

Title	5-fold transitive permutation groups in which the stabilizer of five points has a normal Sylow 2-subgroup
Author(s)	Yoshizawa, Mitsuo
Citation	Osaka Journal of Mathematics. 1978, 15(2), p. 343-350
Version Type	VoR
URL	https://doi.org/10.18910/3911
rights	
Note	

Osaka University Knowledge Archive : OUKA

<https://ir.library.osaka-u.ac.jp/>

Osaka University

5-FOLD TRANSITIVE PERMUTATION GROUPS IN WHICH THE STABILIZER OF FIVE POINTS HAS A NORMAL SYLOW 2-SUBGROUP

MITSUO YOSHIZAWA

(Received March 9, 1977)

1. Introduction

In this paper we shall prove the following theorem.

Theorem. *Let G be a 5-fold transitive permutation group on a set $\Omega = \{1, 2, \dots, n\}$. Let P be a Sylow 2-subgroup of G_{12345} . If P is a nonidentity normal subgroup of G_{12345} , then G is one of the following groups: S_7 , A_9 or M_{24} .*

The idea of the proof of the theorem is derived from Oyama [7].

In order to prove the theorem, we shall use the following two lemmas, which will be proved in Sections 3 and 4.

Lemma 1. *Let G be a permutation group on $\Omega = \{1, 2, \dots, n\}$ satisfying the following three conditions.*

- (i) *For any five points $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and α_5 in Ω , the order of $G_{\alpha_1 \dots \alpha_5}$ is even.*
- (ii) *For any five points $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and α_5 in Ω , a Sylow 2-subgroup of $G_{\alpha_1 \dots \alpha_5}$ is normal in $G_{\alpha_1 \dots \alpha_5}$.*
- (iii) *Any involution in G fixes at most seven points.*

Then G is S_7 or A_9 .

Lemma 2. *Let G be a permutation group on $\Omega = \{1, 2, \dots, n\}$ satisfying the following four conditions.*

- (i) *For any five points $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and α_5 in Ω , the order of $G_{\alpha_1 \dots \alpha_5}$ is even.*
- (ii) *For any five points $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and α_5 in Ω , a Sylow 2-subgroup of $G_{\alpha_1 \dots \alpha_5}$ is normal in $G_{\alpha_1 \dots \alpha_5}$.*
- (iii) *Any involution in G fixes at most nine points.*
- (iv) *For any 2-subgroup X fixing exactly nine points, $N(X)^{I(X)} \leq A_9$.*

Then G is S_7 or A_9 .

The author thanks Professor Eiichi Bannai for his kind advice.

We shall use the same notation as in [3].

2. Proof of the Theorem

Let G be a group satisfying the assumption of the theorem.

Let P be the unique Sylow 2-subgroup of G_{12345} . If P is semiregular on $\Omega-I(P)$ or $|I(P)| > 6$, then G is S_7, A_9 or M_{24} by [2], [3], [4] and [5]. Hence from now on we assume that P is not semiregular on $\Omega-I(P)$ and that $|I(P)| \leq 6$, and we prove that this case does not arise. If $|I(P)| = 6$, then $|I(G_{12345})| = 6$, a contradiction to [1]. Hence $|I(P)| = 5$.

Let r be $\text{Max}|I(a)|$, where a ranges over all involutions in G . Since P is not semiregular on $\Omega-I(P)$, we have $r \geq 7$.

Suppose $r = 7$. Let t be a point of a minimal orbit of P in $\Omega-I(P)$. It is easily seen that $N(P_t)^{I(P_t)} = S_7$. By [6], we have a contradiction.

Suppose $r = 9$. Let Q be a subgroup of P such that the order of Q is maximal among all subgroups of P fixing exactly nine points. By Lemma 1, $N(Q)^{I(Q)} = A_9$. Again by [6], we have a contradiction. Thus we have $r \geq 11$.

Let Q be a subgroup of P such that the order of Q is maximal among all subgroups of P fixing more than nine points. Set $N = N(Q)^{I(Q)}$. Then N satisfies the following conditions.

