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5-FOLD TRANSITIVE PERMUTATION GROUPS IN WHICH THE STABILIZER OF FIVE POINTS HAS A NORMAL SYLOW 2-SUBGROUP

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1. Introduction

In this paper we shall prove the following theorem.

Theorem. Let $G$ be a 5-fold transitive permutation group on a set $\Omega = \{1, 2, \ldots, n\}$. Let $P$ be a Sylow 2-subgroup of $G_{12345}$. If $P$ is a nonidentity normal subgroup of $G_{12345}$, then $G$ is one of the following groups: $S_7$, $A_9$ or $M_{24}$.

The idea of the proof of the theorem is derived from Oyama [7].

In order to prove the theorem, we shall use the following two lemmas, which will be proved in Sections 3 and 4.

Lemma 1. Let $G$ be a permutation group on $\Omega = \{1, 2, \ldots, n\}$ satisfying the following three conditions.

(i) For any five points $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and $\alpha_5$ in $\Omega$, the order of $G_{\alpha_1\ldots\alpha_5}$ is even.
(ii) For any five points $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and $\alpha_5$ in $\Omega$, a Sylow 2-subgroup of $G_{\alpha_1\ldots\alpha_5}$ is normal in $G_{\alpha_1\ldots\alpha_5}$.
(iii) Any involution in $G$ fixes at most seven points.

Then $G$ is $S_7$ or $A_9$.

Lemma 2. Let $G$ be a permutation group on $\Omega = \{1, 2, \ldots, n\}$ satisfying the following four conditions.

(i) For any five points $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and $\alpha_5$ in $\Omega$, the order of $G_{\alpha_1\ldots\alpha_5}$ is even.
(ii) For any five points $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and $\alpha_5$ in $\Omega$, a Sylow 2-subgroup of $G_{\alpha_1\ldots\alpha_5}$ is normal in $G_{\alpha_1\ldots\alpha_5}$.
(iii) Any involution in $G$ fixes at most nine points.
(iv) For any 2-subgroup $X$ fixing exactly nine points, $N(X)^{(x)} \leq A_9$.

Then $G$ is $S_7$ or $A_9$.

The author thanks Professor Eiichi Bannai for his kind advice.

We shall use the same notation as in [3].
2. Proof of the Theorem

Let $G$ be a group satisfying the assumption of the theorem.

Let $P$ be the unique Sylow 2-subgroup of $G$. If $P$ is semiregular on $\Omega - I(P)$ or $|I(P)| > 6$, then $G$ is $S_7$, $A_9$ or $M_{24}$ by [2], [3], [4] and [5]. Hence from now on we assume that $P$ is not semiregular on $\Omega - I(P)$ and that $|I(P)| \leq 6$, and we prove that this case does not arise. If $|I(P)| = 6$, then $|I(G_{12345})| = 6$, a contradiction to [1]. Hence $|I(P)| = 5$.

Let $r$ be $\max |I(a)|$, where $a$ ranges over all involutions in $G$. Since $P$ is not semiregular on $\Omega - I(P)$, we have $r \geq 7$.

Suppose $r = 7$. Let $t$ be a point of a minimal orbit of $P$ in $\Omega - I(P)$. It is easily seen that $N(P_t)^{I(P_t)} = S_7$. By [6], we have a contradiction.

Suppose $r = 9$. Let $Q$ be a subgroup of $P$ such that the order of $Q$ is maximal among all subgroups of $P$ fixing exactly nine points. By Lemma 1, $N(Q)^{I(P)} = A_9$. Again by [6], we have a contradiction. Thus we have $r \geq 11$.

Let $Q$ be a subgroup of $P$ such that the order of $Q$ is maximal among all subgroups of $P$ fixing more than nine points. Set $N = N(Q)^{I(P)}$. Then $N$ satisfies the following conditions.

(i) $N$ is a permutation group on $I(Q)$, and its degree is not less than eleven.
(ii) For any five points $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and $\alpha_5$ in $I(Q)$, the order of $N_{\alpha_1 \cdots \alpha_5}$ is even.
(iii) For any five points $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and $\alpha_5$ in $I(Q)$, a Sylow 2-subgroup of $N_{\alpha_1 \cdots \alpha_5}$ is normal in $N_{\alpha_1 \cdots \alpha_5}$.
(iv) Any involution fixes at most nine points.

