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Some Remarks on Quasi-Frobenius Modules

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Recently Azumaya [1] introduced the concept of quasi-Frobenius two-sided modules to establish certain duality theorems for injective modules and also showed that it is a natural extension of the notion of quasi-Frobenius rings.

The purpose of the present note is to supplement the above paper by giving some characterizations of quasi-Frobenius modules. These results are regarded as generalizations of the known properties of quasi-Frobenius rings.

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§ 1. Quasi-Frobenius Modules.

Let A and A^* be two rings with unit elements and Q a two-sided A-A^*-module. Let M be a left A-module. Then the set M^* of all A-homomorphisms of M into Q forms a right A^*-module under the following definitions:

$$x(\varphi+\psi)=x\varphi+x\psi^{1)}, \quad x(\varphi a^*)=(x\varphi)a^*$$

for $x\in M$ and $a^*\in A^*$. This module is called the *right-dual module* of M with respect to Q. We may similarly define the *left-dual module* for any right A^*-module. Now, Azumaya calls Q a *quasi-Frobenius two-sided A-A^*-module* if i) Q is faithful (with respect to both A and A^*) and ii) for every maximal left ideal $\mathfrak{l}$ of A and for every maximal right ideal $\mathfrak{r}$ of A^* the right annihilator $r_Q(\mathfrak{l})$ and the left annihilator $l_Q(\mathfrak{r})$ of $\mathfrak{l}$ and $\mathfrak{r}$ in Q are A^*-irreducible and A-irreducible respectively, provided they are non-zero.

It should be noted that the above condition ii) is equivalent to the following condition: ii′) for every irreducible A-submodule L of Q and for every irreducible A^*-submodule R of Q the right- and the left-dual

1) Let M_1 and M_2 be two left A-(or right A^*-)modules. For any A-(or A^*-)homomorphism $\varphi: M_1\to M_2$ and any element $x\in M_1$, we denote by $x\varphi$ (or φx) the image of x by φ. If further ψ is an A-(or A^*-)homomorphism of M_2 into a third left A-(or right A^*-)module M_3, then we shall denote by $\varphi\circ\psi$ (or $\psi\circ\varphi$) the composite mapping $x\to(x\varphi)\psi$ (or $x\to\psi(\varphi x)$).

modules of L and R with respect to Q are A^*- and A-irreducible respectively.

THEOREM 1. *Let Q be a quasi-Frobenius two-sided A-A^*-module. Then Au, $u \in Q$, is A-irreducible if and only if uA^* is A^*-irreducible, and moreover, in this case, they are isomorphic to the dual modules of each other.*

Proof. Suppose that Q is quasi-Frobenius and Au is A-irreducible. Since Au is isomorphic to the factor module $A-l_A(u)$, $r_Q(l_A(u))$ must, as the right-dual module of $A-l_A(u)$, be A^*-irreducible and hence necessarily coincides with uA^*.

THEOREM 2. *Let Q be a quasi-Frobenius two-sided A-A^*-module and M an irreducible left A-module such that its right-dual module M^* with respect to Q is non-zero. Then the A-endomorphism ring of M is isomorphic to the A^*-endomorphism ring of M^*.*

Proof. With each A-endomorphism φ of M we can associate an A^*-endomorphism φ^*[2] of M^* by setting $\varphi^* g = \varphi \circ g$, $g \in M^*$, that is,

$$(*) \qquad x(\varphi^* g) = (x\varphi)g \qquad x \in M,\ g \in M^* .$$

And it is easy to see that $(\varphi \circ \psi)^* = \varphi^* \circ \psi^*$ holds for any A-endomorphisms φ and ψ of M. Thus the mapping $\varphi \rightarrow \varphi^*$ is a homomorphism of the A-endomorphism ring of M into the A^*-endomorphism ring of M^*. Since M is A-irreducible and M^* is non-zero whence A^*-irreducible, M may be looked upon as the left-dual module of M^* with respect to Q, and therefore the above equality $(*)$ shows that φ coincides with the dual-mapping of φ^*. Thus, the above homomorphism gives an isomorphism between the A-endomorphism ring of M and the A^*-endomorphism ring of M^*.

From now on, *we shall assume the left and the right minimum conditions for A and A^* respectively* unless otherwise stated. Then it should be noted, according to [1, Theorem 6], that every quasi-Frobenius two-sided A-A^*-module always contains an isomorphic image of every irreducible left A-module as well as that of every irreducible right A^*-module.

