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5-FOLD TRANSITIVE PERMUTATION GROUPS IN WHICH THE STABILIZER OF FIVE POINTS HAS A NORMAL SYLOW 2-SUBGROUP

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1. Introduction

In this paper we shall prove the following theorem.

Theorem. *Let G be a 5-fold transitive permutation group on a set $\Omega=\{1,2,\cdots,n\}$. Let P be a Sylow 2-subgroup of G_{12345}. If P is a nonidentity normal subgroup of G_{12345}, then G is one of the following groups: S_7, A_9 or M_{24}.*

The idea of the proof of the theorem is derived from Oyama [7].

In order to prove the theorem, we shall use the following two lemmas, which will be proved in Sections 3 and 4.

Lemma 1. *Let G be a permutation group on $\Omega=\{1,2,\cdots,n\}$ satisfying the following three conditions.*

(i) *For any five points $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and α_5 in Ω, the order of $G_{\alpha_1\cdots\alpha_5}$ is even.*

(ii) *For any five points α_1, α_2, α_3, α_4 and α_5 in Ω, a Sylow 2-subgroup of $G_{\alpha_1\cdots\alpha_5}$ is normal in $G_{\alpha_1\cdots\alpha_5}$.*

(iii) *Any involution in G fixes at most seven points.*

Then G is S_7 or A_9.

Lemma 2. *Let G be a permutation group on $\Omega=\{1,2,\cdots,n\}$ satisfying the following four conditions.*

(i) *For any five points $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and α_5 in Ω, the order of $G_{\alpha_1\cdots\alpha_5}$ is even.*

(ii) *For any five points α_1, α_2, α_3, α_4 and α_5 in Ω, a Sylow 2-subgroup of $G_{\alpha_1\cdots\alpha_5}$ is normal in $G_{\alpha_1\cdots\alpha_5}$.*

(iii) *Any involution in G fixes at most nine points.*

(iv) *For any 2-subgroup X fixing exactly nine points, $N(X)^{I(X)}\leqq A_9$.*

Then G is S_7 or A_9.

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We shall use the same notation as in [3].

2. Proof of the Theorem

Let G be a group satisfying the assumption of the theorem.

Let P be the unique Sylow 2-subgroup of G_{12345}. If P is semiregular on Ω-$I(P)$ or $|I(P)|>6$, then G is S_7, A_9 or $M_{24}$by [2], [3], [4] and [5]. Hence from now on we assume that P is not semiregular on Ω-$I(P)$ and that $|I(P)|\leqq 6$, and we prove that this case does not arise. If $|I(P)|=6$, then $|I(G_{12345})|=6$, a contradiction to [1]. Hence $|I(P)|=5$.

Let r be $\mathrm{Max}|I(a)|$, where a ranges over all involutions in G. Since P is not semiregular on Ω-$I(P)$, we have $r\geqq 7$.

Suppose $r=7$. Let t be a point of a minimal orbit of P in Ω-$I(P)$. It is easily seen that $N(P_t)^{I(P_t)}=S_7$. By [6], we have a contradiction.

Suppose $r=9$. Let Q be a subgroup of P such that the order of Q is maximal among all subgroups of P fixing exactly nine points. By Lemma 1, $N(Q)^{I(Q)}=A_9$. Again by [6], we have a contradiction. Thus we have $r\geqq 11$.

Let Q be a subgroup of P such that the order of Q is maximal among all subgroups of P fixing more than nine points. Set $N=N(Q)^{I(Q)}$. Then N satisfies the following conditions.

(i) *N is a permutation group on $I(Q)$, and its degree is not less than eleven.*
(ii) *For any five points α_1, α_2, α_3, α_4 and α_5 in $I(Q)$, the order of $N_{\alpha_1\cdots\alpha_5}$ is even.*
(iii) *For any five points α_1, α_2, α_3, α_4 and α_5 in $I(Q)$, a Sylow 2-subgroup of $N_{\alpha_1\cdots\alpha_5}$ is normal in $N_{\alpha_1\cdots\alpha_5}$.*
(iv) *Any imvolution fixes at most nine points.*

By Lemma 1, N has an involution fixing exactly nine points. Let X be any 2-subgroup of N fixing exactly nine points. Set $\Delta=I(X)$. Let S be the Sylow 2-subgroup of G_Δ. Since $I(S)=\Delta$, we have $N_G(S)^{I(S)}=A_9$ by Lemma 1. Since S is a characteristic subgroup of G_Δ, N satisfies the following condition.

