

$$\begin{aligned}&|\mathcal{E}_t(R_n, T_t)| \\ &\leq 2\iint |R_n(x)-R_n(y)|\,[T(t,x)-T(t,y)]_+ K(t,dx,dy) \\ &\leq 2\{\iint (R_n(x)-R_n(y))^2 T(t,x)K(t,dx,dy)\}^{1/2} \\ &\quad \times \{\iint ([1-T(t,y)/T(t,x)]_+)^2 T(t,x)K(t,dx,dy)\}^{1/2}\end{aligned}$$

$$\leq c\{\sup_x \int (R_n(x+z)-R_n(x))^2|z|^{-d-\alpha}dz\}^{1/2}$$
$$\times\{\iint (\log(T(t,x)/T(t,y))[T(t,x)-T(t,y)]_+K(t,dx,dy)\}^{1/2}$$
$$\leq c\mathcal{E}_t(\log T_t, T_t)^{1/2},$$

where we used the symmetry $k(t, x, y)=k(t, y, x)$ and the inequality $1-\tau\leq \log(1/\tau)$. Therefore

$$|M(t)-M(s)|\leq\int_s^t c\mathcal{E}_\tau(\log T_\tau, T_\tau)^{1/2}d\tau,$$

and we obtain (2.8). q.e.d.

Now we shall prove Theorem 1. From Lemma 2.1, there is a constant b such that $T(t, x)\leq e^b t^{-d/\alpha}$. Since

$$Q(t)\geq-\int T(t,x)\log(e^b t^{-d/\alpha})dx=(d/\alpha)\log t-b,$$

the function $g(t)=Q(t)-(d/\alpha)\log t+b$ is non-negative. Define, for $t>0$, a function $h(t)$ by

$$h(t)=\int_1^t \mathcal{E}_\tau(\log T_\tau, T_\tau)d\tau-(d/\alpha)\log t.$$

By (2.7) we have $g(t)-g(s)\geq h(t)-h(s)$ for any $0<s<t$. From (2.8), we have

$$(d/dt)M(t)\leq c(d/\alpha t+h'(t))^{1/2}$$
$$=ct^{-1/2}(1+(\alpha t/d)h'(t))^{1/2}\leq ct^{-1/2}(1+(\alpha t/2d)h'(t)).$$

Using integration by parts, for $0<\varepsilon<1$,

$$M(1)-M(\varepsilon)\leq c\int_\varepsilon^1 (1/\sqrt{t}+\sqrt{t}\,h'(t))dt$$
$$\leq c\{2+[\sqrt{t}\,h(t)]_\varepsilon^1-\int_\varepsilon^1 h(t)/2\sqrt{t}\,dt\}$$
$$=c\{2+\sqrt{\varepsilon}(h(1)-h(\varepsilon))+\int_\varepsilon^1 (h(1)-h(t))/2\sqrt{t}\,dt\}$$
$$\leq c\{2+\sqrt{\varepsilon}(g(1)-g(\varepsilon))+\int_\varepsilon^1 (g(1)-g(t))/2\sqrt{t}\,dt\}$$
$$\leq c\{2+\sqrt{\varepsilon}\,g(1)+\int_\varepsilon^1 g(1)/2\sqrt{t}\,dt\}=c(2+g(1)).$$

Since $M(+0)=0$, we have $M(1)\leq c(1+g(1))$. On the other hand, from inequality (2.6),

$$M(1)\geq c\exp[(\beta/d)Q(1)]\wedge\exp[(\beta'/d)Q(1)]$$
$$=c\exp[(\beta/d)g(1)]\wedge\exp[(\beta'/d)g(1)]=c\exp[(\beta/d)g(1)]$$

for any $0<\beta<\alpha/2<\beta'<1$. Hence we have

$$c\exp[(\beta/d)g(1)]\leq M(1)\leq c(1+g(1)).$$

This implies that $g(1)\leq c$ and $M(1)\leq c$. From Remark in Section 1,

$$\int r(|x|)T(\lambda\|1,x)dx\leq c$$

for any $\lambda>0$. Hence we conclude that

$$\int r(t^{-1/\alpha}|x|)T(t,x)dx=\int r(|x|)T(t^{1/\alpha}\|1,x)dx\leq c.$$

3. The overlap estimate

In this section, we shall show that there is some overlap

$$\int S(0,x_1;t,x)\wedge S(0,x_2;t,x)dx$$

of functions $S(0,x_i;t,x)$ with nearby source points x_1 and x_2. The idea of the proof is the same as that in Nash [14], but the proof requires some new lemmas.

Let $U(t,x)=U_t(x)$ be the function defined by (1.10). Let $\tilde{\mathcal{E}}_t(\cdot,\cdot)$ and $\tilde{A}_t$ denote

$$\mathcal{E}_1(t^{1/\alpha}\|\cdot,\cdot)\quad\text{and}\quad A_1(t^{1/\alpha})$$

respectively which are associated with the function $k(t,t^{1/\alpha}x,t^{1/\alpha}y)$. Then

$$t(\partial/\partial t)U_t(x)=(\partial/\partial(\log t))U_t(x)$$
$$=(1/\alpha)\partial\cdot(xU_t(x))+\tilde{A}_tU_t(x),\tag{3.1}$$

where $\partial=(\partial/\partial x)=((\partial/\partial x_1),\cdots,(\partial/\partial x_d))$. Now define a probability density function $P(x)$ by

$$P(x)=C(1+|x|^2)^{-(d+\alpha+1)/2},$$

and set

$$G_\delta(t)=-\int P(x)\log(U_t(x)+\delta)dx\tag{3.2}$$

for $0<\delta<1$. Although the definition of $P(x)$ is technical, the function $P(x)$ is the simplest one among probability density functions having required properties.

From (3.1),

$$
\begin{aligned}
t(d/dt)G_\delta(t) &= -\int P\,(U_t+\delta)^{-1}\{(1/\alpha)\partial\cdot(xU_t)+\tilde{A}_tU_t\}\,dx \\
&= -(d/\alpha)\int P\,(U_t+\delta)^{-1}U_t dx-(1/\alpha)\int P\,x\cdot\partial(\log(U_t+\delta))dx \\
&\quad +\tilde{\mathcal{E}}_t(P(U_t+\delta)^{-1},\ U_t) \\
&\le (1/\alpha)\int(\partial\cdot xP(x))\log(U_t+\delta)dx+\tilde{\mathcal{E}}_t(P(U_t+\delta)^{-1},\ U_t) \\
&= -(d/\alpha)G_\delta(t)+(1/\alpha)\int(\partial P(x)\cdot x)\log(U_t+\delta)dx \\
&\quad +\tilde{\mathcal{E}}_t(P(U_t+\delta)^{-1},\ U_t)\,.
\end{aligned}
$$

Since $\partial P(x)\cdot x\le 0$ and $\log(U_t+\delta)\ge\log\delta$, we have

$$
t(d/dt)G_\delta\le-(d/\alpha)G_\delta-c\log\delta+\tilde{\mathcal{E}}_t(P(U_t+\delta)^{-1},\ U_t)\,. \tag{3.3}
$$

To estimate the last term of the above inequality, we need following two lemmas which are elemental but play essential roles in this section.

Lemma 3.1. *For any real values θ and ω,*

$$
\mathrm{sh}(\theta-\omega)\ \mathrm{sh}\,\omega\le((\mathrm{sh}\,\theta)/\theta)(\theta-\omega)\omega\,. \tag{3.4}
$$

Proof. Observe that

$$
\begin{aligned}
&\mathrm{sh}\,(\theta-\omega)\ \mathrm{sh}\,\omega \\
&= \lim_{N\to\infty}\{\prod_{n=1}^{N}\mathrm{ch}((\theta-\omega)/2^n)\ \mathrm{ch}\,(\omega/2^n)\}4^N\,\mathrm{sh}\,((\theta-\omega)/2^N)\ \mathrm{sh}\,(\omega/2^N) \\
&= \{\prod_{n=1}^{\infty}\mathrm{ch}\,((\theta-\omega)/2^n\ \mathrm{ch}\,(\omega/2^n)\}(\theta-\omega)\omega \\
&= \{\prod_{n=1}^{\infty}2^{-1}(\mathrm{ch}\,(\theta/2^n)+\mathrm{ch}\,((\theta-2\omega)/2^n))\}(\theta-\omega)\omega\,,
\end{aligned}
$$

and

$$
\mathrm{sh}\,\theta=\lim_{N\to\infty}\{\prod_{n=1}^{N}\mathrm{ch}(\theta/2^n)\}2^N\,\mathrm{sh}(\theta/2^N)=(\prod_{n=1}^{\infty}\mathrm{ch}(\theta/2^n))\,\theta\,.
$$

In case $(\theta-\omega)\omega>0$, it suffices to prove that $\mathrm{ch}((\theta-2\omega)/2^n)<\mathrm{ch}(\theta/2^n)$ or $\mathrm{ch}(\theta-2\omega)<\mathrm{ch}\,\theta$. This inequality holds good, for $(\theta-\omega)\omega>0$ implies

$$
|\theta-2\omega|=|(\theta-\omega)-\omega|<|(\theta-\omega)+\omega|=|\theta|\,.
$$

Similarly, in case $(\theta-\omega)\omega<0$, we have $\mathrm{ch}(\theta-2\omega)/2^n)>\mathrm{ch}(\theta/2^n)$. Hence inequality (3.4) is valid in both cases. q.e.d.

