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<th><strong>Title</strong></th>
<th>Interface regularity for Maxell and Stokes systems</th>
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<tr>
<td><strong>Author(s)</strong></td>
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<tr>
<td><strong>Citation</strong></td>
<td>Osaka Journal of Mathematics. 40(4) P.925-P.943</td>
</tr>
<tr>
<td><strong>Issue Date</strong></td>
<td>2003-12</td>
</tr>
<tr>
<td><strong>Text Version</strong></td>
<td>publisher</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="https://doi.org/10.18910/3867">https://doi.org/10.18910/3867</a></td>
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<tr>
<td><strong>DOI</strong></td>
<td>10.18910/3867</td>
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1. Introduction

The purpose of the present paper is to study the interface regularity of three-dimensional Maxwell and Stokes systems. To our knowledge, not so much regards have been taken in this topic, but actually the solenoidal condition provides the regularity across interface to a specified component of the unknown vector field.

Let \( \Omega \subset \mathbb{R}^3 \) be a bounded domain with Lipschitz boundary \( \partial \Omega \), and \( M \subset \mathbb{R}^3 \) be a \( C^2 \) hypersurface cutting \( \Omega \) transversally. Then, it holds that

\[
M \cap \Omega \neq \emptyset \tag{1}
\]

with the open subsets \( \Omega_\pm \) of \( \Omega \). First, we take the Maxwell system in magnetostatics,

\[
\begin{align*}
\nabla \times B &= J \\
\nabla \cdot B &= 0
\end{align*}
\]

in \( \Omega_\pm \), where \( B = (B^1(x), B^2(x), B^3(x)) \) and \( J = (J^1(x), J^2(x), J^3(x)) \) stand for the three-dimensional vector fields, indicating the magnetic field and the total current density, respectively. Here and henceforth, \( \nabla = (\partial_1, \partial_2, \partial_3) \) denotes the gradient operator and \( \times \) and \( \cdot \) are the outer and the inner products in \( \mathbb{R}^3 \), so that \( \nabla \times \) and \( \nabla \cdot \) are the operations of the rotation and the divergence, respectively.

In the context of magnetoencephalography, Suzuki, Watanabe, and Shimogawara [2] studied the case when the interface is given by the boundary \( \partial D \) of a smooth bounded domain \( D \subset \mathbb{R}^3 \). Namely, from the properties of the layer potential, it showed that if \( J \) is piecewise continuous on \( \mathbb{R}^3 \setminus \partial D \) and system (2) has a solution \( B \in C(\mathbb{R}^3)^3 \cap C^1(\mathbb{R}^3 \setminus \partial D)^3 \) for \( \Omega_- = D \) and \( \Omega_+ = \mathbb{R}^3 \setminus D \), then

\[
[\nabla(n \cdot B)]^+_- = 0 \quad \text{on} \quad \partial D
\]

follows, regardless with the continuity of \( J \) across \( \partial D \). Here, \( n \) denotes the outer unit
normal vector to \( \partial D \), \( [A]^+ = A_+ - A_- \), and

\[
A_+(\xi) = \lim_{x \to \xi, x \in \mathbb{R}^3 \setminus D} A(x), \quad A_-(\xi) = \lim_{x \to \xi, x \in D} A(x)
\]

for \( \xi \in \partial D \). In this paper we study its local version, that is, the case where the bounded domain \( \Omega \) is given with the interface \( \mathcal{M} \cap \Omega \) as in (1).

To state the result, we take preliminaries on function spaces from Girault and Raviart [1]. Namely, let \( D \subset \mathbb{R}^3 \) be a bounded domain with Lipschitz boundary \( \partial D \) and \( n \) be the unit normal vector to \( \partial D \). For \( p \in [1, \infty] \), \( L^p(D) \) denotes the standard \( L^p \) space on \( D \) provided with the norm \( \| \cdot \|_{L^p(D)} \), and the Sobolev space \( H^m(D) \) is defined by

\[
H^m(D) = \left\{ u \in L^2(D) \mid \partial^\alpha u \in L^2(D) \quad \text{for} \quad |\alpha| \leq m \right\}
\]

for a positive integer \( m \), where \( \partial^\alpha = \partial^{\alpha_1}_{x_1} \partial^{\alpha_2}_{x_2} \partial^{\alpha_3}_{x_3} \) for the multi-index \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \). Given \( \sigma \in (0, 1) \), we say that \( u \in H^{m+n\sigma}(D) \) if \( u \in H^m(D) \) and

\[
\int_D \int_D \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^2}{|x - y|^{n+2\sigma}} \, dx \, dy < +\infty
\]

for any \( \alpha \) in \( |\alpha| = m \) and \( n = 3 \). The space \( H^m(\Gamma) \) is defined similarly with \( n = 2 \) through the local chart of \( \Gamma \), where \( s \in [0, 1] \) and \( \Gamma \subset \partial D \) is a relatively open connected set. Then, we set \( H^{-s}(\Gamma) = H^0_0(\Gamma)' \), where \( H^0_0(\Gamma) \) denotes the closure in \( H^s(\Gamma) \) of the space composed of Lipschitz continuous functions on \( \Gamma \) with compact supports. Thus, we have \( H^0_0(\Gamma) = H^m(\Gamma) \) if \( \Gamma \subset \partial D \) is a closed surface, and in particular, it holds that \( H^{1/2}(\partial D) = H^0_0(\partial D) \). We also put

\[
H(\text{div}, D) = \left\{ u \in L^2(D)^3 \mid \nabla \cdot u \in L^2(D) \right\}
\]

and

\[
H(\text{rot}, D) = \left\{ u \in L^2(D)^3 \mid \nabla \times u \in L^2(D)^3 \right\}.
\]

Then, any \( u \in H(\text{div}, D) \) admits the trace \( n \cdot u|_{\partial D} \in H^{-1/2}(\partial D) \), and Green’s formula

\[
\left( (u, \nabla \varphi) \right)_D + \left( \nabla \cdot u, \varphi \right)_D = \left( n \cdot u, \varphi \right)_{\partial D}
\]

holds for \( \varphi \in H^1(D) \). Here and henceforth, \( \langle \cdot, \cdot \rangle_D \) and \( \langle \cdot, \cdot \rangle \) denote \( L^2(D) \) and \( L^2(D)^3 \) inner products, respectively, and \( \langle \cdot, \cdot \rangle_{\partial D} \) the duality pairing between \( H^{-1/2}(\partial D) \) and \( H^{1/2}(\partial D) = H^0_0(\partial D) \). Let us note here that the standard trace theorem guarantees \( \varphi|_{\partial D} \in H^{1/2}(\partial D) \) for \( \varphi \in H^1(D) \). Similarly, any \( u \in H^1(\text{rot}, D) \) admits the trace \( n \times u|_{\partial D} \in H^{-1/2}(\partial D)^3 \), and the Stokes formula

\[
\left( (\nabla \times u, w) \right)_D - \left( (u, \nabla \times w) \right)_D = \left( (n \times u, w) \right)_{\partial D}
\]
holds for $w \in H^1(D)^3$, where $\langle \langle \cdot, \cdot \rangle \rangle_{\partial D}$ denotes the duality pairing between $H^{-1/2}(\partial D)^3$ and $H^{1/2}(\partial D)^3$.

