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# ON FROBENIUS SYSTEMS

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**Abstract**

By a Frobenius system on a finite group $G$, we mean the data, for each maximal solvable subgroup $M$ of $G$, of a normal subgroup $\mathcal{F}(M)$ of $M$, satisfying some of the properties of a Frobenius kernel, and subject to certain additional conditions. We prove that a finite group with a Frobenius system is either solvable (in which case we get a complete description), or isomorphic to $SL_2(K)$ (for $K$ a finite field of characteristic 2) or to a Suzuki group. The respective possibilities for the mapping $\mathcal{F}$ are then determined. This extends a previous result of ours (Nagoya Math. J. **165** (2002), 117–121) by removing the condition that each $M/\mathcal{F}(M)$ be abelian. Curiously enough, the Feit-Thompson Theorem is used in the proof.

## 0. Introduction

In this paper, we shall classify all Frobenius systems on finite groups, modulo a very weak (and natural) nondegeneracy hypothesis (condition $(FS4^{'})$ below). Our result contains as a particular case Theorem 0.1 of [4]. The possibility of such a generalization was suggested to the author by Arad and Herfort's paper ([1]), in which are studied finite groups possessing at least one $CC$-subgroup, i.e. a nontrivial proper subgroup that contains the centralizer of each of its nonidentity elements.

In fact, we only need a particular case of Arad and Herfort's result (Theorem A, (ii), p.2089 in [1]), that is due to Suzuki ([6], Theorem 1; see also [2]); in particular we do *not* need the full Classification of the Finite Simple Groups, but only Suzuki's classification of $ZT$-groups (see [6]). We also use the Brauer-Suzuki Theorem on finite groups with a generalized quaternion Sylow subgroup and the Feit-Thompson Theorem in order to deal with a particularly troublesome configuration.

We shall keep, unless the contrary be mentioned, the notations used in [3] and [4].

## 1. Definitions and statement of the result

By a *Frobenius system* (*kernel system* in [3]) on the finite group $G$, we shall mean, as in [4], p.117, a mapping $\mathcal{F}$ from the set $\mathcal{MS}(G)$ of maximal solvable subgroups of $G$ to the power set $\mathcal{P}(G)$ of $G$, such that the following axioms are satisfied, for all $M \in \mathcal{MS}(G)$:

$(FS1)$ $\mathcal{F}(M)$ is a normal subgroup of $M$;

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$(FS2)$ $\forall a \in M \setminus \mathcal{F}(M),\ C_{\mathcal{F}(M)}(a) = \{1\}$;
$(FS3)$ $\forall g \in G \setminus M,\ \mathcal{F}(M) \cap \mathcal{F}(M)^g = \{1\}$.
(cf. axioms (1), (2), (3) of Definition 1.1 in [3], p.72).

There is a natural notion of isomorphism for groups with a Frobenius system (see [4], p.118). In addition, two families of Lie-type groups over fields of characteristic two do possess a canonical Frobenius system: the Suzuki groups $Sz(2^{2n+1})$ $(n \geq 1)$, and the special linear groups $SL_2(\mathbf{F}_{2^n})$ $(n \geq 2)$; the respective Frobenius systems (defined in [4], p.118) will be denoted by $\mathcal{F}_{(n)}$ (resp. $\mathcal{F}_{\mathbf{F}_{2^n}}$).

We can now state our main result:

**Theorem 1.1.** *Let $\mathcal{F}$ be a Frobenius system on the finite group $G$, such that the following condition hold*:
$(FS4')$ *For each $M \in \mathcal{MS}(G)$, $\mathcal{F}(M) \neq \{1\}$ or $M$ is abelian.*

*Then one* (*and, of course, only one*) *of the following holds*:
(1) *$G$ is abelian and $\mathcal{F}(G) = \{1\}$;*
(2) *$G$ is a nonidentity solvable group and $\mathcal{F}(G) = G$;*
(3) *$G$ is a solvable Frobenius group, and $\mathcal{F}(G)$ is the Frobenius kernel of $G$;*
(4) *$(G, \mathcal{F})$ is isomorphic to $(SL_2(\mathbf{F}_{2^n}), \mathcal{F}_{\mathbf{F}_{2^n}})$, for some $n \geq 2$;*
(5) *$(G, \mathcal{F})$ is isomorphic to $(Sz(2^{2n+1}), \mathcal{F}_{(n)})$, for some $n \geq 1$.*
*Conversely, each of the possibilities* (1), . . . ,(5) *gives rise to a Frobenius system satisfying* $(FS4')$.