We first prove the following

THEOREM 3. *Let Q be a faithful two-sided A-A^*-module. Then Q is quasi-Frobenius if and only if every irreducible left A-module and every*

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2) φ^* is called *the dual mapping of* φ with respect to Q.

irreducible right A^*-module are isomorphic to their double dual modules[3] *with respect to Q respectively.*

Proof. The "only if" part is clear. To prove the "if" part, consider an irreducible left A-module M and let M_1 be a maximal A^*-submodule of M^* (the existence of M_1 being assured by the right minimum condition for A^*). Then, since $l_M(M_1)$ is isomorphic to the left-dual module of the irreducible A^*-module M^*-M_1, it follows $l_M(M_1)\neq 0$ whence $l_M(M_1)=M$. But this implies evidently $M_1=0$, that is, M^* is A^*-irreducible. Similarly, the left-dual module of every irreducible right A^*-module is A-irreducible, and this shows that Q is quasi-Frobenius.

Next, we shall give another characterization of a quasi-Frobenius two-sided A-A^*-module, which may be regarded as a generalization of [2, Theorems 6 and 8].

THEOREM 4. *Let Q be a faithful two-sided A-A^*-module. In order that Q is quasi-Frobenius it is necessary and sufficient that for every irreducible left A-submodule L and for every irreducible right A^*-submodule R of Q the annihilator relations*

$$l_Q(r_{A^*}(L))=L \quad \textit{and} \quad r_Q(l_A(R))=R$$

hold. And, if this is the case, for any $u\in Q$ the left A-submodule Au and the right A^*-submodule uA^* of Q are isomorphic to the dual modules of each other.

Proof. The necessity of the first part follows from [1, Corollary to Proposition 2].

To prove the sufficiency, we consider a maximal left ideal $\mathfrak{l}$ of A such that $r_Q(\mathfrak{l})\neq 0$. Let R be an irreducible right A^*-submodule of $r_Q(\mathfrak{l})$. Then $l_A(R)=\mathfrak{l}$, since $l_A(R)$ is a proper left ideal containing $\mathfrak{l}$. Hence, $r_Q(\mathfrak{l})=r_Q(l_A(R))=R$, and thus $r_Q(\mathfrak{l})$ is A^*-irreducible. Similarly, $l_Q(\mathfrak{r})$ is either 0 or A-irreducible for every maximal right ideal $\mathfrak{r}$ of A^*.

If we put $\mathfrak{l}=l_A(u)$, then Au is A-isomorphic to the factor module $A-\mathfrak{l}$. Thus, the right-dual module of Au is A^*-isomorphic to $r_Q(\mathfrak{l})$. Take any non-zero element v from $r_Q(\mathfrak{l})$. Then the mapping $au\rightarrow av$, $a\in A$, is obviously an A-homomorphism of Au into Q, and hence by [1, Theorem 6] there exists an element a^* of A^* such that $ua^*=v$. Thus $uA^*=r_Q(\mathfrak{l})$, and this proves our theorem.

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3) Let M be a left A-module and M^* the right-dual module of M with respect to Q. Then by *the double dual module of M* we mean the left-dual module $(M^*)^*$ of M^* with respect to Q. The double dual module of a right A^*-module is defined similarly.

Let N and N^* be the radicals of A and A^* respectively. Then, as is well-known, for any two-sided A-A^*-module Q the A-socle of Q, that is, the sum of all irreducible left A-submodules of Q coincides with $r_Q(N)$. Similarly, the A^*-socle of Q coincides with $l_Q(N^*)$.

Then we have the following theorem which corresponds to [2, Theorem 9].

THEOREM 5. *Let Q be a quasi-Frobenius two-sided A-A^*-module and let N and N^* be the radicals of A and A^* respectively. Then $r_Q(N^\nu) = l_Q(N^{*\nu})$ for every $\nu = 1, 2, \cdots$.*

Proof. We shall prove this theorem by induction on ν. For $\nu = 1$, this was shown in [1, Theorem 1]. Let $\nu > 1$ and assume that $r_Q(N^{\nu-1}) = l_Q(N^{*\nu-1})$. Evidently $l_Q(N^{*\nu})N^* \subseteqq l_Q(N^{*\nu-1})$. Hence $N^{\nu-1}l_Q(N^{*\nu})N^* \subseteqq N^{\nu-1}l_Q(N^{*\nu-1}) = N^{\nu-1}r_Q(N^{\nu-1}) = 0$, $N^{\nu-1}l_Q(N^{*\nu}) \subseteqq l_Q(N^*) = r_Q(N)$, and $N^\nu l_Q(N^{*\nu}) \subseteqq N r_Q(N) = 0$, that is, $l_Q(N^{*\nu}) \subseteqq r_Q(N^\nu)$. Similarly we have $r_Q(N^\nu) \subseteqq l_Q(N^{*\nu})$, whence $r_Q(N^\nu) = l_Q(N^{*\nu})$.

