QUANTUM THEORY OF A MEMBRANE

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Quantum Theory of a Membrane

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Abstract

A relativistic membrane is quantized according to the Batalin-Fradkin-Vilkovisky method, and BRST invariant path integral formulas for the membrane are given both in a covariant gauge and in a time-like gauge. The Regge trajectory (spin-mass relation) for quantum states of the membrane is studied in a semiclassical approximation by using the path integral formula. It is confirmed that spins of massless states are exactly calculable in a semiclassical approximation, and then investigated whether massless states are able to be generated in the membrane model. The analysis shows that no massless state is generated in the spectrum if the space-time dimensions are integer. The result implies that, unlike the string, the membrane model cannot be a unification theory of all interactions.

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§1 Introduction

The string theory has a lot of interesting features when observed from a theoretical point of view. Various particle states including gauge bosons and a graviton are created dynamically [1]. Renormalizability of the string theory implies the possibility of constructing a renormalizable quantum gravitation theory, which no local theory ever has achieved [2].

Consistency conditions of the string theory impose many restrictions on models. The space-time dimension is allowed to be 26 for the bosonic string [3] and 10 for the supersymmetric one [4]. They are called the critical dimensions. The anomalyless condition is also crucial. The possible gauge symmetries without anomalies of superstring theories are SO(32) and $E_8 \times E_8$ [5]. When we study string theories, however, we cannot help but thinking that some miracles play the role in the anomaly cancellations and the finiteness of theory in the critical dimensions [6].

Among many of possible extended objects, the string is merely an example. It is then natural for us to ask whether other extended object models such as a membrane, an elastic ball, etc. might share the miraculous properties with the string and could be a candidate for the unifying theory of all interactions. This is our motivation of studying the quantum theory of a membrane.

Historically although quite a many of papers on relativistic quantum theories of extended objects have been published, only few properties are known. This is because of high nonlinearity and complicated gauge symmetries of the theories.

The objects of this work are first to give a well defined foundation to the quantization of a nonlinear membrane theory and then to study whether a membrane theory can provide massless particles at any critical dimensions.

The latter problem is studied by inspecting the spin-mass relation (the Regge trajectory) of quantum states of extended objects. The relation determines the spin of massless particles if they exist. The massless particles play the main role in phenomenology of unification theory of interactions. The massive particles make less effect to the low energy phenomenology, since they are as heavy as the Planck mass in the unification theory. Thus the relation is crucial to examine whether it is possible to build a unification theory based on these models.

There are various ways to determine the critical dimension in the string theory. One of them is as follows. When we use the quantization procedure in which the Lorentz symmetry is not manifest, the critical dimension is determined by the restoration condition of this symmetry [7]. The necessary condition to restore the Lorentz symmetry is satisfied when the angular momentum quantum number of particle states reproduces the spectrum of an irreducible representation of the Lorentz group. Hence the spin-mass relation provides a necessary condition for the critical dimension.

In extended object models, a simple dimensional analysis provides a form of the spin-mass relation. The form of relation tells us that the spin of massless particle states can be obtained exactly by a semiclassical approximation as will be discussed in our text. This calculation can be performed by using the technique explored by Dashen, Hasslacher and Neveu [8]: The effects of quantum fluctuations are estimated by the path integration over the quantum fluctuations around classical solutions.

In order to get the spin-mass relation of the extended object, we must quantize a nonlinear model which has reparametrization invariance of the world manifold swept out by the object. The gauge symmetry of the reparametrization invariance is manifested as the algebra of the first class constraints associated with the invariance in canonical formalism. If the algebra is not closed, the ordinary covariant approach in Lagrange formalism fails in the quantum theory [9]. The extended object models have non-closure algebra when the freedom of the objects is strictly larger than one [10], e.g., the membrane etc.. Thus we will use the Batalin-Fradkin-Vilkovisky (BFV) method [11] to quantize the extended object model since this method overcomes the difficulty. It is applicable to Lagrangians which are not quadratic in the velocity. It also applies to general gauge fixing terms.

We quantize a simple membrane model according to the BFV method, justify a path integral measure in a semiclassical method, and demonstrate that the membrane cannot generate massless particles if space-time dimensions are restricted within integers [12]. Although the result is negative, it should be stressed that (i) this is a new result, and so far only a concrete property deduced from a quantum theory of membrane, (ii) our method is applicable to any other extended object, and (iii) the conclusion is useful for the study of massive states if the model is applied to other than unification theory. The justification of path integral measure we use is also a new development of this work.

In §2, general aspects of the spin-mass relation of the extended object will be studied. In §3, we briefly summarize the BFV quantization method, and the gauge fixed effective actions of the membrane are obtained both in a covariant gauge and in a time-like gauge. In §4, the intercept of the Regge trajectory of the membrane is calculated, and we obtain the relation between the spin of massless states and the space-time dimension. Conclusions and discussions are presented in §5. The appendix A is provided for the definition of the higher-order structure function of the non-closed algebra. In the appendix B, we will calculate the energy spectra of the membrane in the time-like gauge. The appendix C is devoted to the analytic continuation of generalized zeta functions which appear in the calculation of zero point energy of the membrane.

§2 The spin-mass relation of extended objects

The n-dimensional extended object is described by the following Nambu-Goto type action

$$S_n = -\frac{\kappa_n}{2\pi}\int d^n\sigma d\tau\ [\ \det\ \partial_\alpha X^\mu \partial_\beta X_\mu]^{\frac{1}{2}}\ , \tag{2.1}$$

or by the action corresponding to the Polyakov action [13] of a string

$$\overline{S}_n = -\frac{\kappa_n}{4\pi}\int d^n\sigma d\tau\ \sqrt{-g}\ [\ 1-n + g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu\ ]\ . \tag{2.2}$$

where κ_n is a tension parameter of the extended object. One can find the equivalence between S_n and $\overline{S}_n$ by substituting $g_{\alpha\beta}$ for a solution $g_{\alpha\beta}$ which is obtained from the equation of motion of $\delta\overline{S}_n/\delta g_{\alpha\beta} = 0$. Eq. (2.1) shows that the classical motion of the extended object is realized to minimize the volume of the world manifold. This is a natural extension from that of a particle action which is proportional to the length of the world line.

Three dimensionful parameters appear in this system. One of these is the tension parameter κ_n which has dimension $[ML^{1-n}T^{-1}]$ where M, L and T represent mass, length and time, respectively. And the others are light velocity and the Planck constant $\hbar$.

In such a system, one can find easily the following relation between the angular momentum J and the mass M by the dimensional analysis,

$$J = A(\kappa_n)^{-\frac{1}{n}}(cM)^{\frac{n+1}{n}} + B\hbar\ , \tag{2.3}$$

where A and B are dimensionless numbers. In the classical theory only first term of the r.h.s. of Eq. (2.3) survives. The second term appears in the quantum theory. When M is set to be zero, this second term is identified with the spin of massless particle states. It should be noticed here that the second term B is independent of the tension parameter κ_n if no extra dimensionful parameter is introduced to the system. In this case, the dimensionless number B can be calculated in $\kappa_n \to \infty$ limit without loss of generality, so that the calculation in the semiclassical approximation where $\kappa_n/\hbar \to \infty$ becomes exact. So we can obtain the precise value of the spin as far as massless particle states are concerned. This observation is a crucial point in the present approach.

The relation of Eq. (2.3) is obtained by studying the resonance poles which appear in matrix elements of the propagator with angular momentum j. This calculation for the membrane will be provided in §4.3.

The above method to get the spin-mass relation relies on the assumption that the S matrix elements are well defined in the quantum system defined by Eq. (2.1) or Eq. (2.2). However, the action S_n or $\overline{S}_n$ is in general unrenormalizable as $n+1$ dimensional field theory for $n \geqq 2$. If we regard the action of Eq. (2.1) as the effective one, there should be a physical cut-off parameter Λ and the ultraviolet divergence should be regularized by Λ. Then, the quantum calculation becomes well defined. In §4, it will be found that the calculation of B in Eq. (2.3) is equivalent to the calculation of the Casimir energy of vibration modes. The Casimir energy is shown to be independent of the cut-off Λ in the case of the string in Ref. [14]. So we assume that this is true in the case $n \geqq 2$ also. As will be shown in the Appendix C, even in the membrane theory, there exists a cut-off independent regularization.

§3 Quantization of a membrane

§3.1 Methods of quantization for gauge theories

The action of Eq. (2.1) or Eq. (2.2) possesses the reparametrization invariance of the coordinate on the world manifold. When quantizing such a gauge theory, we usually use the method based on the Hamilton formalism or the "covariant approach" in the Lagrange formalism: The procedure of the former is straightforward if the gauge fixing is restricted to the canonical one. "Canonical" gauge means that the gauge fixing function does not involve any time derivatives of the canonical variables, while the latter method is convenient to take covariant gauge fixing.

Both methods are equivalent as far as the gauge algebra is closed. The Lagrangian method, however, fails if the algebra is not closed [9]. If one uses the covariant approach, auxiliary fields must be introduced to close the algebra [15]. However, there is no systematic way to find these auxiliary fields. Hence the covariant approach is not general.

The covariant quantization of the string has been successfully performed by using the covariant approach in Ref. [12] since the algebra of the reparametrization is closed only when $n=1$. For $n\geqq2$, the algebra is no longer closed.

In this paper to quantize the membrane we adopt a new method explored by Batalin, Fradkin and Vilkovisky [11]. This method realizes the BRST transformation in the framework of the Hamilton formalism and provides the BRST invariant partition function independent of the gauge fixing. The following subsection is devoted to reviewing the BFV method.