Lemma 3.2. (i) *Let $2\theta(x,y)=\log(P(x)/P(y))$. Then $|\theta(x,y)|\le c|x-y|$ and*

(3.5) $$\iint P(x)\theta(x, y)^2|x-y|^{-d-\alpha}dx\,dy<\infty\,.$$

(ii) *For any function* $f(x)$ *satisfying* $\int P(x)f(x)dx<\infty$,

(3.6) $$\int P(x)\{f(x)-\int f(y)P(y)dy\}^2dx$$
$$\leq c\int P(x)\{\int (f(x+z)-f(x))^2(1+|z|^2)^{-1/2}|z|^{-d-\alpha}dz\}\,dx\,.$$

Proof. (i) From the mean value theorem, we have

$$4|\theta(x, y)|\leq(d+\alpha+1)|x-y|,$$

so that

$$|\theta(x, y)|\leq c|x-y|\wedge(\log(1+|x|)+\log(1+|y|))\,.$$

Therefore

$$\iint P(x)\theta(x, x+z)^2|z|^{-d-\alpha}dx\,dz$$
$$\leq c\iint_{|z|\leq 1}P(x)|z|^{-d+2-\alpha}dx\,dz$$
$$+c\iint_{|z|>1}P(x)(\log(1+|x|))^2|z|^{-d-\alpha}dx\,dz$$
$$+c\iint_{|z|>1}P(x)(\log(1+|x+z|))^2|z|^{-d-\alpha}dx\,dz<\infty\,.$$

(ii) Using the Schwarz inequality,

$$\int P(x)\{f(x)-\int f(y)P(y)dy\}^2dx$$
$$=\int P(x)\{\int (f(x)-f(y))P(y)\,dy\}^2\,dx$$
$$\leq\int P(x)\{\int (f(x)-f(y))^2P(y)\,dy\}\,dx$$
$$=2\iint_{|x|<|y|}P(x)(f(x)-f(y))^2P(y)\,dx\,dy\,.$$

If $|x|<|y|$, then $|x-y|\leq 2|y|$, so that

$$P(y)\leq c(1+|y|^2)^{-1/2}|y|^{-d-\alpha}$$
$$\leq c(1+|x-y|^2)^{-1/2}|x-y|^{-d-\alpha}\,.$$

Therefore we have (3.6). q.e.d.

Now fix $t>0$ and $0<\delta<1$, and let

$$2\theta(x, y) = \log P(x) - \log P(y)\,,$$
$$2\omega(x, y) = \log(U_t(x)+\delta) - \log(U_t(y)+\delta)\,,$$
$$\tilde{k}(x, y) = k(t, t^{1/\alpha}x, t^{1/\alpha}y)\,.$$

Then we observe that

$$\{P(x)(U_t(x)+\delta)^{-1} - P(y)(U_t(y)+\delta)^{-1}\}(U_t(x)-U_t(y)) = \sqrt{4P(x)P(y)}\,\mathrm{sh}\,(\theta(x, y)-\omega(x, y))\,\mathrm{sh}(\omega(x, y))\,.$$

Applying inequality (3.4), we have

$$\begin{aligned}
&\tilde{\mathcal{E}}_t(P(U_t+\delta)^{-1}, U_t)\\
&\leq \iint 2\sqrt{P(x)P(y)}((\mathrm{sh}\,\theta)/\theta)(\theta-\omega)\omega\tilde{k}\,|x-y|^{-d-\alpha}dx\,dy\\
&= \iint (P(x)+P(y))((\mathrm{th}\,\theta)/\theta)(\theta-\omega)\omega\tilde{k}\,|x-y|^{-d-\alpha}dx\,dy\\
&= \iint 2P(x)((\mathrm{th}\,\theta)/\theta)(\theta-\omega)\omega\tilde{k}\,|x-y|^{-d-\alpha}dx\,dy\\
&\leq -2c_1\iint P(x)((\mathrm{th}\,\theta)/\theta)\,\omega^2|x-y|^{-d-\alpha}dx\,dy\\
&\quad +2\{\iint P(x)((\mathrm{th}\,\theta)/\theta)\theta^2 c_2|x-y|^{-d-\alpha}dx\,dy\}^{1/2}\\
&\quad \times\{\iint P(x)((\mathrm{th}\,\theta)/\theta)\,\omega^2 c_2|x-y|^{-d-\alpha}dx\,dy\}^{1/2}.
\end{aligned}$$

Therefore, from (3.5),

$$\begin{aligned}
&\tilde{\mathcal{E}}_t(P(U_t+\delta)^{-1}, U_t)\\
&\leq -2c_1\iint P(x)\omega^2((\mathrm{th}\,\theta)/\theta)\,|x-y|^{-d-\alpha}dx\,dy\\
&\quad +c\,\{\iint P(x)\omega^2((\mathrm{th}\,\theta)/\theta)\,|x-y|^{-d-\alpha}dx\,dy\}^{1/2}\\
&\leq c-c_1\iint P(x)\omega^2((\mathrm{th}\,\theta)/\theta)\,|x-y|^{-d-\alpha}dx\,dy\,,
\end{aligned}$$

where we used that $-2s+c\sqrt{s}\leq c^2/2-s$ $(s\geq 0)$. Since

$$(\mathrm{th}\theta\,(x, y))/\theta(x, y)\geq c\,(1+\theta(x, y)^2)^{-1/2}\geq c\,(1+|x-y|^2)^{-1/2}\,,$$

applying inequality (3.6) for the function

$$f(x) = \log(U_t(x)+\delta)\,,$$

we obtain the estimate

$$\tilde{\mathcal{E}}_t(P(U_t+\delta)^{-1},\ U_t)\leq c-c\int P(x)(\log(U_t(x)+\delta)+G_\delta(t))^2dx\,. \tag{3.7}$$

Lemma 3.3. *For all sufficiently large G, say for $G\geq c_0$,*

$$\int P(x)(\log(U_t(x)+\delta)+G)^2dx\geq(c+cG)^2\,. \tag{3.8}$$

Proof. Let $h(u)=u^{-1}(\log(u+\delta)+G)^2$, where G is a constant satisfying $e^{-G}-\delta=u_0>0$. If $u\geq u_1=e^{2-G}$, then $u_1>u_0$ and

$$2u/(u+\delta)-\log(u+\delta)-G<2-\log u_1-G=0\,,$$

so that

$$h'(u)=u^{-2}(\log(u+\delta)+G)\,(2u/(u+\delta)-\log(u+\delta)-G)<0\,.$$

Therefore $h(u)$ is strictly decreasing on $[u_1,\infty)$. By Theorem 1 (i), there is a constant u_2 depending only on α, d, c_1 and c_2 such that $U_t(x)\leq u_2$. Hence

$$\begin{aligned}&\int P(x)(\log(U_t+\delta)+G)^2dx\\ &\geq\int P(x)\min\{h(u);\ u_1\leq u\leq u_1\vee u_2\}\,U_t\,I_{(u_1<U_t)}\,dx\\ &=h(u_1\vee u_2)\int PU_t\,I_{(u_1<U_t)}\,dx\,.\end{aligned}$$