Let $\bar{A}$ denote the (semi-simple) factor ring A/N, and let $\bar{A} = \bar{A}_1 \oplus \bar{A}_2 \oplus \cdots \oplus \bar{A}_k$ be its direct decomposition into orthogonal simple two-sided ideals $\bar{A}_\kappa$. Let $\bar{e}_\kappa$ be, for each κ, a primitive idempotent element in $\bar{A}_\kappa$. Then the k modules $\bar{A}\bar{e}_1, \bar{A}\bar{e}_2, \cdots, \bar{A}\bar{e}_k$ exhaust, up to isomorphisms, all irreducible left A-modules. There exists, for each κ, an idempotent representative $e_\kappa (\in A)$ of the coset $\bar{e}_\kappa$. Then k idempotent elements $e_1, e_2, \cdots, e_k$ are all primitive and non-isomorphic, and any primitive idempotent element of A is isomorphic to one of them. Furthermore, if we denote by $f(\kappa)$ the capacity of $\bar{A}\bar{e}_\kappa$, i.e., the dimension of $\bar{A}\bar{e}_\kappa$ over its endomorphism division ring $\bar{e}_\kappa\bar{A}\bar{e}_\kappa$, A is as left A-module isomorphic to the direct sum $\sum_{\kappa=1}^k \oplus (Ae_\kappa)^{f(\kappa)}$. Similarly, we shall denote by $e^*_1, e^*_2, \cdots, e^*_l$ a complete system of non-isomorphic primitive idempotent elements in A^* and by $g(\lambda)$, for each $\lambda = 1, 2, \cdots, l$, the capacity of the irreducible right A^*-module $\overline{e^*_\lambda}\bar{A}^*$.

LEMMA. *Let Q be a two-sided A-A^*-module. Then the right- and the left-dual modules of the irreducible left A- and the irreducible right A^*-modules $\bar{A}\bar{e}_\kappa^*$ and $\overline{e^*_\lambda}\bar{A}$ with respect to Q are isomorphic to $e_\kappa r_Q(N)$ and $l_Q(N^*)e^*_\lambda$ respectively.*

Proof. If we put $\mathfrak{l}_\kappa = A(1-e_\kappa) + N$ then $\mathfrak{l}_\kappa$ is a maximal left ideal of A such that $A - \mathfrak{l}_\kappa$ is A-isomorphic to $\bar{A}\bar{e}_\kappa$. Hence $r_Q(\mathfrak{l}_\kappa) = r_Q(1-e_\kappa) \cap r_Q(N) = e_\kappa Q \cap r_Q(N) = e_\kappa r_Q(N)$ is A^*-isomorphic to the right-dual module of $\bar{A}\bar{e}_\kappa$ with respect to Q. Similarly, $l_Q(N^*)e^*_\lambda$ is, for each λ, A-isomorphic to the left-dual module of $\overline{e^*_\lambda}\bar{A}^*$. This proves our lemma.

From this lemma it follows now immediately

THEOREM 6. *Let Q be a faithful two-sided A-A^*-module. Then Q is quasi-Frobenius if and only if $e_\kappa r_Q(N)$ and $l_Q(N^*)e^*_\lambda$ are, for each κ and λ, A^*- and A-irreducible respectively.*

Now, let Q be a quasi-Frobenius two-sided A-A^*-module. If we associate with each irreducible left A-module its right-dual module with respect to Q, we have a one-to-one correspondence between irreducible left A-modules and irreducible right A^*-modules. This means that $k=l$ and, if we order $e^*_1, e^*_2, \cdots, e^*_k$ suitably, $\overline{e^*_\kappa \bar{A}^*}$ is, for each κ, isomorphic to the right-dual module of $\bar{A}\bar{e}_\kappa$. We shall henceforth retain such ordering of e^*_κ. It follows then from the above lemma that

$$e_\kappa r_Q(N) \cong \overline{e^*_\kappa \bar{A}^*}, \qquad l_Q(N^*)e^*_\kappa \cong \bar{A}\bar{e}_\kappa$$

for each κ. On the other hand, since $r_Q(N)=l_Q(N^*)$ (Theorem 5), $e_\kappa r_Q(N)=e_\kappa l_Q(N^*)$ and $l_Q(N^*)e^*_\kappa=r_Q(N)e^*_\kappa$ are necessarily unique simple A^*- and A-submodules of $e_\kappa Q$ and Qe^*_κ respectively. These facts together give a second proof of the necessity of [1, Theorem 12].