§3.2 The Batalin-Fradkin-Vilkovisky method

Let us briefly review the Dirac canonical formalism [17]. We define the Hamiltonian from the Lagrangian $L(q_1,\ldots,q_N;\dot{q}_1,\ldots,\dot{q}_N)$ as

$$H(q_1,\ldots,q_N;p_1,\ldots,p_N) \equiv p_i\dot{q}^i - L\ , \qquad (3.2.1)$$

where

$$p_i \equiv \frac{\partial L}{\partial \dot{q}^i}(q_1,\ldots,q_N;\dot{q}_1,\ldots,\dot{q}_N)\,, \quad \text{for } i=1,\cdots,N \qquad (3.2.2)$$

Eq. (3.2.2) gives the transformation from $(q_1,\cdots,q_N;\dot{q}_1,\cdots,\dot{q}_N)$ to the phase space $(q_1,\ldots,q_N;p_1,\ldots,p_N)$. The inverse transformation does not always exist. If the transformation of Eq. (3.2.2) is degenerate, i.e.,

$$\det\left|\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right| = 0\ , \qquad (3.2.3)$$

one says that $L(q;\dot{q})$ is the singular Lagrangian.

In the case of the singular Lagrangian, the trajectory of the motion is embedded into a subspace of the phase space. If the dimension of the subspace is $2N-k$, there are independent constraints

$$\phi_A(q,p)=0. \qquad (A=1,\ldots,k) \qquad (3.2.4)$$

The constraints are divided into two classes. To define the classes we introduce the following $k\times k$ matrix

$$G_{AB} \equiv \{\phi_A,\phi_B\} \qquad (3.2.5)$$

where $\{\ ,\ \}$ denotes the Poisson bracket. If the rank of G_{AB} is

r, there are r constraints $\{\overline{\phi}_\alpha\}$ $(\alpha=1,\ldots,r)$ of $\{\phi_A\}$ $(A=1,\ldots,k)$ corresponding to r nonzero eigenvalue of G_{AB}. These $\overline{\phi}_\alpha$'s are called the second class constraints. The others $\{\tilde{\phi}_\alpha\}$ of $\{\phi_A\}$ are called the first class constraints.

One can solve the second constraints for any observables $F(q,p)$ by using the Lagrange multiplier method. Let us define the new observable F^* as

$$F^* \equiv F + \mu^\alpha \overline{\phi}_\alpha + \lambda^\alpha \tilde{\phi}_\alpha, \qquad (3.2.6)$$

where $\tilde{\phi}_\alpha$ refers a constraint that is associated with zero eigenvalue of $G_{\alpha\beta}$. From the consistency condition,

$$\{F^*, \phi_A\} = 0, \quad (A=1,\ldots,k) \qquad (3.2.7)$$

on the constraint hypersurfaces we see that

$$\mu^\alpha = \overline{G}^{-1\alpha\beta} \{F, \overline{\phi}_\beta\}, \quad (\alpha,\beta=1,\ldots,r) \qquad (3.2.8)$$

where $\overline{G}_{\alpha\beta}$ is a $r\times r$ nondegenerate submatrix of G_{AB}. However, λ^α's are arbitrary. This arbitrariness of λ^α's comes from the gauge symmetry of the action. From now on we assume that all the constraints are the first class, i.e., $r=0$, since the second class constraints can be always eliminated by determining μ's as (3.2.8).

The Poisson bracket between the first class constraints becomes a linear combination of the first class constraints, i.e.,

$$\{\tilde{\phi}_\alpha, \tilde{\phi}_\beta\} = C_{\alpha\beta}{}^\gamma \tilde{\phi}_\gamma, \quad (\alpha,\beta,\gamma=1,\ldots,k) \qquad (3.2.9)$$

where $C_{\alpha\beta}{}^\gamma$ is the structure function. The infinitesimal transformation generated by $\tilde{\phi}_\alpha$,

$$\delta_\varepsilon F = \varepsilon^\alpha \{F, \tilde{\phi}_\alpha\}, \tag{3.2.10}$$

is strongly open when $C_{\alpha\beta}{}^\gamma$ involves the canonical variables while it is weakly closed. Indeed we see that

$$\begin{aligned} \delta_\varepsilon \delta_\eta F - \delta_\eta \delta_\varepsilon F &= \{F, \varepsilon^\beta \eta^\alpha \{\tilde{\phi}_\alpha, \tilde{\phi}_\beta\}\} \\ &= \varepsilon^\beta \eta^\alpha C_{\alpha\beta}{}^\gamma \{F, \tilde{\phi}_\alpha\} + \varepsilon^\beta \eta^\alpha \{F, C_{\alpha\beta}{}^\gamma\} \tilde{\phi}_\gamma \ . \end{aligned} \tag{3.2.11}$$

where the second term of the r.h.s. breaks the strong closure of the algebra. Such open algebra generally generates a quasigroup [18]. If $C_{\alpha\beta}{}^\gamma$'s are constants, the algebra is strongly closed and a certain Lie group is generated as in the case of the Yang-Mills gauge theories.

For the developments which follow, let us extend the phase space (q^i, p_i). The Hamiltonian H^*, defined by Eq. (3.2.6) where H is substituted for F, includes arbitrary variables λ^α. We regard this Lagrange multiplier λ^α as the canonical variables, and define π_α as the canonical conjugate momentum of λ^α. The arbitrariness of λ^α implies the gauge transformation $\lambda^\alpha \to \lambda^\alpha + \varepsilon^\alpha$. This gauge transformation is generated by the new first class constraints

$$\pi_\alpha = 0, \ (\ \alpha = 1, \ldots, k) \tag{3.2.12}$$

The first class constraints $\{\tilde{\phi}_\alpha\}$ and $\{\pi_\alpha\}$ will collectively be denoted by G_a $(a=1,\ldots,2k)$.

The extension of the phase space is completed by introducing the ghost variables η^a and its canonical conjugate momenta $\mathscr{P}_a$,

(a=1,...2k), where η^a and $\mathscr{P}_a$ are Grassmann numbers. Thus the fully extended phase space is spanned by $\{q^i, p_i; \lambda^\alpha, \pi_\alpha; \eta^a, \mathscr{P}_a\}$.

Let us return to the problem of how to treat the open algebra of the first class constraints G_a. To understand the non-closure property of the algebra, we introduce series of structure functions $\overset{(n)}{U}{}_{a_1 \cdots a_{n+1}}{}^{b_1 \cdots b_n}$ (n=0,1,2,$\cdots$) associated with the original structure functions $C_{ab}{}^c$. While the definition of $\overset{(n)}{U}{}_{a_1 \cdots a_{n+1}}{}^{b_1 \cdots b_n}$ is presented in the Appendix A, here we briefly mention the relation between $\overset{(n)}{U}{}_{\alpha_1 \cdots \alpha_{n+1}}{}^{\beta_1 \cdots \beta_n}$'s and the non-singular transformation $M_a{}^b$ of the constraints G_a onto new constraints F_a such that

$$F_a = M_a{}^b \, G_b, \tag{3.2.13}$$

$$\{F_a, F_b\} = 0, \tag{3.2.14}$$

where we note that the hypersurfaces determined by F_a=0 are isomorphic to those determined by G_a=0, and the algebra generated by F_a's is abelian, therefore, strongly closed.

The existence of such a transformation $M_a{}^b$ is due to the Darboux's theorem to the system of total differential equations,

$$dG_a = 0. \quad (a=1,\cdots 2k) \tag{3.2.15}$$

The theorem tells us that there exists a local coordinate $\{ z^a \}$ where Eq. (3.2.15) is equivalent to

$$dz^a = 0. \quad (a=1,\cdots 2k) \tag{3.2.16}$$

The integration of Eq. (3.2.16) provides the new constraints,

$$F_a \equiv z^a - g^a(z^{a'}) = 0 \ , \qquad (3.2.17)$$
$$(a=1,\cdots 2k;\ a'=2k+1,\cdots 2N)$$

where g^a is an arbitrary function of $z^{a'}$. Eq. (3.2.17) determines the hypersurfaces isomorphic to those determined by $G_a=0$.

The Poisson brackets among F_a's should be proportional to F_a's, since F_a's are first class constraints. On the other hand, F_a's are in z^a $(a=1,\cdots 2k)$, so that the brackets cannot involve z_a. Therefore they must vanish identically.

The transformation $M_a{}^b$ of Eq. (3.2.13) is determined by differential equations which is obtained by substituting $M_a{}^b G_b$ for F_a in Eq. (3.2.14). The existence theorem of F_a tells us that these differential equations are integrable. This integrability is proved by using the Jacobi identity of the Poisson brackets among G_a's and its associated identities, which appear in the Appendix A as the definitions of $\overset{(n)}{U}{}_{a_1\cdots a_{n+1}}{}^{b_1\cdots b_n}$'s (Eq. (A.8), (A.11), (A.12) and (A.14)). Accordingly the existence of $\overset{(n)}{U}{}_{a_1\cdots a_{n+1}}{}^{b_1\cdots b_n}$ is closely related to the existence of transformation of the constraints G_a onto F_a.

If all the structure functions of order strictly greater than s, i.e., $\overset{(n)}{U}{}_{a_1\cdots a_{n+1}}{}^{b_1\cdots b_n} = 0$ for $n>s$, one says that a set of constraints and of the associated structure functions is of rank s. The rank of a set of constraints can be changed since we can substitute constraints $K_a(G_a)$ such as $K_a(0)=0$ for G_a. Indeed we have seen above that G_a's can always be transformed to F_a's such that $\{F_a, F_b\}=0$ and the rank of F_a's is zero.

The reason why there are classes of abelian, (closed) nonabelian and open gauge algebra in field theories is that natural basis of the fields are determined by the propositions of locality, manifest covariance, etc., to the theories. Generally the transformation of Eq. (3.2.14) becomes nonlocal.