- (i) N is a permutation group on $I(Q)$, and its degree is not less than eleven.
- (ii) For any five points $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and α_5 in $I(Q)$, the order of $N_{\alpha_1 \dots \alpha_5}$ is even.
- (iii) For any five points $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and α_5 in $I(Q)$, a Sylow 2-subgroup of $N_{\alpha_1 \dots \alpha_5}$ is normal in $N_{\alpha_1 \dots \alpha_5}$.
- (iv) Any involution fixes at most nine points.

By Lemma 1, N has an involution fixing exactly nine points. Let X be any 2-subgroup of N fixing exactly nine points. Set $\Delta = I(X)$. Let S be the Sylow 2-subgroup of G_Δ . Since $I(S) = \Delta$, we have $N_G(S)^{I(S)} = A_9$ by Lemma 1. Since S is a characteristic subgroup of G_Δ , N satisfies the following condition.

- (v) For any 2-subgroup X fixing exactly nine points, $N_N(X)^{I(X)} \leq A_9$.

Considering the permutation group N , we have a final contradiction by Lemma 2.

3. Proof of Lemma 1

Let G be a permutation group satisfying the assumptions of Lemma 1. If G has no involution fixing seven points, then G is S_7 or A_9 by [8, Lemma 6] and [2]. Hence from now on we assume that G has an involution fixing exactly seven points, and we prove Lemma 1 by way of contradiction. We may assume that G has an involution a fixing exactly $1, 2, \dots, 7$ and

$$a = (1)(2) \dots (7)(8\ 9) \dots$$

Set $T=C(a)_{89}$.

(1) For any three points i, j and k in $I(a)$, there is an involution in T_{ijk} . Any involution in T is not the identity on $I(a)$.

Proof. Since a normalizes G_{89ijk} and G_{89ijk} is of even order, G_{89ijk} has an involution x commuting with a . Then $x \in T_{ijk}$. Since $|I(a)|=7$ and $I(x) \cong \{8, 9\}$, any involution in T is not the identity on $I(a)$ by (iii).

(2) For any three points i, j and k in $I(a)$, a Sylow 2-subgroup of T_{ijk} is normal in T_{ijk} , and so a Sylow 2-subgroup of $T_{ijk}^{I(a)}$ is normal in $T_{ijk}^{I(a)}$.

Proof. Let S be a Sylow 2-subgroup of T_{ijk} . Since S is a Sylow 2-subgroup of $C(a)_{89ijk}$, S is a normal subgroup of $C(a)_{89ijk}$ by (ii).

We have the following property from (2).

(3) If $x_1^{I(a)}$ and $x_2^{I(a)}$ are involutions in $T^{I(a)}$ with $|I(x_1^{I(a)}) \cap I(x_2^{I(a)})| \geq 3$, then $x_1^{I(a)}x_2^{I(a)}$ is a 2-element of $T^{I(a)}$.

(4) Since $|I(a)|=7$, $T^{I(a)}$ is one of the following groups.

- (a) $T^{I(a)}$ is intransitive and has an orbit of length one or two.
- (b) $T^{I(a)}$ is intransitive and has an orbit of length three.
- (c) $T^{I(a)}$ is primitive.

(5) The case (a) does not hold.

Proof. Suppose $T^{I(a)}$ has an orbit of length one or two. We may assume that either $\{1\}$ or $\{1, 2\}$ is such an orbit. By (1), T_{234} has an involution x_1 . We may assume that

$$x_1 = (1)(2)(3)(4)(5\ 6)(7) \dots$$

Similarly T_{235} has an involution x_2 of the form

$$x_2 = (1)(2)(3)(4)(5)(6\ 7) \dots, (1)(2)(3)(5)(4\ 6)(7) \dots \text{ or} \\ (1)(2)(3)(5)(4\ 7)(6) \dots$$

