

Title	Some remarks on quasi-Frobenius modules
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Citation	Osaka Mathematical Journal. 10(2) p.213-p.220
Issue Date	1958
oaire:version	VoR
URL	https://doi.org/10.18910/3910
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Some Remarks on Quasi-Frobenius Modules

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Recently Azumaya [1] introduced the concept of quasi-Frobenius two-sided modules to establish certain duality theorems for injective modules and also showed that it is a natural extension of the notion of quasi-Frobenius rings.

The purpose of the present note is to supplement the above paper by giving some characterizations of quasi-Frobenius modules. These results are regarded as generalizations of the known properties of quasi-Frobenius rings.

The writer wishes to express his gratitude to Professor G. Azumaya for his helpful suggestions.

§ 1. Quasi-Frobenius Modules.

Let A and A^* be two rings with unit elements and Q a two-sided A - A^* -module. Let M be a left A -module. Then the set M^* of all A -homomorphisms of M into Q forms a right A^* -module under the following definitions:

$$x(\varphi + \psi) = x\varphi + x\psi^{1)}, \quad x(\varphi a^*) = (x\varphi)a^*$$

for $x \in M$ and $a^* \in A^*$. This module is called the *right-dual module* of M with respect to Q . We may similarly define the *left-dual module* for any right A^* -module. Now, Azumaya calls Q a *quasi-Frobenius two-sided A - A^* -module* if i) Q is faithful (with respect to both A and A^*) and ii) for every maximal left ideal \mathfrak{l} of A and for every maximal right ideal \mathfrak{r} of A^* the right annihilator $r_Q(\mathfrak{l})$ and the left annihilator $l_Q(\mathfrak{r})$ of \mathfrak{l} and \mathfrak{r} in Q are A^* -irreducible and A -irreducible respectively, provided they are non-zero.

It should be noted that the above condition ii) is equivalent to the following condition: ii') for every irreducible A -submodule L of Q and for every irreducible A^* -submodule R of Q the right- and the left-dual

1) Let M_1 and M_2 be two left A -(or right A^* -)modules. For any A -(or A^* -)homomorphism $\varphi: M_1 \rightarrow M_2$ and any element $x \in M_1$, we denote by $x\varphi$ (or φx) the image of x by φ . If further ψ is an A -(or A^* -)homomorphism of M_2 into a third left A -(or right A^* -)module M_3 , then we shall denote by $\varphi \circ \psi$ (or $\psi \circ \varphi$) the composite mapping $x \rightarrow (x\varphi)\psi$ (or $x \rightarrow \psi(\varphi x)$).

modules of L and R with respect to Q are A^* - and A -irreducible respectively.

THEOREM 1. *Let Q be a quasi-Frobenius two-sided A - A^* -module. Then Au , $u \in Q$, is A -irreducible if and only if uA^* is A^* -irreducible, and moreover, in this case, they are isomorphic to the dual modules of each other.*

Proof. Suppose that Q is quasi-Frobenius and Au is A -irreducible. Since Au is isomorphic to the factor module $A-l_A(u)$, $r_Q(l_A(u))$ must, as the right-dual module of $A-l_A(u)$, be A^* -irreducible and hence necessarily coincides with uA^* .

THEOREM 2. *Let Q be a quasi-Frobenius two-sided A - A^* -module and M an irreducible left A -module such that its right-dual module M^* with respect to Q is non-zero. Then the A -endomorphism ring of M is isomorphic to the A^* -endomorphism ring of M^* .*

Proof. With each A -endomorphism φ of M we can associate an A^* -endomorphism $\varphi^{*2)}$ of M^* by setting $\varphi^*g = \varphi \circ g$, $g \in M^*$, that is,

$$(*) \quad x(\varphi^*g) = (x\varphi)g \quad x \in M, g \in M^* .$$

And it is easy to see that $(\varphi \circ \psi)^* = \varphi^* \circ \psi^*$ holds for any A -endomorphisms φ and ψ of M . Thus the mapping $\varphi \rightarrow \varphi^*$ is a homomorphism of the A -endomorphism ring of M into the A^* -endomorphism ring of M^* . Since M is A -irreducible and M^* is non-zero whence A^* -irreducible, M may be looked upon as the left-dual module of M^* with respect to Q , and therefore the above equality (*) shows that φ coincides with the dual-mapping of φ^* . Thus, the above homomorphism gives an isomorphism between the A -endomorphism ring of M and the A^* -endomorphism ring of M^* .