(v) *For any 2-subgroup X fixing exactly nine points, $N_N(X)^{I(X)}\leqq A_9$.*

Considering the permutation group N, we have a final contradiction by Lemma 2.

3. Proof of Lemma 1

Let G be a permutation group satisfying the assumptions of Lemma 1. If G has no involution fixing seven points, then G is S_7 or A_9 by [8, Lemma 6] and [2]. Hence from now on we assume that G has an involution fixing exactly seven points, and we prove Lemma 1 by way of contradiction. We may assume that G has an involution a fixing exactly $1,2,\cdots,7$ and

$$a=(1)(2)\cdots(7)(8\ 9)\cdots.$$

Set $T=C(a)_{89}$.

(1) *For any three points i, j and k in $I(a)$, there is an involution in T_{ijk}. Any involution in T is not the identity on $I(a)$.*

Proof. Since a normalizes G_{89ijk} and G_{89ijk} is of even order, G_{89ijk} has an involution x commuting with a. Then $x\in T_{ijk}$. Since $|I(a)|=7$ and $I(x)\supseteqq\{8,9\}$, any involution in T is not the identity on $I(a)$ by (iii).

(2) *For any three points i, j and k in $I(a)$, a Sylow 2-subgroup of T_{ijk} is normal in T_{ijk}, and so a Sylow 2-subgroup of $T_{ijk}^{I(a)}$ is normal in $T_{ijk}^{I(a)}$.*

Proof. Let S be a Sylow 2-subgroup of T_{ijk}. Since S is a Sylow 2-subgroup of $C(a)_{89ijk}$, S is a normal subgroup of $C(a)_{89ijk}$ by (ii).

We have the following property from (2).

(3) *If $x_1^{I(a)}$ and $x_2^{I(a)}$ are involutions in $T^{I(a)}$ with $|I(x_1^{I(a)})\cap I(x_2^{I(a)})|\geqq 3$, then $x_1^{I(a)}x_2^{I(a)}$ is a 2-element of $T^{I(a)}$.*

(4) *Since $|I(a)|=7$, $T^{I(a)}$ is one of the following groups.*
(a) *$T^{I(a)}$ is intransitive and has an orbit of length one or two.*
(b) *$T^{I(a)}$ is intransitive and has an orbit of length three.*
(c) *$T^{I(a)}$ is primitive.*

(5) *The case* (a) *does not hold.*

Proof. Suppose $T^{I(a)}$ has an orbit of length one or two. We may assume that either $\{1\}$ or $\{1, 2\}$ is such an orbit. By (1), T_{234} has an involution x_1. We may assume that

$$x_1=(1)(2)(3)(4)(5\ 6)(7)\cdots.$$

Similarly T_{235} has an involution x_2 of the form

$$x_2=(1)(2)(3)(4)(5)(6\ 7)\cdots,\ (1)(2)(3)(5)(4\ 6)(7)\cdots \text{ or}$$
$$(1)(2)(3)(5)(4\ 7)(6)\cdots.$$

If the first or the second alternative holds, then $|I(x_1^{I(a)})\cap I(x_2^{I(a)})|=4$, and $x_1^{I(a)}x_2^{I(a)}$ is not a 2-element, a contradiction to (3). Thus $x_2=(1)(2)(3)(5)(4\ 7)(6)\cdots$. Again by (1), T_{245} has an involution x_3 of the form

$$x_3=(1)(2)(4)(5)(3)(6\ 7)\cdots,\ (1)(2)(4)(5)(3\ 6)(7)\cdots \text{ or}$$
$$(1)(2)(4)(5)(3\ 7)(6)\cdots.$$

In every case, we get a contradiction to (3) by considering either $x_1^{I(a)}x_3^{I(a)}$ or $x_2^{I(a)}x_3^{I(a)}$.

(6) *The case* (*b*) *does not hold.*

Proof. Suppose $T^{I(a)}$ has an orbit of length three. We may assume that $\{1,2,3\}$ is such an orbit of length three. By (5), $\{4,5,6,7\}$ is a $T^{I(a)}$-orbit. By (1), T_{456} has an involution x_1. We may assume that

$$x_1=(1\ 2)(3)(4)(5)(6)(7)\cdots.$$

Since $\{1,2,3\}$ is a $T^{I(a)}$-orbit, T has an element y of the form

$$y=(1\ 2\ 3)\cdots.$$

Set $x_2=x_1^y$, then $x_2=(2\ 3)(1)(4)(5)(6)(7)\cdots$. So, $|I(x_1^{I(a)})\cap I(x_2^{I(a)})|=4$, and $x_1x_2=(1\ 3\ 2)(4)(5)(6)(7)\cdots$, which is a contradiction. Hence $T^{I(a)}$ has no orbit of length three.