Clearly, axiom $(FS4')$ follows from axiom $(FS4)$ in [4] (and *a fortiori* from axiom (5) in [3], p.73); therefore, Theorem 0.1 in [4] follows at once from Theorem 1.1 (we only have to consider case (3), when $(FS4)$ yields that the Frobenius complement in $G$ is abelian, hence cyclic by the same argument as in [4], p.120).

## 2. Proof of Theorem 1.1

Some parts of our proof will be very close to the corresponding ones in [4].

Let $(G, \mathcal{F})$ satisfy $(FS1)$, $(FS2)$, $(FS3)$ and $(FS4')$. If, for some $M \in \mathcal{MS}(G)$, one has $\mathcal{F}(M) = \{1\}$, then (by $(FS4')$) $M$ is abelian, whence (by [4], Lemma 1.1, p.118) $G = M$, and we obtain (1). Therefore we may assume that:

$$\text{(2.1)} \qquad \text{For each} \quad M \in \mathcal{MS}(G), \quad \mathcal{F}(M) \neq \{1\}.$$

Let us now assume that, for some $M \in \mathcal{MS}(G)$, $\mathcal{F}(M) = M$. If $M = G$, then $G$ is solvable, $\mathcal{F}(G) = G$ and we are in case (2); if $M \neq G$, it follows from $(FS3)$ that $M$ is a solvable Frobenius complement in $G$, and we reach a contradiction as in [4], p.120, l.6. Therefore, we may assume that:

$$\text{(2.2)} \qquad \forall M \in \mathcal{MS}(G), \quad \{1\} \neq \mathcal{F}(M) \neq M.$$

As in [4], p.120, it now follows from [3], Proposition 1.5, p.73, that:

(2.3) For each $M \in \mathcal{MS}(G)$, $\mathcal{F}(M)$ is nilpotent.

(As in the argument preciding (2.2), we do not need here Thompson's Theorem on the nilpotency of Frobenius kernels, but only the far more elementary fact that *solvable* Frobenius kernels are nilpotent). If $G$ is solvable, then $\mathcal{MS}(G) = \{G\}$, and Lemma 1.3 from [3] yields that $\mathcal{F}(G)$ is a Frobenius kernel in $G$, and thence (3) holds. Therefore we may assume that:

(2.4) $G$ is not solvable.

Let now $S \in Syl_2(G)$, and let $M \in \mathcal{MS}(G)$ contain $S$. If $\mathcal{F}(M)$ has even order, then, as by [3], Lemma 1.3 and (2.2) $\mathcal{F}(M)$ is a $CC$-subgroup of $G$, Theorem 1 of [6] yields that either:

(1) $G$ is a Frobenius group, and $\mathcal{F}(M)$ is its kernel, or
(2) $G$ is a Frobenius group, and $\mathcal{F}(M)$ is its complement, or
(3) $G$ is a $ZT$-group.

But (2) would imply that $\mathcal{F}(M) = N_G(\mathcal{F}(M)) = M$ (cf. [3], Lemma 2.5 (i), p.75) contradicting (2.2), and (1) would imply that $G = N_G(\mathcal{F}(M)) = M$ would be solvable, contradicting (2.4). Therefore one has (3), whence, by [6], $G \simeq SL_2(\mathbf{F}_{2^n})$ $(n \geq 2)$ or $G \simeq Sz(2^{2n+1})$ $(n \geq 1)$, and one may conclude as in [4], p.120, that case (4) or case (5) of Theorem 1.1 holds.

Thus it may be supposed that:

(2.5) $\mathcal{F}(M)$ has odd order.