From now on, we shall assume the left and the right minimum conditions for A and A^* respectively unless otherwise stated. Then it should be noted, according to [1, Theorem 6], that every quasi-Frobenius two-sided A - A^* -module always contains an isomorphic image of every irreducible left A -module as well as that of every irreducible right A^* -module.

We first prove the following

THEOREM 3. *Let Q be a faithful two-sided A - A^* -module. Then Q is quasi-Frobenius if and only if every irreducible left A -module and every*

2) φ^* is called the dual mapping of φ with respect to Q .

irreducible right A^* -module are isomorphic to their double dual modules³⁾ with respect to Q respectively.

Proof. The "only if" part is clear. To prove the "if" part, consider an irreducible left A -module M and let M_1 be a maximal A^* -submodule of M^* (the existence of M_1 being assured by the right minimum condition for A^*). Then, since $l_M(M_1)$ is isomorphic to the left-dual module of the irreducible A^* -module $M^* - M_1$, it follows $l_M(M_1) \neq 0$ whence $l_M(M_1) = M$. But this implies evidently $M_1 = 0$, that is, M^* is A^* -irreducible. Similarly, the left-dual module of every irreducible right A^* -module is A -irreducible, and this shows that Q is quasi-Frobenius.

Next, we shall give another characterization of a quasi-Frobenius two-sided A - A^* -module, which may be regarded as a generalization of [2, Theorems 6 and 8].

THEOREM 4. *Let Q be a faithful two-sided A - A^* -module. In order that Q is quasi-Frobenius it is necessary and sufficient that for every irreducible left A -submodule L and for every irreducible right A^* -submodule R of Q the annihilator relations*

$$l_Q(r_{A^*}(L)) = L \quad \text{and} \quad r_Q(l_A(R)) = R$$

hold. And, if this is the case, for any $u \in Q$ the left A -submodule Au and the right A^* -submodule uA^* of Q are isomorphic to the dual modules of each other.

Proof. The necessity of the first part follows from [1, Corollary to Proposition 2].

To prove the sufficiency, we consider a maximal left ideal I of A such that $r_Q(I) \neq 0$. Let R be an irreducible right A^* -submodule of $r_Q(I)$. Then $l_A(R) = I$, since $l_A(R)$ is a proper left ideal containing I . Hence, $r_Q(I) = r_Q(l_A(R)) = R$, and thus $r_Q(I)$ is A^* -irreducible. Similarly, $l_Q(r)$ is either 0 or A -irreducible for every maximal right ideal r of A^* .

If we put $I = l_A(u)$, then Au is A -isomorphic to the factor module $A - I$. Thus, the right-dual module of Au is A^* -isomorphic to $r_Q(I)$. Take any non-zero element v from $r_Q(I)$. Then the mapping $au \rightarrow av$, $a \in A$, is obviously an A -homomorphism of Au into Q , and hence by [1, Theorem 6] there exists an element a^* of A^* such that $ua^* = v$. Thus $uA^* = r_Q(I)$, and this proves our theorem.

3) Let M be a left A -module and M^* the right-dual module of M with respect to Q . Then by the double dual module of M we mean the left-dual module $(M^*)^*$ of M^* with respect to Q . The double dual module of a right A^* -module is defined similarly.

Let N and N^* be the radicals of A and A^* respectively. Then, as is well-known, for any two-sided A - A^* -module Q the A -socle of Q , that is, the sum of all irreducible left A -submodules of Q coincides with $r_Q(N)$. Similarly, the A^* -socle of Q coincides with $l_Q(N^*)$.

Then we have the following theorem which corresponds to [2, Theorem 9].