If the first or the second alternative holds, then $|I(x_1^{I(a)}) \cap I(x_2^{I(a)})|=4$, and $x_1^{I(a)}x_2^{I(a)}$ is not a 2-element, a contradiction to (3). Thus $x_2=(1)(2)(3)(5)(4\ 7)(6) \dots$. Again by (1), T_{245} has an involution x_3 of the form

$$x_3 = (1)(2)(4)(5)(3)(6\ 7) \dots, (1)(2)(4)(5)(3\ 6)(7) \dots \text{ or} \\ (1)(2)(4)(5)(3\ 7)(6) \dots$$

In every case, we get a contradiction to (3) by considering either $x_1^{I(a)}x_3^{I(a)}$ or $x_2^{I(a)}x_3^{I(a)}$.

(6) *The case (b) does not hold.*

Proof. Suppose $T^{I(a)}$ has an orbit of length three. We may assume that $\{1, 2, 3\}$ is such an orbit of length three. By (5), $\{4, 5, 6, 7\}$ is a $T^{I(a)}$ -orbit. By (1), T_{456} has an involution x_1 . We may assume that

$$x_1 = (1\ 2)(3\ 4)(5\ 6)(7\ \dots).$$

Since $\{1, 2, 3\}$ is a $T^{I(a)}$ -orbit, T has an element y of the form

$$y = (1\ 2\ 3)\ \dots.$$

Set $x_2 = x_1^y$, then $x_2 = (2\ 3)(1\ 4)(5\ 6)(7\ \dots)$. So, $|I(x_1^{I(a)}) \cap I(x_2^{I(a)})| = 4$, and $x_1 x_2 = (1\ 3\ 2)(4\ 5)(6\ 7)\ \dots$, which is a contradiction. Hence $T^{I(a)}$ has no orbit of length three.

(7) *We show that the case (c) does not hold, and complete the proof of Lemma 1.*

Proof. Suppose $T^{I(a)}$ is primitive. By (1), we have $T^{I(a)} \geq A_7$ (cf. e.g. [10]). Therefore for any involution x in G fixing exactly seven points, $C(x)^{I(x)} \geq A_7$.

Let Γ be any subset of Ω with $|\Gamma| = 5$. Set $\Gamma = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$. By (i), $G_{\alpha_1 \dots \alpha_5}$ has an involution. If $G_{\alpha_1 \dots \alpha_5}$ has an involution x_1 fixing exactly seven points, then $C(x_1)^{I(x_1)} \geq A_7$. Hence $G_{\{\alpha_1, \dots, \alpha_5\}} = S_5$. Suppose that $G_{\alpha_1 \dots \alpha_5}$ has no involution fixing seven points. Let x_2 be an involution in $G_{\alpha_1 \dots \alpha_5}$. Let $x_2 = (\alpha_1) \dots (\alpha_5)(\beta_1\ \beta_2)\ \dots$. It is easily seen that $C(x_2)_{\beta_1 \beta_2}^{I(x_2)} = S_5$. Hence $G_{\{\alpha_1, \dots, \alpha_5\}} = S_5$. Thus we have $G_{\{\alpha_1, \dots, \alpha_5\}} = S_5$ in either case. Therefore by [9, Lemma 3], G is 4-fold transitive on Ω .

Let x be an involution in G fixing seven points. Let S be the Sylow 2-subgroup of $G_{I(x)}$. Since $C(x)^{I(x)} \geq A_7$, we have $N(S)^{I(S)} \geq A_7$. By [6], we get a contradiction.

Thus we complete the proof of Lemma 1.

4. Proof of Lemma 2

Let G be a permutation group satisfying the assumptions of Lemma 2. If G has no involution fixing nine points, then G is S_7 or A_9 by Lemma 1. Hence from now on we assume that G has an involution fixing exactly nine points, and we prove Lemma 2 by way of contradiction. We may assume that G has an involution a fixing $1, 2, \dots, 9$ and

$$a = (1\ 2)\ \dots\ (9\ 10\ 11)\ \dots.$$

Set $T = C(a)_{10\ 11}$.