We are in a position to introduce the BRST operator based on the Hamilton formalism. When the set of constraints is of rank s, we define the BRST operator Ω such that

$$\Omega \equiv \sum_{n=0}^{s} \eta^{a_1} \cdots \eta^{a_{n+1}} \overset{(n)}{U}{}_{a_{n+1}\cdots a_1}{}^{b_1\cdots b_n} \mathcal{P}_{b_n} \cdots \mathcal{P}_{b_1}. \qquad (3.2.18)$$

where $\overset{(n)}{U}$ is the nth-order structure function defined in the Appendix A. This expression has been decided by the following two conditions. First, Ω should generate the transformation which is the gauge transformation with the odd Grassmann parity parameter η^a instead of the even parity gauge parameter ε^a, in the zeroth order approximation of $\mathcal{P}_a$, i.e.,

$$\Omega \;|_{\mathcal{P}_a=0} = \eta^a G_a. \qquad (3.2.19)$$

Secondly, Ω should be nilpotent, i.e.,

$$\{\Omega,\Omega\} = 0. \qquad (3.2.20)$$

The BRST symmetry was first discovered in the gauge fixed action of the quantum Yang-Mills field theory [19]. This symmetry plays the main role when one deals the theory in the covariant manner. The construction of the covariant string field theory relies strictly upon the BRST symmetry [20]. We will find that the BFV method also relies upon the BRS symmetry generated by Ω.

The BRST invariant Hamiltonian $H_{(\mathrm{inv})}$ is decided in the extended phase space by imposing

$$\text{i)} \quad H_{(\mathrm{inv})} \Big|_{\mathcal{P}_a=0} = H, \tag{3.2.21}$$

$$\text{ii)} \quad \{H_{(\mathrm{inv})}, \Omega\} = 0, \tag{3.2.22}$$

where H is the naive Hamiltonian defined by Eq. (3.2.1).

Now we present the quantization method provided by Batalin, Fradkin and Vilkovisky. They proved the following theorem [11]:

Theorem (3.2.1): If one defines that

$$Z = \int \mathcal{D}q\mathcal{D}p\mathcal{D}\lambda\mathcal{D}\pi\mathcal{D}\eta\mathcal{D}\mathcal{P}\exp(iS_{\mathrm{eff}}), \tag{3.2.23}$$

$$S_{\mathrm{eff}} = \int dt \, [\dot{q}\cdot p + \dot{\lambda}\cdot\pi - \dot{\eta}\cdot\mathcal{P} - H_{\mathrm{eff}}], \tag{3.2.24}$$

$$H_{\mathrm{eff}} = H_{(\mathrm{inv})} - \{\Psi, \Omega\}, \tag{3.2.25}$$

then the partition function Z of Eq. (3.2.23) is invariant under the BRST transformations and independent of the gauge fixing function Ψ. The proof of this theorem is rather easy. First, one sees that the functional measure of Eq. (3.2.23) and the S_{eff} except the term involving Ψ are BRST invariant because the BRST transformation is realized as a canonical transformation generated by Ω in the extended phase space. The nilpotency of Ω keeps the Ψ part of the S_{eff} invariant for the BRST transformation. For the transformation $\Psi \rightarrow \Psi + \delta\Psi$, the change of H_{eff} is absorbed by the transformation of the integration variables such that

$$\delta z_A = i\{\Omega, z_A\}\delta\Psi, \tag{3.2.26}$$

where z_A stands for the canonical variables which appear in the path integral of Eq. (3.2.23).

Finally one can quantize the theories with the gauge freedom by using the partition function Z defined by Eq. (3.2.23). In the following subsection, we will use this BFV method to quantize the membrane. S_{eff} of Eq. (3.2.24) will be obtained for the membrane in a covariant gauge and in a time-like gauge. The W.K.B. approximation will be also discussed in the case of time-like gauge.

§3.3 Quantization of a membrane

We apply the canonical formalism mentioned in the above subsection to a relativistic membrane. The action of a membrane is described by Eq. (2.1) or (2.2) where n=2. The canonical conjugate momentum to X^{μ} is as follows;

$$P_{\mu} = \frac{\kappa_2}{2\pi} \sqrt{-g} \; (g^{00}\dot{X}_{\mu} + g^{0i}\partial_i X_{\mu}) \; , \tag{3.3.1}$$

where

$$g_{\alpha\beta} = \partial_{\alpha}X^{\mu}\partial_{\beta}X_{\mu} \; , \tag{3.3.2}$$

$$g \quad = \det \; g_{\alpha\beta} \; , \tag{3.3.3}$$

and $g^{\alpha\beta}$ is the inverse matrix of $g_{\alpha\beta}$. (Here we use the indices (α,β) for (0,1,2) and (i,j) for (1,2).) One finds that there are following three constraints,

$$\mathcal{H}_0 \equiv \frac{2\pi}{\kappa_2} P_{\mu}P^{\mu} + \frac{2\pi}{\kappa_2} \tilde{g} = 0, \tag{3.3.4}$$

$$\mathcal{H}_i \equiv P_\mu \partial_i X^\mu = 0, \quad (i=1,2) \tag{3.3.5}$$

where $\tilde{g}_{ij}$ corresponds to the space part submatrix of $g_{\alpha\beta}$, and $\tilde{g}$ is the determinant of $\tilde{g}_{ij}$. The classical Hamiltonian H defined by Eq. (3.2.1) is zero and has thus strongly vanishing brackets with the constraints.

$\mathcal{H}_0$ and $\mathcal{H}_i$ $(i=1,2)$ constitute the first class constraints. The algebra of the constraints is given by

$$\begin{aligned}\{\mathcal{H}_0(\sigma),\mathcal{H}_0(\sigma')\} = [&\tilde{g}(\sigma)\tilde{g}^{ij}(\sigma)\mathcal{H}_i(\sigma) \\ &+ \tilde{g}(\sigma')\tilde{g}^{ij}(\sigma')\mathcal{H}_i]\ \partial_j\delta(\sigma,\sigma').\end{aligned} \tag{3.3.6}$$

$$\{\mathcal{H}_0(\sigma),\mathcal{H}_i(\sigma')\} = [\mathcal{H}_0(\sigma) + \mathcal{H}(\sigma')]\ \partial_i\delta(\sigma,\sigma'). \tag{3.3.7}$$

$$\{\mathcal{H}_i(\sigma),\mathcal{H}_j(\sigma')\} = \mathcal{H}_j(\sigma)\partial_i\delta(\sigma,\sigma') + \mathcal{H}_i(\sigma)\partial_j\delta(\sigma,\sigma'), \tag{3.3.8}$$

where ∂_i represents the partial differentiation with respect to σ^i. This algebra reflects the gauge group, namely, the diffeomorphism group of the world manifold swept out by the membrane. The same group appears in quantum gravity theory, where $\mathcal{H}_0$ and $\mathcal{H}_i$'s are called "super Hamiltonian" and "super momentum", respectively.

From Eq. (3.3.6), (3.3.7) and (3.3.8), the structure functions are given by

$$\begin{aligned}C_{0''0'}{}^{i} = \tilde{g}(\sigma)\tilde{g}^{ij}(\sigma)\ [\ &\partial_j\delta(\sigma,\sigma')\delta(\sigma,\sigma'') \\ &- \delta(\sigma,\sigma')\partial_j\delta(\sigma,\sigma'')\ ],\end{aligned} \tag{3.3.9}$$

$$C_{0''i'}{}^{0} = \partial_i\delta(\sigma,\sigma')\delta(\sigma,\sigma'') - \delta(\sigma,\sigma')\partial_i\delta(\sigma,\sigma''), \tag{3.3.10}$$

$$C_{i''j'}{}^{k} = \delta^k_j\partial_i\delta(\sigma,\sigma')\delta(\sigma,\sigma'') - \delta^k_i\delta(\sigma,\sigma')\partial_j\delta(\sigma,\sigma''), \tag{3.3.11}$$

and the others vanish. Here α, α' and α'' denote the indices (0,1 and 2) of vectors fixed to the position of σ, σ' and σ'', respectively. From Eq. (3.3.9), we see that this algebra is not closed. Explicit calculation shows that the second order structure functions are as follows;

$$\overset{(2)}{U}{}_{\underline{0}\ \underline{0}'\underline{0}''}^{\ \ ij'} = -\frac{6\pi}{\kappa_2}\tilde{g}(\sigma)\tilde{g}^{jk}(\sigma)\tilde{g}^{j\ell}(\sigma) \times \delta(\sigma,\sigma')[\delta(\sigma,\underline{\sigma})\partial_k\delta(\sigma,\underline{\sigma}')\partial_\ell\delta(\sigma,\underline{\sigma}'')]_A , \tag{3.3.12}$$

and the other components are equal to zero. All structure functions of order greater than two vanish. Hence the set of constraints is of rank two and the gauge algebra is open in the membrane theory, while the string theory has a set of constraints of rank one and closed gauge algebra that is similar similar to Eq. (3.3.6), (3.3.7) and (3.3.8) where structure functions $C_{\alpha\beta}{}^{\gamma}$'s according to Eq. (3.3.9), (3.3.10) and (3.3.11) are independent of the canonical variables.

