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## ON THE SMALL HULLS OF A COMMUTATIVE RING

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This short note is a continuous one of [1]. We always assume that every ring  $R$  is a commutative ring with identity 1 and every ring homomorphism  $f$  is unitary, i.e.  $f(1)$  is the identity. Let  $R$  and  $R'$  be rings and  $f: R \rightarrow R'$  a ring homomorphism. Then every  $R'$ -module may be regarded as an  $R$ -module through  $f$ . If  $f(R)$  is a small submodule of  $R'$  as an  $R$ -module, we say  $f$  being small or  $R$  being small in  $R'$  [1]. In this note, we shall find the smallest small homomorphism of  $R$ .

We shall make use of the same notations as in [1]. We define the smallest small homomorphism as follows:

DEFINITION. Let  $R$  and  $\tilde{R}$  be rings and  $f: R \rightarrow \tilde{R}$  a ring homomorphism. We say  $\tilde{R}$  being a small hull of  $R$  if the following conditions are satisfied.

- 1)  $f$  is small.
- 2) For any small (ring) homomorphism  $g: R \rightarrow R'$ , there exists a unique (ring) homomorphism  $h: \tilde{R} \rightarrow R'$  such that  $g = hf$ .

We say  $\tilde{R}$  being a local small hull of  $R$  if the following conditions are satisfied.

$\tilde{R}$  is a local ring and the properties 1), 2) above are satisfied whenever  $R'$  is a local ring ( $R$  is not necessarily local).

It is clear that (local) small hulls of  $R$  are unique up to isomorphism if they exist.

**Proposition 1.** We assume  $R$  has a (local) small hull  $f: R \rightarrow \tilde{R}$ . Then for any (local) ring  $R'$  and a ring homomorphism  $g: R \rightarrow R'$   $g$  is small if and only if there exists a homomorphism  $h: \tilde{R} \rightarrow R'$  such that  $g = hf$ .

Proof. It is clear from the definition and [1], Remark 3.

**Proposition 2.** Let  $R$  be a ring. Then the trivial ring 0 is a small hull of  $R$  if and only if  $\text{K. dim } R = 0$ .

Proof. We assume  $\text{K. dim } R = 0$  and  $f: R \rightarrow R'$  a ring homomorphism. Let  $M'$  be a maximal ideal in  $R'$ . If  $f \neq 0$ ,  $M' \cap R$  is maximal. Hence  $f$  is not small

by [1], Theorem 1. The converse is clear from Proposition 1 and [1], Proposition 9.

From the definition of the small hull, we have

**Proposition 3.** *If  $R$  has a small hull, every small ring extension of  $R$  contains the unique minimal small extension of  $R$ .*

From now on, we assume that a (local) small hull means a non-trivial one unless otherwise stated.

**Theorem 4.** *Let  $R$  be a commutative ring. Then  $R$  has a local small hull if and only if there exists a unique maximal one  $\mathfrak{P}$  among non-maximal prime ideals in  $R$ . In this case,  $\mu_{\mathfrak{P}}: R \rightarrow R_{\mathfrak{P}}$  is a local small hull and a small hull, too.*

Proof. "If" part.  $\mu_{\mathfrak{P}}: R \rightarrow R_{\mathfrak{P}}$  is small by [1], Theorem 1. Let  $R'$  be a local ring with maximal ideal  $M'$  and  $g: R \rightarrow R'$  a small homomorphism. Since  $p = R \cap M'$  is not maximal by [1], Theorem 1,  $p \subseteq \mathfrak{P}$ . Therefore, there exists a unique homomorphism  $h: R_{\mathfrak{P}} \rightarrow R'$  such that  $h\mu_{\mathfrak{P}} = g$ , since  $R' = R'_{M'}$ . Hence,  $R_{\mathfrak{P}}$  is a local small hull of  $R$ .