Let $V(x)=U_t(x)$ if $u_1<U_t(x)$, $V(x)=0$ if $u_1\geq U_t(x)$ and

$$\beta=(1/2)\int V(x)dx\,,$$

which depend on t and G. By Theorem 1 (ii), there is a constant b depending only on α, d, c_1 and c_2 such that

$$\int r(|x|)U_t(x)dx\leq b\,.$$

Since

$$\begin{aligned}\int_{r(|x|)\leq b/\beta}V(x)dx&=2\beta-\int_{r(|x|)>b/\beta}V(x)\,dx\\ &\geq2\beta-(\beta/b)\int r(|x|)V(x)dx\geq2\beta-\beta=\beta\,,\end{aligned}$$

we have the inequality

$$\int P(x)V(x)dx\geq\beta\min\{P(x);\ r(|x|)\leq b/\beta\}\,.$$

The function $V_1(x)=U_t(x)-V(x)$ satisfies

$$V_1(x)\leq u_1 \quad \text{and} \quad \int r(|x|)V_1(x)\,dx\leq b\,.$$

From the Neymann-Pearson Lemma in statistics, if $V_0(x)=u_1$ for $|x|\leq R$, $V_0(x)=0$ for $|x|>R$ and

$$\int r(|x|)V_0(x)dx = u_1\int_{|x|\leq R} r(|x|)dx = b\,,$$

then

$$u_1\int_{|x|\leq R} dx = \int V_0(x)\,dx\geq\int V_1(x)dx = 1-2\beta\,.$$

Observe that the constant R depends on u_1 or G and $R=R(G)\uparrow\infty$ as $G\uparrow\infty$. Since

$$(1-2\beta)/b\leq(\int_{|x|\leq R} dx)/(\int_{|x|\leq R} r(|x|)dx)\downarrow 0$$

as $R\uparrow\infty$ or $G\uparrow\infty$, the constant β is bounded below, say $\beta\geq 1/2$ as long as $G\geq c_0$. We may suppose $2-c_0<\log u_2$. Then, for $G\geq c_0$,

$$\begin{aligned}&\int P(x)(\log(U_t+\delta)+G)^2dx\\ &\quad\geq h(u_1\vee u_2)\beta\min\{P(x);\, r(|x|)\leq b/\beta\}\\ &\quad\geq ch(u_1\vee u_2) = ch(u_2)\\ &\quad\geq(c+cG)^2\,.\end{aligned}$$

q.e.d.

Note that Theorem 1 is essential in the proof of the above lemma. Returning to inequalities (3.3) and (3.7) and using inequality (3.8), we see that there are positive constants c_3 and c_4 depending only on α, d, c_1 and c_2 such that, as long as $G_\delta(t)\geq c_0$,

$$t(d/dt)G_\delta(t)\leq c_3\log(1/\delta)-c_4(G_\delta(t))^2\,. \tag{3.9}$$

Suppose $G_\delta(t_0)\geq c_0$ and $c_3\log(1/\delta)-c_4G_\delta(t_0)^2\leq-\varepsilon<0$ are satisfied. Then we have from (3.9) that $G_\delta(t_0)\leq G_\delta(t)-\varepsilon\log(t_0/t)$ for $0<t<t_0$, which implies that $G_\delta(t)\to\infty$ as $t\to 0$. But this is a conrtadiction, for $G_\delta(t)\leq\log(1/\delta)$. Therefore it must be satisfied that

$$G_\delta(t)\leq c_0\vee|(c_3/c_4)\log(1/\delta)|^{1/2}\,,$$

or, for all sufficiently small δ,

$$G_\delta(t)\leq c\sqrt{\log(1/\delta)}\,. \tag{3.10}$$

Theorem 3. *There is a strictly positive function $\phi(\sigma)$ on $[0, \infty)$ which is decreasing and depends only on α, d, c_1 and c_2 such that*

$$\int S(0, x_1; t, x) \wedge S(0, x_2; t, x)dx \geq \phi(t^{-1/\alpha}|x_1-x_2|)\,. \tag{3.11}$$

Proof. Let

$$U^{(i)}(t, x) = t^{d/\alpha} S(0, t^{1/\alpha}x_i; t, t^{1/\alpha}x)$$

for $i=1$ or 2 and $x_i \in \boldsymbol{R}^d$. Then (3.11) is equivalent to

$$\int U^{(1)}(t, x) \wedge U^{(2)}(t, x)dx \geq \phi(|x_1-x_2|)\,. \tag{3.12}$$

We apply (3.10) for $U(t, x)=U^{(i)}(t, x+x_i)$, obtaining

$$\begin{aligned} &-\int P(x-x_i) \log (U^{(i)}(t, x)+\delta)\, dx \\ &\quad = -\int P(x) \log (U^{(i)}(t, x+x_i)+\delta)dx \leq c\sqrt{\log (1/\delta)}\,. \end{aligned}$$

Since the inequality

$$a_1b_1+a_2b_2 \leq (a_1 \wedge a_2)(b_1 \wedge b_2)+(a_1 \vee a_2)(b_1 \vee b_2)$$

holds good for any a_i and b_i, we have

$$\begin{aligned} &-\int \{\min_i P(x-x_i) \cdot \min_i \log (U^{(i)}+\delta) \\ &\qquad +\max_i P(x-x_i) \cdot \max_i \log (U^{(i)}+\delta)\}\, dx \\ &\quad \leq -\sum_i \int P(x-x_i) \log (U^{(i)}+\delta)\, dx \leq c_5\sqrt{\log (1/\delta)}\,. \end{aligned}$$

We observe that

$$\begin{aligned} &\int \max_i P(x-x_i) \cdot \max_i \log (U^{(i)}+\delta)dx \\ &\quad \leq \int (\max_i P(x-x_i))(U^{(1)}+U^{(2)})\, dx \\ &\quad \leq c_6 \int (U^{(1)}+U^{(2)})dx = 2c_6\,, \\ &\int \min_i P(x-x_i) \cdot \min_i \log (U^{(i)}+\delta)\, dx \\ &\quad \leq \int (\min_i P(x-x_i)) \{\log \delta + \log ((U^{(1)} \wedge U^{(2)})/\delta+1)\}\, dx \\ &\quad \leq \log \delta \int \min_i P(x-x_i)\, dx + (c_6/\delta) \int U^{(1)} \wedge U^{(2)} dx\,. \end{aligned}$$

Therefore we have

$$-(c_6/\delta)\int U^{(1)}\wedge U^{(2)}dx$$
$$\leq c_5\sqrt{\log(1/\delta)}+2c_6+\log\delta\int\min_i P(x-x_i)dx$$

for all sufficiently small δ, say $\delta\leq e^{-N}$. Let

$$w=w(|x_1-x_2|)=\int\min_i P(x-x_i)dx\,.$$

Then we obtain the estimate

$$c_6\int U^{(1)}\wedge U^{(2)}dx\geq\max_{\beta\geq N}e^{-\beta}(w(|x_1-x_2|)\beta-c_5\sqrt{\beta}-2c_6)\,.$$

Since the function $w(\sigma)$ is positive and decreasing, the function

$$\phi(\sigma)=(1/c_6)\max_{\beta\geq N}e^{-\beta}\{w(\sigma)\beta-c_4\sqrt{\beta}-2c_6\} \tag{3.13}$$

is also positive and decreasing, and

$$\int U^{(1)}\wedge U^{(2)}dx\geq\phi(|x_1-x_2|)\,.$$ q.e.d.

4. The continuity estimate

In this section we shall prove Theorem 2 by iterative use of (3.11). Although the proof is rather complicated, its outline is similar to [14] except for the use of Lemma 4.3. Theorem 2 is nothing but a consequence of Theorem 1 and Theorem 3, and the difficulty arises from the form of the function $r(\sigma)$.

Fix two points $x_1\neq x_2$, and set

$$S_+(t,x)=[S(0,x_1;t,x)-S(0,x_2;t,x)]\vee 0\,,$$
$$S_-(t,x)=[S(0,x_2;t,x)-S(0,x_1;t,x)]\vee 0\,.$$

Since $S_++S_-=|S(0,x_1;t,x)-S(0,x_2;t,x)|$ and $S_+-S_-=S(0,x_1;t,x)-S(0,x_2;t,x)$, we have

$$\begin{aligned}F(t)&=\int S_+(t,x)dx=\int S_-(t,x)dx\\&=(1/2)\int|S(0,x_1;t,x)-S(0,x_2;t,x)|dx\\&=1-\int S(0,x_1;t,x)\wedge S(0,x_2;t,x)dx\,,\end{aligned} \tag{4.1}$$

defining $F(t)$. From Theorem 3, the function $\psi(\sigma)=1-\phi(\sigma)$ is strictly less than 1 and increasing, and satisfies

$$F(t)\leq\psi(t^{-1/\alpha}|x_1-x_2|)\,. \tag{4.2}$$

Let $W(s, y, z)=S_+(s, y)S_-(s, z)/F(s)$. Then, for $s<t$,

$$\begin{aligned}
&S(0, x_1; t, x)-S(0, x_2; t, x)\\
&=\int (S(0, x_1; s, y)-S(0, x_2; s, y))S(s, y; t, x)dy\\
&=\int (S_+(s, y)-S_-(s, y))S(s, y; t, x)dy\\
&=\int S_+(s, y)S(s, y; t, x)dy-\int S_-(s, z)S(s, z; t, x)dz\\
&=\iint W(s, y, z)(S(s, y; t, x)-S(s, z; t, x))dy\,dz\,.
\end{aligned}$$

Hence from (3.11) or (4.2),

$$\begin{aligned}
(4.3)\qquad F(t)&\le(1/2)\iiint W(s, y, z)\,|\,S(s, y; t, x)-S(s, z; t, x)\,|\,dy\,dz\,dx\\
&\le\iint W(s, y, z)\psi((t-s)^{-1/\alpha}\,|\,y-z\,|\,)dy\,dz\,.
\end{aligned}$$

Since

$$\iint W(s, y, z)\psi dy\,dz\le\iint W(s, y, z)dy\,dz = F(s)\,,$$

we have $F(t)\le F(s)$. Inequality (4.3) is the key to the iterative argument.