Now, to discuss the interface regularity of the solution $B$ to the Maxwell system (2), we take that

$$\Gamma_\pm = \partial \Omega_\pm \cap M$$

with $\partial \Omega_\pm$ being the boundary of $\Omega_\pm$. This means that $\Gamma_+$ and $\Gamma_-$ coincide as sets, but they are regarded as the parts of the boundaries of $\Omega_+$ and $\Omega_-$, respectively. Henceforth, $n$ denotes the outer unit normal vector to $\Gamma_-$ so that $-n$ is the outer unit normal vector to $\Gamma_+$. Henceforth, $C^2$ extension of the vector field $\eta$ defined on $\Gamma = \mathcal{M} \cap \Omega$ is always taken to $\Omega$. Furthermore, given a function $A(x)$ on $\Omega_\pm$, we set

$$[A]^\pm_\pm = A_+ - A_- \quad \text{on} \quad \Gamma,$$

where $A_\pm(\xi) = \lim_{x \to \xi, x \in \Omega_\pm} A(x)$ for $\xi \in \Gamma$ are usually taken in the sense of traces to $\Gamma_\pm$.

Suppose that $B$ and $J$ are in $L^2(\Omega_\pm)^3$ and satisfy (2). This means that those relations hold piecewisely in $\Omega_\pm$ in the sense of distributions $\mathcal{D}'(\Omega_\pm)$, that is,

$$\int_{\Omega_\pm} B \cdot \nabla \times C = \int_{\Omega_\pm} J \cdot C \quad \text{and} \quad \int_{\Omega_\pm} B \cdot \nabla \varphi = 0$$

for any $C \in C_0^\infty(\Omega_\pm)^3$ and $\varphi \in C_0^\infty(\Omega_\pm)$. Unless otherwise stated, those vector fields $B \in L^2(\Omega_\pm)$ and $J \in L^2(\Omega_\pm)^3$ are identified with the elements in $L^2(\Omega)^3$.

Relation (2) for $B \in L^2(\Omega_\pm)^3$ and $J \in L^2(\Omega_\pm)^3$ implies that $B \in H(\text{rot}, \Omega_\pm) \cap H(\text{div}, \Omega_\pm)$, which assures the well-definedness of

$$n \times B|_{\Gamma_\pm} \in H^{-1/2}(\Gamma_\pm)^3 \quad \text{and} \quad n \cdot B|_{\Gamma_\pm} \in H^{-1/2}(\Gamma_\pm),$$

and hence $B|_{\Gamma_\pm} \in H^{-1/2}(\Gamma_\pm)^3$ follows. Furthermore,

$$(3) \quad [n \times B]^\pm_\pm = 0 \quad \text{and} \quad [n \cdot B]^\pm_\pm = 0$$

if and only if

$$(4) \quad \nabla \times B = J \in L^2(\Omega)^3 \quad \text{and} \quad \nabla \cdot B = 0 \in L^2(\Omega)$$

as distributions in $\Omega$, respectively. If both relations of (3) are satisfied, then $B \in H^{1}_{\text{loc}}(\Omega)^3$ follows, because $B \in H^{1}_{\text{loc}}(\Omega)^3$ is equivalent to $[B]^\pm = 0$ on $\Gamma$ for $B \in H^1(\Omega_\pm)^3$. This fact is also obtained by Corollary 1.2.10 of [1],

$$(5) \quad H(\text{rot}, \Omega) \cap H(\text{div}, \Omega) \subset H^{1}_{\text{loc}}(\Omega)^3,$$
as (4) for $B \in L^2(\Omega)^3$ implies $B \in H^1_{loc}(\Omega)^3$.

Our first result is stated as follows. Let us note again that $n$ defined on $\mathcal{M}$ is extended to a $C^2$ vector field in $\Omega$, and $n \cdot B \in H^1_{loc}(\Omega)$ follows from $B \in H^1(\Omega)^3$.

**Theorem 1.** If $B \in H^1(\Omega)^3$ and $J \in H(\text{rot}, \Omega_{\pm})$ satisfy (2), then it holds that $n \cdot B \in H^2_{loc}(\Omega)$.

In the above theorem, $B$ solves (2) in $\Omega$ as a distribution, because it is assumed to be in $H^1(\Omega)^3$. That is,

$$\int_{\Omega} B \cdot \nabla \times C = \int_{\Omega} J \cdot C \quad \text{and} \quad \int_{\Omega} B \cdot \nabla \varphi = 0$$

hold for any $C \in C_0^\infty(\Omega)^3$ and $\varphi \in C_0^\infty(\Omega)$. On the other hand, $J \in H(\text{rot}, \Omega_{\pm})$ belongs to $J \in H(\text{rot}, \Omega)$ if and only if $[n \times J]_+ = 0$ on $\Gamma$. If this condition is satisfied furthermore, then it holds that

$$-\Delta B = \nabla \times J \in L^2(\Omega)^3$$

(as distributions in $\Omega$), because $\nabla \times B = J \in H(\text{rot}, \Omega)$ and $\nabla \cdot B = 0 \in L^2(\Omega)^3$ are valid similarly in $\Omega$. Then, $B \in H^2_{loc}(\Omega)^3$ is obtained from the elliptic regularity. Thus, Theorem 1 says, in contrast, that even if $n \times J$ has an interface on $\Gamma = \mathcal{M} \cap \Omega$, the normal component $n \cdot B$ of $B$ gains the regularity in one rank. It is not difficult to suspect that the solenoidal condition $\nabla \cdot B = 0$ in $\Omega$ plays an essential role in such a regularity.

In this connection, it may be worth noting that the assumption of Theorem 1 does not permit the interface to $n \cdot J$. In fact, equation (2) holds in $\Omega$ as we have seen, and therefore,

$$\nabla \cdot J = \nabla \cdot (\nabla \times B) = 0$$

follows there. This implies $J \in H(\text{div}, \Omega)$, and hence we have $[n \cdot J]_+ = 0$ on $\Gamma$ in particular.

Theorem 1 can be applicable to the stationary Stokes system;

(6)$$\begin{cases}
-\Delta u + \nabla p = f \\
\nabla \cdot u = 0
\end{cases} \quad \text{in} \quad \Omega_{\pm}$$

and the stationary Navier-Stokes system;

(7)$$\begin{cases}
-\Delta u + (u \cdot \nabla) v + \nabla p = f \\
\nabla \cdot v = 0
\end{cases} \quad \text{in} \quad \Omega_{\pm},$$

where $v = (u^1(x), u^2(x), u^3(x))$ denotes the velocity of fluid, $p = p(x)$ the pressure, and
$f(x) = (f^1(x), f^2(x), f^3(x))$ the external force. We have the following theorem, where $\omega = \nabla \times v$ indicates the vorticity of fluid.

**Theorem 2.** If $v \in H^2(\Omega_\pm)^3$, $p \in H^1(\Omega_\pm)$, and $f \in H(\text{rot}, \Omega_\pm)$ satisfy (6) or (7) and if $\omega = \nabla \times v$ is in $H^1(\Omega)^3$, then it holds that $n \cdot \omega \in H^2_{\text{loc}}(\Omega)$.

We note that $v \in H^2_0(\Omega_\pm)^3$ implies $\omega = \nabla \times v \in H^1(\Omega_\pm)^3$, and hence the assumption $\omega \in H^1(\Omega)^3$ means $[\omega]_+ = 0$ on $\Gamma$. It is equivalent to saying that $\omega = \nabla \times v \in H^1(\Omega)^3$ as a distribution in $\Omega$, with $v$ regarded as an element in $L^2(\Omega)^3$.

In the above theorem, system of equations is supposed to hold piecewisely in $\Omega_\pm$, and $v$, $\nabla v$, $p$, and $f$ may have interfaces on $\Gamma = M \cap \Omega$. Nevertheless, it says that the normal component $n \cdot \omega$ of vorticity $\omega$ gains the regularity in one rank if $[\omega]_+ = 0$ holds on $\Gamma = \Omega \cap M$ for $\omega = \nabla \times v \in H^1(\Omega)^3$.

