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## ON FROBENIUS SYSTEMS

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### Abstract

By a Frobenius system on a finite group  $G$ , we mean the data, for each maximal solvable subgroup  $M$  of  $G$ , of a normal subgroup  $\mathcal{F}(M)$  of  $M$ , satisfying some of the properties of a Frobenius kernel, and subject to certain additional conditions. We prove that a finite group with a Frobenius system is either solvable (in which case we get a complete description), or isomorphic to  $SL_2(K)$  (for  $K$  a finite field of characteristic 2) or to a Suzuki group. The respective possibilities for the mapping  $\mathcal{F}$  are then determined. This extends a previous result of ours (Nagoya Math. J. **165** (2002), 117–121) by removing the condition that each  $M/\mathcal{F}(M)$  be abelian. Curiously enough, the Feit-Thompson Theorem is used in the proof.

### 0. Introduction

In this paper, we shall classify all Frobenius systems on finite groups, modulo a very weak (and natural) nondegeneracy hypothesis (condition  $(FS4')$  below). Our result contains as a particular case Theorem 0.1 of [4]. The possibility of such a generalization was suggested to the author by Arad and Herfort's paper ([1]), in which are studied finite groups possessing at least one  $CC$ -subgroup, i.e. a nontrivial proper subgroup that contains the centralizer of each of its nonidentity elements.

In fact, we only need a particular case of Arad and Herfort's result (Theorem A, (ii), p.2089 in [1]), that is due to Suzuki ([6], Theorem 1; see also [2]); in particular we do *not* need the full Classification of the Finite Simple Groups, but only Suzuki's classification of  $ZT$ -groups (see [6]). We also use the Brauer-Suzuki Theorem on finite groups with a generalized quaternion Sylow subgroup and the Feit-Thompson Theorem in order to deal with a particularly troublesome configuration.

We shall keep, unless the contrary be mentioned, the notations used in [3] and [4].

### 1. Definitions and statement of the result

By a *Frobenius system* (*kernel system* in [3]) on the finite group  $G$ , we shall mean, as in [4], p.117, a mapping  $\mathcal{F}$  from the set  $\mathcal{MS}(G)$  of maximal solvable subgroups of  $G$  to the power set  $\mathcal{P}(G)$  of  $G$ , such that the following axioms are satisfied, for all  $M \in \mathcal{MS}(G)$ :

(FS1)  $\mathcal{F}(M)$  is a normal subgroup of  $M$ ;

(FS2)  $\forall a \in M \setminus \mathcal{F}(M), C_{\mathcal{F}(M)}(a) = \{1\}$ ;

(FS3)  $\forall g \in G \setminus M, \mathcal{F}(M) \cap \mathcal{F}(M)^g = \{1\}$ .

(cf. axioms (1), (2), (3) of Definition 1.1 in [3], p.72).

There is a natural notion of isomorphism for groups with a Frobenius system (see [4], p.118). In addition, two families of Lie-type groups over fields of characteristic two do possess a canonical Frobenius system: the Suzuki groups  $Sz(2^{2n+1})$  ( $n \geq 1$ ), and the special linear groups  $SL_2(\mathbb{F}_{2^n})$  ( $n \geq 2$ ); the respective Frobenius systems (defined in [4], p.118) will be denoted by  $\mathcal{F}_{(n)}$  (resp.  $\mathcal{F}_{\mathbb{F}_{2^n}}$ ).

We can now state our main result:

**Theorem 1.1.** *Let  $\mathcal{F}$  be a Frobenius system on the finite group  $G$ , such that the following condition hold:*

(FS4') *For each  $M \in \mathcal{MS}(G)$ ,  $\mathcal{F}(M) \neq \{1\}$  or  $M$  is abelian.*

*Then one (and, of course, only one) of the following holds:*

- (1)  $G$  is abelian and  $\mathcal{F}(G) = \{1\}$ ;
- (2)  $G$  is a nonidentity solvable group and  $\mathcal{F}(G) = G$ ;
- (3)  $G$  is a solvable Frobenius group, and  $\mathcal{F}(G)$  is the Frobenius kernel of  $G$ ;
- (4)  $(G, \mathcal{F})$  is isomorphic to  $(SL_2(\mathbb{F}_{2^n}), \mathcal{F}_{\mathbb{F}_{2^n}})$ , for some  $n \geq 2$ ;
- (5)  $(G, \mathcal{F})$  is isomorphic to  $(Sz(2^{2n+1}), \mathcal{F}_{(n)})$ , for some  $n \geq 1$ .