Let $\lambda=(3+\psi(2))/4<1$ and t_n be the least time such that $F(t_n)=\lambda^n$, if t_n exists. Set $x_0=(x_1+x_2)/2$, and define

$$\begin{aligned}
M_+(t) &= \int r(\,|\,y-x_0\,|\,)S_+(t, y)dy\,,\\
M_-(t) &= \int r(\,|\,z-x_0\,|\,)S_-(t, z)dz\,,\\
M_n &= M_+(t_n)\vee M_-(t_n)\,.
\end{aligned}$$

Lemma 4.1. *Let $r^{-1}(\cdot)$ denote the inverse function of $r(\cdot)$. Then*

$$(4.4)\qquad t_{n+1}-t_n\le(r^{-1}(2\lambda^{-n}M_n))^{\alpha}\,.$$

Proof. Define $S'_+(t, y)=S_+(t, y)$ if $r(\,|\,y-x_0\,|\,)\le2\lambda^{-n}M_n$, otherwise $S'_+(t, y)=0$. Then

$$\begin{aligned}
&\int (S_+(t_n, y)-S'_+(t_n, y))\,dy\\
&\quad\le2^{-1}\lambda^n M_n^{-1}\int r(\,|\,y-x_0\,|\,)S_+(t_n, y)dy\le\lambda^n/2\,,
\end{aligned}$$

so that

$$\int S'_+(t_n, y)dy \geq F(t_n) - \lambda^n/2 = \lambda^n/2 .$$

Define $S'_-(t, z)$ similarly and let

$$W'_n(y, z) = \lambda^{-n} S'_+(t_n, y) S'_-(t_n, z) .$$

Using (4.3) with $s=t_n$, we have

$$\begin{aligned} F(t) &\leq \iint \{(W(t_n, y, z) - W'_n(y, z)) + W'_n(y, z)\} \\ &\quad \times \psi((t-t_n)^{-1/\alpha} |y-z|) dy\, dz \\ &\leq \iint (W(t_n, y, z) - W'_n(y, z)) dy\, dz \\ &\quad + \psi(2(t-t_n)^{-1/\alpha} r^{-1}(2\lambda^{-n} M_n)) \iint W'_n(y, z) dy\, dz , \end{aligned}$$

for the reason that if $W'_n(y, z) > 0$, then $|y-x_0| \leq r^{-1}(2\lambda^{-n} M_n)$ and $|z-x_0| \leq r^{-1}(2\lambda^{-n} M_n)$, so that $|y-z| \leq 2r^{-1}(2\lambda^{-n} M_n)$. Hence

$$\begin{aligned} F(t) &\leq \iint W(t_n, y, z) dy\, dz - (1-\psi) \int W'_n(y, z) dy\, dz \\ &= \lambda^n - (1-\psi)\lambda^{-n} \int S'_+(t_n, y) dy \int S'_-(t_n, z) dz \\ &\leq \lambda^n - (1-\psi)\lambda^{-n} (\lambda^n/2)^2 \\ &= (\lambda^n/4) \{3 + \psi(2(t-t_n)^{-1/\alpha} r^{-1}(2\lambda^{-n} M_n))\} . \end{aligned}$$

Let $t = t_n + (r^{-1}(2\lambda^{-n} M_n))^\alpha$. Then

$$F(t) \leq (\lambda^n/4)(3 + \psi(2)) = \lambda^{n+1} .$$

Therefore $t_{n+1} \leq t = t_n + (r^{-1}(2\lambda^{-n} M_n))^\alpha$. q.e.d.

Now we shall define

$$J(\sigma) = \sigma(2 + (\log \sigma)^2)^{1/2}, \quad \hat{J}(\sigma) = \sigma(1 + (\log \sigma)^2)^{-1/2}, \tag{4.5}$$

which are strictly increasing on $(0, \infty)$. It is easy to show that

$$\begin{aligned} &J(\sigma\sigma') \leq cJ(\sigma)J(\sigma') , \quad \hat{J}(\sigma\sigma') \leq cJ(\sigma)\hat{J}(\sigma') , \\ &\hat{J}(J(\sigma)) \geq c\sigma \quad \text{or} \quad \hat{J}^{-1}(\sigma) \leq cJ(\sigma) . \end{aligned}$$

Since $cr(\sigma) \leq \hat{J}(\sigma^{\alpha/2}) \leq cr(\sigma)$, we have

$$cr^{-1}(\sigma) \leq (\hat{J}^{-1}(\sigma))^{2/\alpha} \leq cr^{-1}(\sigma) ,$$

so that

$$(r^{-1}(\sigma))^\alpha \leq c(J(\sigma))^2 .$$

Observe that $J(\sigma)^2=\sigma^2(2+(\log\sigma)^2)$ is a convex function.

Lemma 4.2. *We have the inqeuality*

$$M_{n+1}-M_n\leq cJ(\sqrt{t_{n+1}-t_n})\lambda^n . \tag{4.6}$$

Proof. Let $t>s$. Since

$$\begin{aligned}S_+(t, x) &= \{\int (S_+(s, y)-S_-(s, y))S(s, y; t, x)dy\}\vee 0\\ &\leq\int S_+(s, y)S(s, y; t, x)dy\end{aligned}$$

and $r(\sigma)$ is concave, we have

$$\begin{aligned}M_+(t)&\leq\iint r(|x-x_0|)S_+(s, y)S(s, y; t, x)dxdy\\ &\leq\iint\{r(|y-x_0|)+r(|x-y|)\}S_+(s, y)S(s, y; t, x)dxdy\\ &=M_+(s)+\int S_+(s, y)\{\int r(|x-y|)S(s, y; t, x)dx\}dy\\ &\leq M_+(s)+\{\sup_y\int r(|x-y|)S(s, y; t, x)dx\}F(s) .\end{aligned}$$

From (1.8),

$$\int\hat{J}(\sqrt{|x-y|^\alpha/(t-s)})S(s, y; t, x)dx\leq c ,$$

so that

$$\begin{aligned}&\int r(|x-y|)S(s, y; t, x)dx\\ &\leq c\int\hat{J}(\sqrt{t-s}\cdot\sqrt{|x-y|^\alpha/(t-s)})S(s, y; t, x)dx\\ &\leq cJ(\sqrt{t-s})\int\hat{J}(\sqrt{|x-y|^\alpha/(t-s)})S(s, y; t, x)dx\\ &\leq cJ(\sqrt{t-s}) .\end{aligned}$$

Hence we have $M_+(t)\leq M_+(s)+cJ(\sqrt{t-s})F(s)$. Let $s=t_n$ and $t=t_{n+1}$. Then

$$M_+(t_{n+1})\leq M_+(t_n)+cJ(\sqrt{t_{n+1}-t_n})\lambda^n .$$

Similarly we have

$$M_-(t_{n+1})\leq M_-(t_n)+cJ(\sqrt{t_{n+1}-t_n})\lambda^n ,$$

so that we obtain (4.6). q.e.d.