On the other hand, all components of $\omega$ gain the interface regularity, if $v$, $p$, $f$ are free from the interface, so that if $v \in H^2(\Omega)^3$, $p \in H^1(\Omega)$, and $f \in H(\text{rot}, \Omega)$ hold in (6), then $\omega \in H^2_{\text{loc}}(\Omega)^3$ follows. In fact, in this case system (6) holds in $\Omega$, and hence

$$\nabla \times \nabla p = 0 \in L^2(\Omega), \quad \text{and} \quad -\Delta \omega = \nabla \times f \in L^2(\Omega)$$

follow in turn as distributions in $\Omega$. Then, the elliptic regularity guarantees for $\omega \in H^1(\Omega)^3$ to be in $\omega \in H^2_{\text{loc}}(\Omega)$ from the last inclusion.

The interface regularity of $p$, the pressure of fluid, follows similarly from the standard regularity. Namely, if $v \in H^2(\Omega)^3$, $p \in H^1(\Omega)$, and $f \in H(\text{div}, \Omega)$ satisfy (6), then it follows that $p \in H^2_{\text{loc}}(\Omega)$. In fact, then we have

$$\nabla p = \Delta v + f \quad \text{and} \quad \nabla \cdot v = 0$$

in $\Omega$ (as distributions again), and hence

$$\Delta p = \nabla \cdot f \in L^2(\Omega)$$

follows similarly. Thus, we obtain $p \in H^2_{\text{loc}}(\Omega)$ from the elliptic regularity.

Those standard regularities are valid even to (7), because $v \in H^2(\Omega)^3$ implies $v \in L^\infty(\Omega)^3$ and $\partial v / \partial x_j \in L^4(\Omega)^3$ for $j = 1, 2, 3$ by Sobolev’s imbedding theorem, and therefore, $(v \cdot \nabla)v \in H^1(\Omega)^3$ follows from

$$\frac{\partial}{\partial x_j} (v \cdot \nabla v) = \frac{\partial v}{\partial x_j} \cdot \nabla v + v \cdot \nabla \frac{\partial v}{\partial x_j} \in L^2(\Omega).$$

In other words, even in (7), $v \in H^2(\Omega)$, $p \in H^1(\Omega)$, and $f \in H(\text{rot}, \Omega)$ imply $v \in H^2_{\text{loc}}(\Omega)^3$ and $v \in H^2(\Omega)$, $p \in H^1(\Omega)$, and $f \in H(\text{div}, \Omega)$ imply $p \in H^2_{\text{loc}}(\Omega)$.

We confirm again that, in contrast with those standard results, Theorem 2 assures the interface regularity gain in one rank for $n \cdot \omega$, only from the piecewise regularity.
of the data. This is actually the case even for the velocity itself as the following theorem shows, where $C^3$ extension of $n$ is taken to $\Omega$. See the remark after the following theorem concerning the non-standard interface regularity for $p$.

**Theorem 3.** If $\mathcal{M} \subset \mathbb{R}^3$ is $C^3$ and $v \in H^2(\Omega)^3$, $p \in H^2(\Omega_\pm)$, and $f \in H^1(\Omega_\pm)^3$ satisfy (6) or (7), then

$$\nabla (\cdot) |_{\pm} v \in H^{-1/2}(\Gamma)^3$$

The corresponding standard regularity to the above theorem is obvious, so that $v \in H^2(\Omega)^3$, $p \in H^2(\Omega)$, and $f \in H^1(\Omega)^3$ imply $v \in H_{\text{loc}}^3(\Omega)$ in (6) or (7).

In this theorem, similarly to the previous one, (6) or (7) does not hold in $\Omega$ as a system, because $p \in H^1(\Omega)$ is not required in spite of $v \in H^2(\Omega)^3$. However, if we add $f \in H(\text{div}, \Omega)$ in (6) to the assumptions of Theorem 3, then

$$\left[ \frac{\partial p}{\partial n} \right]^+ = 0 \quad \text{on} \quad \Gamma$$

follows from

$$\nabla p = f + \Delta v \quad \text{in} \quad \Omega_\pm,$$

because $v \in H^2(\Omega)^3$ and $n \cdot v \in H^1_{\text{loc}}(\Omega)$ imply $n \cdot \Delta v \in H^1_{\text{loc}}(\Omega)$. The same fact holds similarly to (7), as $(v \cdot \nabla) v \in H^1(\Omega)^3$ holds by $v \in H^2(\Omega)^3$. Later, we shall show that (8) is valid under the assumptions of Theorem 2 and $f \in H(\text{div}, \Omega)$.

Relation (8) implies $\nabla \cdot (\nabla p) \in L^2(\Omega)$ if $\nabla p \in H^1(\Omega_\pm)^3$ is regarded as an element in $L^2(\Omega)^3$. However, in contrast with the standard case described before Theorem 3, this does not mean $\Delta p \in L^2(\Omega)$ because $p \in H^2(\Omega_\pm)$ itself may have the interface, and $\nabla p \in L^2(\Omega)^3$ does not hold in $\Omega$ when the distributional derivative is taken to $p \in L^2(\Omega)$ in $\Omega$.

This paper is composed of three sections. In Section 2, a key lemma is provided. Then, Theorems 1, 2, 3 are proven in Section 3. Those theorems have component-wise versions and Sections 4 are devoted to that topic. The final section is the concluding remark.

### 2. Key lemma

In this section, we are concentrated on the Maxwell system (2) and show the following lemma. It is a fundamental tool for the proof of theorems.

**Lemma 2.1.** If $B \in L^2(\Omega_\pm)^3$ and $J \in H(\text{rot}, \Omega_\pm)$ satisfy (2), then

$$\nabla (n \cdot B)|_{\pm} \in H^{-1/2}(\Gamma)^3$$
is well-defined and it holds that

\[
\langle \langle \nabla (n \cdot B), C \rangle \rangle_+ - \langle \langle \nabla \cdot (n B), C \rangle \rangle_+ = \langle \langle B, (n \cdot \nabla) C \rangle \rangle_+ - \langle \langle (n \times B, \nabla \times C) \rangle \rangle_- - \langle \langle n \cdot B, \nabla \cdot C \rangle \rangle_- 
\]

(9)

for any \( C \in C_0^\infty(\Omega)^3 \), where \( \langle \langle \cdot, \cdot \rangle \rangle_+ = \langle \langle \cdot, \cdot \rangle \rangle_{\Gamma_+} - \langle \langle \cdot, \cdot \rangle \rangle_{\Gamma_-} \).

Proof. As is described in introduction, it follows from (2) and \( B, J \in L^2(\Omega^\pm) \) that \( B \in H(\text{rot}, \Omega^\pm), n \times B|_{\Gamma^\pm} \in H^{-1/2}(\Gamma^\pm)^3, B \in H(\text{div}, \Omega^\pm), \) and \( n \cdot B|_{\Gamma^\pm} \in H^{-1/2}(\Gamma^\pm). \) It also holds by (5) that \( B \in H^1_{\text{loc}}(\Omega^\pm)^3 \).