Our next task is to determine the fermionic generator Ω of the BRST transformation. In the present case, the extended phase space is spanned by $\{X^\mu, P_\mu; \lambda^\alpha, \pi_\alpha; \eta^a, \mathscr{P}_a\}$ as is defined in §3.2. Here we split the ghosts as follows;

$$\eta^a = (-i\mathscr{P}^\alpha,\ C^\alpha), \tag{3.3.12}$$

$$\mathscr{P}_a = (i\bar{C}_\alpha,\ \bar{\mathscr{P}}_\alpha). \tag{3.3.13}$$

The variables C^α, $\bar{C}_\alpha$ are real and are conjugate to $\bar{\mathscr{P}}_\alpha$, $\mathscr{P}^\alpha$. According to the general expansion of Ω provided by Eq. (3.2.18), we obtain

$$\Omega = \int d^2\sigma \; [\; -i\mathscr{P}^\alpha \pi_\alpha + C^\alpha \mathscr{H}_\alpha + \tilde{g}\tilde{g}^{k\ell} C^0 C^0{}_{,\ell}\bar{\mathscr{P}}_k + C^j C^k{}_{,j}\bar{\mathscr{P}}_k$$

$$+ C^0 C^i{}_{,i}\bar{\mathscr{P}}_0 - C^0{}_{,i} C^i \bar{\mathscr{P}}_0 + \frac{6\pi}{\kappa_2}\tilde{g}\tilde{g}^{ik}\tilde{g}^{j\ell} C^0{}_{,\ell} C^0{}_{,k}\bar{\mathscr{P}}_j\bar{\mathscr{P}}_i \;] . \tag{3.3.14}$$

The quantization of the membrane is performed by using the partition function defined by Eq. (3.2.23), (3.2.24) and (3.2.25). In the following we study details of two cases; one a "covariant gauge" and the other a "time-like gauge".

First we choose the form of Ψ as

$$\Psi = i\bar{C}_\alpha \chi^\alpha + \bar{\mathscr{P}}_\alpha \lambda^\alpha \; . \tag{3.3.15}$$

Let us take χ as

$$\chi^0 = \frac{1}{\beta}(\lambda^0 - 1) \; , \tag{3.3.16}$$

$$\chi^i = \frac{1}{\beta}\lambda^i \; . \tag{3.3.17}$$

Taking Eq. (3.3.14) into account, one then computes $\{\Psi,\Omega\}$ and obtains

$$\{\Psi,\Omega\} = -\pi_\alpha\chi^\alpha - \lambda^\alpha\mathscr{H}_\alpha - A^\alpha\bar{\mathscr{P}}_\alpha - \frac{1}{\beta}\bar{C}_\alpha\mathscr{P}^\alpha$$
$$+ B^{ji}\bar{\mathscr{P}}_j\bar{\mathscr{P}}_i - i\bar{\mathscr{P}}_\alpha\bar{\mathscr{P}}^\alpha - \lambda^0{}_{,i}C^i\bar{C}_0 - \lambda^i C^0{}_{,i}\bar{C}_0 \; , \tag{3.3.18}$$

where

$$A^0 = \partial_i(\lambda^0 C^i) \; , \tag{3.3.19}$$

$$A^k = \partial_i(\lambda^i C^k) - \tilde{g}\tilde{g}^{k\ell}\partial_\ell(\lambda^0 C^0) \; , \tag{3.3.20}$$

$$B^{ji} = \frac{6\pi}{\kappa_2}\tilde{g}\tilde{g}^{ik}\tilde{g}^{j\ell}\,(\lambda^0 C^0{}_{,\ell}C^0{}_{,k} + 2\lambda^0{}_{,[\ell}C^0{}_{,k]}C^0) \; . \tag{3.3.21}$$

If one makes the change of variables

$$\pi_\alpha = \beta\pi_\alpha' \ , \ \overline{C}_\alpha = \beta\overline{C}_\alpha' \ , \tag{3.3.22}$$

whose super-Jacobian is one and takes the limit $\beta\to 0$, then one finds

$$S_{eff} = \int d\tau d^2\sigma \ \dot{X}^\mu P_\mu - \chi^\alpha\pi_\alpha - \mathscr{P}_\alpha\lambda^\alpha + \overline{C}_k\overline{C}_\ell B^{k\ell} - i\overline{C}_\alpha(A^\alpha - \dot{C}^\alpha)$$
$$- i\overline{\mathscr{P}}_0\mathscr{P}^0 + \overline{\mathscr{P}}_k\overline{\mathscr{P}}_\ell B^{k\ell} + \tfrac{1}{4}\mathscr{P}^k\mathscr{P}^\ell B^{-1}{}_{k\ell} \ . \tag{3.3.23}$$

The integral over π_α gives $\delta(\chi^\alpha)$, which fixes the value of λ_α. The Integrals over P_μ, $\mathscr{P}^\alpha$ and $\overline{\mathscr{P}}_\alpha$ are Gaussian and are easily performed. We obtain

$$Z = \int \mathscr{D}X^\mu \mathscr{D}C^\alpha \mathscr{D}\overline{C}_\alpha \ \exp(\ i\overline{S}_{eff} \ ) \ , \tag{3.3.24}$$

$$\overline{S}_{eff} = \int d\tau d^2\sigma \ [\ - \frac{\kappa_2}{4\pi} \ \dot{X}^\mu\dot{X}_\mu + \frac{\kappa_2}{4\pi} \ \tilde{g}$$
$$+ \ i\overline{C}_0(\dot{C}^0 - C^i{}_{,i}) + i\overline{C}_k(\dot{C}^k - \tilde{g}\tilde{g}^{k\ell}C^0{}_{,\ell})$$
$$- \frac{\kappa_2}{4\pi} \ \tilde{g}\tilde{g}^{ik}g^{j\ell}\overline{C}_j\overline{C}_i C^0{}_{,\ell}C^0{}_{,k} \ ] \ . \tag{3.3.25}$$

The first four terms have the same form of the covariant gauge action of a string provided in Ref. [16]. The last term is characteristic to the non-closed constraints algebra. This four ghosts interaction term does not fall within the scope of the Faddeev-Popov method.

For the practical purpose of calculation of Z, the "time-like gauge" will also be investigated. Instead of Eq. (3.3.16) and (3.3.17), let us adopt

$$\chi^0 = -(X^0 - \tau) , \tag{3.3.26}$$

$$\chi^i = -(X^i - x^i) , \qquad (i=1,2) \tag{3.3.27}$$

where x^i is a function of (τ,σ). The computation of S_{eff} defined in Eq. (3.2.24) is similar to the case of covariant gauge. We make the variables change of Eq. (3.3.22) and take the limit $\beta \to 0$. The integrals over P_μ, C^α, $\bar{C}_\alpha$, $\mathcal{P}^\alpha$, $\bar{\mathcal{P}}_\alpha$, λ^i become Gaussian and are performed easily. The integration over π_α gives $\delta(\chi^\alpha)$. Then we obtain

$$Z = \int \mathcal{D}X \mathcal{D}\lambda^0 \prod_\alpha \delta(\chi^\alpha) \, (\lambda^0)^{\frac{D}{2}} \tilde{g}^{-\frac{1}{2}}$$
$$\times \exp\left[- \int d\tau d^2\sigma \left[\frac{\kappa_2}{4\pi} \lambda^0 \tilde{g} + \frac{\kappa_2}{4\pi} (\lambda^0)^{-1} g \tilde{g}^{-1} \right] \right]. \tag{3.3.28}$$

If we evaluate the r.h.s. of Eq. (3.3.28) in the W.K.B. approximation provided κ_2 is large, the integration over λ^0 can be performed by stationary phase and we get

$$Z \underset{WKB}{\propto} \int \mathcal{D}X \, \Pi\delta(\chi^\alpha) \exp\left[i\frac{\kappa_2}{2\pi} \int d\tau d^2\sigma \sqrt{-g} \right] . \tag{3.3.29}$$

As a result we have proven that, in the large stress constant limit ($\kappa_2 \to \infty$), the usual Lagrangian form of the simple path integral measure as shown in (3.3.29) works for the membrane model in the time-like gauge. As we have argued in §2, the intercept of the Regge trajectory B is independent of κ_2, hence the value obtained in the W.K.B. method should give an exact value for B. In the next section we will exploit (3.3.29) to calculate the spin-mass relation for the membrane.

§4 Calculation of spin-mass relation

§4.1 Spin-mass relation of a membrane

We will read the spin-mass relation from resonance poles in the following propagator,

$$\langle j|\frac{1}{E-H}|j\rangle = \mathrm{Tr}[\ \delta(j-J)\frac{1}{E-H}\ ]$$

$$= \int\frac{d(\Delta\theta)}{2\pi}\ \exp(i\Delta\theta j)\ D_E(\Delta\theta)\ , \tag{4.1.1}$$

$$D_E(\Delta\theta) = \int_0^\infty dT\ \exp(iET)\ D(T,\Delta\theta)\ , \tag{4.1.2}$$

and

$$D(T,\Delta\theta) = \mathrm{Tr}\ [\ \exp(i\Delta\theta J - iHT)\ ]. \tag{4.1.3}$$

where H, J and $|j\rangle$ represent the operator Hamiltonian of a membrane, a certain angular momentum operator (defined later) generating the rotation in a certain plane and the eigenstates of J with eigenvalue j, respectively. We evaluate $D(T,\Delta\theta)$ of Eq. (4.1.2) by the path integral [8,21]. In the previous section, we have seen that the path integral quantization of a membrane in the time-like gauge can be performed by using Eq. (3.3.29) provided that the W.K.B. approximation is valid. Therefore $D(T,\Delta\theta)$ may be written as

$$D(T,\Delta\theta) = \int dX(0)\int_{X(0)}^{X(T)=e^{i\Delta\theta J}\cdot X(0)} \mathscr{D}X\quad \Pi\delta(\chi)\exp(iS_2(X))\ , \tag{4.1.4}$$

where $S_2(X)$ is given by Eq. (2.1). We evaluate the r.h.s. of Eq. (4.1.4) by the path integration over the fluctuations around the classical solutions, i.e.,

$$D(T,\Delta\theta) = \int dX_{\mathrm{cl}}(0) \sum_{X_{\mathrm{cl}}} \exp(iS_2[X_{\mathrm{cl}}]) \times \int_{\mathrm{cyclic}} \mathscr{D}\eta \exp(iS^{(Q)}[X_{\mathrm{cl}};\eta]) \tag{4.1.5}$$

where X_{cl} represents a classical solution of the equation of motion,

$$\partial_\alpha(\sqrt{-g}g^{\alpha\beta}\partial_\beta X^\mu) = 0 , \tag{4.1.6}$$

$$g_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu . \tag{4.1.7}$$

and $S^{(Q)}$ denotes the action which governs the quantum fluctuation η around X_{cl}. The explicit form of $S^{(Q)}$ is shown in the Appendix B.

