

Title	On Frobenius systems
Author(s)	Lescot, Paul
Citation	Osaka Journal of Mathematics. 2007, 44(4), p. 887-891
Version Type	VoR
URL	https://doi.org/10.18910/3906
rights	
Note	

Osaka University Knowledge Archive : OUKA

<https://ir.library.osaka-u.ac.jp/>

Osaka University

ON FROBENIUS SYSTEMS

PAUL LESCOT

(Received February 27, 2006, revised November 24, 2006)

Abstract

By a Frobenius system on a finite group G , we mean the data, for each maximal solvable subgroup M of G , of a normal subgroup $\mathcal{F}(M)$ of M , satisfying some of the properties of a Frobenius kernel, and subject to certain additional conditions. We prove that a finite group with a Frobenius system is either solvable (in which case we get a complete description), or isomorphic to $SL_2(K)$ (for K a finite field of characteristic 2) or to a Suzuki group. The respective possibilities for the mapping \mathcal{F} are then determined. This extends a previous result of ours (Nagoya Math. J. **165** (2002), 117–121) by removing the condition that each $M/\mathcal{F}(M)$ be abelian. Curiously enough, the Feit-Thompson Theorem is used in the proof.

0. Introduction

In this paper, we shall classify all Frobenius systems on finite groups, modulo a very weak (and natural) nondegeneracy hypothesis (condition $(FS4')$ below). Our result contains as a particular case Theorem 0.1 of [4]. The possibility of such a generalization was suggested to the author by Arad and Herfort's paper ([1]), in which are studied finite groups possessing at least one CC -subgroup, i.e. a nontrivial proper subgroup that contains the centralizer of each of its nonidentity elements.

In fact, we only need a particular case of Arad and Herfort's result (Theorem A, (ii), p.2089 in [1]), that is due to Suzuki ([6], Theorem 1; see also [2]); in particular we do *not* need the full Classification of the Finite Simple Groups, but only Suzuki's classification of ZT -groups (see [6]). We also use the Brauer-Suzuki Theorem on finite groups with a generalized quaternion Sylow subgroup and the Feit-Thompson Theorem in order to deal with a particularly troublesome configuration.

We shall keep, unless the contrary be mentioned, the notations used in [3] and [4].

1. Definitions and statement of the result

By a *Frobenius system* (*kernel system* in [3]) on the finite group G , we shall mean, as in [4], p.117, a mapping \mathcal{F} from the set $\mathcal{MS}(G)$ of maximal solvable subgroups of G to the power set $\mathcal{P}(G)$ of G , such that the following axioms are satisfied, for all $M \in \mathcal{MS}(G)$:

(FS1) $\mathcal{F}(M)$ is a normal subgroup of M ;

(FS2) $\forall a \in M \setminus \mathcal{F}(M), C_{\mathcal{F}(M)}(a) = \{1\}$;

(FS3) $\forall g \in G \setminus M, \mathcal{F}(M) \cap \mathcal{F}(M)^g = \{1\}$.

(cf. axioms (1), (2), (3) of Definition 1.1 in [3], p.72).

There is a natural notion of isomorphism for groups with a Frobenius system (see [4], p.118). In addition, two families of Lie-type groups over fields of characteristic two do possess a canonical Frobenius system: the Suzuki groups $Sz(2^{2n+1})$ ($n \geq 1$), and the special linear groups $SL_2(\mathbf{F}_{2^n})$ ($n \geq 2$); the respective Frobenius systems (defined in [4], p.118) will be denoted by $\mathcal{F}_{(n)}$ (resp. $\mathcal{F}_{\mathbf{F}_{2^n}}$).

We can now state our main result:

Theorem 1.1. *Let \mathcal{F} be a Frobenius system on the finite group G , such that the following condition hold:*

(FS4') *For each $M \in \mathcal{MS}(G)$, $\mathcal{F}(M) \neq \{1\}$ or M is abelian.*

Then one (and, of course, only one) of the following holds:

(1) *G is abelian and $\mathcal{F}(G) = \{1\}$;*

(2) *G is a nonidentity solvable group and $\mathcal{F}(G) = G$;*

(3) *G is a solvable Frobenius group, and $\mathcal{F}(G)$ is the Frobenius kernel of G ;*

(4) *(G, \mathcal{F}) is isomorphic to $(SL_2(\mathbf{F}_{2^n}), \mathcal{F}_{\mathbf{F}_{2^n}})$, for some $n \geq 2$;*

(5) *(G, \mathcal{F}) is isomorphic to $(Sz(2^{2n+1}), \mathcal{F}_{(n)})$, for some $n \geq 1$.*

Conversely, each of the possibilities (1), . . . , (5) gives rise to a Frobenius system satisfying (FS4').