Now, in use of \( J \in H(\text{rot}, \Omega^\pm), \) we have

\[
n \times J|_{\Gamma^\pm} = n \times (\nabla \times B)|_{\Gamma^\pm} \in H^{-1/2}(\Gamma^\pm)^3.\]

Furthermore, \( \nabla \cdot B = 0 \) in \( \Omega^\pm \) implies \(-\Delta B = \nabla \times J \in L^2(\Omega^\pm)^3\), and hence

\[
\frac{\partial B}{\partial n} = (n \cdot \nabla) B|_{\Gamma^\pm} \in H^{-1/2}(\Gamma^\pm)^3
\]

is well-defined for \( B \in H^1_{\text{loc}}(\Omega^\pm)^3 \) by Corollary 1.2.6 of [1]. Thus, through the (distributional) identity

\[
\langle n \cdot \nabla) B + n \times (\nabla \times B) = \nabla (n \cdot B) - (\nabla \cdot n) B,
\]

valid for \( n \in C^1(\Omega) \) and \( B \in L^2(\Omega)^3 \), it follows that

\[
\nabla (n \cdot B)|_{\Gamma^\pm} \in H^{-1/2}(\Gamma^\pm)^3.
\]

Henceforth, we set

\[
\nabla B \otimes \nabla C = \sum_{i,j=1}^3 \frac{\partial B^i}{\partial x_j} \frac{\partial C^j}{\partial x_j}
\]

for \( B = (B^1, B^2, B^3) \) and \( C = (C^1, C^2, C^3) \in C_0^\infty(\Omega)^3 \). Then, it holds that

\[
\int\Omega \nabla B \otimes \nabla C = -\langle \langle (n \cdot \nabla) B, C \rangle \rangle_+ - (\langle \langle \nabla B, C \rangle \rangle_\Omega - (\Delta B^i, C^i)_{\Omega^\pm}
\]

(11)

In fact, because \( \mp n \) is the outer unit normal vector to \( \Gamma^\pm \), Green’s formula, described in the previous section, guarantees that

\[
\langle \nabla B^i, C^i \rangle_{\Omega^\pm} = -\left\langle \left\langle \frac{\partial B^i}{\partial n}, C^i \right\rangle \right\rangle_{\Gamma^\pm} - (\Delta B^i, C^i)_{\Omega^\pm}
\]
for $i = 1, 2, 3$. This implies (11).

Here, equality (10) is applied to the first term of the right-hand side of (11). We have

$$-\langle \langle (n \cdot \nabla) B, C \rangle \rangle^+_n = \langle \langle n \times (\nabla \times B), C \rangle \rangle^+_n - \langle \langle \nabla \cdot (n \cdot B), C \rangle \rangle^-_n + \langle \langle (\nabla \cdot n) B, C \rangle \rangle^-_n.$$  

Since $-\nabla \times (\nabla \times B) = \Delta B$ holds in $\Omega_\pm$, the Stokes formula now gives that

$$\langle \langle n \times (\nabla \times B), C \rangle \rangle^+_n = -\langle \langle (\nabla \times (\nabla \times B), C \rangle \rangle^+_\Omega + \langle \langle (\nabla \times B, \nabla \times C) \rangle \rangle^+_\Omega$$

$$= \langle \langle \Delta B, C \rangle \rangle^+_\Omega + \langle \langle (\nabla \times B, \nabla \times C) \rangle \rangle^+_\Omega.$$

Those relations are summarized as

$$\langle \langle \nabla \cdot (n \cdot B), C \rangle \rangle^+_n - \langle \langle \nabla (n \cdot B), C \rangle \rangle^-_n = \int_{\Omega} \nabla B \otimes \nabla C - \langle \langle (\nabla \times B, \nabla \times C) \rangle \rangle^+_\Omega. \tag{12}$$

On the other hand, we have

$$\int_{\Omega} \nabla B \otimes \nabla C = -\langle \langle B, (n \cdot \nabla) C \rangle \rangle^+_n - \langle \langle B, \Delta C \rangle \rangle^+_\Omega$$

similarly to (11). Combining this with (12), we obtain

$$\langle \langle \nabla (n \cdot B), C \rangle \rangle^+_n - \langle \langle (\nabla \cdot n) B, C \rangle \rangle^+_n = \langle \langle B, (n \cdot \nabla) C \rangle \rangle^+_n + \langle \langle (B, \Delta C) \rangle \rangle^+_\Omega + \langle \langle (\nabla \times B, \nabla \times C) \rangle \rangle^+_\Omega. \tag{13}$$

Now, we take the Helmholtz decomposition of $C$. We put

$$C = C_0 + \nabla p, \tag{14}$$

where $p$ is a scalar field defined in $\Omega$ satisfying

$$-\Delta p = \nabla \cdot C \text{ in } \Omega, \quad \frac{\partial p}{\partial n} = n \cdot C (= 0) \text{ on } \partial \Omega.$$

First, we have $p \in C^\infty(\Omega)$ and $\nabla \times C_0 = \nabla \times C \in C^\infty_0(\Omega)^3$. This implies that

$$\langle \langle (\nabla \times B, \nabla \times C) \rangle \rangle^+_\Omega = \langle \langle (\nabla \times B, \nabla \times C_0) \rangle \rangle^+_\Omega.$$

On the other hand, we have $\Delta p = -\nabla \cdot C \in C^\infty_0(\Omega)$ and hence

$$\langle \langle (B, \Delta C) \rangle \rangle^+_\Omega = \langle \langle (B, \Delta C_0 + \nabla (\Delta p)) \rangle \rangle^+_\Omega.$$
\[ = ((B, \Delta C_0) + (\nabla \cdot B, \Delta p)_\Omega + \langle n \cdot B, \Delta p \rangle^+_{\Omega}) \]
\[ = ((B, \Delta C_0) + (n \cdot B, \nabla \cdot C)^+_{\Omega}). \]

Finally, we have for \( L = \nabla \times C_0 \in C^\infty_0(\Omega)^3 \) that \( \Delta C_0 = -\nabla \times L \) by \( \nabla \cdot C_0 = 0 \) in \( \Omega \) and hence
\[ \left((B, \Delta C_0)\right)_\Omega = -\left((B, \nabla \times L)\right)_\Omega \]
\[ = -\left((\nabla \times B, L)\right)_\Omega - \left\langle (n \times B, L) \right\rangle^+ \]
\[ = -\left((\nabla \times B, \nabla \times C_0)\right)_\Omega - \left\langle (n \times B, \nabla \times C_0) \right\rangle^+_{\Omega}. \]

Those relations are summarized as
\[ \left((B, \Delta C)\right)_\Omega + \left((\nabla \times B, \nabla \times C)\right)_\Omega \]
\[ = \left((B, \Delta C_0)\right)_\Omega + \left((\nabla \times B, \nabla \times C_0)\right)_\Omega - \left\langle n \cdot B, \nabla \cdot C \right\rangle^- \]
\[ = -\left\langle (n \times B, \nabla \times C_0) \right\rangle^+ - \left\langle n \cdot B, \nabla \cdot C \right\rangle^- \]
\[ = -\left\langle (n \times B, \nabla \times C) \right\rangle^+ - \left\langle n \cdot B, \nabla \cdot C \right\rangle^+. \]

Therefore, (9) follows from (13). The proof is complete. \( \square \)

3. Proof of Theorems

First, we give the following.

Proof of Theorem 1. Since \( [B]^+ = 0 \) on \( \Gamma \), we have by making use of (9) that
\[ \left\langle (\nabla (n \cdot B), C) \right\rangle^+ = 0 \]
for any \( C \in \{C^\infty_0(\Omega)\}^3 \) by (9). This implies that
\[ \left\langle (\nabla (n \cdot B)) \right\rangle^+ = 0 \quad \text{on} \quad \Gamma, \]

On the other hand, we have \( B \in H^1_{\text{loc}}(\Omega)^3 \) and \( -\Delta B = \nabla \times J \in L^2(\Omega^\pm) \). Hence \( \Delta (n \cdot B) \in L^2(\Omega^\pm) \) follows in \( \Omega^\pm \) as distributions. Combining this with (15), we get \( \Delta (n \cdot B) \in L^2(\Omega) \) with \( n \cdot B \in H^1(\Omega) \), and the elliptic regularity guarantees that \( n \cdot B \in H^2_{\text{loc}}(\Omega) \).