"Only if" part. We assume  $f: R \rightarrow \tilde{R}$  is a local small hull. Let  $p$  be a non-maximal prime ideal in  $R$ . Then  $\mu_p: R \rightarrow R_p$  is small. Hence, there exists a unique homomorphism  $h: \tilde{R} \rightarrow R_p$  such that  $\mu_p = hf$ . Put  $R_0 = h(\tilde{R})$  and  $M_0 = h(\tilde{M})$ , where  $\tilde{M}$  is the maximal ideal of  $\tilde{R}$ . Since  $f$  is small,  $\mathfrak{P} = f^{-1}(\tilde{M})$  is not maximal by [1], Theorem 1. Furthermore,  $p = \mu_p^{-1}(pR_p) = (hf)^{-1}(pR_p) = (hf)^{-1}(pR_p \cap R_0) \subseteq (hf)^{-1}(M_0) = f^{-1}(\tilde{M}) = \mathfrak{P}$ . We shall show that  $R_{\mathfrak{P}}$  is also a small hull of  $R$ . Let  $g: R \rightarrow R'$  be a small homomorphism and  $s \in R - \mathfrak{P}$ . If  $g(s)R' \neq R'$ , we can take a maximal ideal  $M'$  in  $R'$  containing  $g(s)R'$ . Then  $s \in R \cap M'$ . Since  $g$  is small,  $R \cap M' \subseteq \mathfrak{P}$ , which is a contradiction. Hence,  $g(s)R' = R'$  i.e.  $g(s)$  is unit in  $R'$ . Therefore, we obtain a unique homomorphism  $h: R_{\mathfrak{P}} \rightarrow R'$  such that  $h\mu_{\mathfrak{P}} = g$ .

The following theorem shows that the existence of a small hull does not guarantee the existence of a local small hull.

**Theorem 5.** *Let  $R$  be a commutative noetherian ring. Then  $R$  has a small hull if and only if  $\text{K. dim } R = 1$ . In this case,  $R_S$  is a small hull of  $R$ , where  $S = R - \bigcup_{i=1}^r p_i$  and  $\{p_i\}_{i=1}^r$  is the set of minimal but not maximal prime ideals.*

Proof. We assume that  $R$  had a small hull and  $\text{K. dim } R \geq 2$ . Let  $M$  be a maximal ideal and  $M \supseteq P' \supseteq P$  a chain of prime ideals in  $R$ . Then  $R_1 = R/P$  is a noetherian domain of  $\text{K. dim } \geq 2$ . Let  $R_1^*$  be the integral closure of  $R_1$ . Then we can take a maximal ideal  $M_1^*$  in  $R_1^*$  such that  $M_1^* \cap R_1 = M/P$  and height  $M_1^* \geq 2$  ([2], p. 30). Since  $R_1^*_{M_1^*}$  is a Krull ring by [2], Theorem 33.10,

$R_1^*_{M_1^*} = \bigcap_{p^*} R_1^*_{p^*}$ , where the  $p^*$  ranges over all prime ideals in  $M_1^*$  of height 1 and so the  $p^*$  is not maximal. Hence,  $R_1^*$  is small in  $R_1^*_{p^*}$ . Therefore,  $R$  is small in  $R_1^*_{p^*}$  by [2], p. 30 and [1], Proposition 4. Accordingly,  $R$  is small in  $R_1^*_{M_1^*}$  by Proposition 3. On the other hand,  $M_1^* R_1^*_{M_1^*} \cap R = M_1^* \cap R_1 \cap R = M/P \cap R = M$ . Therefore,  $R$  is not small in  $R_1^*_{M_1^*}$  by [1], Theorem 1. Thus, we have proved  $\text{K. dim } R \leq 1$ . Therefore,  $\text{K. dim } R = 1$  by Proposition 2. Conversely, we assume  $\text{K. dim } R = 1$ . Let  $g: R \rightarrow R'$  be a small homomorphism and  $M'$  a maximal ideal in  $R'$ . Since  $\text{K. dim } R = 1$  and  $M' \cap R$  is not maximal by [1], Theorem 1,  $R - (M' \cap R) \subset S$ . Therefore, there exists a unique homomorphism  $h: R_S \rightarrow R'$  such that  $h\mu_S = g$  as in the proof of Theorem 4. Accordingly,  $R_S$  is a small hull of  $R$ , since  $R$  is small in  $R_S$  by [1], Theorem 1.