Inequality (4.4) will bound the sequence $\{t_n\}$ after we obtain a bound on the sequence $\{M_n\}$. From (4.4),

$$\sqrt{t_{n+1}-t_n} \leq c(r^{-1}(\lambda^{-n}M_n))^{\alpha/2} \leq cJ(\lambda^{-n}M_n) .$$

Since $J(J(\sigma)) \leq c\sigma(1+(\log\sigma)^2)$,

$$J(\sqrt{t_{n+1}-t_n}) \leq c(\lambda^{-n}M_n)\{1+(\log(\lambda^{-n}M_n))^2\} .$$

Combining this and (4.6), we have

$$M_{n+1} \leq M_n\{c+c(\log(\lambda^{-n}M_n))^2\} .$$

Now let $\xi_n=(1/2)\log(\lambda^{-n}M_n)$. Then we have the inequality

$$(\exp\xi_{n+1})^2 \leq (\exp\xi_n)^2(c+c(\xi_n)^2) ,$$

or the difference inequality

$$\xi_{n+1}-\xi_n \leq \log(c+c|\xi_n|) . \tag{4.7}$$

Lemma 4.3. *If* $p\geq 1$, $q\geq 1/2$ *and*

$$\xi_{n+1} \leq \xi_n+\log(1+p|\xi_n|)+q$$

for all $n\geq 0$, *then*

$$\xi_n \leq \xi_0+n\{\log(1+p|\xi_0|)+2(\log(np)+q)\} . \tag{4.8}$$

Proof. We shall verify (4.8) by induction. Set $q_n=2(\log(np)+q)$. Suppose that (4.8) is valid for $n=m\geq 1$. Then

$$\begin{aligned}
&\{\xi_0+(m+1)(\log(1+p|\xi_0|)+q_{m+1})\}-\xi_{m+1}\\
&\quad\geq \xi_0+(m+1)\log(1+p|\xi_0|)+(m+1)q_{m+1}\\
&\qquad -\xi_m-\log(1+p|\xi_m|)-q\\
&\quad\geq (m+1)\log(1+p|\xi_0|)+(m+1)q_{m+1}\\
&\qquad -m(\log(1+p|\xi_0|)-mq_m\\
&\qquad -\log\{1+p|\xi_0+m\log(1+p|\xi_0|)+mq_m|\}-q\\
&\quad\geq (m+1)q_{m+1}-mq_m-q+\log(1+p|\xi_0|)\\
&\qquad -\log\{1+p|\xi_0|+mp(p|\xi_0|+q_m)\}\\
&\quad= (m+1)(q_{m+1}-q_m)+(q_m-q)\\
&\qquad -\log\{1+mp(q_m+p|\xi_0|)/(1+p|\xi_0|)\}\\
&\quad\geq m(q_{m+1}-q_m)+(q_m-q)-\log(1+mpq_m) ,
\end{aligned}$$

for $q_m\geq 2(\log p+q)\geq 1$. Since

$$\begin{aligned}
(mp)^2e^q-e^{-q} &= 2mp\,\mathrm{sh}(\log(mp)+q)\\
&\geq 2mp(\log(mp)+q) = mpq_m ,
\end{aligned}$$

we have

$$q_m - q = \log((mp)^2 e^q) > \log(mpq_m) .$$

Hence

$$\begin{aligned} & m(q_{m+1}-q_m)+(q_m-q)-\log(1+mpq_m) \\ & \geq m(q_{m+1}-q_m)+\log(mpq_m)-\log(1+mpq_m) \\ & = 2m\log(1+1/m)-\log(1+1/mpq_m) \\ & > 1-1/mpq_m \geq 0 . \end{aligned}$$

Therefore (4.8) is valid for $n=m+1$. q.e.d.

Applying the absolute inequality: $XY \leq (e^X-1)+(1-Y+Y\log Y)$ for $X=\log(1+p|\xi_0|)$ and $Y=2np$, we have

$$\begin{aligned} n\log(1+p|\xi_0|) & \leq (1/2p)\{p|\xi_0|+(1-2np+2np\log(2np))\} \\ & = |\xi_0|/2+n\log n+n(\log(2p)-1)+1/2p . \end{aligned}$$

Hence (4.8) can be replaced by

$$\xi_n \leq \xi_0 + |\xi_0|/2 + n(3\log n + C) + 1/2p , \tag{4.9}$$

where $C=\log(2p^3)+2q-1$. From (4.7) and (4.9), we have

$$\lambda^{-n} M_n = \exp(2\xi_n) \leq c(cn^6)^n \exp(2\xi_0 + |\xi_0|) .$$

Since $F(+0)=1=\lambda^0$, we see that $t_0=0$ and

$$\begin{aligned} \exp(2\xi_0) = M_0 & = \int r(|y-x_0|)\delta(y-x_1)\,dy \\ & = r(|x_1-x_2|/2) \leq c|x_1-x_2|^{\alpha/2} . \end{aligned}$$

If $|x_1-x_2| \leq 2$, then $\xi_0 \leq 0$, so that

$$\lambda^{-n} M_n \leq c(cn^6)^n |x_1-x_2|^{\alpha/4} . \tag{4.10}$$

Theorem 4. *Let $\Phi(\sigma)$ be defined by* (1.9). *Then*

$$\int |S(0, x_1; 1, x) - S(0, x_2; 1, x)|\,dx \leq c\,\Phi(|x_1-x_2|)^c . \tag{4.11}$$

Proof. Combining (4.4) and (4.10), we obtain

$$\begin{aligned} \sqrt{t_{n+1}-t_n} & \leq (r^{-1}(2\lambda^{-n} M_n))^{\alpha/2} \leq cJ(\lambda^{-n} M_n) \\ & \leq cJ((an^6)^n |x_1-x_2|^{\alpha/4}) , \end{aligned}$$

where a is a positive constant depending only on α, d, c_1 and c_2. Since the function $J(\sigma)^2$ is convex,

$$\begin{aligned} t_m &= \textstyle\sum_{n=0}^{m-1} (t_{n+1}-t_n) \\ &\leq c \textstyle\sum_{n=0}^{m-1} J((an^6)^n |x_1-x_2|^{\alpha/4})^2 \\ &\leq cJ((am^6)^m |x_1-x_2|^{\alpha/4})^2 \textstyle\sum_{n=0}^{m-1} (an^6)^n/(am^6)^m \\ &\leq [bJ((am^6)^m |x_1-x_2|^{\alpha/4})]^2 , \end{aligned}$$

where b is also a constant depending only on α, d, c_1 and c_2. Therefore if

$$(am^6)^m |x_1-x_2|^{\alpha/4} \leq J^{-1}(1/b) ,$$

then $t_m \leq 1$, so that $F(1) \leq F(t_m) = \lambda^m$.

Let $m = m(\sigma)$ be the integer such that

$$m \leq (\alpha/24)(\log \sigma^{-1}/\log\log \sigma^{-1}) < m+1 .$$

It is a routine work to show that $(am(\sigma)^6)^{m(\sigma)} \sigma^{\alpha/4}$ tends to 0 as $\sigma \downarrow 0$, so that there is a constant δ, $0<\delta<e^{-e}$, satisfying

$$(am(\sigma)^6)^{m(\sigma)} \sigma^{\alpha/4} \leq J^{-1}(1/b) \qquad \text{for all} \quad 0<\sigma<\delta .$$

If $|x_1-x_2|<\delta$, then

$$\begin{aligned} F(1) \leq \lambda^m &= \lambda^{-1}(e^{-m-1})^{\log(1/\lambda)} \\ &\leq \lambda^{-1} \Phi(|x_1-x_2|)^{(\alpha/24)\log(1/\lambda)} , \end{aligned}$$

where $m = m(|x_1-x_2|)$. This implies (4.11). q.e.d.

Now we shall prove Theorem 2 (i). From Theorem 4 and Remark in Section 1, we have for $0<\tau$,

$$\begin{aligned} &\int |S(0, x; \tau, x_1) - S(0, x; \tau, x_2)| dx \\ &= \int |S(0, x_1; \tau, x) - S(0, x_2; \tau, x)| dx \\ &= \int |S(\tau^{1/\alpha} \| 0, \tau^{-1/\alpha} x_1; 1, y) - S(\tau^{1/\alpha} \| 0, \tau^{-1/\alpha} x_2; 1, y)| dy \\ &\leq c\, \Phi(\tau^{-1/\alpha} |x_1-x_2|)^c , \end{aligned}$$

so that for $0<s<t$,

$$\begin{aligned} &\int |S(s, x; t, x_1) - S(s, x; t, x_2)| dx \\ &\qquad \leq c\, \Phi((t-s)^{-1/\alpha} |x_1-x_2|)^c . \end{aligned}$$

Hence from Theorem 1 (i), we have

$$\begin{aligned} &|T(t, x_1) - T(t, x_2)| \\ &\quad \leq \int T(s, x) |S(s, x; t, x_1) - S(s, x; t, x_2)| dx \end{aligned}$$

$$\leq cs^{-d/\alpha}\int |S(s, x; t, x_1)-S(s, x; t, x_2)|dx$$
$$\leq cs^{-d/\alpha}\Phi((t-s)^{-1/\alpha}|x_1-x_2|)^c ,$$

which completes the proof.