We try to obtain the leading trajectory, i.e., the one on which the spin takes maximum for a given amount of energy. Rigid rotator solutions are expected to provide the leading trajectory because no energy is given to vibrational modes which do not generate angular momentum [22].

In a synchronous gauge where $X^0=\tau$ and $\dot{X}\partial_i X=0$ ($i=1,2$), a rigid rotator solution corresponding to the rotation in X^1-X^2 plane as well as in X^3-X^4 plane†), is given by

$$X^1_{\mathrm{cl}} = f(\sigma_1,\sigma_2)\cos\omega_1\tau ,$$

$$X^2_{\mathrm{cl}} = f(\sigma_1,\sigma_2)\sin\omega_1\tau ,$$

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†) In the four dimensional space-time ($D=4$) rigid rotator solutions are not allowed because of the transversal condition $\dot{X}\partial_i X=0$. Following arguments are valid for $D\geqq 5$.

$$X^3_{\rm cl} = g(\sigma_1,\sigma_2)\ \cos\omega_2\tau\ ,$$

$$X^4_{\rm cl} = g(\sigma_1,\sigma_2)\ \sin\omega_2\tau\ ,$$

$$X^i_{\rm cl} = 0\ , \qquad \text{for}\quad 5 \leqq i \leqq D-1 \tag{4.1.8}$$

where

$$1 - \omega_1^2 f^2 - \omega_2^2 g^2 = 0 \tag{4.1.9}$$

should be satisfied at the boundary. If one evaluates the integral over $X_{cl}(0)$ in Eq. (4.1.2) by stationary phase approximation, the stationary point is determined by

$$\frac{\partial S_2[X_{\rm cl}]}{\partial X_{\rm cl}(0)} + \frac{\partial S_2[X_{\rm cl}]}{\partial X_{\rm cl}(T)}\frac{\partial X_{\rm cl}(T)}{\partial X_{\rm cl}(0)} = 0\ , \tag{4.1.10}$$

From Eq. (4.1.10) and (4.1.4), one finds that

$$P_{\rm cl}(T) = e^{i\Delta\theta J}\ P_{\rm cl}(0), \tag{4.1.11}$$

and

$$X_{\rm cl}(T) = e^{i\Delta\theta J}\ X_{\rm cl}(0). \tag{4.1.12}$$

These equations require periodicity of $X_{\rm cl}(\tau)$, i.e.,

$$\omega_1 = p\omega,\ \omega_2 = q\omega, \tag{4.1.13}$$

where (p,q) are relatively prime integers and

$$\begin{aligned}\omega &= (2\pi\ell + \Delta\theta)/T, \quad (\ell=0,\pm1,\pm2,\cdots)\\ &\equiv \frac{2\pi}{\tau_0}\ .\end{aligned} \tag{4.1.14}$$

In this case, the angular momentum operator J is expressed as

$$J = pJ_{12} + qJ_{34} \ . \qquad (4.1.15)$$

In a period τ_0, the membrane revolves p and q times in X^1-X^2 and X^3-X^4 plane, respectively.

Note that the path integral in the r.h.s. of Eq. (4.1.5) includes a zero mode integration corresponding to the mode of the zero stability angle, namely,

$$\eta \propto \dot{X}_{\text{cl}}(\tau,\sigma) \ . \qquad (4.1.16)$$

The zero mode arises from the time translational invariance of the equation of motion of Eq. (4.1.6). So we perform the integration with respect to this mode by fixing $\eta(0)=\eta(T)=0$. The remaining path integral part of Eq. (4.1.5) can be rewritten as a new form by using the operator Hamiltonian $H^{(\mathrm{Q})}$ associated with $S^{(\mathrm{Q})}$. Eventually $D(T,\Delta\theta)$ is able to be expressed as

$$D(T,\Delta\theta) = \sum_{\ell=-\infty}^{\infty} \exp(iS[X_{\text{cl}}^{(\ell)}]) \times \int_0^{2\pi} d\theta(0) \left| \frac{\partial^2 S_{\text{cl}}}{\partial\theta(0)\partial\theta(T)} \right|^{\frac{1}{2}} \mathrm{Tr}'^{(\mathrm{Q})}[\ \exp(-iTH^{(\mathrm{Q})})\ ] \ , \qquad (4.1.17)$$

where $\mathrm{Tr}'^{(\mathrm{Q})}$ denotes the trace over quantum excitations without the zero mode. In the evaluation of Eq. (4.1.3), we have used the classical period τ_0 as an integration parameter instead of T which is related by Eq. (4.1.14), and then performed the integration by stationary phase method to get

$$D_E(\Delta\theta) = \mathrm{Tr}'^{(\mathrm{Q})}[\ 2\pi\frac{d\alpha(E)}{dE}\cdot\frac{\exp(i\Delta\theta\alpha(E))}{1-\exp(2\pi i\alpha(E))}\ ] \ , \qquad (4.1.18)$$

where

$$\alpha(E) = \frac{2}{3}\sqrt{\frac{2pq}{\kappa_2}}\,(E^{\frac{3}{2}} - \frac{3}{2}H^{(Q)}E^{\frac{1}{2}})\,. \qquad (4.1.19)$$

Hence

$$\langle j|\frac{1}{E-H}|j\rangle = \mathrm{Tr}'^{(Q)}[\,2\pi\frac{d\alpha(E)}{dE}\cdot\frac{\delta(j-\alpha(E))}{1-\exp(2\pi i\alpha(E))}\,]\,. \qquad (4.1.20)$$

The mass spectrum of the membrane is given by inspecting resonance poles of the propagator, i.e.,

$$j = \alpha(E)\,. \qquad (4.1.21)$$

An average angular momentum $(pJ_{12}+qJ_{34})/(p+q) = \alpha/(p+q)$ takes maximum for $p=q$, since α is proportional to $\sqrt{pq}$. The minimum value of $H^{(Q)}$ is equal to the ground state expectation value $\langle 0|H^{(Q)}|0\rangle$ which is nothing but the zero point energy. Hence we obtain

$$\max\left(\frac{J_{p,q}}{p+q}\right) = \frac{2}{3}\sqrt{\frac{2}{\kappa_2}}\,[E^{\frac{3}{2}} - \frac{3}{2}E^{\frac{1}{2}}\langle 0|H^{(Q)}(\omega=\sqrt{\tfrac{\kappa_2}{2E}})|0\rangle]\,. \qquad (4.1.22)$$

Noting that $H^{(Q)}$ is proportional to $\omega\hbar$ in our approximation, one finds that Eq. (4.1.22) has the same form of Eq. (2.3). In the following subsection, we will calculate the the zero point energy $\langle 0|H^{(Q)}|0\rangle$ by using the ζ-function regularization and obtain the spin-mass relation.

§4.2 Spin of massless particle states created from a string

It is instructive to review the calculation of the zero point energy of a string before calculating that of a membrane.

The zero point energy E_{zero} of a bosonic string is given by

$$E_{\text{zero}} = d_{\text{eff}} \cdot r \sum_{n=1}^{\infty} \tfrac{1}{2}\, \omega_n \,, \tag{4.2.1}$$

where d_{eff} represent the number of phonon polarization, i.e.,

$$d_{\text{eff}} = (D-2), \tag{4.2.2}$$

with the integer r which takes

$$r = \begin{cases} 1 & \text{(for open string)} \\ 2 & \text{(for closed string)} \end{cases} , \tag{4.2.3}$$

and

$$\omega_n = \frac{\kappa_1}{2E}\, n \,. \tag{4.2.4}$$

where E denotes the total energy of the string. E_{zero} is divergent in the expression of Eq. (4.2.1). We have to use a certain regularization to get finite physical quantity. Brink and Nielsen [14] regularized the infinite sum in the r.h.s. of Eq. (4.2.1) by using a regulator function, and extracted the regulator independent part as follows;

$$E_{\text{zero}} = \frac{d_{\text{eff}}}{24}\, \frac{\kappa_1}{2E} \,. \tag{4.2.5}$$

The result of Eq. (4.2.5) can also be obtained by using the ζ-function regularization. In this regularization, the infinite sum of Eq. (4.2.1) is defined, by using the Riemann ζ-function, as follows;

$$\sum_{n=1}^{\infty} n = \zeta_R(-1) \ , \qquad (4.2.6)$$

where

$$\zeta_R(\nu) = \sum_{n=1}^{\infty} n^{-\nu} \ . \qquad (4.2.7)$$

In the expression of Eq. (4.2.7), $\zeta_R(\nu)$ is well defined in Re $\nu>1$. The value of the function at Re $\nu\leqq 1$ is defined by analytic continuation from Re $\nu>1$. The value of $\zeta_R(-1)$ is known to be -1/12. Hence this ζ-function regularization provides the same result of Eq. (4.2.5).