Let Q be a quasi-Frobenius two-sided A-A^*-module. Suppose that a left A-module M and a right A^*-module M^* form an orthogonal pair[4] with respect to Q, and suppose further that either M or M^* satisfies both the maximum and the minimum conditions for A- or A^*-submodules respectively. Then, for any A-submodules M_1 and M_2 of M such that $M_1 \supseteqq M_2$, the factor modules M_1-M_2 and $r_{M^*}(M_2)-r_{M^*}(M_1)$ form also an orthogonal pair with respect to Q in the natural manner, and hence, by [1, Proposition 2], they are the dual modules of each other (for this, we may not necessarily assume the minimum conditions for A and A^*).

In particular, if M_2 is a maximal A-submodule of M_1 and $M_1-M_2 \cong \bar{A}\bar{e}_\kappa$ then $r_{M^*}(M_1)$ is a maximal A^*-submodule of $r_{M^*}(M_2)$ and $r_{M^*}(M_2)-r_{M^*}(M_1) \cong \overline{e^*_\kappa \bar{A}^*}$, which is a generalization of [2, Theorem 9].

§ 2. Frobenius Modules.

We shall call, following G. Azumaya [1], a quasi-Frobenius two-sided A-A^*-module Q to be *Frobenius* if, for any irreducible left A-module M, the capacity of M coincides with that of the right-dual module M^* of M with respect to Q, that is, $f(\kappa)=g(\kappa)$ for each κ.

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4) Let M be a left A-module and M^* a right A^*-module. Suppose that for any $x \in M$ and $y \in M^*$ there corresponds a product xy in Q such that

$$(x+x')y=xy+x'y, \qquad x(y+y')=xy+xy',$$
$$a(xy)=(ax)y, \qquad (xy)a^*=x(ya^*)$$

for $x, x' \in M$; $y, y' \in M^*$; $a \in A$; $a^* \in A^*$, and moreover $xM^*=0$, $x \in M$, and $My=0$, $y \in M^*$, imply always $x=0$ and $y=0$ then we shall, following Azumaya [1], say that M and M^* form an *orthogonal pair* with respect to Q.

Let L be an irreducible left A-module. Then we denote by $d_A(L)$ the capacity of L, that is, $d_A(L)=f(\kappa)$ if $L\cong \bar{A}\bar{e}_\kappa$. And generally, for any left A-module L having a composition series

$$L=L_0\supset L_1\supset\cdots\supset L_{s-1}\supset L_s=0\,,$$

we put

$$d_A(L)=\textstyle\sum_{i=1}^s d_A(L_{i-1}-L_i)\,.$$

It is clear that for any A-submodule L' of L we have $d_A(L)=d_A(L-L')+d_A(L')$. For an A^*-module R possessing a composition series we may also define $d_{A^*}(R)$ in the similar manner.

THEOREM 7. *Let Q be a quasi-Frobenius two-sided A-A^*-module. If Q is Frobenius, then for every finitely generated left A-module M we have $d_A(M)=d_{A^*}(M^*)$, where M^* is the right-dual module of M with respect to Q. Conversely, if the above equality holds for every irreducible left A-module M, then Q is Frobenius.*

Proof. Suppose that Q is Frobenius and M is a finitely generated left A-module. Then, by [1, Theorem 8], the right-dual module M^* of M with respect to Q is also finitely generated with respect to A^* and M coincides with the left-dual module of M^* with respect to Q. Let

$$M=M_0\supset M_1\supset\cdots\supset M_{s-1}\supset M_s=0$$

be a composition series of M. Then, as we have seen at the end of the preceding section,

$$M^*=r_{M^*}(M_s)\supset r_{M^*}(M_{s-1})\supset\cdots\supset r_{M^*}(M_0)=0$$

gives a composition series of M^*, and moreover, if $M_{i-1}-M_i$ is A-isomorphic to $\bar{A}\bar{e}_\kappa$ then $r_{M^*}(M_i)-r_{M^*}(M_{i-1})$ is A^*-isomorphic to $\overline{e^*_\kappa A^*}$. Thus we have

$$d_A(M_{i-1}-M_i)=f(\kappa)=g(\kappa)=d_{A^*}(r_{M^*}(M_i)-r_{M^*}(M_{i-1}))\,,$$

and hence $d_A(M)=d_{A^*}(M^*)$.