Next we shall prove Theorem 2 (ii). Let $t'>t>s>0$. For any positive constant σ, we have

$$\begin{aligned}
&|T(t, x)-T(t', x)| \\
&\quad = |\int (T(t, x)-T(t, z))S(t, z; t', x)dz| \\
&\quad \leq \int |T(t, x)-T(t, x+y)|S(t, x; t', x+y)dy \\
&\quad \leq \sup\{|T(t, x)-T(t, x+y)| ; |y|\leq \sigma(t-s)^{1/\alpha}\} \\
&\qquad + ct^{-d/\alpha}\int_{|y|>\sigma(t-s)^{1/\alpha}} S(t, x; t', x+y)dy \\
&\quad \leq cs^{-d/\alpha}\Phi(\sigma)^c + ct^{-d/\alpha}\int \hat{J}(\sigma^{-\alpha/2}(t-s)^{-1/2}|y|^{\alpha/2})S(t, x; t', x+y)dy \\
&\quad \leq cs^{-d/\alpha}\{\Phi(\sigma)^c + J(\sqrt{(t'-t)/(t-s)}\,\sigma^{-\alpha/2}) \\
&\qquad \times \int \hat{J}((t'-t)^{-1/2}|y|^{\alpha/2})S(t, x; t', x+y)dy\} \\
&\quad \leq cs^{-d/\alpha}\{\Phi(\sigma)^c + J(\sqrt{(t'-t)/(t-s)}\,\sigma^{-\alpha/2})\} ,
\end{aligned}$$

where we used Theorem 1 and Theorem 2 (i). Here, let

$$\begin{aligned}
\eta &= -\log((t'-t)/(t-s)) > e , \\
\alpha \log \sigma &= -\eta + \eta/\log \eta .
\end{aligned}$$

Then we have

$$\begin{aligned}
J(\sqrt{(t'-t)/(t-s)}\,\sigma^{-\alpha/2}) &= J(e^{-\eta/(2\log \eta)}) \leq e^{-c\eta/\log \eta} , \\
\Phi(\sigma) &= \Phi(\exp[(\eta/\log \eta-\eta)/\alpha]) \leq e^{-c\eta/\log \eta} .
\end{aligned}$$

Hence

$$\begin{aligned}
&|T(t, x)-T(t', x)| \\
&\quad \leq cs^{-d/\alpha}e^{-c\eta/\log \eta} = cs^{-d/\alpha}\Phi(e^{-\eta})^c \\
&\quad = cs^{-d/\alpha}\Phi((t'-t)/(t-s))^c ,
\end{aligned}$$

which completes the proof.

5. The estimates for weak solutions

In the preceding sections, the function $k(t, x, y)$ is assumed to be smooth in x and y. The purpose of this section is to remove the smoothness condition on the function $k(t, x, y)$, and only condition (1.2) and (1.5) are assumed in what follows. The parabolic equation $\partial u/\partial t = A_t u$ must be considered in the weak sense. The existence of fundamental solution of the weak solution will also be proved in this section.

A function $u_t(x) = u(t, x)$ is said to be a weak solution of the Cauchy problem $(\partial/\partial t)u_t = A_{\zeta+t}u_t$, $u_{0+} = \phi$ if

$$u_t \in L^2(\boldsymbol{R}^d, dx), \quad \lim_{t \downarrow 0} \|u_t - \phi\|_{L^2} = 0,$$

and u_t is a weak solution of the parabolic equation associated with the Dirichlet form $\mathcal{E}_{\zeta+t}(\cdot, \cdot)$ (see Section 1).

Lemma 5.1. *For any $\zeta \in \boldsymbol{R}^1_+$ and $\phi \in L^2(\boldsymbol{R}^d, dx)$, there exists at most a weak solution of the Cauchy problem $(\partial/\partial t)u_t = A_{\zeta+t}u_t$, $u_{0+} = \phi$.*

Proof. Let $\rho(x)$ be a smooth function such that $0 \leq \rho(x) = \rho(-x) \leq 1$, $\rho(x) = 1$ for $|x| \leq 1$ and $\rho(x) = 0$ for $|x| \geq 2$, and set $\rho_n(x) = \rho(x/n)$, $\tilde{\rho}_n = \mathcal{F}^{-1}\rho_n$. To simplify the proof, we shall suppose $\zeta = 0$. Let $u_t(x)$ be a weak solution of the Cauchy problem $(\partial/\partial t)u_t = A_t u_t$, $u_{0+} = 0$. Define

$$u_t^{(n)}(x) = \mathcal{F}^{-1}[\mathcal{F}u_t(\xi)\rho_n(\xi)](x) = (u_t * \tilde{\rho}_n)(x).$$

Then $\sup_t \|\partial^\nu u_t^{(n)}\|_{L^2} < \infty$ for any $\nu \in \boldsymbol{Z}^d_+$. By the approximation procedure $u_t^{(n)}\rho_m \to u_t^{(n)}$ and $\tilde{\rho}_n(x - \cdot)\rho_m \to \tilde{\rho}_n(x - \cdot)$ as $m \to \infty$, we have

$$(\partial/\partial\tau)(u_\tau, u_\sigma^{(n)})_{L^2} = -\mathcal{E}_\tau(u_\tau, u_\sigma^{(n)}),$$
$$(\partial/\partial\tau)u_\tau^{(n)}(x) = (\partial/\partial\tau)(u_\tau, \tilde{\rho}_n(x - \cdot))_{L^2} = -\mathcal{E}_\tau(u_\tau, \tilde{\rho}_n(x - \cdot)).$$

Let $\{\delta_m(t)\}$ be a sequence of non-negative test functions on $\boldsymbol{R}^1$ such that $\delta_m(t) \to \delta(t)$ as $m \to \infty$. Then we have

$$(\partial/\partial\tau)[(u_\tau, u_\sigma^{(n)})_{L^2}\delta_m(\tau - \sigma)] + \mathcal{E}_\tau(u_\tau, u_\sigma^{(n)})\delta_m(\tau - \sigma)$$
$$= (u_\tau, u_\sigma^{(n)})_{L^2}(\partial/\partial\tau)\delta_m(\tau - \sigma).$$

Integrating by σ and letting $m \to \infty$, we see that

$$\begin{aligned}
&(u_t, u_t^{(n)})_{L^2} - (u_s, u_s^{(n)})_{L^2} + \int_s^t \mathcal{E}_\tau(u_\tau, u_\tau^{(n)})d\tau \\
&= \int_s^t (u_\tau, (\partial/\partial\tau)u_\tau^{(n)})_{L^2} d\tau \\
&= -\int_s^t \int u_\tau(x)\mathcal{E}_\tau(u_\tau, \tilde{\rho}_n(x - \cdot))d\tau\, dx \\
&= -\int_s^t \mathcal{E}_\tau(u_\tau, u_\tau^{(n)})d\tau
\end{aligned}$$

for any $0<s<t$. Since

$$\sup_{s\leq\tau\leq t}\|u_\tau-u_\tau^{(n)}\|_{L^2}^2+\int_s^t\mathcal{E}_\tau(u_\tau-u_\tau^{(n)},\ u_\tau-u_\tau^{(n)})d\tau$$
$$\leq c\sup_{s\leq\tau\leq t}\|(1-\rho_n)\mathcal{F}u_\tau\|_{L^2}^2+c\int_s^t\int|\xi|^\alpha(1-\rho_n)^2|\mathcal{F}u_\tau|^2d\tau d\xi\to 0$$
$$\text{as}\quad n\to\infty\,,$$

we have

$$\|u_t\|_{L^2}^2-\|u_s\|_{L^2}^2+2\int_s^t\mathcal{E}_\tau(u_\tau,\ u_\tau)d\tau=0\,.$$

Therefore $\|u_t\|_{L^2}\leq\|u_s\|_{L^2}\leq\|u_{0+}\|_{L^2}=0$. q.e.d.