By using Eq. (4.2.5), one obtains the following spin-mass relation of the string

$$\begin{aligned} J &= \frac{1}{r} M^2 - 2ME_{\text{zero}} \\ &= \frac{1}{r} M^2 - \frac{D-2}{24} r \ . \end{aligned} \qquad (4.2.8)$$

where D is the dimension of space-time. The second term corresponds to the spin of a massless ($M^2 = 0$) particle.

The particle states generated by a quantum string should obey the representation of the Lorentz group, i.e., J should be an integer, since the action of the string is Lorentz invariant. From this consistency condition to the massless particle states, one gets

$$D = 24n + 2 \ , \qquad (4.2.9)$$

where n is an integer. Eq. (4.2.9) provides a necessary condition to the critical dimension. Indeed the quantized bosonic string theory is consistent if and only if D=26.

In the following subsection, we will apply the ζ-function regularization for the calculation of the zero point energy of the membrane, and consider the critical dimension in the same way as above.

§4.3 Spin of massless particle states created from a membrane

The Hamiltonian $H^{(Q)}$ for the quantum fluctuation of a membrane has the following form,

$$H^{(Q)} = \sum_{i} \sum_{n,\ell} \omega \, \lambda_{n,\ell}^{(i)} \left(N_{n,\ell}^{(i)} + \frac{1}{2} \right) , \qquad (4.3.1)$$

where (n,ℓ), i, $N_{n,\ell}^{(i)}$ and ω represent the quantum numbers of excitation, polarization of the phonon, the number operator of the modes and the angular frequency defined in Eq. (4.1.22), respectively. The polarization consists of transverse modes $(i=5,\cdots,D-1)$ and longitudinal modes $(i=1,3)$ where "transverse" and "longitudinal" refer to the direction of the motion of the classical solution. The spectra $\lambda_{n,\ell}^{(i)}$'s are as follows;

$$\lambda_{n,\ell}^{(2)} = 2\sqrt{(n+|\ell|+3/4)(n+3/4)+7/16} , \qquad (4.3.2)$$

$$\lambda_{n,\ell}^{(4)} = \lambda_{n,\ell}^{(i)} = 2\sqrt{(n+|\ell|+1/4)(n+1/4)-1/16} , \qquad (4.3.3)$$

$$\text{for } i = 5, \cdots, D-1$$

Detail calculation is provided in the Appendix B.

Using Eq. (4.1.22), we obtain the spin of massless particle states on the leading trajectory as follows;

$$j = \lim_{E\to 0} \sqrt{\frac{2}{\kappa_2}}\, E^{\frac{1}{2}}\, E_{\rm zero}(\omega = \sqrt{\frac{\kappa_2}{2E}})$$

$$= -\frac{1}{2}\sum_i \sum_{\ell, n} \lambda^{(i)}_{\ell, n} \ . \qquad (4.3.4)$$

Here, we set $p=q=1$ in Eq. (4.1.22). The infinite sum in the r.h.s is divergent. Therefore we have to regularize it and extract a finite part by using the ζ-function regularization.

The infinite sum of Eq. (4.3.4) will be given by an analytic continuation of the following form;

$$Z(\nu; b, c) = \sum_{n=0}^{\infty} \sum_{\ell=-\infty}^{\infty} [(n+|\ell|+c)(n+c) + b]^{-\frac{\nu}{2}}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} [(n+c)(m+c) + b]^{-\frac{\nu}{2}} \ . \qquad (4.3.5)$$

As will be shown in the Appendix C, we have succeeded in getting an integral representation of the r.h.s. of Eq. (4.3.5). Then we have made an analytic continuation of Z in ν from $\nu>2$ to $\nu=-1$ to get the values of the r.h.s. of Eq. (4.3.4). We have obtained the values of Z corresponding to the sum of Eq. (4.3.4) by numerical calculation, namely,

$$Z(-1; \tfrac{3}{4}, \tfrac{7}{16}) = 0.02882848\cdots \qquad (4.3.6)$$

$$Z(-1; \tfrac{1}{4}, -\tfrac{1}{16}) = -0.1392569\cdots \qquad (4.3.7)$$

Hence the spin of massless particle states on leading trajectory is given by

$$j = - Z(-1;\tfrac{3}{4},\tfrac{7}{16}) - (D-4)\ Z(-1;\tfrac{1}{4},-\tfrac{1}{16}) \tag{4.3.8}$$

It is very difficult to prove in general that Eq. (4.3.8) can be or cannot be an integer for some integer D. For practical purpose, however, massless particles have to have either $j=1$ or 2. To get the spin 2 massless particle (the graviton), we find from Eq. (4.3.8)

$$D = 18.568962\cdots \tag{4.3.9}$$

and to get the spin 1 (the gauge boson),

$$D = 11.387989\cdots \tag{4.3.10}$$

In either case the dimensions of space-time have to be far from an integer. We are confident in these numbers at least down to six places of decimals. Hence the membrane does not provide massless particles in any integer dimensions.

§5 Conclusions and discussion

We have discussed a possibility that a higher dimensionally extended object model can generate the massless particle states which play an important role in building a unification theory. It has been shown that the calculation in semiclassical approximation is enough to obtain a criterion whether massless particle states are generated by an extended object model.

The Nambu-Goto type action of a membrane is not quadratic in the velocity. Accordingly, the usual Lagrangian form of the path integral is not applicable. Furthermore, it is required to treat carefully the gauge freedom since the gauge algebra is open. Hence we have used the BFV method to quantize the membrane, and it has been shown that the usual Lagrangian form can be used as far as we take the "time-like" gauge in the semiclassical approximation.

We have calculated the spin of massless particle states on the leading trajectory of a membrane. And it has been shown that, for the membrane model, massless particles cannot be generated in integer space-time dimensions. It may be impossible to generate massless particles also in other higher dimensionally extended object models since it is expected that the behavior of generalized ζ-function in other models is similar to that of the membrane rather than to that of the string; there is a distinct geometrical difference between the string model and the other extended models. The string model has conformal invariance as well as reparametrization invariance of the world manifold, while the other models possess only the latter invariance.

Additional symmetry imposed on the extended object models might make it possible to create massless particle states. Particularly we suspect the supersymmetry as a candidate for the following reason. The spin of massless states is generally related to the vacuum energy, as has been shown in Eq. (4.3.4). Since the space-time dimension D appears in the vacuum energy as the number of freedom of vibrational modes, the spin of massless states becomes a function of the space-time dimension. This relation has led to the inconsistency for the membrane model as shown in §4.3. However, this mechanism does not work in the case of the vanishing vacuum energy, which is achieved by the supersymmetry. Thus massless states might exist in these models. This problem is, however, unable to be covered in this paper.

We think that our study on a quantum membrane is applicable to other various problems; the bag model of hadron with the quarks confined, the domain wall in grand unification models with membrane-like excitation, the induced gravity theory in three dimensional space-time, etc.. The bag model and the domain wall have a structure of closed membrane (boundaryless), though the effects with no boundary are not studied in this paper. Three dimensional induced gravity theory is also interesting since this theory has the same gauge symmetry as the membrane model has, and it is reported that quantum gravity in three dimensions is soluble [23].

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Appendix A. Definition of structure functions

Let us define the operator δ_2 which maps antisymmetric tensor $F^{\alpha_1 \cdots \alpha_q}$ of order $q \geqq 1$ on the antisymmetric tensor of order $q-1$, such that

$$(\delta_2 F)^{\alpha_1 \cdots \alpha_{q-1}} = F^{\alpha_1 \cdots \alpha_q} \tilde{\phi}_{\alpha_q}. \tag{A.1}$$

where $\tilde{\phi}_{\alpha_q}$ is the first class constraints. One easily finds that this operator δ_2 has the properties such that

$$\delta_2{}^2 F = 0. \tag{A.2}$$

We can prove the following theorem†);

Theorem (A.1):

$$\delta_2 E = 0 \quad \text{-->} \quad \exists\, K;\ E = \delta_2 K.$$

where E and K are an antisymmetric tensor of order q and of order $q+1$, respectively.

Let us define

$$\overset{(1)}{U}_{\alpha\beta}{}^{\gamma} = -\tfrac{1}{2}\, C_{\alpha\beta}{}^{\gamma} \tag{A.3}$$

which is called the "first-order structure function" associated with the $\tilde{\phi}_\alpha$. The first order structure function cannot be arbitrary by virtue of the Jacobi identity,

$$\Big[\ \{\tilde{\phi}_\alpha, \{\tilde{\phi}_\beta, \tilde{\phi}_\gamma\}\}\ \Big]_{\mathrm{A}} = 0, \tag{A.4}$$

where $(\,)_{\mathrm{A}}$ denotes antisymmetrization for the indices α, β and γ.

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†) The proof of this theorem is provided in Ref. [10].

By using Eq. (3.2.9), one can rewrite Eq. (A.4) as

$$\left[\overset{(1)}{D}_{\alpha\beta\gamma}{}^{\delta} \right]_{A} \tilde{\phi}_{\delta} = 0, \tag{A.5}$$

and with use of the operator δ_2 defined by Eq. (A.1),

$$\delta_2 \overset{(1)}{D} = 0, \tag{A.6}$$

where

$$\overset{(1)}{D}_{\alpha\beta\gamma}{}^{\delta} = \{ \overset{(1)}{U}_{\alpha\beta}{}^{\delta}, \tilde{\phi}_{\gamma} \} + \overset{(1)}{U}_{\alpha\beta}{}^{\eta} \overset{(1)}{U}_{\gamma\eta}{}^{\delta} . \tag{A.7}$$

From the theorem (A.1), there exists the antisymmetric tensor $\overset{(2)}{U}_{\alpha\beta\gamma}{}^{\delta\varepsilon}$ such that

$$\left[\overset{(1)}{D}_{\alpha\beta\gamma}{}^{\delta} \right]_{A} = 2 \overset{(2)}{U}_{\alpha\beta\gamma}{}^{\delta\varepsilon} \tilde{\phi}_{\varepsilon}. \tag{A.8}$$

This antisymmetric tensor is called the "second-order structure function".