THEOREM 5. *Let Q be a quasi-Frobenius two-sided A - A^* -module and let N and N^* be the radicals of A and A^* respectively. Then $r_Q(N^\nu) = l_Q(N^{*\nu})$ for every $\nu = 1, 2, \dots$.*

Proof. We shall prove this theorem by induction on ν . For $\nu = 1$, this was shown in [1, Theorem 1]. Let $\nu > 1$ and assume that $r_Q(N^{\nu-1}) = l_Q(N^{*(\nu-1)})$. Evidently $l_Q(N^{*\nu})N^* \subseteq l_Q(N^{*(\nu-1)})$. Hence $N^{\nu-1}l_Q(N^{*\nu})N^* \subseteq N^{\nu-1}l_Q(N^{*(\nu-1)}) = N^{\nu-1}r_Q(N^{\nu-1}) = 0$, $N^{\nu-1}l_Q(N^{*\nu}) \subseteq l_Q(N^*) = r_Q(N)$, and $N^\nu l_Q(N^{*\nu}) \subseteq Nr_Q(N) = 0$, that is, $l_Q(N^{*\nu}) \subseteq r_Q(N^\nu)$. Similarly we have $r_Q(N^\nu) \subseteq l_Q(N^{*\nu})$, whence $r_Q(N^\nu) = l_Q(N^{*\nu})$.

Let \bar{A} denote the (semi-simple) factor ring A/N , and let $\bar{A} = \bar{A}_1 \oplus \bar{A}_2 \oplus \dots \oplus \bar{A}_k$ be its direct decomposition into orthogonal simple two-sided ideals \bar{A}_κ . Let \bar{e}_κ be, for each κ , a primitive idempotent element in \bar{A}_κ . Then the k modules $\bar{A}\bar{e}_1, \bar{A}\bar{e}_2, \dots, \bar{A}\bar{e}_k$ exhaust, up to isomorphisms, all irreducible left A -modules. There exists, for each κ , an idempotent representative $e_\kappa (\in A)$ of the coset \bar{e}_κ . Then k idempotent elements e_1, e_2, \dots, e_k are all primitive and non-isomorphic, and any primitive idempotent element of A is isomorphic to one of them. Furthermore, if we denote by $f(\kappa)$ the capacity of $\bar{A}\bar{e}_\kappa$, i.e., the dimension of $\bar{A}\bar{e}_\kappa$ over its endomorphism division ring $\bar{e}_\kappa\bar{A}\bar{e}_\kappa$, A is as left A -module isomorphic to the direct sum $\sum_{\kappa=1}^k \oplus (Ae_\kappa)^{f(\kappa)}$. Similarly, we shall denote by $e^*_1, e^*_2, \dots, e^*_l$ a complete system of non-isomorphic primitive idempotent elements in A^* and by $g(\lambda)$, for each $\lambda = 1, 2, \dots, l$, the capacity of the irreducible right A^* -module $\bar{e}^*_\lambda\bar{A}^*$.

LEMMA. *Let Q be a two-sided A - A^* -module. Then the right- and the left-dual modules of the irreducible left A - and the irreducible right A^* -modules $\bar{A}\bar{e}_\kappa^*$ and $\bar{e}^*_\lambda\bar{A}$ with respect to Q are isomorphic to $e_\kappa r_Q(N)$ and $l_Q(N^*)e^*_\lambda$ respectively.*

Proof. If we put $I_\kappa = A(1 - e_\kappa) + N$ then I_κ is a maximal left ideal of A such that $A - I_\kappa$ is A -isomorphic to $\bar{A}\bar{e}_\kappa$. Hence $r_Q(I_\kappa) = r_Q(1 - e_\kappa) \cap r_Q(N) = e_\kappa Q \cap r_Q(N) = e_\kappa r_Q(N)$ is A^* -isomorphic to the right-dual module of $\bar{A}\bar{e}_\kappa$ with respect to Q . Similarly, $l_Q(N^*)e^*_\lambda$ is, for each λ , A -isomorphic to the left-dual module of $\bar{e}^*_\lambda\bar{A}^*$. This proves our lemma.