By Lemma 1, $N$ has an involution fixing exactly nine points. Let $X$ be any 2-subgroup of $N$ fixing exactly nine points. Set $\Delta = I(X)$. Let $S$ be the Sylow 2-subgroup of $G_\Delta$. Since $I(S) = \Delta$, we have $N_G(S)^{I(\Delta)} = A_9$ by Lemma 1. Since $S$ is a characteristic subgroup of $G_\Delta$, $N$ satisfies the following condition.

(v) For any 2-subgroup $X$ fixing exactly nine points, $N_N(X)^{I(X)} \leq A_9$.

Considering the permutation group $N$, we have a final contradiction by Lemma 2.

3. Proof of Lemma 1

Let $G$ be a permutation group satisfying the assumptions of Lemma 1. If $G$ has no involution fixing seven points, then $G$ is $S_7$ or $A_9$ by [8, Lemma 6] and [2]. Hence from now on we assume that $G$ has an involution fixing exactly seven points, and we prove Lemma 1 by way of contradiction. We may assume that $G$ has an involution $a$ fixing exactly $1, 2, \cdots, 7$ and $8$ and

$$a = (1) (2) \cdots (7) (8 9) \cdots.$$
5-Fold Transitive Permutation Groups

Set \( T = C(a)_9 \).

(1) For any three points \( i, j \) and \( k \) in \( I(a) \), there is an involution in \( T_{ijk} \). Any involution in \( T \) is not the identity on \( I(a) \).

Proof. Since \( a \) normalizes \( G_{89ij} \) and \( G_{90ij} \) is of even order, \( G_{89ij} \) has an involution \( x \) commuting with \( a \). Then \( x \in T_{ijk} \). Since \( |I(a)| = 7 \) and \( I(x) \subseteq \{8, 9\} \), any involution in \( T \) is not the identity on \( I(a) \) by (iii).

(2) For any three points \( i, j \) and \( k \) in \( I(a) \), a Sylow 2-subgroup of \( T_{ijk} \) is normal in \( T_{ijk} \), and so a Sylow 2-subgroup of \( T^{(a)} \) is normal in \( T^{(a)} \).

Proof. Let \( S \) be a Sylow 2-subgroup of \( T_{ijk} \). Since \( S \) is a Sylow 2-subgroup of \( C(a)_{89ij} \), \( S \) is a normal subgroup of \( C(a)_{89ij} \) by (ii).

We have the following property from (2).

(3) If \( x_1^{(a)} \) and \( x_2^{(a)} \) are involutions in \( T^{(a)} \) with \( |I(x_1^{(a)}) \cap I(x_2^{(a)})| \geq 3 \), then \( x_1^{(a)}x_2^{(a)} \) is a 2-element of \( T^{(a)} \).

(4) Since \( |I(a)| = 7 \), \( T^{(a)} \) is one of the following groups.

(a) \( T^{(a)} \) is intransitive and has an orbit of length one or two.
(b) \( T^{(a)} \) is intransitive and has an orbit of length three.
(c) \( T^{(a)} \) is primitive.

(5) The case (a) does not hold.

Proof. Suppose \( T^{(a)} \) has an orbit of length one or two. We may assume that either \( \{1\} \) or \( \{1, 2\} \) is such an orbit. By (1), \( T_{234} \) has an involution \( x_1 \). We may assume that

\[
x_1 = (1)(2)(3)(4)(5)(6)(7) \ldots.
\]

Similarly \( T_{235} \) has an involution \( x_2 \) of the form

\[
x_2 = (1)(2)(3)(4)(5)(67) \ldots, (1)(2)(3)(5)(46)(7) \ldots \text{ or } (1)(2)(5)(47)(6) \ldots.
\]

If the first or the second alternative holds, then \( |I(x_1^{(a)}) \cap I(x_2^{(a)})| = 4 \), and \( x_1^{(a)}x_2^{(a)} \) is not a 2-element, a contradiction to (3). Thus \( x_2 = (1)(23)(5)(47)(6) \ldots \). Again by (1), \( T_{235} \) has an involution \( x_3 \) of the form

\[
x_3 = (1)(2)(4)(5)(3)(67) \ldots, (1)(2)(4)(5)(36)(7) \ldots \text{ or } (1)(2)(4)(5)(37)(6) \ldots.
\]

In every case, we get a contradiction to (3) by considering either \( x_1^{(a)}x_2^{(a)} \) or \( x_2^{(a)}x_3^{(a)} \).
(6) The case (b) does not hold.