The third-order structure function is defined as follows. Taking the Poisson bracket of Eq. (A.8) with $\tilde{\phi}_{\alpha}$, one obtains, by using Eq. (3.2.9) and (A.7), that

$$\left[\overset{(2)}{D}_{\alpha\beta\gamma\delta}{}^{\varepsilon\kappa} \tilde{\phi}_{\kappa} \right]_{A} = 0, \tag{A.9}$$

where the function $\overset{(2)}{D}_{\alpha\beta\gamma\delta}{}^{\varepsilon\kappa}$ contains $\overset{(n)}{U}$ ($n \leqq 2$) and its Poisson bracket. Eq. (A.9) says that

$$\delta_2 \overset{(2)}{D} = 0. \tag{A.10}$$

From the theorem (A.1), one obtains the "third-order structure function" $\overset{(3)}{U}_{\alpha\beta\gamma\delta}{}^{\varepsilon\kappa\rho}$ such that

$$\left[\overset{(2)}{D}_{\alpha\beta\gamma\delta}{}^{\varepsilon\kappa} \right]_{A} = 3 \overset{(3)}{U}{}_{\alpha\beta\gamma\delta}{}^{\varepsilon\kappa\rho} \tilde{\phi}_{\rho}. \tag{A.11}$$

The definition of the higher order structure function is similar to that of the third-order structure function. Provided that all the structure functions of order less than n are well known, then the following equation is well defined,

$$\overset{(n-2)}{D}{}_{\alpha_1 \cdots \alpha_n}{}^{\beta_1 \cdots \beta_{n-2}} = (n-1) \overset{(n-1)}{U}{}_{\alpha_1 \cdots \alpha_n}{}^{\beta_1 \cdots \beta_{n-1}} \tilde{\phi}_{\beta_{n-1}}. \tag{A.12}$$

Taking the Poisson bracket of Eq. (A.12) with $\tilde{\phi}_\alpha$, one obtains the identity such that

$$\overset{(n-1)}{D}{}_{\alpha_1 \cdots \alpha_{n+1}}{}^{\beta_1 \cdots \beta_{n-1}} \tilde{\phi}_{\beta_{n-1}} = 0. \tag{A.13}$$

From the theorem (A.1), one gets the nth-order structure function defined as

$$\overset{(n-1)}{D}{}_{\alpha_1 \cdots \alpha_{n+1}}{}^{\beta_1 \cdots \beta_{n-1}} = n \overset{(n)}{U}{}_{\alpha_1 \cdots \alpha_{n+1}}{}^{\beta_1 \cdots \beta_n} \tilde{\phi}_{\beta_n}. \tag{A.14}$$

Appendix B. Energy spectra of a membrane

We define the quantum fluctuation Z^μ around the classical solution (3.1) as follows,

$$X^\mu = X^\mu_{\mathrm{cl}} + R_{12}(\omega_1 \tau) R_{34}(\omega_2 \tau) Z^\mu \tag{B.1}$$

where $R_{ij}(\theta)$ is a rotation operator in i-j plane by θ. To the second order of Z^μ, the action (2.10) for n=2 becomes

$$S[X] = S_{\mathrm{cl}}[X^\mu_{\mathrm{cl}}] + S^{(\mathrm{Q})}[Z^\mu; X^\mu_{\mathrm{cl}}] , \tag{B.2}$$

$$S^{(\mathrm{Q})} = S^{(\mathrm{Q})}_{/\!/} + S^{(\mathrm{Q})}_{\perp} , \tag{B.3}$$

$$S_{/\!/}^{(Q)} = -\frac{\kappa_2}{4\pi}\int d\tau df dg\sqrt{A}\Big[-\frac{2\omega_1\omega_2 fg}{A^2}(\dot{Z}^2 + \omega_1 Z^1)(\dot{Z}^4 + \omega_2 Z^3)$$

$$-\frac{1-\omega_2^2 g^2}{A^2}(\dot{Z}^2 + \omega_1 Z^1)^2 \quad -\frac{1-\omega_2^2 f^2}{A^2}(\dot{Z}^4 + \omega_2 Z^3)^2$$

$$+\frac{2}{A}(\dot{Z}^1 - \omega_1 Z^2)\{\omega_1 f\frac{\partial}{\partial f}Z^2 + \omega_2 g\frac{\partial}{\partial f}Z^4\}$$

$$+\frac{2}{A}(\dot{Z}^3 - \omega_2 Z^4)\{\omega_1 f\frac{\partial}{\partial g}Z^2 + \omega_2 g\frac{\partial}{\partial g}Z^4\} \tag{B.4}$$

$$-\frac{2}{A}(\dot{Z}^2 + \omega_1 Z^1)\{\omega_1 f\frac{\partial}{\partial f}Z^1 + \omega_1 f\frac{\partial}{\partial g}Z^3\}$$

$$-\frac{2}{A}(\dot{Z}^4 + \omega_2 Z^3)\{\omega_2 g\frac{\partial}{\partial f}Z^1 + \omega_2 g\frac{\partial}{\partial g}Z^3\}$$

$$+\frac{1-\omega_2^2 g^2}{A}\{\left(\frac{\partial Z^2}{\partial f}\right)^2 + \left(\frac{\partial Z^2}{\partial g}\right)^2\} \quad + \frac{1-\omega_1^2 g^2}{A}\{\left(\frac{\partial Z^4}{\partial f}\right)^2 + \left(\frac{\partial Z^4}{\partial g}\right)^2\}$$

$$+\frac{2\omega_1\omega_2 fg}{A}\{\left(\frac{\partial Z^2}{\partial f}\right)\left(\frac{\partial Z^4}{\partial f}\right) + \left(\frac{\partial Z^2}{\partial g}\right)\left(\frac{\partial Z^4}{\partial g}\right)\}$$

$$+2\{\left(\frac{\partial Z^1}{\partial f}\right)\left(\frac{\partial Z^3}{\partial g}\right) - \left(\frac{\partial Z^1}{\partial g}\right)\left(\frac{\partial Z^3}{\partial f}\right)\}\quad\Big],$$

$$S_{\perp}^{(Q)} = -\frac{\kappa_2}{4\pi}\int d\tau df dg\sqrt{A}\sum_{i=5}^{D-1}\left\{-\frac{1}{A}\left(\dot{Z}^i\right)^2 + \left(\frac{\partial Z^i}{\partial f}\right)^2 + \left(\frac{\partial Z^i}{\partial g}\right)^2\right\}, \tag{B.5}$$

$$A \quad = 1 - \omega_1^2 f^2 - \omega_2^2 g^2 . \tag{B.6}$$

where the integral variables f and g are what appear in the classical solution (4.1.8), and they are in the range,

$$0 \leqq 1 - \omega_1^2 f^2 \; - \omega_2^2 g^2 \leqq 1 . \tag{B.7}$$

The variables (σ_1, σ_2) on a membrane have been taken to (f, g) themselves as new variables. In the following, we consider only when $\omega_1=\omega_2=\omega$, since this case provides the leading trajectory.

The equations of motion for the transverse components Z^i $(5\leqq i\leqq D-1)$ are

$$\left(\frac{\partial^2}{\partial\tau^2} - D\right) Z^i = 0 . \tag{B.8}$$

where

$$D \equiv (1-\omega^2 f^2-\omega^2 g^2)\left(\frac{\partial^2}{\partial f^2} + \frac{\partial^2}{\partial g^2}\right) - \omega^2\left(f\frac{\partial}{\partial f} + g\frac{\partial}{\partial g}\right) , \tag{B.9}$$

and

$$\sqrt{1-\omega^2 f^2-\omega^2 g^2} \left(f\frac{\partial}{\partial f} + g\frac{\partial}{\partial g}\right) Z^i = 0 \tag{B.10}$$

on the boundary. Using polar variables (θ,r) in (f,g) plane, and using Z^i defined as

$$Z^i = e^{iE^i\tau} e^{i\ell\theta} R^i(r) , \tag{B.11}$$

we obtain the following ordinary differential equation for $R(r)$,

$$(1-\omega^2 r^2)\left(\frac{d^2}{dr^2} + (1-\omega^2 r^2)\frac{d}{rdr} - \frac{\ell^2}{r^2}\right) R^i(r) = -(E^i)^2 R^i(r) . \tag{B.12}$$

To satisfy the boundary condition (B.10), $R^i(r)$ has to be finite at $r=0$ and $r=1/\omega$. Using power series expansion in r and the boundary condition, we obtain the energy spectra,

$$E_\perp = 2\omega\sqrt{(n+|\ell|+1/4)(n+1/4)-1/16} . \tag{B.13}$$

$$n = 0,\ 1,\ 2,\dots$$

$$\ell = 0,\pm1,\pm2,\dots$$

Next, we consider the longitudinal components Z^1, Z^2, Z^3 and Z^4. The variation of $S^{(Q)}$ by Z^1 provides,

$$\frac{2}{A}(g-f)\frac{\partial Z^3}{\partial f} = 0 \ , \tag{B.14}$$

and

$$\frac{1}{\sqrt{A}}\ \{f(\dot{Z}^{2\,2}+ \omega Z^{1\,2}) + g(\dot{Z}^{4\,2}+ \omega Z^{3\,2})\} = 0 \ , \tag{B.15}$$

on the boundary. Similarly the variation by Z^3 provides the same equations but in which Z^1 is replaced by Z^3, Z^2 by Z^4, and f by g. Z^1 and Z^3 have no dynamical mode due to reparametrization invariance of f and g.

