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On Homeomorphisms which are Regular Except for a Finite Number of Points

By Tatsuo Homma and Shin’ichi Kinoshita

Introduction

All spaces considered in this paper are separable metric. Let \( h \) be a homeomorphism of a set \( X \) onto itself. Then \( p \in X \) is called regular\(^1\) under \( h \), if for each \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that if \( d(p, x) < \delta \), then \( d(h^n(p), h^n(x)) < \varepsilon \) for every integer \( n \). If \( p \in X \) is not regular under \( h \), then \( p \) is called irregular.

A set \( X \) will be called a \( C^* \)-set if \( X - A \) is connected for any \( A \) which consists of a finite number of points of \( X \). For example any \( n \)-manifold \((n \geq 2)\) is a \( C^* \)-set. Then one of the purpose of this paper is to prove the following

**Theorem I.** Let \( X \) be a compact \( C^* \)-set and \( h \) a homeomorphism of \( X \) onto itself. If \( h \) is regular at every \( x \in X \) except for a finite number of points, then the number of points which are irregular under \( h \) is at most two.

We shall also prove the following

**Theorem II.\(^2\)** Let \( X \) be a compact \( C^* \)-set and \( h \) a homeomorphism of \( X \) onto itself such that

(i) \( h \) is irregular at \( a, b \ (\neq) \in X \),
(ii) \( h \) is regular at every \( x \in X - (a \cup b) \).

Then either (1) for each \( x \in X - b \) \( h^n(x) \) converges to \( a \) when \( n \to \infty \) and for each \( x \in X - a \) \( h^n(x) \) converges to \( b \) when \( n \to -\infty \), or (2) for each \( x \in X - a \) \( h^n(x) \) converges to \( b \) when \( n \to \infty \) and for each \( x \in X - b \) \( h^n(x) \) converges to \( a \) when \( n \to -\infty \).

§ 1.

Let \( X \) be a set and \( h \) a homeomorphism of \( X \) onto itself. Let \( R(h) \) be the set of all points which are regular under \( h \) and \( I(h) \) the set of all points which are irregular under \( h \). Then

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1) Introduced by B. v. Kerékjártó [5].
2) This is a converse theorem of Theorem 1 of the authors [3].
Furthermore let \( A(h) \) be the set of all points which are regular and almost periodic\(^3\) under \( h \) and \( N(h) \) the set of all points which are regular and not almost periodic under \( h \). Then
\[
R(h) = A(h) \cup N(h) \quad \text{and} \quad A(h) \cap N(h) = \emptyset.
\]

**Lemma 1.** Let \( p \in R(h) \). Then \( p \in A(h) \) if and only if for each \( \varepsilon > 0 \) there exists a natural number \( n \) such that \( d(p, h^n(p)) < \varepsilon \).

**PROOF.** It is clear that the condition is sufficient. We shall prove that the condition is necessary. Let \( \varepsilon > 0 \). Since \( p \in R(h) \), there exists \( \delta > 0 \) such that if \( d(p, x) < \delta \), then \( d(h^n(p), h^n(x)) < \varepsilon \) for every integer \( n \). Since \( p \in A(h) \), there exists an integer \( N(\phi) \) such that \( d(p, h^n(p)) < \delta \). If \( \delta > 0 \), then the proof is already complete. If \( N < 0 \), then \( d(h^{-N}(p), p) < \varepsilon \), which completes the proof.

Similarly we have the following

**Lemma 1'.** Let \( p \in R(h) \). Then \( p \in A(h) \) if and only if for each \( \varepsilon > 0 \) there exists a natural number \( n \) such that \( d(p, h^{-n}(p)) < \varepsilon \).

**Lemma 2.** Let \( p \in R(h) \). If \( (\lim_{n \to \pm \infty} h^n(p)) \cap R(h) = \emptyset \), then \( p \in A(h) \).

**PROOF.** Let \( q \in (\lim_{n \to \pm \infty} h^n(p)) \cap R(h) \). Given \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that if \( d(q, x) < \delta \), then \( d(h^n(q), h^n(x)) < \frac{\varepsilon}{2} \) for every integer \( n \). Since \( q \in h^n(p) \), there exist integers \( m_1 \) and \( m_2 \) (\( m_1 \geq m_2 \)) such that \( d(q, h^{m_2}(p)) < \delta \) and \( d(q, h^{m_2}(p)) < \delta \). Then
\[
d(p, h^{m_2-m_1}(p)) \leq d(p, h^{-m_1}(q)) + d(h^{-m_1}(q), h^{m_2-m_1}(p)) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\]
which completes the proof.