(7) *We show that the case* (*c*) *does not hold, and complete the proof of Lemma* 1.

Proof. Suppose $T^{I(a)}$ is primitive. By (1), we have $T^{I(a)}\geqq A_7$ (cf.e.g.[10]). Therefore for any involution x in G fixing exactly seven points, $C(x)^{I(x)}\geqq A_7$.

Let Γ be any subset of Ω with $|\Gamma|=5$. Set $\Gamma=\{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5\}$. By (i), $G_{\alpha_1\cdots\alpha_5}$ has an involution. If $G_{\alpha_1\cdots\alpha_5}$ has an involution x_1 fixing exactly seven points, then $C(x_1)^{I(x_1)}\geqq A_7$. Hence $G_{\{\alpha_1,\cdots,\alpha_5\}}^{\{\alpha_1,\cdots,\alpha_5\}}=S_5$. Suppose that $G_{\alpha_1\cdots\alpha_5}$ has no involution fixing seven points. Let x_2 be an involution in $G_{\alpha_1\cdots\alpha_5}$. Let $x_2=(\alpha_1)\cdots(\alpha_5)(\beta_1\ \beta_2)\cdots$. It is easily seen that $C(x_2)_{\beta_1\beta_2}^{I(x_2)}=S_5$. Hence $G_{\{\alpha_1,\cdots,\alpha_5\}}^{\{\alpha_1,\cdots,\alpha_5\}}=S_5$. Thus we have $G_{\{\alpha_1,\cdots,\alpha_5\}}^{\{\alpha_1,\cdots,\alpha_5\}}=S_5$ in either case. Therefore by [9, Lemma 3], G is 4-fold transitive on Ω.

Let x be an involution in G fixing seven points. Let S be the Sylow 2-subgroup of $G_{I(x)}$. Since $C(x)^{I(x)}\geqq A_7$, we have $N(S)^{I(S)}\geqq A_7$. By [6], we get a contradiction.

Thus we complete the proof of Lemma 1.

4. Proof of Lemma 2

Let G be a permutation group satisfying the assumptions of Lemma 2. If G has no involution fixing nine points, then G is S_7 or A_9 by Lemma 1. Hence from now on we assume that G has an involution fixing exactly nine points, and we prove Lemma 2 by way of contradiction. We may assume that G has an involution a fixing $1,2,\cdots,9$ and

$$a=(1)(2)\cdots(9)(10\ 11)\cdots.$$

Set $T=C(a)_{10\,11}$.

(1) *For any three points i,j and k in $I(a)$, there is an involution in T_{ijk}. Any involution in T is not the identity on $I(a)$.*

(2) *For any three points i,j and k in I(a), a Sylow 2-subgroup of* T_{ijk} *is normal in* T_{ijk}, *and so a Sylow 2-subgroup of* $T_{ijk}^{I(a)}$ *is normal in* $T_{ijk}^{I(a)}$.

(3) *If* $x_1^{I(a)}$ *and* $x_2^{I(a)}$ *are involutions in* $T^{I(a)}$ *with* $|I(x_1^{I(a)})\cap I(x_2^{I(a)})|\geqq 3$, *then* $x_1^{I(a)}x_2^{I(a)}$ *is a 2-element of* $T^{I(a)}$.

The proofs of (1), (2) and (3) are similar to the proofs of (1), (2) and (3) in Section 3 respectively.

(4) *Since* $|I(a)|=9$, $T^{I(a)}$ *is one of the following groups.*
(a) $T^{I(a)}$ *is intransitive and has an orbit of length one or two.*
(b) $T^{I(a)}$ *is either an intransitive group with an orbit of length three, or a transitive but imprimitive group with three blocks of length three.*
(c) $T^{I(a)}$ *is intransitive and has an orbit of length four.*
(d) $T^{I(a)}$ *is primitive.*