Proof. Suppose \( T^{(e)} \) has an orbit of length three. We may assume that \( \{1,2,3\} \) is such an orbit of length three. By (5), \( \{4,5,6,7\} \) is a \( T^{(e)} \)-orbit. By (1), \( T_{1456} \) has an involution \( x_1 \). We may assume that

\[
x_1 = (1 \ 2 \ 3) (4 \ 5 \ 6) (7) \cdots.
\]

Since \( \{1,2,3\} \) is a \( T^{(e)} \)-orbit, \( T \) has an element \( y \) of the form

\[
y = (1 \ 2 \ 3) \cdots.
\]

Set \( x_2 = x_1^2 \), then \( x_2 = (2 \ 3) (1 \ 4 \ 5) (6) (7) \cdots \). So, \( |I(x_1^{(e)}) \cap I(x_2^{(e)})| = 4 \), and \( x_1 x_2 = (1 \ 3 \ 2) (4 \ 5 \ 6) (7) \cdots \), which is a contradiction. Hence \( T^{(e)} \) has no orbit of length three.

(7) We show that the case (c) does not hold, and complete the proof of Lemma 1.

Proof. Suppose \( T^{(e)} \) is primitive. By (1), we have \( T^{(e)} \supseteq A_7 \) (cf.e.g.[10]). Therefore for any involution \( x \) in \( G \) fixing exactly seven points, \( C(x)^{(e)} \supseteq A_7 \).

Let \( \Gamma \) be any subset of \( \Omega \) with \( |\Gamma| = 5 \). Set \( \Gamma = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\} \). By (i), \( G_{\alpha_1 \cdots \alpha_5} \) has an involution. If \( G_{\alpha_1 \cdots \alpha_5} \) has an involution \( x_1 \) fixing exactly seven points, then \( C(x_1)^{(e)} \supseteq A_7 \). Hence \( G_{\alpha_1 \cdots \alpha_5}^{\alpha_1} = S_5 \). Suppose that \( G_{\alpha_1 \cdots \alpha_5} \) has no involution fixing seven points. Let \( x_2 \) be an involution in \( G_{\alpha_1 \cdots \alpha_5} \). Let \( x_2 = (\alpha_1) \cdots (\alpha_5) (\beta_1 \beta_2) \cdots \). It is easily seen that \( C(x_2)^{(e)} \supseteq S_5 \). Hence \( G_{\alpha_1 \cdots \alpha_5}^{\alpha_1} = S_5 \). Thus we have \( G_{\alpha_1 \cdots \alpha_5}^{\alpha_1} = S_5 \) in either case. Therefore by [9, Lemma 3], \( G \) is 4-fold transitive on \( \Omega \).

Let \( x \) be an involution in \( G \) fixing seven points. Let \( S \) be the Sylow 2-subgroup of \( G_{(e)} \). Since \( C(x)^{(e)} \supseteq A_7 \), we have \( N(S)^{(e)} \supseteq A_7 \). By [6], we get a contradiction.

Thus we complete the proof of Lemma 1.

4. Proof of Lemma 2

Let \( G \) be a permutation group satisfying the assumptions of Lemma 2. If \( G \) has no involution fixing nine points, then \( G \) is \( S_7 \) or \( A_9 \) by Lemma 1. Hence from now on we assume that \( G \) has an involution fixing exactly nine points, and we prove Lemma 2 by way of contradiction. We may assume that \( G \) has an involution \( a \) fixing 1, 2, \cdots, 9 and

\[
a = (1 \ 2 \ \cdots \ (9 \ 10 \ 11) \cdots.
\]

Set \( T = C(a)_{1011} \).

(1) For any three points \( i, j \) and \( k \) in \( I(a) \), there is an involution in \( T_{ijk} \). Any involution in \( T \) is not the identity on \( I(a) \).
For any three points \(i, j\) and \(k\) in \(I(a)\), a Sylow 2-subgroup of \(T_{ijk}\) is normal in \(T_{ij}\), and so a Sylow 2-subgroup of \(T_{ijk}'\) is normal in \(T_{ij}'\).

If \(x_1^{(a)}\) and \(x_2^{(a)}\) are involutions in \(T^{(a)}\) with \(\mid I(x_1^{(a)}) \cap I(x_2^{(a)}) \mid \geq 3\), then \(x_1^{(a)}x_2^{(a)}\) is a 2-element of \(T^{(a)}\).

The proofs of (1), (2) and (3) are similar to the proofs of (1), (2) and (3) in Section 3 respectively.