(1) *For any three points i, j and k in $I(a)$, there is an involution in T_{ijk} . Any involution in T is not the identity on $I(a)$.*

(2) For any three points i, j and k in $I(a)$, a Sylow 2-subgroup of T_{ijk} is normal in T_{ijk} , and so a Sylow 2-subgroup of $T_{ijk}^{I(a)}$ is normal in $T_{ijk}^{I(a)}$.

(3) If $x_1^{I(a)}$ and $x_2^{I(a)}$ are involutions in $T^{I(a)}$ with $|I(x_1^{I(a)}) \cap I(x_2^{I(a)})| \geq 3$, then $x_1^{I(a)}x_2^{I(a)}$ is a 2-element of $T^{I(a)}$.

The proofs of (1), (2) and (3) are similar to the proofs of (1), (2) and (3) in Section 3 respectively.

(4) Since $|I(a)|=9$, $T^{I(a)}$ is one of the following groups.

(a) $T^{I(a)}$ is intransitive and has an orbit of length one or two.

(b) $T^{I(a)}$ is either an intransitive group with an orbit of length three, or a transitive but imprimitive group with three blocks of length three.

(c) $T^{I(a)}$ is intransitive and has an orbit of length four.

(d) $T^{I(a)}$ is primitive.

(5) The case (a) does not hold.

Proof. Suppose $T^{I(a)}$ has an orbit of length one or two. We may assume that either $\{1\}$ or $\{1,2\}$ is such an orbit. By (1), T_{234} has an involution x_1 . By the assumption (iv), we may assume that

$$x_1 = (1)(2)(3)(4)(5)(67)(89) \dots .$$

Similarly T_{236} has an involution x_2 . We may assume without loss of generality that

$$\begin{aligned} x_2^{I(a)} = & (1)(2)(3)(6)(7)(45)(89) \dots \alpha , \\ & (1)(2)(3)(6)(7)(48)(59) \dots \beta , \\ & (1)(2)(3)(6)(8)(79)(45) \dots \gamma , \\ & (1)(2)(3)(6)(8)(74)(59) \dots \delta , \\ & (1)(2)(3)(6)(4)(57)(89) \dots \varepsilon \quad \text{or} \\ & (1)(2)(3)(6)(4)(78)(59) \dots \zeta . \end{aligned}$$

If $x_2^{I(a)}$ is of the form δ , ε or ζ , then $|I(x_1^{I(a)}) \cap I(x_2^{I(a)})| \geq 3$, and $x_1^{I(a)}x_2^{I(a)}$ is not a 2-element, a contradiction to (3). Hence $x_2^{I(a)}$ is of the form α , β or γ . T_{269} has an involution x_3 . $x_3^{I(a)}$ is of the form

$$\begin{aligned} x_3^{I(a)} = & (1)(2)(6)(9)(3)(45)(78) \dots \textcircled{1} , \\ & (1)(2)(6)(9)(3)(47)(58) \dots \textcircled{2} , \\ & (1)(2)(6)(9)(3)(48)(57) \dots \textcircled{3} , \\ & (1)(2)(6)(9)(4)(35)(78) \dots \textcircled{4} , \\ & (1)(2)(6)(9)(4)(37)(58) \dots \textcircled{5} , \\ & (1)(2)(6)(9)(4)(38)(57) \dots \textcircled{6} , \end{aligned}$$

$$\begin{aligned}
& (1)(2)(6)(9)(5)(34)(78) \cdots \textcircled{7}, \\
& (1)(2)(6)(9)(5)(37)(48) \cdots \textcircled{8}, \\
& (1)(2)(6)(9)(5)(38)(47) \cdots \textcircled{9}, \\
& (1)(2)(6)(9)(7)(34)(58) \cdots \textcircled{10}, \\
& (1)(2)(6)(9)(7)(35)(48) \cdots \textcircled{11}. \\
& (1)(2)(6)(9)(7)(38)(45) \cdots \textcircled{12}, \\
& (1)(2)(6)(9)(8)(34)(57) \cdots \textcircled{13}, \\
& (1)(2)(6)(9)(8)(35)(47) \cdots \textcircled{14} \text{ or} \\
& (1)(2)(6)(9)(8)(37)(45) \cdots \textcircled{15}.
\end{aligned}$$