REMARK. If  $R$  is a valuation ring of  $\text{K. dim } R \geq 2$ , then  $R_{\mathfrak{P}}$  is a small hull of  $R$  for a prime ideal  $\mathfrak{P}$  of depth 1. Hence, Theorem 5 is not true if  $R$  is not noetherian.

**Proposition 6.** *Let  $R$  be a domain. Then the quotient field  $Q$  of  $R$  is a (local) small hull of  $R$  if and only if  $\text{K. dim } R = 1$ .*

Proof. It is clear from Theorem 4 and [1], Proposition 11.

**Proposition 7.** *Let  $R$  be a local ring with principal maximal ideal  $(m)$ . Then if  $m$  is nilpotent, the trivial ring is a small hull of  $R$  and if  $m$  is not nilpotent,  $R_{(m^t)}$  is a small hull of  $R$ .*

Proof. The first part is clear from Proposition 2. If  $g: R \rightarrow R'$  is small,  $g(m)R' = R'$  by [1], Theorem 1. Hence,  $R_{(m^t)}$  is a small hull of  $R$ .

**Proposition 8.** *Let  $R$  be a domain. We assume that every non-zero prime ideal contains a prime element (e.g. U.F.D., see [2], p. 42). Then  $R$  has a small hull if and only if  $\text{K. dim } R = 1$ . In this case,  $R$  is noetherian.*

Proof. If  $\text{K. dim } R = 1$ ,  $Q$  is a small hull of  $R$  by Proposition 6. Conversely, we assume that  $R$  has a small hull. Let  $M$  be a maximal ideal and  $R_1 = R_M$ . Then  $R_M$  satisfies the same condition as in the proposition. Let  $m$  be a prime element in  $M_1 = MR_1$ . Let  $n$  be any non-zero element in  $M_1$ . We assume  $nR_1 \not\subseteq mR_1$ . Let  $x$  be in  $R_{1(m^t)} \cap R_{1(n^s)}$ . Then  $x = a/m^t = b/n^s$ ;  $a, b \in R_1$ . Since  $mR_1$  is prime,  $a \in mR_1$ . Therefore,  $R_1 = R_{1(m^t)} \cap R_{1(n^s)}$ . Since  $R_1$  is small in  $R_{1(m^t)}$  and  $R_{1(n^s)}$ ,  $R$  is small in  $R_1$  by Proposition 3 and [1], Proposition 4. On the other hand,  $R$  is not small in  $R_1$  by [1], Theorem 1. Therefore,  $M_1 = (m)$ . Since every non-zero prime ideal contains a prime element,  $M_1$  is a unique non-zero prime ideal. Therefore,  $\text{K. dim } R = 1$  and  $R$  is noetherian by [2], Theorem 3.4.

A Krull ring  $R$  satisfies

(\*) any principal ideal  $aR$  ( $\neq 0$ ) of  $R$  is the intersection of a finite number of primary ideals of height 1.

**Proposition 9.** *Let  $R$  be a domain satisfying (\*). Then  $R$  has a small hull if and only if  $K.\dim R=1$ .*

*Proof.* If  $K.\dim R=1$ ,  $\mathcal{Q}$  is a small hull of  $R$  from Proposition 6. We assume  $R$  has a small hull and  $K.\dim R \geq 2$ . If we use the same argument as in the proof of Theorem 5, we may assume  $R$  is local. Then  $R = \bigcap_p R_p$  by [2], p. 115, where the  $p$  ranges over all prime ideals of height 1. Since  $K.\dim R \geq 2$ ,  $R$  is small in  $R_p$ , which is a contradiction from Proposition 3.

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