On the other hand, the equation of motion for Z^2 and Z^4 are as follows,

$$\left\{ \left[\frac{\partial^2}{\partial \tau^2}+\omega^2-D\right]\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - 2\omega^2\begin{bmatrix} f\frac{\partial}{\partial f} & g\frac{\partial}{\partial f} \\ f\frac{\partial}{\partial g} & g\frac{\partial}{\partial g}\end{bmatrix} \right\} \begin{bmatrix} Z^2 \\ Z^4 \end{bmatrix} = 0 \ , \tag{B.16}$$

with the boundary condition,

$$\frac{1}{\sqrt{A}}\ \Big\{ \omega f^2(\dot{Z}^1-\omega Z^2) + \omega fg(\dot{Z}^3-\omega Z^4) \tag{B.17}$$

$$+ (1-\omega^2 g^2)(f\frac{\partial}{\partial f} + g\frac{\partial}{\partial g})Z^2 + \omega^2 fg(f\frac{\partial}{\partial f} + g\frac{\partial}{\partial g})Z^4 \Big\} = 0 \ ,$$

and the similar equation with Z^1 replaced by Z^3, Z^2 by Z^4, and f by g. Combining (B.15) and (B.17), we find that Z^2 and Z^4 have to be regular on the boundary. Let Z^2 and Z^4 be as follows,

$$\begin{pmatrix} Z^2 \\ Z^4 \end{pmatrix} = e^{iE_{/\!/}\tau} \begin{pmatrix} A_{m,k}(r)\chi^{(\pm)}_{2m,k}(\theta) + B_{m,k}(r)\chi^{(\pm)}_{2m-1,k}(\theta) \\ \pm A_{m,k}(r)\chi^{(\pm)}_{2m,k}(\pi/2-\theta) \pm B_{m,k}(r)\chi^{(\pm)}_{2m-1,k}(\pi/2-\theta) \end{pmatrix} \tag{B.18}$$

$$m = 0,\ 1,\ 2,\ldots$$

$$k = 0,\ 1,\ 2,\ 3$$

$$\chi^{(\pm)}_{n,k}(\theta) = \cos((2n+k)\theta + \tfrac{3}{4}\pi k \pm \tfrac{\pi}{4}) . \tag{B.19}$$

Then we obtain the following coupled ordinary differential equations,

$$[(1-\omega^2 r^2)\left\{\frac{d^2}{dr^2} + \frac{d}{rdr} - \frac{1}{r^2}(4m+k)\right\} + E_{/\!/}^2 - \omega^2(4m+k+1)]A_{m,k}(r)$$

$$+ \omega^2(r\frac{d}{dr} - 4m - k - 2)B_{m,k}(r) = 0 , \tag{B.20}$$

$$[(1-\omega^2 r^2)\left\{\frac{d^2}{dr^2} + \frac{d}{rdr} - \frac{1}{r^2}(4m+k-2)\right\} + E_{/\!/}^2 + \omega^2(4m+k-3)]B_{m,k}(r)$$

$$+ \omega^2(r\frac{d}{dr} + 4m + k)A_{m,k}(r) = 0 , \tag{B.21}$$

Imposing $A_{m,k}(r)$ and $B_{m,k}(r)$ to be regular at $r=0$ and $r=1/\omega$, we look for eigenvalues by well known power expansion method and find the two sets of eigenvalues,

$$E_{/\!/} = 2\omega\sqrt{(n+|\ell|+3/4)(n+3/4) + 7/16} \tag{B.22}$$

and

$$E_{/\!/} = 2\omega\sqrt{(n+|\ell|+1/4)(n+1/4) - 1/16} \tag{B.23}$$

where $n = 0,\ 1,\ 2,\ldots$ and $\ell = 0,\pm1,\pm2,\ldots$.

Appendix C. Analytic continuation of $Z(\nu;b,c)$

We perform the analytic continuation of $Z(\nu;b,c)$ defined by (3.15) with respect to ν. Note first the summand is expressed as

$$\{(n+c)(m+c)+b\}^{-\frac{\nu}{2}} = \frac{1}{\Gamma(\frac{\nu}{2})}\int dt\, t^{\frac{\nu}{2}-1} \quad e^{-t(m+c)}e^{-\frac{bt}{n+c}}(n+c)^{-\frac{\nu}{2}} \quad . \tag{C.1}$$

Performing the sum with respect to n, then we obtain

$$\sum_{n=0}^{\infty} e^{-\frac{bt}{n+c}}(n+c)^{-\frac{\nu}{2}} = \sum_{k=0}^{\infty}\frac{1}{k!}(-bt)^k\sum_{n=0}^{\infty}(n+c)^{-\frac{\nu}{2}-k}$$

$$= (tb)^{-\frac{\nu}{4}+\frac{1}{2}}\int_0^\infty ds\, s^{\frac{\nu}{4}-\frac{1}{2}}\ \frac{e^{-sc}}{1-e^{-s}}\ J_{\frac{\nu}{2}-1}(2\sqrt{tbs}). \tag{C.2}$$

Here we has used the following parameter integral formula,

$$\sum_{n=0}^{\infty}(n+c)^{-\alpha} = \sum_{n=0}^{\infty}\frac{1}{\Gamma(\alpha)}\int_0^\infty ds\, s^{\alpha-1}e^{-s(n+c)}$$

$$= \frac{1}{\Gamma(\alpha)}\int_0^\infty ds\, s^{\alpha-1}\frac{e^{-sc}}{1-e^{-s}} \ . \tag{C.3}$$

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu\sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{2k}}{k!\ \Gamma(k+\nu+1)} \ . \tag{C.4}$$

where $J_\nu(z)$ is the ν-th Bessel function. Using (C.1) and (C.2), we obtain

$$Z(\nu;b,c) = \frac{b^{-\frac{\nu}{4}+\frac{1}{2}}}{\Gamma(\frac{\nu}{2})} \int_0^\infty dt \int_0^\infty ds(ts)^{\frac{\nu}{4}-\frac{1}{2}} \frac{e^{-tc}}{1-e^{-t}} \frac{e^{-sc}}{1-e^{-s}} J_{\frac{\nu}{2}-1}(2\sqrt{tbs}) \tag{C.5}$$

$$= \frac{2b^{-\frac{\nu}{4}+\frac{1}{2}}}{\Gamma(\frac{\nu}{2})} \int_0^\infty dy y^{\nu-3} f_\nu(y) \ , \tag{C.6}$$

where

$$f_\nu(y) = \int_0^1 dz(1-z)^{-\frac{1}{2}} z^{\frac{\nu}{2}-\frac{1}{2}} g_\nu(z,y) \ , \tag{C.7}$$

$$g_\nu(z,y) = \tag{C.8}$$

$$(\sqrt{z}y)^{-\frac{\nu}{2}+1} J_{\frac{\nu}{2}-1}(2\sqrt{bzy^2}) \frac{zy^2 \ e^{2(1-c)y}}{(e^{y(1+\sqrt{1-z})}-1)(e^{y(1-\sqrt{1-z})}-1)}$$

Eq. (C.6) is well defined at $\nu > 2$. Performing the analytic continuation with respect to ν down to around $\nu = -1$, we obtain

$$f_\nu(y) = \frac{(\nu-1)(\nu+1)}{(\nu-2)\nu} \int_0^1 dz z^{\frac{\nu}{2}}(1-z)^{-\frac{1}{2}} g_\nu(z,y)$$

$$- \frac{4(\nu+1)}{(\nu-2)\nu} \int_0^1 dz z^{\frac{\nu}{2}}(1-z)^{\frac{1}{2}} \frac{d}{dz} g_\nu(z,y)$$

$$+ \frac{4}{(\nu-2)\nu} \int_0^1 dz z^{\frac{\nu}{2}}(1-z)^{\frac{3}{2}} \frac{d^2}{dz^2} g_\nu(z,y) \ , \tag{C.9}$$

$$Z(\nu;b,c) = \frac{1}{(\nu-2)(\nu-1)\nu(\nu+1)} \int dy y^{\nu+1} \frac{d^4}{dy^4} f_\nu(y) \quad . \tag{C.10}$$

Let $\nu=-1+\varepsilon$, then

$$Z(-1+\varepsilon;b,c) = \frac{b^{3/4}}{6\sqrt{\pi}} \left[\frac{A_0}{\varepsilon} + A_1 + O(\varepsilon) \right] , \tag{C.11}$$

$$A_0 = -\left[\frac{d^3}{dy^3} f_{-1}(y) \right]_{y=0} , \tag{C.12}$$

$$A_1 = \int_0^\infty dy \log(y) \frac{d^4}{dy^4} f_{-1}(y) - \left[\frac{\partial^3}{\partial y^3} \frac{\partial}{\partial \nu} f_\nu(y) \right]_{\substack{y=0\\ \nu=-1}} . \tag{C.13}$$

Using (C.8) and (C.9), we have found that A_0 vanishes and therefore $Z(-1;b,c)$ is finite. In evaluation of A_1 in eq. (C.13), the second term is analytically calculable, but the first term is not. So we have computed the first term numerically. The following is the result. When $c=5/4$ and $b=-1/16$,

$$A_1 = -0.3647609 + \frac{7}{2}\sqrt{\pi} = 5.838828 \quad . \tag{C.14}$$

When c=3/4 and b=7/16,

$$A_1 = -3.754627 + \frac{3}{2}\, 7^{\frac{1}{4}}\sqrt{\pi} = 0.5699207 \quad . \tag{C.15}$$

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