More precisely, because \( \nabla (n \cdot B)|_{\Gamma^\pm} \in H^{-1/2}(\Gamma^\pm)^3 \) satisfies (15), Green’s formula now gives that
\[ \int_\Omega \Delta (n \cdot B) \psi dx = \int_\Omega (n \cdot B) \Delta \psi dx \]
for any \( \psi \in C^\infty_0(\Omega) \). This means that for \( f \in L^2(\Omega) \) defined by
\[ f = \begin{cases} \Delta (n \cdot B)|_{\Omega} & \text{in} \ \Omega^+, \\ \Delta (n \cdot B)|_{\Omega} & \text{in} \ \Omega^-, \end{cases} \]
it follows that $\Delta(n \cdot B) = f \in L^2(\Omega)$ in $\Omega$ (as distributions). The proof is complete. \hfill \Box

Now, we study the Stokes system (6):

$$
\begin{align*}
-\Delta v + \nabla p &= f \\
\nabla \cdot v &= 0 \\
\end{align*}
$$

in $\Omega$. We give the following.

Proof of Theorem 2 to (6). Recall that $v \in H^2(\Omega_\pm)^3$, $\nabla p \in L^2(\Omega_\pm)^3$, and $f \in H(\text{rot}, \Omega_\pm)$ satisfy (6), and that $\omega = \nabla \times v$ is in $H^1(\Omega)^3$. Then, we have

$$
\begin{align*}
\nabla \times \omega &= J \\
\nabla \cdot \omega &= 0 \\
\end{align*}
$$

(16) in $\Omega_\pm$ for

$$
J = f - \nabla p \in L^2(\Omega_\pm)^3.
$$

Here, we have $\nabla \times J = \nabla \times f \in L^2(\Omega_\pm)^3$, and hence $J \in H(\text{rot}, \Omega_\pm)$ follows. Then, Theorem 2 for (6) is a direct consequence of Theorem 1. \hfill \Box

Under the assumption of Theorem 2, relation (16) holds with $\omega \in H^1(\Omega)^3$ and $J = f - \nabla p \in H(\text{rot}, \Omega_\pm)$. As is noticed in introduction, this implies $[n \cdot J]^- = 0$ on $\Gamma$ as a compatibility condition. Therefore, $(\partial p/\partial n) = 0$ on $\Gamma$ is obtained if $f \in H(\text{div}, \Omega)$ is imposed furthermore. Namely, relation (8) holds with the well-definedness of $\partial p/\partial n \in H^{-1/2}(\Gamma_\pm)$ under the assumptions of Theorem 2 and $f \in H(\text{div}, \Omega)$. The same fact is true for (7), from the proof of this theorem to that case.

Proof of Theorem 3 to (6). Recall that $v \in H^2(\Omega)^3$, $\nabla p \in H^1(\Omega_\pm)^3$, and $f \in H^1(\Omega_\pm)^3$ satisfy (6). Then, we have

$$
\begin{align*}
\nabla \times \left( \frac{\partial v}{\partial x_j} \right) &= \frac{\partial \omega}{\partial x_j} \\
\nabla \cdot \left( \frac{\partial v}{\partial x_j} \right) &= 0 \\
\end{align*}
$$

(17) in $\Omega_\pm$ for $j = 1, 2, 3$, where $\omega = \nabla \times v$.

Now, we shall show that

$$
\frac{\partial \omega}{\partial x_j} \in H(\text{rot}, \Omega_\pm).
$$

(18)
In fact, if this is the case, then Theorem 1 applied to (17) guarantees that \( n \cdot (\partial v/\partial x_j) \in H^2_{\text{div}}(\Omega) \), and then, the desired conclusion, \( n \cdot v \in H^3_{\text{div}}(\Omega) \) follows.

For this purpose, first, we note that \( \omega = \nabla \times v \in H^1(\Omega)^3 \) holds by \( v \in H^2(\Omega)^3 \), which implies that \( \partial \omega/\partial x_j \in L^2(\Omega)^3 \). On the other hand, from (6) we have

\[
\nabla \times \left( \frac{\partial \omega}{\partial x_j} \right) = \frac{\partial}{\partial x_j} (\nabla \times \omega) = -\frac{\partial}{\partial x_j} \Delta v = \frac{\partial}{\partial x_j} (f - \nabla p) \quad \text{in} \quad \Omega_{\pm}
\]

and hence \( \nabla \times \left( \frac{\partial \omega}{\partial x_j} \right) \in L^2(\Omega_{\pm})^3 \) holds by \( f \in H^1(\Omega_{\pm})^3 \) and \( p \in H^2(\Omega_{\pm}) \). This means (18), and thus the proof is complete.

As is noticed in the above proof, relation (17) holds for \( v \in H^2(\Omega)^3 \) and \( \omega = \nabla \times v \in H^1(\Omega)^3 \). Then, we have \( \partial \omega/\partial x_j \in H(\text{div}, \Omega) \) similarly to \( J \in H(\text{rot}, \Omega) \) for (2), and \([n \cdot \partial \omega/\partial x_j]^+ = 0\) follows on \( \Gamma \) together with the well-definedness of \( n \cdot \partial \omega/\partial x_j \in H^{-1/2}(\Gamma_{\pm})^3 \). Thus, if \( \mathcal{M} \) is \( C^1 \), \( v \in H^2(\Omega)^3 \), and \( \omega = \nabla \times v \), then it holds that \( [\nabla (n \cdot \omega)]^+ = 0 \) on \( \Gamma \) with \( \nabla (n \cdot \omega) \in H^{-1/2}(\Gamma_{\pm})^3 \).

This section is concluded by the study of the Navier-Stokes system (7):

\[
\begin{aligned}
-\Delta v + (v \cdot \nabla) v + \nabla p &= f \\
\nabla \cdot v &= 0
\end{aligned}
\quad \text{in} \quad \Omega_{\pm}.
\]

Proof of Theorem 2 to (7). System (7) is identified with (6) if \( f \) is replaced by \( f - (v \cdot \nabla)v \). Therefore, we have only to show that the condition

\[
F \equiv f - (v \cdot \nabla)v \in H(\text{rot}, \Omega_{\pm})
\]

follows from the assumption for this theorem to prove.

In fact, we have \( v \in H^2(\Omega_{\pm})^3 \subset L^\infty(\Omega_{\pm})^3 \) and hence \( (v \cdot \nabla)v \in L^2(\Omega_{\pm})^3 \) holds. Furthermore, \( \partial v/\partial x_j \in H^1(\Omega_{\pm})^3 \subset L^4(\Omega_{\pm})^3 \) implies that

\[
\frac{\partial}{\partial x_j} ((v \cdot \nabla)v) = \frac{\partial v}{\partial x_j} \cdot \nabla v + v \cdot \nabla \frac{\partial v}{\partial x_j} \in L^2(\Omega_{\pm})^3.
\]

Those relations guarantee that \( (v \cdot \nabla)v \in H^1(\Omega_{\pm})^3 \), and (19) follows from the assumption to \( f \).

Proof of Theorem 3 to (7). Similarly, we have only to show that

\[
\frac{\partial}{\partial x_j} ((v \cdot \nabla)v) \in L^2(\Omega_{\pm})^3
\]

holds for \( j = 1, 2, 3 \). However, this follows actually from the proof of the previous theorem, and the proof is complete.
4. Component-wise Regularity

In this section, we suppose that \( \mathcal{M} \) is flat.