The converse part is clear.

Now we have the following theorem which corresponds to [2, Theorem 7].

THEOREM 8. *Let Q be a quasi-Frobenius two-sided A-A^*-module. Then Q is Frobenius if and only if either*

$$d_A(\mathfrak{l})+d_{A^*}(r_Q(\mathfrak{l}))=d_{A^*}(Q)$$

holds for every irreducible left ideal $\mathfrak{l}$ *of* A *or*

$$d_{A^*}(\mathfrak{r})+d_A(l_Q(\mathfrak{r}))=d_A(Q)$$

holds for every irreducible right ideal $\mathfrak{r}$ *of* A^*.

Proof. Suppose that Q is Frobenius and $\mathfrak{l}$ is a left ideal of A. If we apply Theorem 7 to $M=\mathfrak{l}$ whence $M^*=Q-r_Q(\mathfrak{l})$, then we have $d_A(\mathfrak{l})+d_{A^*}(r_Q(\mathfrak{l}))=d_{A^*}(Q)$. Similarly, we have $d_{A^*}(\mathfrak{r})+d_A(l_Q(\mathfrak{r}))=d_A(Q)$ for every right ideal $\mathfrak{r}$ of A^*.

The converse part is almost evident from (the second half of) Theorem 7.

Finally, we shall give a characterization of a Frobenius two-sided A-A^*-module.

THEOREM 9. *Let Q be a two-sided A-A^*-module. Then Q is Frobenius if and only if*

i) $d_A(\mathfrak{l})+d_{A^*}(r_Q(\mathfrak{l}))=d_{A^*}(Q)$,

ii) $d_{A^*}(\mathfrak{r})+d_A(l_Q(\mathfrak{r}))=d_A(Q)$,

iii) $d_A(L)+d_{A^*}(r_{A^*}(L))=d_{A^*}(A^*)$,

iv) $d_{A^*}(R)+d_A(l_A(R))=d_A(A)$

hold for every left ideal $\mathfrak{l}$ *of* A, *for every right ideal* $\mathfrak{r}$ *of* A^*, *for every left* A-*submodule* L *of* Q *and for every right* A^*-*submodule* R *of* Q.

Proof. The "only if" part can be proved in the similar way as in Theorem 8.

Conversely, let L be a left A-submodule of Q. Then, $r_{A^*}(L)$ is a right ideal of A^*, and hence by ii), $d_{A^*}(r_{A^*}(L))+d_A(l_Q(r_{A^*}(L)))=d_A(Q)$. On the other hand, by putting $\mathfrak{r}=A^*$ in ii), we have $d_{A^*}(A^*)=d_A(Q)$. These, combined with iii), yield $d_A(L)=d_A(l_Q(r_{A^*}(L)))$. But since $l_Q(r_{A^*}(L))\supseteqq L$, we have necessarily $l_Q(r_{A^*}(L))=L$. Similarly we have $r_Q(l_A(R))=R$ for every right A^*-submodule R of Q.

Now, if we put $L=Q$ in iii), then we have $d_A(Q)+d_{A^*}(r_{A^*}(Q))=d_{A^*}(A^*)$. Since $d_A(Q)=d_{A^*}(A^*)$, it follows $d_{A^*}(r_{A^*}(Q))=0$, that is, $r_{A^*}(Q)=0$. Thus Q is faithful with respect to A^*. Similarly, Q is also faithful with respect to A. Hence Q is quasi-Frobenius by Theorem 4. Our theorem now follows from Theorem 8.

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References

[1] G. Azumaya: A duality theory for injective modules, forthcoming.
[2] T. Nakayama: On Frobeniusean algebras, II, Ann. Math. **42** (1941), 1–21.
[3] K. Morita and H. Tachikawa: Character modules, submodules of a free module, and quasi-Frobenius rings, Math. Zeitschr. **65** (1956), 414–428.