(5) *The case* (a) *does not hold.*

Proof. Suppose $T^{I(a)}$ has an orbit of length one or two. We may assume that either {1} or {1,2} is such an orbit. By (1), T_{234} has an involution x_1. By the assumption (iv), we may assume that

$$x_1=(1)(2)(3)(4)(5)(6\ 7)(8\ 9)\cdots .$$

Similarly T_{236} has an involution x_2. We may assume without loss of generality that

$$\begin{aligned} x_2^{I(a)} = &(1)(2)(3)(6)(7)(4\ 5)(8\ 9)\cdots \alpha\ ,\\ &(1)(2)(3)(6)(7)(4\ 8)(5\ 9)\cdots \beta\ ,\\ &(1)(2)(3)(6)(8)(7\ 9)(4\ 5)\cdots \gamma\ ,\\ &(1)(2)(3)(6)(8)(7\ 4)(5\ 9)\cdots \delta\ ,\\ &(1)(2)(3)(6)(4)(5\ 7)(8\ 9)\cdots \varepsilon \quad \text{or}\\ &(1)(2)(3)(6)(4)(7\ 8)(5\ 9)\cdots \zeta\ . \end{aligned}$$

If $x_2^{I(a)}$ is of the form δ, ε or ζ, then $|I(x_1^{I(a)})\cap I(x_2^{I(a)})|\geqq 3$, and $x_1^{I(a)}x_2^{I(a)}$ is not a 2-element, a contradiction to (3). Hence $x_2^{I(a)}$ is of the form α, β or γ. T_{269} has an involution x_3. $x_3^{I(a)}$ is of the form

$$\begin{aligned} x_3^{I(a)} = &(1)(2)(6)(9)(3)(4\ 5)(7\ 8)\cdots ①\ ,\\ &(1)(2)(6)(9)(3)(4\ 7)(5\ 8)\cdots ②\ ,\\ &(1)(2)(6)(9)(3)(4\ 8)(5\ 7)\cdots ③\ ,\\ &(1)(2)(6)(9)(4)(3\ 5)(7\ 8)\cdots ④\ ,\\ &(1)(2)(6)(9)(4)(3\ 7)(5\ 8)\cdots ⑤\ ,\\ &(1)(2)(6)(9)(4)(3\ 8)(5\ 7)\cdots ⑥\ , \end{aligned}$$

$$(1)(2)(6)(9)(5)(3\ 4)(7\ 8)\cdots ⑦,$$
$$(1)(2)(6)(9)(5)(3\ 7)(4\ 8)\cdots ⑧,$$
$$(1)(2)(6)(9)(5)(3\ 8)(4\ 7)\cdots ⑨,$$
$$(1)(2)(6)(9)(7)(3\ 4)(5\ 8)\cdots ⑩,$$
$$(1)(2)(6)(9)(7)(3\ 5)(4\ 8)\cdots ⑪.$$
$$(1)(2)(6)(9)(7)(3\ 8)(4\ 5)\cdots ⑫,$$
$$(1)(2)(6)(9)(8)(3\ 4)(5\ 7)\cdots ⑬,$$
$$(1)(2)(6)(9)(8)(3\ 5)(4\ 7)\cdots ⑭ \text{ or}$$
$$(1)(2)(6)(9)(8)(3\ 7)(4\ 5)\cdots ⑮.$$

If $x^{I(a)}$ is of the form ②, ③, ⑤, ⑥, ⑧ or ⑨, then $|I(x_1^{I(a)})\cap I(x_3^{I(a)})|=3$, and $x_1^{I(a)}x_3^{I(a)}$ is not a 2-element, which is a contradiction. Suppose $x_3^{I(a)}$ is of the form ⑩. Then $x_1x_3=(1)(2)(3\ 4)(5\ 8\ 9)(6\ 7)\cdots$, and $(x_1x_3)^2=(1)(2)(3)(4)(5\ 9\ 8)(6)(7)\cdots$. Set $y=(x_1x_3)^2$ and $x_4=x_1^y$. Then $x_4=(1)(2)(3)(4)(9)(6\ 7)(5\ 8)\cdots$. So, $|I(x_1^{I(a)})\cap I(x_4^{I(a)})|=4$, and $x_1x_4=(1)(2)(3)(4)(5\ 8\ 9)(6)(7)\cdots$, which is a contradiction. If $x_3^{I(a)}$ is of the form ⑪, ⑫, ⑬, ⑭ or ⑮, we have a contradiction by the same argument as in the case ⑩. Hence $x_3^{I(a)}$ is of the form ①, ④ or ⑦.

Suppose $x_2^{I(a)}$ is of the form α or γ. Since $x_3^{I(a)}$ is of the form ①, ④ or ⑦, we get a contradiction by considering $x_2^{I(a)}x_3^{I(a)}$.