If $x^{I(a)}$ is of the form $\textcircled{2}$, $\textcircled{3}$, $\textcircled{5}$, $\textcircled{6}$, $\textcircled{8}$ or $\textcircled{9}$, then $|I(x_1^{I(a)}) \cap I(x_3^{I(a)})| = 3$, and $x_1^{I(a)}x_3^{I(a)}$ is not a 2-element, which is a contradiction. Suppose $x_3^{I(a)}$ is of the form $\textcircled{10}$. Then $x_1x_3 = (1)(2)(34)(589)(67) \cdots$, and $(x_1x_3)^2 = (1)(2)(3)(4)(598)(6)(7) \cdots$. Set $y = (x_1x_3)^2$ and $x_4 = x_3^y$. Then $x_4 = (1)(2)(3)(4)(9)(67)(58) \cdots$. So, $|I(x_1^{I(a)}) \cap I(x_4^{I(a)})| = 4$, and $x_1x_4 = (1)(2)(3)(4)(589)(6)(7) \cdots$, which is a contradiction. If $x_3^{I(a)}$ is of the form $\textcircled{11}$, $\textcircled{12}$, $\textcircled{13}$, $\textcircled{14}$ or $\textcircled{15}$, we have a contradiction by the same argument as in the case $\textcircled{10}$. Hence $x_3^{I(a)}$ is of the form $\textcircled{1}$, $\textcircled{4}$ or $\textcircled{7}$.

Suppose $x_2^{I(a)}$ is of the form α or γ . Since $x_3^{I(a)}$ is of the form $\textcircled{1}$, $\textcircled{4}$ or $\textcircled{7}$, we get a contradiction by considering $x_2^{I(a)}x_3^{I(a)}$.

Suppose $x_2^{I(a)}$ is of the form β . If $x_3^{I(a)}$ is of the form $\textcircled{1}$ or $\textcircled{4}$, we get a contradiction by considering $x_2^{I(a)}x_3^{I(a)}$. Suppose $x_3^{I(a)}$ is of the form $\textcircled{7}$. Then $x_2x_3 = (1)(2)(6)(59)(3478) \cdots$. Set $x_5 = (x_2x_3)^2$, then $x_5 = (1)(2)(6)(5)(9)(37)(48) \cdots$. So, $|I(x_1^{I(a)}) \cap I(x_5^{I(a)})| = 3$, and $x_1x_5 = (1)(2)(5)(376)(489) \cdots$, which is a contradiction. Thus $T^{I(a)}$ has neither orbit of length one nor orbit of length two.

(6) *The case (b) does not hold.*

Proof. Suppose $T^{I(a)}$ has an orbit of length three or three blocks of length three. We may assume that $\{1, 2, 3\}$ is such an orbit or a block.

Assume that $T^{I(a)}$ has three orbits of length three or three blocks of length three. We may assume that $\{1, 2, 3\}$, $\{4, 5, 6\}$ and $\{7, 8, 9\}$ are the orbits or the blocks. T_{124} has an involution x_1 . By the assumption (iv),

$$x_1 = (1)(2)(3)(4)(56) \cdots.$$

Similarly T_{125} has an involution x_2 of the form

$$x_2 = (1)(2)(3)(5)(46) \cdots.$$

So, $|I(x_1^{I(a)}) \cap I(x_2^{I(a)})| \geq 3$, and $x_1x_2 = (1)(2)(3)(465) \cdots$, which is a contradiction.