Clearly, axiom (FS4') follows from axiom (FS4) in [4] (and *a fortiori* from axiom (5) in [3], p.73); therefore, Theorem 0.1 in [4] follows at once from Theorem 1.1 (we only have to consider case (3), when (FS4) yields that the Frobenius complement in G is abelian, hence cyclic by the same argument as in [4], p.120).

2. Proof of Theorem 1.1

Some parts of our proof will be very close to the corresponding ones in [4].

Let (G, \mathcal{F}) satisfy (FS1), (FS2), (FS3) and (FS4'). If, for some $M \in \mathcal{MS}(G)$, one has $\mathcal{F}(M) = \{1\}$, then (by (FS4')) M is abelian, whence (by [4], Lemma 1.1, p.118) $G = M$, and we obtain (1). Therefore we may assume that:

(2.1) For each $M \in \mathcal{MS}(G)$, $\mathcal{F}(M) \neq \{1\}$.

Let us now assume that, for some $M \in \mathcal{MS}(G)$, $\mathcal{F}(M) = M$. If $M = G$, then G is solvable, $\mathcal{F}(G) = G$ and we are in case (2); if $M \neq G$, it follows from (FS3) that M is a solvable Frobenius complement in G , and we reach a contradiction as in [4], p.120, l.6. Therefore, we may assume that:

(2.2) $\forall M \in \mathcal{MS}(G), \{1\} \neq \mathcal{F}(M) \neq M$.

As in [4], p.120, it now follows from [3], Proposition 1.5, p.73, that:

$$(2.3) \quad \text{For each } M \in \mathcal{MS}(G), \mathcal{F}(M) \text{ is nilpotent.}$$

(As in the argument preceding (2.2), we do not need here Thompson's Theorem on the nilpotency of Frobenius kernels, but only the far more elementary fact that *solvable* Frobenius kernels are nilpotent). If G is solvable, then $\mathcal{MS}(G) = \{G\}$, and Lemma 1.3 from [3] yields that $\mathcal{F}(G)$ is a Frobenius kernel in G , and thence (3) holds. Therefore we may assume that:

$$(2.4) \quad G \text{ is not solvable.}$$

Let now $S \in \text{Syl}_2(G)$, and let $M \in \mathcal{MS}(G)$ contain S . If $\mathcal{F}(M)$ has even order, then, as by [3], Lemma 1.3 and (2.2) $\mathcal{F}(M)$ is a *CC*-subgroup of G , Theorem 1 of [6] yields that either:

- (1) G is a Frobenius group, and $\mathcal{F}(M)$ is its kernel, or
- (2) G is a Frobenius group, and $\mathcal{F}(M)$ is its complement, or
- (3) G is a *ZT*-group.

But (2) would imply that $\mathcal{F}(M) = N_G(\mathcal{F}(M)) = M$ (cf. [3], Lemma 2.5 (i), p.75) contradicting (2.2), and (1) would imply that $G = N_G(\mathcal{F}(M)) = M$ would be solvable, contradicting (2.4). Therefore one has (3), whence, by [6], $G \simeq SL_2(\mathbb{F}_{2^n})$ ($n \geq 2$) or $G \simeq Sz(2^{2n+1})$ ($n \geq 1$), and one may conclude as in [4], p.120, that case (4) or case (5) of Theorem 1.1 holds.

Thus it may be supposed that:

$$(2.5) \quad \mathcal{F}(M) \text{ has odd order.}$$

Let us assume that G possesses a nontrivial normal solvable subgroup N_0 , and let N denote a minimal normal subgroup of G contained in N_0 ; then N is an elementary abelian p -group for some prime p . Let $P \in \text{Syl}_p(G)$, and let $M_1 \in \mathcal{MS}(G)$, $M_1 \supseteq P$; then $N \subseteq P \subseteq M_1$. Therefore

$$\begin{aligned} [N, \mathcal{F}(M_1)] &\subseteq [N, G] \cap [M_1, \mathcal{F}(M_1)] \\ &\subseteq N \cap \mathcal{F}(M_1). \end{aligned}$$