Suppose $x_2^{I(a)}$ is of the form β. If $x_3^{I(a)}$ is of the form ① or ④, we get a contradiction by considering $x_2^{I(a)}x_3^{I(a)}$. Suppose $x_3^{I(a)}$ is of the form ⑦. Then $x_2x_3=(1)(2)(6)(5\ 9)(3\ 4\ 7\ 8)\cdots$. Set $x_5=(x_2x_3)^2$, then $x_5=(1)(2)(6)(5)(9)(3\ 7)(4\ 8)\cdots$. So, $|I(x_1^{I(a)})\cap I(x_5^{I(a)})|=3$, and $x_1x_5=(1)(2)(5)(3\ 7\ 6)(4\ 8\ 9)\cdots$, which is a contradiction. Thus $T^{I(a)}$ has neither orbit of length one nor orbit of length two.

(6) *The case* (*b*) *does not hold.*

Proof. Suppose $T^{I(a)}$ has an orbit of length three or three blocks of length three. We may assume that $\{1,2,3\}$ is such an orbit or a blcok.

Assume that $T^{I(a)}$ has three orbits of length three or three blocks of length three. We may assume that $\{1,2,3\}$, $\{4,5,6\}$ and $\{7,8,9\}$ are the orbits or the blocks. T_{124} has an involution x_1. By the assumption (iv),

$$x_1=(1)(2)(3)(4)(5\ 6)\cdots.$$

Similarly T_{125} has an involution x_2 of the form

$$x_2=(1)(2)(3)(5)(4\ 6)\cdots.$$

So, $|I(x_1^{I(a)})\cap I(x_2^{I(a)})|\geqq 3$, and $x_1x_2=(1)(2)(3)(4\ 6\ 5)\cdots$, which is a contradiction.

By (5) and the above, we have that $\{1, 2, 3\}$ and $\{4, 5, 6, 7, 8, 9\}$ are the

$T^{I(a)}$-orbits. Since $3 \,\Big|\, |\{4, 5, \cdots, 9\}|$, we may assume that T has an element y of the form

$$y=(4\ 5\ 6)\cdots .$$

T_{789} has an involution x_1. We may assume that

$$x_1=(1\ 2)(3)(4\ 5)(6)(7)(8)(9)\cdots .$$

Set $x_2=x_1^y$, then $x_2=(5\ 6)(4)(7)(8)(9)\cdots$. So, $|I(x_1^{I(a)})\cap I(x_2^{I(a)})|\geqq 3$, and $x_1x_2=(4\ 6\ 5)(7)(8)(9)\cdots$, which is a contradiction. Thus $T^{I(a)}$ has neither orbit of length three nor block of length three.

(7) *The case* (c) *does not hold.*

Proof. Suppose $T^{I(a)}$ has an orbit of length four. We may assume that $\{1,2,3,4\}$ is a $T^{I(a)}$-orbit. By (5) and (6), $\{5,6,7,8,9\}$ is a $T^{I(a)}$-orbit. Since $5 \,\Big|\, |\{5, 6, 7, 8, 9\}|$, we may assume that T has an element y of the form

$$y=(1)(2)(3)(4)(5\ 6 7\ 8\ 9)\cdots .$$

T_{123} has an involution x_1. We may assume that x_1 is of the form

$$\begin{aligned} x_1=&(1)(2)(3)(4)(5)(6\ 7)(8\ 9)\cdots,\\ &(1)(2)(3)(4)(5)(6\ 8)(7\ 9)\cdots \text{ or}\\ &(1)(2)(3)(4)(5)(6\ 9)(7\ 8)\cdots .\end{aligned}$$

Set $x_2=x_1^y$. Then x_2 is of the following form respectively:

$$\begin{aligned} x_2=&(1)(2)(3)(4)(6)(7\ 8)(5\ 9)\cdots,\\ &(1)(2)(3)(4)(6)(7\ 9)(5\ 8)\cdots \text{ or}\\ &(1)(2)(3)(4)(6)(5\ 7)(8\ 9)\cdots .\end{aligned}$$

In any case, we get a contradiction by considering $x_1^{I(a)}x_2^{I(a)}$.

(8) *We show that the case* (d) *does not hold, and complete the proof of Lemma* 2.

Proof. If $T^{I(a)}$ is primitive, then by (1) and the assumption (iv), we have $T^{I(a)}=A_9$(cf.e.g.[10]). But this contradicts (2). Thus $T^{I(a)}$ is not primitive.

Thus we complete the proof of Lemma 2.

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