From this lemma it follows now immediately

THEOREM 6. *Let Q be a faithful two-sided A - A^* -module. Then Q is quasi-Frobenius if and only if $e_\kappa r_Q(N)$ and $l_Q(N^*)e^*_\lambda$ are, for each κ and λ , A^* - and A -irreducible respectively.*

Now, let Q be a quasi-Frobenius two-sided A - A^* -module. If we associate with each irreducible left A -module its right-dual module with respect to Q , we have a one-to-one correspondence between irreducible left A -modules and irreducible right A^* -modules. This means that $k=l$ and, if we order $e^*_{\kappa_1}, e^*_{\kappa_2}, \dots, e^*_{\kappa_k}$ suitably, $\overline{e^*_{\kappa}A^*}$ is, for each κ , isomorphic to the right-dual module of $\overline{A\bar{e}_\kappa}$. We shall henceforth retain such ordering of e^*_{κ} . It follows then from the above lemma that

$$e_\kappa r_Q(N) \cong \overline{e^*_{\kappa}A^*}, \quad l_Q(N^*)e^*_{\kappa} \cong \overline{A\bar{e}_\kappa}$$

for each κ . On the other hand, since $r_Q(N) = l_Q(N^*)$ (Theorem 5), $e_\kappa r_Q(N) = e_\kappa l_Q(N^*)$ and $l_Q(N^*)e^*_{\kappa} = r_Q(N)e^*_{\kappa}$ are necessarily unique simple A^* - and A -submodules of $e_\kappa Q$ and Qe^*_{κ} respectively. These facts together give a second proof of the necessity of [1, Theorem 12].

Let Q be a quasi-Frobenius two-sided A - A^* -module. Suppose that a left A -module M and a right A^* -module M^* form an orthogonal pair⁴⁾ with respect to Q , and suppose further that either M or M^* satisfies both the maximum and the minimum conditions for A - or A^* -submodules respectively. Then, for any A -submodules M_1 and M_2 of M such that $M_1 \supseteq M_2$, the factor modules $M_1 - M_2$ and $r_{M^*}(M_2) - r_{M^*}(M_1)$ form also an orthogonal pair with respect to Q in the natural manner, and hence, by [1, Proposition 2], they are the dual modules of each other (for this, we may not necessarily assume the minimum conditions for A and A^*).

In particular, if M_2 is a maximal A -submodule of M_1 and $M_1 - M_2 \cong \overline{A\bar{e}_\kappa}$ then $r_{M^*}(M_1)$ is a maximal A^* -submodule of $r_{M^*}(M_2)$ and $r_{M^*}(M_2) - r_{M^*}(M_1) \cong \overline{e^*_{\kappa}A^*}$, which is a generalization of [2, Theorem 9].

§ 2. Frobenius Modules.

We shall call, following G. Azumaya [1], a quasi-Frobenius two-sided A - A^* -module Q to be *Frobenius* if, for any irreducible left A -module M , the capacity of M coincides with that of the right-dual module M^* of M with respect to Q , that is, $f(\kappa) = g(\kappa)$ for each κ .

4) Let M be a left A -module and M^* a right A^* -module. Suppose that for any $x \in M$ and $y \in M^*$ there corresponds a product xy in Q such that

$$\begin{aligned} (x+x')y &= xy+x'y, & x(y+y') &= xy+xy', \\ a(xy) &= (ax)y, & (xy)a^* &= x(ya^*) \end{aligned}$$

for $x, x' \in M$; $y, y' \in M^*$; $a \in A$; $a^* \in A^*$, and moreover $xM^* = 0$, $x \in M$, and $My = 0$, $y \in M^*$, imply always $x = 0$ and $y = 0$ then we shall, following Azumaya [1], say that M and M^* form an *orthogonal pair* with respect to Q .

Let L be an irreducible left A -module. Then we denote by $d_A(L)$ the capacity of L , that is, $d_A(L) = f(\kappa)$ if $L \cong \bar{A}\bar{e}_\kappa$. And generally, for any left A -module L having a composition series

$$L = L_0 \supset L_1 \supset \dots \supset L_{s-1} \supset L_s = 0,$$

we put

$$d_A(L) = \sum_{i=1}^s d_A(L_{i-1} - L_i).$$

It is clear that for any A -submodule L' of L we have $d_A(L) = d_A(L - L') + d_A(L')$. For an A^* -module R possessing a composition series we may also define $d_{A^*}(R)$ in the similar manner.