If $N \cap \mathcal{F}(M_1) = \{1\}$, then N centralizes $\mathcal{F}(M_1)$; for $x \in \mathcal{F}(M_1)^\sharp$, one has $N \subseteq C_G(x) \subseteq \mathcal{F}(M_1)$ (by [3], Lemma 1.3) whence $N = N \cap \mathcal{F}(M_1) = \{1\}$, a contradiction. Thus $N \cap \mathcal{F}(M_1) \neq \{1\}$; let now $x \in (N \cap \mathcal{F}(M_1))^\sharp$. One has $N \subseteq C_G(x)$ (as N is abelian), and $C_G(x) \subseteq \mathcal{F}(M_1)$ (as above), whence $N \subseteq \mathcal{F}(M_1)$. But now, for each $y \in G$, one has

$$\{1\} \neq N = N^y \subseteq \mathcal{F}(M_1) \cap \mathcal{F}(M_1)^y$$

whence, according to (FS3), $y \in M_1$; thus $G = M_1$ is solvable, contradicting (2.4).

Therefore:

(2.6) G has no nontrivial solvable normal subgroup.

In particular, by the Feit-Thompson Theorem:

(2.7) $O_{2'}(G) = \{1\}$.

By (2.2), $\mathcal{F}(M) \neq \{1\}$; let q be a prime factor of $|\mathcal{F}(M)|$, and let $Z(\mathcal{F}(M)_q)$ denote the q -component (i.e. the subgroup of elements whose order is a power of q) of the *finite abelian* group $Z(\mathcal{F}(M))$. Then S acts freely on the elementary abelian q -group $\Omega_1(Z(\mathcal{F}(M)_q))$ (as, for each $y \in \Omega_1(Z(\mathcal{F}(M)_q))^\pm$, one has $C_G(y) \subseteq \mathcal{F}(M)$, whence $C_S(y) = \{1\}$ by (2.5)); therefore $H =_{\text{def}} SB = S \ltimes B$ is a Frobenius group with kernel B and complement S . It now follows from 12.6.15 (ii), p.356 in [5] that either

(1) S is cyclic,

or:

(2) S is a generalized quaternion group.

In case (1), G is 2-nilpotent, hence (by (2.7)) $G = SO_{2'}(G) = S$ is solvable, contradicting (2.4). Therefore we have case (2), whence, by the Brauer-Suzuki Theorem, one has (denoting by t the unique involution in S):

$$G = C_G(t)O_{2'}(G) = C_G(t).$$

But then one has $t \in Z(G)$, whence $\langle t \rangle$ is a nontrivial solvable normal subgroup of G , contradicting (2.6). Thus is the proof concluded. \square

3. Errata

I am taking the opportunity to correct some misprints in [4]:

- p.117, 1.5 of the abstract: instead of “group”, read “groups”;
- p.117, 1.4 of the main text: remove “Frobenius”;
- p.117, 1.8 of the main text: instead of “1”, read “{1}”;
- p.118, 1.3: instead of “ 2^{r+1} ”, read “ 2^{n+1} ”;
- p.120, 1.-9: instead of “1440”, read “720”;
- p.120, 1.-1: replace the line by “ $\mathcal{F}_{\mathbb{F}_{2^n}}$ (resp. $\mathcal{F}_{(n)}$)”.

References

- [1] Z. Arad and W. Herfort: *Classification of finite groups with a CC-subgroup*, Comm. Algebra **32** (2004), 2087–2098.
- [2] Z. Arad and W. Herfort: *A classification of locally finite and profinite groups with a centralizer condition*, Comm. Algebra **10** (1982), 1749–1764.
- [3] P. Lescot: *Kernel systems on finite groups*, Nagoya Math. J. **163** (2001), 71–85.

- [4] P. Lescot: *Families of solvable Frobenius subgroups in finite groups*, Nagoya Math. J. **165** (2002), 117–121.
- [5] W.R. Scott: *Group Theory*, second edition, Dover, New York, 1987.
- [6] M. Suzuki: *Two characteristic properties of (ZT)-groups*, Osaka J. Math. **15** (1963), 143–150.

INSSET

Université de Picardie

48 Rue Raspail

02100 Saint-Quentin

France

Phone 00 33 (0)3 23 62 89 61

Fax 00 33 (0)3 23 62 89 35

LAMFA

Faculté de Mathématiques et Informatique

Université de Picardie

80039 Amiens Cédex

France

e-mail: paul.lescot@u-picardie.fr