Let $\{\delta_m(x)\}$ be a sequence of non-negative test functions on $\boldsymbol{R}^d$ such that the support of $\delta_m(x)$ decreases to $\{0\}$ as $m\to\infty$ and $\int\delta_m(x)dx=1$. Let $\mathcal{E}_{m,t}(\cdot,\ \cdot)$ denote the Dirichlet form associated with the function

$$k_m(t,\ x,\ y)=\iint k(t,\ \xi,\ \eta)\delta_m(x-\xi)\delta_m(y-\eta)d\xi d\eta\,. \tag{5.1}$$

Then each k_m satisfies condition (1.2) and (1.5), and $(\partial/\partial x)^\mu(\partial/\partial y)^\nu k_m$ are bounded and continuous for all $\mu,\ \nu\in\boldsymbol{Z}_+^d$. Let $S_m(s,\ x;\ t,\ y)$ be the fundamental solution of the parabolic equation $\partial u/\partial t=A_{m,t}u$ associated with the Dirichlet form $\mathcal{E}_{m,t}(\cdot,\ \cdot)$. Since $S_m(s,\ x;\ t,\ y)$ is the transition function of the Markov process with pre-generator $(A_{m,t},\ C_0^\infty(\boldsymbol{R}^d))$, we see that $S_m(s,\ x;\ t,\ y)=S_m(s,\ y;\ t,\ x)\geq 0$ and $\int S_m(s,\ x;\ t,\ y)dy=1$. Applying Theorem 1 (i) and Theorem 2 for the set of functions

$$\{S_m(\zeta,\ z;\ \zeta+t,\ z+x);\ m,\ \zeta,\ z\}\cup\{S_m(\zeta-t,\ z-x;\ \zeta,\ z);\ m,\ \zeta,\ z\}\ ,$$

we see that the functions $\{S_m(s,\ x;\ t,\ y)\}$ are uniformly bounded and equicontinuous on any compact subset of $\{(s,\ x,\ t,\ y);\ s<t \text{ and } x,\ y\in\boldsymbol{R}^d\}$. From the Ascoli-Arzera theorem, choosing a subsequence $\{m(n)\}\subset\{m\}$ if necessary, we may suppose that $\{S_n(s,\ x;\ t,\ y)\}$ converges to a certain function $S(s,\ x;\ t,\ y)$ as $n\to\infty$ locally uniformly on the set $\{(s,\ x;\ t,\ y);\ s<t \text{ and } x,\ y\in\boldsymbol{R}^d\}$. Obviously $S(s,\ x;\ t,\ y)=S(s,\ y;\ t,\ x)\geq 0$.

Now fix a point $(\zeta,\ z)\in\boldsymbol{R}_+\times\boldsymbol{R}^d$, and let $T_t^{(n)}(x)=S_n(\zeta,\ z;\ \zeta+t,\ z+x)$, $T_t^{(\infty)}(x)=S(\zeta,\ z;\ \zeta+t,\ z+x)$. From Theorem 1 (ii), we see that

$$\int T_t^{(\infty)}(x)dx=\lim_{N\to\infty}(\lim_{n\to\infty}\int_{|x|\leq N}T_t^{(n)}(x)dx)$$
$$=\lim_{N\to\infty}(1-\lim_{n\to\infty}\int_{|x|>N}T_t^{(n)}(x)\,dx)$$
$$\geq\lim_{N\to\infty}(1-c\,r(Nt^{-1/\alpha})^{-1})=1\,,$$

and for any $\varepsilon>0$,

$$\int_{|x|>\varepsilon} T_t^{(\infty)}(x)dx \leq \lim_{n\to\infty} \int_{|x|>\varepsilon} T_t^{(n)}(x)dx$$
$$\leq \lim_{n\to\infty} c\, r(\varepsilon t^{-1/\alpha})^{-1} \to 0 \quad \text{as } t\downarrow 0\,.$$

This implies that $\int T_t^{(\infty)}(x)dx=1$ and $T_t^{(\infty)}(x)\to\delta(x)$ as $t\downarrow 0$. Therefore

$$\int S(s, x; t, y)dy = 1\,, \quad S(s, x; t, y) \to \delta(y-x) \quad \text{as } t\downarrow s\,. \tag{5.2}$$

Let $\mathcal{E}_t^{(n)}(\cdot, \cdot)$ (resp. $\mathcal{E}_t^{(\infty)}(\cdot, \cdot)$) denote the Dirichlet form associated with the function $k^{(n)}(t, x, y)=k_n(\zeta+t, z+x, z+y)$ (resp. $k^{(\infty)}(t, x, y)=k(\zeta+t, z+x, z+y)$). Then

$$\mathcal{E}_t^{(\infty)}(f(z+\cdot), g(z+\cdot)) = \mathcal{E}_{\zeta+t}(f, g)\,.$$

Lemma 5.2. *For any* $1\leq n\leq\infty$,

$$\int_s^\infty \mathcal{E}_\tau^{(n)}(T_\tau^{(n)}, T_\tau^{(n)})d\tau \leq cs^{-d/\alpha}\,. \tag{5.3}$$

Proof. First we shall show the estimate for $n<\infty$. Set $U_t^{(n)}(x)= t^{d/\alpha}T_t^{(n)}(t^{1/\alpha}x)$ (see (1.10)). From inequality (2.7),

$$\int_s^t \mathcal{E}_\tau^{(n)}(T_\tau^{(n)}, \log T_\tau^{(n)})d\tau$$
$$\leq -\int T_t^{(n)}(x) \log T_t^{(n)}(x)dx + \int T_s^{(n)}(x) \log T_s^{(n)}(x)dx$$
$$= \frac{d}{\alpha} \log \frac{t}{s} - \int U_t^{(n)} \log U_t^{(n)}dx + \int U_s^{(n)} \log U_s^{(n)}dx\,.$$

By Theorem 1 and Lemma 2.2,

$$\int U_s^{(n)} \log U_s^{(n)}dx \leq \int U_s^{(n)} \log c\, dx \leq c\,,$$
$$-\int U_t^{(n)} \log U_t^{(n)}dx \leq c \log \int r(|x|)U_t^{(n)}(x)dx + c \leq c\,.$$

Therefore, for any $0<s<t$,

$$\int_s^t \mathcal{E}_\tau^{(n)}(T_\tau^{(n)}, \log T_\tau^{(n)})d\tau \leq \frac{d}{\alpha} \log \frac{t}{s} + c\,.$$

Set $K^{(n)}(t, dx, dy)=(1/2)k^{(n)}(t, x, y)|x-y|^{-d-\alpha}dxdy$. Then

$$\mathcal{E}_t^{(n)}(T_t^{(n)}, T_t^{(n)})$$
$$= 2\iint T_t^{(n)}(x)(T_t^{(n)}(x)-T_t^{(n)}(y))[1-T_t^{(n)}(y)/T_t^{(n)}(x)]_+K^{(n)}$$

$$\leq 2 \iint T_t^{(n)}(x)(T_t^{(n)}(x)-T_t^{(n)}(y)) \log [T_t^{(n)}(x)/T_t^{(n)}(y)]K^{(n)}$$
$$\leq ct^{-d/\alpha}\mathcal{E}_t^{(n)}(T_t^{(n)} \log T_t^{(n)}) .$$

Combining these inequalities and integrating by parts,

$$\int_s^t \mathcal{E}_\tau^{(n)}(T_\tau^{(n)},\ T_\tau^{(n)})d\tau \leq c\int_s^t \tau^{-d/\alpha}\mathcal{E}_\tau^{(n)}(T_\tau^{(n)},\ \log T_\tau^{(n)})d\tau$$
$$\leq ct^{-d/\alpha}\left(c+\frac{d}{\alpha}\log\frac{t}{s}\right)+c\int_s^t \tau^{-d/\alpha}\left(c+\frac{d}{\alpha}\log\frac{\tau}{s}\right)\tau^{-1}d\tau$$
$$\leq cs^{-d/\alpha}\left[\max_{w\geqq 1} w^{-d/\alpha}\left(c+\frac{d}{\alpha}\log w\right)+\int_1^\infty w^{-d/\alpha}\left(c+\frac{d}{\alpha}\log w\right)w^{-1}dw\right]$$
$$=cs^{-d/\alpha} .$$

Therefore (5.3) is proved for $n<\infty$. From the Fatou lemma,

$$\int_s^\infty \mathcal{E}_\tau^{(\infty)}(T_\tau^{(\infty)},\ T_\tau^{(\infty)})d\tau \leq \varliminf_{n\to\infty}\int_s^\infty \mathcal{E}_\tau^{(\infty)}(T_\tau^{(n)},\ T_\tau^{(n)})d\tau$$
$$\leq c\varliminf_{n\to\infty}\int_s^\infty \mathcal{E}_\tau^{(n)}(T_\tau^{(n)},\ T_\tau^{(n)})d\tau \leq cs^{-d/\alpha} .$$

Namely (5.3) is valid also for $n=\infty$. q.e.d.