Since \(\mid I(a) \mid = 9\), \(T^{(a)}\) is one of the following groups.

(a) \(T^{(a)}\) is intransitive and has an orbit of length one or two.
(b) \(T^{(a)}\) is either an intransitive group with an orbit of length three, or a transitive but imprimitive group with three blocks of length three.
(c) \(T^{(a)}\) is intransitive and has an orbit of length four.
(d) \(T^{(a)}\) is primitive.

The case (a) does not hold.

Proof. Suppose \(T^{(a)}\) has an orbit of length one or two. We may assume that either \(\{1\}\) or \(\{1,2\}\) is such an orbit. By (1), \(T_{234}\) has an involution \(x_1\). By the assumption (iv), we may assume that

\[
x_1 = (1) (2) (3) (4) (5) (6 7) (8 9) \ldots.
\]

Similarly \(T_{235}\) has an involution \(x_2\). We may assume without loss of generality that

\[
x_2^{(a)} = (1) (2) (3) (6) (7) (4 5) (8 9) \ldots \alpha , \\
(1) (2) (3) (6) (7) (4 8) (5 9) \ldots \beta , \\
(1) (2) (3) (6) (8) (7 9) (4 5) \ldots \gamma , \\
(1) (2) (3) (6) (8) (7 4) (5 9) \ldots \delta , \\
(1) (2) (3) (6) (4) (5 7) (8 9) \ldots \epsilon \text{ or } \\
(1) (2) (3) (6) (4) (7 8) (5 9) \ldots \zeta .
\]

If \(x_2^{(a)}\) is of the form \(\delta\), \(\epsilon\) or \(\zeta\), then \(\mid I(x_1^{(a)}) \cap I(x_2^{(a)}) \mid \geq 3\), and \(x_1^{(a)}x_2^{(a)}\) is not a 2-element, a contradiction to (3). Hence \(x_2^{(a)}\) is of the form \(\alpha\), \(\beta\) or \(\gamma\). \(T_{269}\) has an involution \(x_3\). \(x_3^{(a)}\) is of the form

\[
x_3^{(a)} = (1) (2) (6) (9) (3) (4 5) (7 8) \ldots \text{1}, \\
(1) (2) (6) (9) (3) (4 7) (5 8) \ldots \text{2}, \\
(1) (2) (6) (9) (3) (4 8) (5 7) \ldots \text{3}, \\
(1) (2) (6) (9) (4) (3 5) (7 8) \ldots \text{4}, \\
(1) (2) (6) (9) (4) (3 7) (5 8) \ldots \text{5}, \\
(1) (2) (6) (9) (4) (3 8) (5 7) \ldots \text{6},
\]
If $x^{I(a)}$ is of the form $\alpha$, $\beta$, $\gamma$, $\delta$, $\varepsilon$, $\zeta$, then $|I(x^{I(a)}) \cap I(x^{I(a)})| = 3$, and $x_{1}^{I(a)}x_{2}^{I(a)}$ is not a 2-element, which is a contradiction. Suppose $x_{3}^{I(a)}$ is of the form $\eta$. Then $x_{1}x_{2}= (1) (2) (3) (4) (5 9) (6 7) \cdots$, and $(x_{1}x_{2})^{2} = (1) (2) (3) (4) (5 9) (8 9) (6 7) \cdots$. Set $y=x_{1}x_{2}$ and $x_{1}=x_{1}$. Then $x_{1}=(1) (2) (3) (4) (9) (6 7) (5 8) \cdots$. So, $|I(x^{I(a)}) \cap I(x^{I(a)})| = 4$, and $x_{1}x_{2}=(1) (2) (3) (4) (5 9) (8 9) (6 7) \cdots$, which is a contradiction. If $x_{4}^{I(a)}$ is of the form $\alpha$, $\beta$, $\gamma$, $\delta$, $\varepsilon$, or $\zeta$, we have a contradiction by the same argument as in the case $\eta$. Hence $x_{4}^{I(a)}$ is of the form $\alpha$, $\beta$, or $\gamma$. Suppose $x_{4}^{I(a)}$ is of the form $\alpha$ or $\gamma$. Since $x_{4}^{I(a)}$ is of the form $\alpha$, $\beta$, or $\gamma$, we get a contradiction by considering $x_{4}^{I(a)}x_{5}^{I(a)}$.