First, we take the Maxwell system (2). As we have seen in Lemma 2.1, in this case \( B \in H^1(\Omega_\pm)^3 \) and \( J \in H(\text{rot}, \Omega_\pm) \) imply \( \nabla(n \cdot B) \in H^{-1/2}(\Gamma_\pm)^3 \). Then, Theorem 1 splits into component-wise versions described in the following theorem. In this connection, we confirm that the traces to \( \Gamma_\pm \) of the first derivatives of any component of \( B \) are also well-defined in this system with \( B \in H^1(\Omega_\pm)^3 \) and \( J \in H(\text{rot}, \Omega_\pm) \). In fact, \( (n \cdot \nabla)B|_{\Gamma_\pm} \in H^{-1/2}(\Gamma_\pm)^3 \) is well-defined by \( -\Delta B = \nabla \times J \in L^2(\Omega_\pm)^3 \) and \( B \in H^1(\Omega_\pm)^3 \) as is indicated in the proof of Lemma 2.1. Next, \( B \in H^1(\Omega_\pm)^3 \) implies \( B|_{\Gamma_\pm} \in H^{1/2}(\Gamma_\pm)^3 \) and hence \( (n \times \nabla)B|_{\Gamma_\pm} \in H^{-1/2}(\Gamma_\pm)^3 \) is also well-defined through the local chart. Those traces are compatible to the ones taken in the proof of Lemma 2.1 and that of the next theorem.

**Theorem 4.** Suppose that the interface \( \mathcal{M} \) is flat, and that \( B \in H^1(\Omega_\pm)^3 \) and \( J \in H(\text{rot}, \Omega_\pm) \) satisfy (2). Then, if \( [n \cdot B]^+ = 0 \) on \( \Gamma \) it holds that \( [(n \times \nabla)(n \cdot B)]^- = 0 \) on \( \Gamma \). Similarly, if \( [n \times B]^+ = 0 \) on \( \Gamma \) we have \( [(n \cdot \nabla)(n \cdot B)]^+ = 0 \) on \( \Gamma \).

**Proof.** In this case \( n \) is a constant vector and we have

\[
B \cdot (n \cdot \nabla) C - n \times B \cdot \nabla \times C = B \cdot \nabla (n \cdot C)
\]

for \( C \in C^\infty_0(\Omega)^3 \). Therefore, equality (9) is reduced to

\[
\langle \langle \nabla (n \cdot B), C \rangle \rangle^+ = \langle \langle B, \nabla (n \cdot C) \rangle \rangle^+ - \langle n \cdot B, \nabla \cdot C \rangle^+
\]

Without loss of generality, we assume \( \mathcal{M} = \{(x_1, x_2, x_3) \mid x_3 = 0\} \) and \( n = (0, 0, 1) \). Then, if \( [n \cdot B]^+ = 0 \) on \( \Gamma \) we have

\[
\langle \langle B, \nabla(n \cdot C) \rangle \rangle^+ = \left\langle \frac{\partial B^1}{\partial x_1}, C^1 \right\rangle^- + \left\langle \frac{\partial B^2}{\partial x_2}, C^2 \right\rangle^- - \left\langle \frac{\partial B^3}{\partial x_1} + \frac{\partial B^2}{\partial x_2}, C^3 \right\rangle^- = \left\langle \frac{\partial B^3}{\partial x_3}, C^3 \right\rangle^-.
\]

Therefore, it follows from (20) that

\[
\left\langle \frac{\partial B^3}{\partial x_1}, C^1 \right\rangle^- + \left\langle \frac{\partial B^3}{\partial x_2}, C^2 \right\rangle^- = 0
\]

for any \( C^1, C^2 \in C^\infty_0(\Omega) \). This implies \( [\partial B^3/\partial x_1]^+ = [\partial B^3/\partial x_2]^+ = 0 \), or equivalently, \( [(n \times \nabla)(n \cdot B)]^- = 0 \) on \( \Gamma \).
If \( [n \times B]^+ = 0 \) on \( \Gamma \), equality (20) is reduced to
\[
\langle \langle \nabla (n \cdot B) , C \rangle \rangle^+ = \langle \langle B , (n \cdot \nabla) C \rangle \rangle^+ - \langle n \cdot B , \nabla \cdot C \rangle^+
\]
\[
= \left\langle B^3, \frac{\partial C^3}{\partial x_3} \right\rangle^+ - \langle B^3, \nabla \cdot C \rangle^+
\]
\[
= - \left\langle B^3, \frac{\partial C^1}{\partial x_1} + \frac{\partial C^2}{\partial x_2} \right\rangle^+_-
\]
\[
= \left\langle \frac{\partial B^3}{\partial x_1}, C^1 \right\rangle^+ + \left\langle \frac{\partial B^3}{\partial x_1}, C^2 \right\rangle^+.
\]
This implies \( [\partial B^3/\partial x_3]^- = 0 \), or equivalently, \( [(n \cdot \nabla)(n \cdot B)]^+ = 0 \) on \( \Gamma \). The proof is complete. \( \square \)

Now, we proceed to the Stokes system (6). We continue to suppose that \( \mathcal{M} \) is flat and take \( \eta = T(0, 0, 1) \) without loss of generality. The following propositions are obtained by applying Theorem 4 to systems (17) with \( j = 3 \) and (16), respectively, and the traces to \( \Gamma_\pm \) in their conclusions are well-defined in \( H^{-1/2}(\Gamma_\pm)^3 \) or \( H^{-1/2}(\Gamma_\pm)^3 \).

**Proposition 4.1.** Assume that \( \mathcal{M} \) is flat and system (6) holds with \( (n \cdot \nabla)v \in H^1(\Omega_\pm)^3 \), \( (n \cdot \nabla)\omega \in L^2(\Omega_\pm)^3 \), \( (n \cdot \nabla)p \in H^1(\Omega_\pm) \), and \( (n \cdot \nabla)f \in L^2(\Omega_\pm)^3 \) for \( \omega = \nabla \times v \). Then, the conditions
\[
[(n \cdot \nabla)(n \cdot v)]^+_\pm = 0 \quad \text{and} \quad [(n \cdot \nabla)(n \times v)]^+_\pm = 0
\]
imply
\[
[(n \times \nabla)(n \cdot \nabla)(n \cdot v)]^+_\pm = 0 \quad \text{and} \quad [(n \cdot \nabla)^2(n \cdot v)]^+_\pm = 0,
\]
respectively, on \( \Gamma \).

**Proof.** In fact, we have (6) and (17) as distributions in \( \Omega_\pm \). In the latter relation with \( j = 3 \), we have \( B = \partial v/\partial x_3 \in H^1(\Omega_\pm)^3 \) and \( J = \partial \omega/\partial x_3 \in H(\text{rot}, \Omega_\pm) \) because
\[
\nabla \times \left( \frac{\partial \omega}{\partial x_3} \right) = \frac{\partial}{\partial x_3}(-\Delta v) = \frac{\partial}{\partial x_3}(f - \nabla p)
\]
holds by the former. Then the assertion is obtained from the previous theorem. \( \square \)

**Proposition 4.2.** Assume that \( \mathcal{M} \) is flat and system (6) holds with \( \omega = \nabla \times v \in H^1(\Omega_\pm)^3 \) and \( f \in H(\text{rot}, \Omega_\pm) \). Then, the conditions
\[
[n \cdot (\nabla \times v)]^- = 0 \quad \text{and} \quad [n \times (\nabla \times v)]^- = 0
\]
imply
\[(n \times \nabla)(n \cdot (\nabla \times u))\] and \[(n \cdot \nabla)(n \cdot (\nabla \times u))\]
respectively, on \(\Gamma\).