Lemma 5.3. *For any test function $f(x)$ and $0<s<t$,*

$$(5.4)\qquad (T_t^{(\infty)}, f)_{L^2}-(T_s^{(\infty)}, f)_{L^2}+\int_s^t \mathcal{E}_\tau^{(\infty)}(T_\tau^{(\infty)}, f)d\tau = 0 .$$

Proof. By condition (1.5)

$$\lim_{\tau\to\sigma}\sup_{n,x,y} |k_n(\tau,\ x,\ y)-k_n(\sigma,\ x,\ y)| = 0 .$$

Since $k_n(\tau, x, y)\to k(\tau, x, y)$ as $n\to\infty$ for a.a. (x, y) and each τ, we have

$$\lim_{n\to\infty}\sup_{s\leq\tau\leq t}\iint_{|x|+|y|\leq N}(k^{(n)}(\tau,\ x,\ y)-k^{(\infty)}(\tau,\ x,\ y))^2 dx\,dy = 0$$

for any N. Then there exists a number $m=m(\varepsilon)$ for each $\varepsilon>0$ such that

$$\sup_{s\leq\tau\leq t}\iint (f(x)-f(y))^2(k^{(n)}(\tau,\ x,\ y)-k^{(m)}(\tau,\ x,\ y))^2|x-y|^{-d-\alpha}dx\,dy \leq \varepsilon^2$$

is satisfied as long as $m\leq n\leq\infty$. Therefore, for any $1\leq j\leq\infty$,

$$\int_s^t |\mathcal{E}_\tau^{(n)}(T_\tau^{(j)}, f)-\mathcal{E}_\tau^{(m)}(T_\tau^{(j)}, f)|\,d\tau$$
$$\leq \int_s^t c\varepsilon\mathcal{E}_\tau^{(j)}(T_\tau^{(j)},\ T_\tau^{(j)})^{1/2}d\tau$$

$$\leq c\mathcal{E}(t-s)^{1/2}\left(\int_s^t \mathcal{E}_\tau^{(j)}(T_\tau^{(j)}, T_\tau^{(j)})d\tau\right)^{1/2}$$
$$\leq c\mathcal{E}(t-s)^{1/2}s^{-d/2\alpha} = c(s, t)\mathcal{E} ,$$

where Lemma 5.2 is used. Then

$$\int_s^t |\mathcal{E}_\tau^{(n)}(T_\tau^{(n)}, f)-\mathcal{E}_\tau^{(\infty)}(T_\tau^{(\infty)}, f)|\, d\tau$$
$$\leq \int_s^t |\mathcal{E}_\tau^{(n)}(T_\tau^{(n)}, f)-\mathcal{E}_\tau^{(m)}(T_\tau^{(n)}, f)|\, d\tau$$
$$+\int_s^t |\mathcal{E}_\tau^{(m)}(T_\tau^{(n)}-T_\tau^{(\infty)}, f)|\, d\tau$$
$$+\int_s^t |\mathcal{E}_\tau^{(m)}(T_\tau^{(\infty)}, f)-\mathcal{E}_\tau^{(\infty)}(T_\tau^{(\infty)}, f)|\, d\tau$$
$$\leq 2c(s, t)\mathcal{E}+\int_s^t |(T_\tau^{(n)}-T_\tau^{(\infty)}, A_\tau^{(m)}f)_{L^2}|\, d\tau$$
$$\leq 2c(s, t)\mathcal{E}+\sup_{\tau, x}|A_\tau^{(m)}f(x)|\cdot\int_s^t\int |T_\tau^{(n)}(x)-T_\tau^{(\infty)}(x)|\, d\tau\, dx .$$

Let $n\to\infty$ first and $\mathcal{E}\downarrow 0$ next. Then we have

$$\lim_{n\to\infty}\int_s^t |\mathcal{E}_\tau^{(n)}(T_\tau^{(n)}, f)-\mathcal{E}_\tau^{(\infty)}(T_\tau^{(\infty)}, f)|\, d\tau = 0 .$$

Since, for any $n<\infty$,

$$(T_t^{(n)}, f)_{L^2}-(T_s^{(n)}, f)_{L^2}+\int_s^t \mathcal{E}_\tau^{(n)}(T_\tau^{(n)}, f)d\tau = 0 ,$$

we see that equality (5.4) is satisfied. q.e.d.

We shall say that a function $\tilde{S}(s,x;t,y)$ is the fundamental solution of the parabolic equation associated with the Dirichlet form $\mathcal{E}_t(\cdot, \cdot)$ if, for any $\phi(x)\in L^1\cap L^2$ and any $\zeta\geq 0$, the function

$$u_t(x) = \int \phi(z)\tilde{S}(\zeta, z; \zeta+t, x)dz$$

is a solution of the Cauchy problem $(\partial/\partial t)u_t=A_{\zeta+t}u_t$, $u_{0+}=\phi$. From Lemma 5.1, the fundamental solution is uniquely determined.

Theorem 5. *Under condition* (1.2) *and* (1.5), *there exists the fundamental solution of the parabolic equation associated with the Dirichlet from* $\mathcal{E}_t(\cdot, \cdot)$.

Proof. From Lemma 5.3, for any test function $f(x)$ and $0<s<t$,

$$(5.5)\qquad (S(\zeta, z; \zeta+t, \cdot), f)_{L^2}-(S(\zeta, z; \zeta+s, \cdot), f)_{L^2}$$
$$+\int_s^t \mathcal{E}_{\zeta+\tau}(S(\zeta, z; \zeta+\tau, \cdot), f)d\tau = 0 .$$

Let $\phi(x)$ be a function in $L^1 \cap L^2$, and define

$$u_t(x) = \int \phi(z) S(\zeta, z; \zeta+t, x) dz = \int S(\zeta, x; \zeta+t, z) \phi(z) dz .$$

Then we have $\|u_t\|_{L^2} \leq \|\phi\|_{L^2}$ and $\|u_t - \phi\|_{L^2} \to 0$ as $t \downarrow 0$ from (5.2). And

$$\begin{aligned} &\int_s^t \mathcal{E}_{\zeta+\tau}(u_\tau, u_\tau) d\tau \\ &\leq \|\phi\|_{L^1}^2 \int_s^t \mathcal{E}_{\zeta+\tau}(S(\zeta, z; \zeta+\tau, \cdot), S(\zeta, z; \zeta+\tau, \cdot)) d\tau \\ &\leq c \|\phi\|_{L^1}^2 s^{-d/\alpha} < \infty \end{aligned}$$

from Lemma 5.2. Therefore we have

$$\int \phi(z) [\int_s^t \mathcal{E}_{\zeta+\tau}(S(\zeta, z; \zeta+\tau, \cdot), f) d\tau] dz = \int_s^t \mathcal{E}_{\zeta+\tau}(u_\tau, f) d\tau .$$

This equality and (5.5) imply that

$$(u_t, f)_{L^2} - (u_s, f)_{L^2} + \int_s^t \mathcal{E}_{\zeta+\tau}(u_\tau, f) d\tau = 0 .$$

Hence the function $S(s, x; t, y)$ is the fundamental solution. q.e.d.

Let $S(s, x; t, y)$ be the fundamental solution of the parabolic equation associated with the Dirichlet form $\mathcal{E}_t(\cdot, \cdot)$. From the manner of construction of the function $S(s, x; t, y)$ in this section, it is immediate to see that each function

$$T(t, x) = S(\zeta, z; \zeta+t, z+x)$$

satisfies the estimates in Theorem 1 and Theorem 2. Namely the assumption that $(\partial/\partial x)^\mu (\partial/\partial y)^\nu k(t, x, y)$ are bounded and continuous for all $\mu, \nu \in \boldsymbol{Z}_+^d$ can be taken off.

Let $P_{s,t}$ be the operator defined by

$$P_{s,t} f(x) = \int S(s, x; t, y) f(y) dy .$$

Then $\{P_{s,t}\}$ satisfies the semi-group property: $P_{s,t} P_{t,u} = P_{s,u}$ for any $0 \leq s < t < u$. It is clear from Theorem 1 that

$$\max_x |P_{s,t} f(x) - f(x)| \to 0 \qquad \text{as} \quad (t-s) \downarrow 0$$

for any $f(x) \in C_0(\boldsymbol{R}^d)$. Since $P_{s,t} 1 = 1$ and $P_{s,t} f \geq 0$ if $f \geq 0$, $\{P_{s,t}\}$ is a Feller semi-group. For any bounded measurable function $f(x)$ on $\boldsymbol{R}^d$, the function $P_{s,t} f(x)$ is continuous on $\{(s, t, x); s < t \text{ and } x \in \boldsymbol{R}^d\}$. Therefore each Dirichlet form $\mathcal{E}_t(\cdot, \cdot)$ satisfying (1.2) and (1.5) determines a strong Feller semi-group $\{P_{s,t}\}$.

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