Suppose $x_{4}^{I(a)}$ is of the form $\beta$. If $x_{3}^{I(a)}$ is of the form $\alpha$ or $\beta$, we get a contradiction by considering $x_{4}^{I(a)}x_{5}^{I(a)}$. Suppose $x_{4}^{I(a)}$ is of the form $\beta$. Then $x_{1}x_{2} = (1) (2) (3) (4) (5 9) (6 7) \cdots$. Set $x_{3} = (x_{1}x_{2})^{2}$, then $x_{3} = (1) (2) (3) (4) (5 9) (3 7) (4 8) \cdots$. So, $|I(x^{I(a)}) \cap I(x^{I(a)})| = 3$, and $x_{1}x_{5} = (1) (2) (5) (3 7 6) (4 8 9) \cdots$, which is a contradiction. Thus $T^{I(a)}$ has neither orbit of length one nor orbit of length two.

(6) The case (b) does not hold.

Proof. Suppose $T^{I(a)}$ has an orbit of length three or three blocks of length three. We may assume that $\{1,2,3\}$ is such an orbit or a block.

Assume that $T^{I(a)}$ has three orbits of length three or three blocks of length three. We may assume that $\{1,2,3\}$, $\{4,5,6\}$ and $\{7,8,9\}$ are the orbits or the blocks. $T_{124}$ has an involution $x_{1}$. By the assumption (iv),

$$x_{1} = (1) (2) (3) (4) (5 6) \cdots .$$

Similarly $T_{125}$ has an involution $x_{2}$ of the form

$$x_{2} = (1) (2) (3) (5) (4 6) \cdots .$$

So, $|I(x_{1}^{I(a)}) \cap I(x_{2}^{I(a)})| \geq 3$, and $x_{1}x_{2} = (1) (2) (3) (4 6 5) \cdots$, which is a contradiction.

By (5) and the above, we have that $\{1,2,3\}$ and $\{4,5,6,7,8,9\}$ are the
$T^{(a)}$-orbits. Since $3 \mid |\{4, 5, \ldots, 9\}|$, we may assume that $T$ has an element $y$ of the form

$$y = (4\ 5\ 6) \ldots .$$

$T_{789}$ has an involution $x_1$. We may assume that

$$x_1 = (1\ 2)\ (3)\ (4\ 5)\ (6)\ (7)\ (8)\ (9) \ldots .$$

Set $x_2 = x_1^2$, then $x_2 = (5\ 6)\ (4)\ (7)\ (8)\ (9) \ldots$. So, $|I(x_1^{(a)}) \cap I(x_2^{(a)})| \geq 3$, and $x_1 x_2 = (4\ 6\ 5)\ (7)\ (8)\ (9) \ldots$, which is a contradiction. Thus $T^{(a)}$ has neither orbit of length three nor block of length three.

(7) The case $(c)$ does not hold.

Proof. Suppose $T^{(a)}$ has an orbit of length four. We may assume that $\{1, 2, 3, 4\}$ is a $T^{(a)}$-orbit. By (5) and (6), $\{5, 6, 7, 8, 9\}$ is a $T^{(a)}$-orbit. Since $5 \mid |\{5, 6, 7, 8, 9\}|$, we may assume that $T$ has an element $y$ of the form

$$y = (1)\ (2)\ (3)\ (4)\ (5\ 6\ 7\ 8\ 9) \ldots .$$

$T_{123}$ has an involution $x_1$. We may assume that $x_1$ is of the form

$$x_1 = (1)\ (2)\ (3)\ (4)\ (5)\ (6\ 7)\ (8\ 9) \ldots ,$$

or

$$x_1 = (1)\ (2)\ (3)\ (4)\ (5)\ (6\ 8)\ (7\ 9) \ldots .$$

Set $x_2 = x_1^2$. Then $x_2$ is of the following form respectively:

$$x_2 = (1)\ (2)\ (3)\ (4)\ (6)\ (7\ 8)\ (5\ 9) \ldots ,$$

or

$$x_2 = (1)\ (2)\ (3)\ (4)\ (6)\ (7\ 9)\ (5\ 8) \ldots .$$

In any case, we get a contradiction by considering $x_1^{(a)} x_2^{(a)}$.

(8) We show that the case $(d)$ does not hold, and complete the proof of Lemma 2.

Proof. If $T^{(a)}$ is primitive, then by (1) and the assumption (iv), we have $T^{(a)} = A_5$ (cf. e.g. [10]). But this contradicts (2). Thus $T^{(a)}$ is not primitive.

Thus we complete the proof of Lemma 2.

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References