*Conversely, each of the possibilities (1), ..., (5) gives rise to a Frobenius system satisfying (FS4').*

Clearly, axiom (FS4') follows from axiom (FS4) in [4] (and *a fortiori* from axiom (5) in [3], p.73); therefore, Theorem 0.1 in [4] follows at once from Theorem 1.1 (we only have to consider case (3), when (FS4) yields that the Frobenius complement in  $G$  is abelian, hence cyclic by the same argument as in [4], p.120).

## 2. Proof of Theorem 1.1

Some parts of our proof will be very close to the corresponding ones in [4].

Let  $(G, \mathcal{F})$  satisfy (FS1), (FS2), (FS3) and (FS4'). If, for some  $M \in \mathcal{MS}(G)$ , one has  $\mathcal{F}(M) = \{1\}$ , then (by (FS4'))  $M$  is abelian, whence (by [4], Lemma 1.1, p.118)  $G = M$ , and we obtain (1). Therefore we may assume that:

$$(2.1) \quad \text{For each } M \in \mathcal{MS}(G), \quad \mathcal{F}(M) \neq \{1\}.$$

Let us now assume that, for some  $M \in \mathcal{MS}(G)$ ,  $\mathcal{F}(M) = M$ . If  $M = G$ , then  $G$  is solvable,  $\mathcal{F}(G) = G$  and we are in case (2); if  $M \neq G$ , it follows from (FS3) that  $M$  is a solvable Frobenius complement in  $G$ , and we reach a contradiction as in [4], p.120, 1.6. Therefore, we may assume that:

$$(2.2) \quad \forall M \in \mathcal{MS}(G), \quad \{1\} \neq \mathcal{F}(M) \neq M.$$

As in [4], p.120, it now follows from [3], Proposition 1.5, p.73, that:

(2.3) For each  $M \in \mathcal{MS}(G)$ ,  $\mathcal{F}(M)$  is nilpotent.

(As in the argument preceding (2.2), we do not need here Thompson's Theorem on the nilpotency of Frobenius kernels, but only the far more elementary fact that *solvable* Frobenius kernels are nilpotent). If  $G$  is solvable, then  $\mathcal{MS}(G) = \{G\}$ , and Lemma 1.3 from [3] yields that  $\mathcal{F}(G)$  is a Frobenius kernel in  $G$ , and thence (3) holds. Therefore we may assume that:

(2.4)  $G$  is not solvable.

Let now  $S \in \text{Syl}_2(G)$ , and let  $M \in \mathcal{MS}(G)$  contain  $S$ . If  $\mathcal{F}(M)$  has even order, then, as by [3], Lemma 1.3 and (2.2)  $\mathcal{F}(M)$  is a CC-subgroup of  $G$ , Theorem 1 of [6] yields that either:

- (1)  $G$  is a Frobenius group, and  $\mathcal{F}(M)$  is its kernel, or
- (2)  $G$  is a Frobenius group, and  $\mathcal{F}(M)$  is its complement, or
- (3)  $G$  is a ZT-group.

But (2) would imply that  $\mathcal{F}(M) = N_G(\mathcal{F}(M)) = M$  (cf. [3], Lemma 2.5 (i), p.75) contradicting (2.2), and (1) would imply that  $G = N_G(\mathcal{F}(M)) = M$  would be solvable, contradicting (2.4). Therefore one has (3), whence, by [6],  $G \simeq SL_2(\mathbb{F}_{2^n})$  ( $n \geq 2$ ) or  $G \simeq Sz(2^{2n+1})$  ( $n \geq 1$ ), and one may conclude as in [4], p.120, that case (4) or case (5) of Theorem 1.1 holds.

Thus it may be supposed that:

(2.5)  $\mathcal{F}(M)$  has odd order.