Proof. We have (16) with \(J = f - \nabla p\), and the assertion is obtained by Theorem 4 and \(\nabla \times J = \nabla \times f\) in \(\Omega_\pm\).

Those propositions assure the extra regularity on the interface to the tangential components of the solution, and in particular, the following theorems hold.

**Theorem 5.** Let \(\mathcal{M}\) be flat, and assume that \(v \in H^2(\Omega_\pm)^3\), \(p \in H^1(\Omega_\pm)\), and \(f \in L^2(\Omega_\pm)^3\) satisfy the Stokes system (6). Assume, furthermore, that \((n \cdot \nabla)p \in H^1(\Omega_\pm)\), \(f \in H(\text{rot}, \Omega_\pm)\), and \((n \cdot \nabla)f \in L^2(\Omega_\pm)\) hold true. Then, if the conditions
\[
[n \times v]_+ = 0 \quad \text{and} \quad [(n \cdot \nabla)(n \cdot v)]_+ = [n \cdot (\nabla \times v)]_+ = 0
\]
are satisfied on \(\Gamma\), it holds that \(n \times v|_\Gamma \in H^{5/2}_{\text{loc}}(\Gamma)^3\).

Proof. In fact, all requirements of piecewise regularity in Propositions 4.1 and 4.2 are satisfied, and therefore, from the assumption across the interface regularity we have
\[
[(n \times \nabla)(n \cdot \nabla)(n \cdot v)]_+ = [(n \times \nabla)(n \cdot (\nabla \times u))]_+ = 0
\]
on \(\Gamma\). Without loss of generality, we continue to take
\[
\mathcal{M} = \{ (x_1, x_2, x_3) \mid x_3 = 0 \} \quad \text{and} \quad n = T(0, 0, 1).
\]

Let
\[
g = \frac{\partial y_1}{\partial x_1} + \frac{\partial y_2}{\partial x_2} \quad \text{and} \quad h = \frac{\partial y_2}{\partial x_1} - \frac{\partial y_1}{\partial x_2}.
\]
Then, we have \(g = -\partial y_3/\partial x_3 = -(n \cdot \nabla)(n \cdot v)\) and hence
\[
\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2} \in H^1_{\text{loc}}(\Omega)
\]
follows from \([(n \times \nabla)(n \cdot \nabla)(n \cdot v)]_+ = 0\). On the other hand, we have \(h = n \cdot (\nabla \times v)\) and hence
\[
\frac{\partial h}{\partial x_1}, \frac{\partial h}{\partial x_2} \in H^1_{\text{loc}}(\Omega)
\]
follows from \([(n \times \nabla)(n \cdot (\nabla \times v))]_\varepsilon^+ = 0\). Those relations imply
\[
\frac{\partial g}{\partial x_1} - \frac{\partial h}{\partial x_2} = \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) v^1 \in H^1_{\text{loc}}(\Omega)
\]
and
\[
\frac{\partial g}{\partial x_2} + \frac{\partial h}{\partial x_1} = \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) v^2 \in H^1_{\text{loc}}(\Omega).
\]
Therefore,
\[
\left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) (n \times v) \big|_\Gamma \in H^{1/2}_{\text{loc}}(\Gamma)^3
\]
holds.

On the other hand, we have \(n \times v|_{\Gamma} \in H^{1/2}_{\text{loc}}(\Gamma)^3\) from the assumption, and hence \(n \times v|_{\Gamma} \in H^{3/2}_{\text{loc}}(\Gamma)^3\) is obtained by the elliptic regularity. The proof is complete. \(\square\)

**Theorem 6.** Suppose, similarly, that \(M\) is flat, that \(v \in H^2(\Omega_\pm)^3\), \(p \in H^1(\Omega_\pm)\), and \(f \in L^2(\Omega_\pm)^3\) satisfy the Stokes system (6), and that \((n \cdot \nabla)p \in H^1(\Omega_\pm)\), \(f \in H(\text{rot}, \Omega_\pm)\), and \((n \cdot \nabla)f \in L^2(\Omega_\pm)\) hold true. Then, if the conditions
\[
[(n \cdot \nabla)(n \times v)]_\varepsilon^+ = [n \times (\nabla \times v)]_\varepsilon^+ = 0
\]
are satisfied on \(\Gamma\), it holds that \((n \cdot \nabla)(n \times v)|_{\Gamma} \in H^{3/2}_{\text{loc}}(\Gamma)^3\).

**Proof.** Under the same notations as in the proof of the previous theorem, we have \(\partial g/\partial x_3\), \(\partial h/\partial x_3 \in H^1_{\text{loc}}(\Omega)\) in this case by Proposition 4.2 and Theorem 5. This implies
\[
\left. \frac{\partial g}{\partial x_3} \right|_{\Gamma}, \left. \frac{\partial h}{\partial x_3} \right|_{\Gamma} \in H^{1/2}_{\text{loc}}(\Gamma),
\]
and hence
\[
\left( \frac{\partial^2 g}{\partial x_1 \partial x_3} - \frac{\partial^2 h}{\partial x_2 \partial x_3} \right) \big|_{\Gamma} = \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) \left. \frac{\partial v^1}{\partial x_3} \right|_{\Gamma} \in H^{-1/2}_{\text{loc}}(\Gamma)
\]
and
\[
\left( \frac{\partial^2 g}{\partial x_2 \partial x_3} - \frac{\partial^2 h}{\partial x_1 \partial x_3} \right) \big|_{\Gamma} = \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) \left. \frac{\partial v^2}{\partial x_3} \right|_{\Gamma} \in H^{-1/2}_{\text{loc}}(\Gamma)
\]
follows. On the other hand, we have
\[
\left. \frac{\partial v^1}{\partial x_3} \right|_{\Gamma}, \left. \frac{\partial v^2}{\partial x_3} \right|_{\Gamma} \in H^{1/2}_{\text{loc}}(\Gamma)
\]
by \([n \cdot \nabla)(n \times v)]^+ = 0\) on \(\Gamma\), and hence \((n \cdot \nabla)(n \times v)|_\Gamma \in H^{3/2}_{\text{loc}}(\Gamma)^3\) follows from the elliptic regularity. The proof is complete.  \(\square\)

The Navier-Stokes system (7) is treated similarly. Actually, this system is reduced to (6) with \(f\) replaced by \(f - (v \cdot \nabla)v\). Then, the nonlinear term \((v \cdot \nabla)v\) is in \(H^1(\Omega)^3\) in the case of \(v \in H^2(\Omega)^3\). Thus, we get the following theorem.