Let us assume that $G$ possesses a nontrivial normal solvable subgroup $N_0$, and let $N$ denote a minimal normal subgroup of $G$ contained in $N_0$; then $N$ is an elementary abelian $p$-group for some prime $p$. Let $P \in Syl_p(G)$, and let $M_1 \in \mathcal{MS}(G)$, $M_1 \supseteq P$; then $N \subseteq P \subseteq M_1$. Therefore

$$\begin{aligned}[N, \mathcal{F}(M_1)] &\subseteq [N, G] \cap [M_1, \mathcal{F}(M_1)] \\ &\subseteq N \cap \mathcal{F}(M_1).\end{aligned}$$

If $N \cap \mathcal{F}(M_1) = \{1\}$, then $N$ centralizes $\mathcal{F}(M_1)$; for $x \in \mathcal{F}(M_1)^\sharp$, one has $N \subseteq C_G(x) \subseteq \mathcal{F}(M_1)$ (by [3], Lemma 1.3) whence $N = N \cap \mathcal{F}(M_1) = \{1\}$, a contradiction. Thus $N \cap \mathcal{F}(M_1) \neq \{1\}$; let now $x \in (N \cap \mathcal{F}(M_1))^\sharp$. One has $N \subseteq C_G(x)$ (as $N$ is abelian), and $C_G(x) \subseteq \mathcal{F}(M_1)$ (as above), whence $N \subseteq \mathcal{F}(M_1)$. But now, for each $y \in G$, one has

$$\{1\} \neq N = N^y \subseteq \mathcal{F}(M_1) \cap \mathcal{F}(M_1)^y$$

whence, according to $(FS3)$, $y \in M_1$; thus $G = M_1$ is solvable, contradicting (2.4).

Therefore:

(2.6) $G$ has no nontrivial solvable normal subgroup.

In particular, by the Feit-Thompson Theorem:

$$O_{2'}(G) = \{1\}. \tag{2.7}$$

By (2.2), $\mathcal{F}(M) \neq \{1\}$; let $q$ be a prime factor of $|\mathcal{F}(M)|$, and let $Z(\mathcal{F}(M)_q)$ denote the $q$-component (i.e. the subgroup of elements whose order is a power of $q$) of the *finite abelian* group $Z(\mathcal{F}(M))$. Then $S$ acts freely on the elementary abelian $q$-group $\Omega_1(Z(\mathcal{F}(M)_q))$ (as, for each $y \in \Omega_1(Z(\mathcal{F}(M)_q))^\sharp$, one has $C_G(y) \subseteq \mathcal{F}(M)$, whence $C_S(y) = \{1\}$ by (2.5)); therefore $H =_{\mathrm{def}} SB = S \ltimes B$ is a Frobenius group with kernel $B$ and complement $S$. It now follows from 12.6.15 (ii), p.356 in [5] that either
(1) $S$ is cyclic,
or:
(2) $S$ is a generalized quaternion group.

In case (1), $G$ is 2-nilpotent, hence (by (2.7)) $G = SO_{2'}(G) = S$ is solvable, contradicting (2.4). Therefore we have case (2), whence, by the Brauer-Suzuki Theorem, one has (denoting by $t$ the unique involution in $S$):

$$G = C_G(t)O_{2'}(G) = C_G(t).$$

But then one has $t \in Z(G)$, whence $\langle t \rangle$ is a nontrivial solvable normal subgroup of $G$, contradicting (2.6). Thus is the proof concluded. $\square$

## 3. Errata

I am taking the opportunity to correct some misprints in [4]:

- p.117, l.5 of the abstract: instead of "group", read "groups";
- p.117, l.4 of the main text: remove "Frobenius";
- p.117, l.8 of the main text: instead of "1", read "$\{1\}$";
- p.118, l.3: instead of "$2^{r+1}$", read "$2^{n+1}$";
- p.120, l.-9: instead of "1440", read "720";
- p.120, l.-1: replace the line by "$\mathcal{F}_{\mathbf{F}_{2^n}}$ (resp. $\mathcal{F}_{(n)}$)".

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**References**


[1] Z. Arad and W. Herfort: *Classification of finite groups with a CC-subgroup*, Comm. Algebra **32** (2004), 2087–2098.

[2] Z. Arad and W. Herfort: *A classification of locally finite and profinite groups with a centralizer condition*, Comm. Algebra **10** (1982), 1749–1764.

[3] P. Lescot: *Kernel systems on finite groups*, Nagoya Math. J. **163** (2001), 71–85.

[4] P. Lescot: *Families of solvable Frobenius subgroups in finite groups*, Nagoya Math. J. **165** (2002), 117–121.
[5] W.R. Scott: Group Theory, second edition, Dover, New York, 1987.
[6] M. Suzuki: *Two characteristic properties of* $(ZT)$*-groups*, Osaka J. Math. **15** (1963), 143–150.

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