Let us assume that  $G$  possesses a nontrivial normal solvable subgroup  $N_0$ , and let  $N$  denote a minimal normal subgroup of  $G$  contained in  $N_0$ ; then  $N$  is an elementary abelian  $p$ -group for some prime  $p$ . Let  $P \in \text{Syl}_p(G)$ , and let  $M_1 \in \mathcal{MS}(G)$ ,  $M_1 \supseteq P$ ; then  $N \subseteq P \subseteq M_1$ . Therefore

$$\begin{aligned} [N, \mathcal{F}(M_1)] &\subseteq [N, G] \cap [M_1, \mathcal{F}(M_1)] \\ &\subseteq N \cap \mathcal{F}(M_1). \end{aligned}$$

If  $N \cap \mathcal{F}(M_1) = \{1\}$ , then  $N$  centralizes  $\mathcal{F}(M_1)$ ; for  $x \in \mathcal{F}(M_1)^\sharp$ , one has  $N \subseteq C_G(x) \subseteq \mathcal{F}(M_1)$  (by [3], Lemma 1.3) whence  $N = N \cap \mathcal{F}(M_1) = \{1\}$ , a contradiction. Thus  $N \cap \mathcal{F}(M_1) \neq \{1\}$ ; let now  $x \in (N \cap \mathcal{F}(M_1))^\sharp$ . One has  $N \subseteq C_G(x)$  (as  $N$  is abelian), and  $C_G(x) \subseteq \mathcal{F}(M_1)$  (as above), whence  $N \subseteq \mathcal{F}(M_1)$ . But now, for each  $y \in G$ , one has

$$\{1\} \neq N = N^y \subseteq \mathcal{F}(M_1) \cap \mathcal{F}(M_1)^y$$

whence, according to (FS3),  $y \in M_1$ ; thus  $G = M_1$  is solvable, contradicting (2.4).

Therefore:

(2.6)  $G$  has no nontrivial solvable normal subgroup.

In particular, by the Feit-Thompson Theorem:

(2.7)  $O_{2'}(G) = \{1\}$ .

By (2.2),  $\mathcal{F}(M) \neq \{1\}$ ; let  $q$  be a prime factor of  $|\mathcal{F}(M)|$ , and let  $Z(\mathcal{F}(M)_q)$  denote the  $q$ -component (i.e. the subgroup of elements whose order is a power of  $q$ ) of the *finite abelian* group  $Z(\mathcal{F}(M))$ . Then  $S$  acts freely on the elementary abelian  $q$ -group  $\Omega_1(Z(\mathcal{F}(M)_q))$  (as, for each  $y \in \Omega_1(Z(\mathcal{F}(M)_q))^\#$ , one has  $C_G(y) \subseteq \mathcal{F}(M)$ , whence  $C_S(y) = \{1\}$  by (2.5)); therefore  $H =_{\text{def}} SB = S \ltimes B$  is a Frobenius group with kernel  $B$  and complement  $S$ . It now follows from 12.6.15 (ii), p.356 in [5] that either

(1)  $S$  is cyclic,

or:

(2)  $S$  is a generalized quaternion group.

In case (1),  $G$  is 2-nilpotent, hence (by (2.7))  $G = SO_{2'}(G) = S$  is solvable, contradicting (2.4). Therefore we have case (2), whence, by the Brauer-Suzuki Theorem, one has (denoting by  $t$  the unique involution in  $S$ ):

$$G = C_G(t)O_{2'}(G) = C_G(t).$$

But then one has  $t \in Z(G)$ , whence  $\langle t \rangle$  is a nontrivial solvable normal subgroup of  $G$ , contradicting (2.6). Thus is the proof concluded.  $\square$

### 3. Errata

I am taking the opportunity to correct some misprints in [4]:

- p.117, 1.5 of the abstract: instead of “group”, read “groups”;
- p.117, 1.4 of the main text: remove “Frobenius”;
- p.117, 1.8 of the main text: instead of “1”, read “{1}”;
- p.118, 1.3: instead of “ $2^{r+1}$ ”, read “ $2^{n+1}$ ”;
- p.120, 1.-9: instead of “1440”, read “720”;
- p.120, 1.-1: replace the line by “ $\mathcal{F}_{\mathbf{F}_{2^n}}$  (resp.  $\mathcal{F}_{(n)}$ )”.

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