**Theorem 7.** Theorems 5 and 6 hold similarly even to system (7).

Now, we shall examine the assumptions and conclusions of Theorems 5 and 6. First, assumptions on the piecewise regularity of those theorems are summarized as

\[(21)\]
\[
p \in H^1(\Omega), \quad (n \cdot \nabla)p \in H^1(\Omega), \\
f \in H(\text{rot}, \Omega), \quad (n \cdot \nabla)f \in L^2(\Omega)^3,
\]

and

\[(22)\] \[v \in H^2(\Omega)^3.\]

On the other hand, the assumptions across interface of Theorems 5 and 6 are

\[(23)\]
\[
[v]^+ = [v]^+ = 0, \quad [\frac{\partial v^3}{\partial x_3}]^+ = 0, \quad [\frac{\partial v^2}{\partial x_1} - \frac{\partial v^1}{\partial x_2}]^+ = 0
\]

and

\[
\left[\frac{\partial v^1}{\partial x_3}\right]^+ = \left[\frac{\partial v^3}{\partial x_3}\right]^+ = 0, \quad \left[\frac{\partial v^3}{\partial x_2} - \frac{\partial v^2}{\partial x_3}\right]^+ = \left[\frac{\partial v^1}{\partial x_3} - \frac{\partial v^3}{\partial x_1}\right]^+ = 0,
\]

respectively. The latter means that

\[(24)\]
\[
\left[\frac{\partial v^2}{\partial x_3}\right]^+ = \left[\frac{\partial v^3}{\partial x_3}\right]^+ = 0, \quad \left[\frac{\partial v^3}{\partial x_2}\right]^+ = \left[\frac{\partial v^3}{\partial x_1}\right]^+ = 0.
\]

The second relations of (23) and (24) are summarized as \(n \cdot v \in H^2(\Omega)\). Then, the first relation of (23) means \(v \in H^1(\Omega)^3\), and the rest are equivalent to \(\omega = \nabla \times v \in H^1(\Omega)^3\) and \((n \cdot \nabla)v \in H^1(\Omega)^3\). Namely, we have

\[(25)\]
\[
v \in H^1(\Omega)^3, \quad \omega = \nabla \times v \in H^1(\Omega)^3
\]

and

\[(26)\]
\[
n \cdot v \in H^2(\Omega), \quad (n \cdot \nabla)v \in H^1(\Omega)^3
\]
as the regularity assumption across the interface. However, (25) implies \( v \in H^2_{loc}(\Omega) \), because \( \nabla \cdot v = 0 \) and \( -\Delta v = \nabla \times \omega \in L^2(\Omega)^3 \) holds in \( \Omega \). Thus, we replace those assumptions on the interface regularity, (25) and (26), simply by \( v \in H^2(\Omega)^3 \).

On the other hand, the conclusions of Theorems 5, 6 assure for \( \psi = v^1, v^2 \) that

\[
\frac{\partial^2 \psi}{\partial x_1^2}, \frac{\partial^2 \psi}{\partial x_1 \partial x_2}, \frac{\partial^2 \psi}{\partial x_3^2}, \frac{\partial^2 \psi}{\partial x_3 \partial x_1}, \frac{\partial^2 \psi}{\partial x_3 \partial x_2} \in H^1_{loc}(\Omega).
\]

Thus, we obtain the following.

**Theorem 8.** Let the interface \( \mathcal{M} \) be flat and \( v \in H^2(\Omega)^3, p \in H^1(\Omega_{\pm}) \).

\((n \cdot \nabla)p \in H^1(\Omega_{\pm}), f \in H(\text{rot}, \Omega_{\pm}), \text{ and } (n \cdot \nabla)f \in L^2(\Omega_{\pm})^3 \) hold in the Stokes or the Navier-Stokes system (6), (7), and let \( \psi \) be any tangential component of \( v \). Then, \((\partial \psi/\partial n)^2|_{\Gamma_{\pm}} \in H^{-1/2}(\Gamma_{\pm}) \) is well-defined, and \( \psi \) belongs to \( H^3_{loc}(\Omega) \) if and only if

\[
(27) \quad \left[ \left( \frac{\partial}{\partial n} \right)^2 \psi \right]^+ = 0
\]

holds on \( \Gamma \).

Proof. We shall describe only on the Stokes system (6), because the Navier-Stokes system (7) is treated similarly. In fact, we have (17) with \( j = 3 \),

\[
\nabla \times \left( \frac{\partial v}{\partial x_3} \right) = \frac{\partial \omega}{\partial x_3} \quad \text{and} \quad \nabla \cdot \left( \frac{\partial v}{\partial x_3} \right) = 0 \quad \text{in} \quad \Omega,
\]

with \( \partial v/\partial x_3 \in H^1(\Omega)^3, \partial \omega/\partial x_3 \in L^2(\Omega)^3 \), and

\[
(28) \quad \nabla \times \left( \frac{\partial \omega}{\partial x_3} \right) = -\Delta \left( \frac{\partial v}{\partial x_3} \right) = \frac{\partial}{\partial x_3} (f - \nabla p) \in L^2(\Omega_{\pm})^3
\]

(as distributions) in \( \Omega_{\pm} \). This implies \( \partial \omega/\partial x_3 \in L^2(\text{rot}, \Omega_{\pm}) \) and therefore, \( \nabla(\partial \omega/\partial x_3) \in H^{-1/2}(\Gamma_{\pm})^3 \) is well-defined for \( j = 1, 2, 3 \), as is noticed at the beginning of \S 4.

Then, the assumption (27) implies \(-\Delta(\partial \psi/\partial x_3) \in L^2(\Omega) \) as distributions in \( \Omega \) with \( \partial \psi/\partial x_3 \in H^1(\Omega)^3 \), because \(-\Delta(\partial \psi/\partial x_3) \in L^2(\Omega_{\pm}) \) holds in \( \Omega_{\pm} \) by (28). Therefore, \( \psi \in H^3_{loc}(\Omega) \) follows from the elliptic regularity. The only if part is obvious, and the proof is complete. \( \square \)

Theorem 3 guarantees the interface regularity of the normal component of \( v \). Because all assumptions of Theorems 3 and 8 are satisfied if \( v \in H^2(\Omega)^3, p \in H^2(\Omega_{\pm}), \)

and \( f \in H^1(\Omega_{\pm})^3 \), we get the following.
**Theorem 9.** If the interface $M$ is flat and $v \in H^2(\Omega)^3$, $p \in H^2(\Omega_{\pm})$, $f \in H^1(\Omega_{\pm})^3$ satisfy the Stokes or the Navier-Stokes system (6), (7), then $H^3(\Omega)$ interface of $v$ can occur only in the normal direction of tangential components. Namely, any second derivative of any component of $v$ is well-defined as an element in $H^{-1/2}(\Gamma_{\pm})$, and

\[(29) \quad \left(\frac{\partial}{\partial n} (n \times v)\right)^+ = 0 \quad \text{on} \quad \Gamma\]

implies $v \in H^3_{loc}(\Omega)^3$.

The assumptions of Theorem 9 holds if $v \in H^2(\Omega)^3 \cap H^3(\Omega_{\pm})^3$ and $p \in H^2(\Omega_{\pm})$. Actually, if $v \in H^2(\Omega)^3 \cap H^3(\Omega_{\pm})^3$ satisfies $\nabla \cdot v = 0$ in $\Omega$, then the Stokes system (6) arises for $p = 0$ and $f = -\Delta v$. Then, we can apply Theorem 9 and obtain the following.

**Theorem 10.** If $M$ is flat and $v \in H^2(\Omega)^3 \cap H^3(\Omega_{\pm})^3$ satisfies $\nabla \cdot v = 0$ in $\Omega$, then the condition (29) implies $v \in H^3(\Omega)$.

In this connection, it should be noted that the assumptions of Theorems 8 or 9 without (27) or (29) does not induce even the piecewise regularity indicated as $v \in H^3(\Omega_{\pm})^3$. In fact, they assure only $v \in H^2(\Omega)^3$ and $-\Delta v \in H^1(\Omega_{\pm})^3$, which guarantees only $v \in H^3_{loc}(\Omega_{\pm})^3$. Thus, the $H^3$ regularity up to $\Gamma_{\pm}$ of $v$ is missed without (29).

5. **Concluding Remarks**

Generizations of Theorems 8, 9, and 10 to the case of the non-flat interface are quite interesting. Actually, they will be regarded as a counter part or the natural extension of Theorem 3 and will be studied in the forthcoming paper.

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**References**


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