By (5) and the above, we have that $\{1, 2, 3\}$ and $\{4, 5, 6, 7, 8, 9\}$ are the

$T^{I(a)}$ -orbits. Since $3 \mid |\{4, 5, \dots, 9\}|$, we may assume that T has an element y of the form

$$y = (4\ 5\ 6) \dots .$$

T_{789} has an involution x_1 . We may assume that

$$x_1 = (1\ 2)(3\ 4)(5\ 6)(7\ 8)(9) \dots .$$

Set $x_2 = x_1^y$, then $x_2 = (5\ 6)(4\ 7)(8\ 9) \dots$. So, $|I(x_1^{I(a)}) \cap I(x_2^{I(a)})| \geq 3$, and $x_1 x_2 = (4\ 6\ 5)(7\ 8)(9) \dots$, which is a contradiction. Thus $T^{I(a)}$ has neither orbit of length three nor block of length three.

(7) *The case (c) does not hold.*

Proof. Suppose $T^{I(a)}$ has an orbit of length four. We may assume that $\{1, 2, 3, 4\}$ is a $T^{I(a)}$ -orbit. By (5) and (6), $\{5, 6, 7, 8, 9\}$ is a $T^{I(a)}$ -orbit. Since $5 \mid |\{5, 6, 7, 8, 9\}|$, we may assume that T has an element y of the form

$$y = (1)(2)(3)(4)(5\ 6\ 7\ 8\ 9) \dots .$$

T_{123} has an involution x_1 . We may assume that x_1 is of the form

$$\begin{aligned} x_1 &= (1)(2)(3)(4)(5)(6\ 7)(8\ 9) \dots , \\ &(1)(2)(3)(4)(5)(6\ 8)(7\ 9) \dots \text{ or} \\ &(1)(2)(3)(4)(5)(6\ 9)(7\ 8) \dots . \end{aligned}$$

Set $x_2 = x_1^y$. Then x_2 is of the following form respectively:

$$\begin{aligned} x_2 &= (1)(2)(3)(4)(6)(7\ 8)(5\ 9) \dots , \\ &(1)(2)(3)(4)(6)(7\ 9)(5\ 8) \dots \text{ or} \\ &(1)(2)(3)(4)(6)(5\ 7)(8\ 9) \dots . \end{aligned}$$

In any case, we get a contradiction by considering $x_1^{I(a)} x_2^{I(a)}$.

(8) *We show that the case (d) does not hold, and complete the proof of Lemma 2.*

Proof. If $T^{I(a)}$ is primitive, then by (1) and the assumption (iv), we have $T^{I(a)} = A_9$ (cf. e.g. [10]). But this contradicts (2). Thus $T^{I(a)}$ is not primitive.

Thus we complete the proof of Lemma 2.

GAKUSHUIN UNIVERSITY

References

- [1] H. Nagao: *On multiply transitive groups IV*, Osaka J. Math. 2 (1965), 327–341.

- [2] T. Oyama: *On multiply transitive groups VII*, Osaka J. Math. **5** (1968), 155–164.
- [3] T. Oyama: *On multiply transitive groups VIII*, Osaka J. Math. **6** (1969), 315–319.
- [4] T. Oyama: *On multiply transitive groups IX*, Osaka J. Math. **7** (1970), 41–56.
- [5] T. Oyama: *On multiply transitive groups XI*, Osaka J. Math. **10** (1973), 379–439.
- [6] T. Oyama: *On multiply transitive groups XII*. Osaka J. Math. **11** (1974), 595–636.
- [7] T. Oyama: *On multiply transitive groups XIII*, Osaka J. Math. **13** (1976), 367–383.
- [8] D. Livingstone and A. Wagner: *Transitivity of finite permutation groups on unordered sets*, Math. Z. **90** (1965), 393–403.
- [9] A. Wagner: *Normal subgroups of triply-transitive permutation groups of odd degree*, Math. Z. **94** (1966), 219–222.
- [10] C.C. Sims: *Computational methods in the study of permutation groups*, (in Computational Problems in Abstract Algebra), Pergamon Press, London, 1970, 169–183.