THEOREM 7. *Let Q be a quasi-Frobenius two-sided A - A^* -module. If Q is Frobenius, then for every finitely generated left A -module M we have $d_A(M) = d_{A^*}(M^*)$, where M^* is the right-dual module of M with respect to Q . Conversely, if the above equality holds for every irreducible left A -module M , then Q is Frobenius.*

Proof. Suppose that Q is Frobenius and M is a finitely generated left A -module. Then, by [1, Theorem 8], the right-dual module M^* of M with respect to Q is also finitely generated with respect to A^* and M coincides with the left-dual module of M^* with respect to Q . Let

$$M = M_0 \supset M_1 \supset \dots \supset M_{s-1} \supset M_s = 0$$

be a composition series of M . Then, as we have seen at the end of the preceding section,

$$M^* = r_{M^*}(M_s) \supset r_{M^*}(M_{s-1}) \supset \dots \supset r_{M^*}(M_0) = 0$$

gives a composition series of M^* , and moreover, if $M_{i-1} - M_i$ is A -isomorphic to $\bar{A}\bar{e}_\kappa$ then $r_{M^*}(M_i) - r_{M^*}(M_{i-1})$ is A^* -isomorphic to $\bar{e}_\kappa^* \bar{A}^*$. Thus we have

$$d_A(M_{i-1} - M_i) = f(\kappa) = g(\kappa) = d_{A^*}(r_{M^*}(M_i) - r_{M^*}(M_{i-1})),$$

and hence $d_A(M) = d_{A^*}(M^*)$.

The converse part is clear.

Now we have the following theorem which corresponds to [2, Theorem 7].

THEOREM 8. *Let Q be a quasi-Frobenius two-sided A - A^* -module. Then Q is Frobenius if and only if either*

$$d_A(I) + d_{A^*}(r_Q(I)) = d_{A^*}(Q)$$

holds for every irreducible left ideal I of A or

$$d_{A^*}(r) + d_A(l_Q(r)) = d_A(Q)$$

holds for every irreducible right ideal r of A^* .

Proof. Suppose that Q is Frobenius and I is a left ideal of A . If we apply Theorem 7 to $M=I$ whence $M^*=Q-r_Q(I)$, then we have $d_A(I) + d_{A^*}(r_Q(I)) = d_{A^*}(Q)$. Similarly, we have $d_{A^*}(r) + d_A(l_Q(r)) = d_A(Q)$ for every right ideal r of A^* .

The converse part is almost evident from (the second half of) Theorem 7.

Finally, we shall give a characterization of a Frobenius two-sided A - A^* -module.

THEOREM 9. *Let Q be a two-sided A - A^* -module. Then Q is Frobenius if and only if*

- i) $d_A(I) + d_{A^*}(r_Q(I)) = d_{A^*}(Q)$,
- ii) $d_{A^*}(r) + d_A(l_Q(r)) = d_A(Q)$,
- iii) $d_A(L) + d_{A^*}(r_{A^*}(L)) = d_{A^*}(A^*)$,
- iv) $d_{A^*}(R) + d_A(l_A(R)) = d_A(A)$

hold for every left ideal I of A , for every right ideal r of A^* , for every left A -submodule L of Q and for every right A^* -submodule R of Q .

Proof. The "only if" part can be proved in the similar way as in Theorem 8.

Conversely, let L be a left A -submodule of Q . Then, $r_{A^*}(L)$ is a right ideal of A^* , and hence by ii), $d_{A^*}(r_{A^*}(L)) + d_A(l_Q(r_{A^*}(L))) = d_A(Q)$. On the other hand, by putting $r=A^*$ in ii), we have $d_{A^*}(A^*) = d_A(Q)$. These, combined with iii), yield $d_A(L) = d_A(l_Q(r_{A^*}(L)))$. But since $l_Q(r_{A^*}(L)) \supseteq L$, we have necessarily $l_Q(r_{A^*}(L)) = L$. Similarly we have $r_Q(l_A(R)) = R$ for every right A^* -submodule R of Q .

Now, if we put $L=Q$ in iii), then we have $d_A(Q) + d_{A^*}(r_{A^*}(Q)) = d_{A^*}(A^*)$. Since $d_A(Q) = d_{A^*}(A^*)$, it follows $d_{A^*}(r_{A^*}(Q)) = 0$, that is, $r_{A^*}(Q) = 0$. Thus Q is faithful with respect to A^* . Similarly, Q is also faithful with respect to A . Hence Q is quasi-Frobenius by Theorem 4. Our theorem now follows from Theorem 8.

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(Received September 15, 1958)

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