**Lemma 3.** For each \( p \in A(h) \), \( (\lim_{n \to \pm \infty} h^n(p)) \cap N(h) = \emptyset \).

**PROOF.** Given \( q \in (\lim_{n \to \pm \infty} h^n(p)) \cap R(h) \), it is easy to see that \( p \in \lim_{n \to \pm \infty} h^n(q) \).

From Lemma 2 it follows that \( q \in A(h) \), which completes the proof.

Now assume that \( p \in A(h) \) and that \( U \) is a neighbourhood of \( p \). Let \( n(p, U) \) be the set of all integers \( n \) such that \( h^n(p) \in U \). Furthermore assume that \( n_0 = 0 \) and that \( n_i < n_{i+1} \). It follows from Lemmas 1 and 1' that \( n_i \) is defined uniquely for every integer \( i \).

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3) Let \( h \) be a homeomorphism of \( X \) onto itself. Then \( x \in X \) is called almost periodic under \( h \), if for each \( \varepsilon > 0 \) there exists an integer \( n=0 \) such that \( d(x, h^n(x)) < \varepsilon \).

4) \( \lim_{n \to \pm \infty} h^n(p) = \{x\} \) for each \( \varepsilon > 0 \) there exist infinitely many integers \( n \) such that \( d(x, h^n(p)) < \varepsilon \).
A homeomorphism $h$ of $X$ onto itself is said to be strongly regular at $p \in X$, if there exists a neighbourhood $U$ of $p$ such that $h$ is regular for every point of $U$. Then we have the following

**Lemma 4.** Let $X$ be locally compact. If $h$ is strongly regular at $p \in A(h)$, then there exists $\varepsilon_0 > 0$ such that $n[p, U_\varepsilon(p)]$ is finite for every $\varepsilon < \varepsilon_0$.

**Proof.** Since $X$ is locally compact, there exists a neighbourhood $U$ of $p$ such that $U$ is compact. From the strong regularity of $h$ at $p$ it follows that there exists a neighbourhood $V$ of $p$ such that $h$ is regular for every point of $V$. Let $\varepsilon_0 > 0$ be such that

$$U_{\varepsilon_0}(p) \subset U \cap V.$$

Let $\varepsilon < \varepsilon_0$. Suppose on the contrary that $n[p, U_\varepsilon(p)]$ is not finite. Then either $\lim_{i \to \infty} m[n_i] = \infty$ or $\lim_{i \to \infty} m[n_{-i}] = \infty$.

First we suppose that $\lim_{i \to \infty} m[n_i] = \infty$. Then there exists a subsequence $\{n_{i_j}\}$ of $\{n_i\}$ such that $\lim_{j \to \infty} m[n_{i_j}] = \infty$. Since $U_\varepsilon(p)$ is compact, there exists a subsequence $\{n_k\}$ of $\{n_{i_j}\}$ such that $\lim_{j \to \infty} h^{n_k}(p) = q$, where $q \in U_\varepsilon(p)$. Since $q \in R(h)$, there exists $\delta > 0$ such that if $d(q, x) < \delta$, then $d(h^n(q), h^n(x)) < \frac{\varepsilon}{3}$ for every integer $n$. Let $K$ be a natural number such that if $k \geq K$, then $d(q, h^{n_k}(p)) < \delta$. Since $p \in A(h)$, there exists a natural number $N$ such that $d(p, h^{n_K+N}(p)) < \frac{\varepsilon}{3}$. Then for each $k \geq K$

$$d(p, h^{n_K+N}(p)) \leq d(p, h^{n_K+N}(p)) + d(h^{n_K+N}(p), h^N(q)) + d(h^N(q), h^{n_K+N}(p)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

But this contradicts $\lim_{k \to \infty} m[n_k] = \infty$.

Now we suppose that $\lim_{i \to \infty} m[n_{-i}] = \infty$. Then there exists a subsequence $\{n_{i_j}\}$ of $\{n_i\}$ such that $\lim_{j \to \infty} m[n_{-i_j}] = \infty$. Since $U_\varepsilon(p)$ is compact, there exists a subsequence $\{n_k\}$ of $\{n_{i_j}\}$ such that $\lim_{j \to \infty} h^{-n_j}(p) = q$, where $q \in U_\varepsilon(p)$. Since $q \in R(h)$, there exists $\delta > 0$ such that if $d(q, x)$

5) $U_\varepsilon(p) = \{x \mid d(p, x) < \varepsilon\}$. 

$<\delta$, then $d(h^n(q), h^n(x))<\frac{\varepsilon}{2}$ for every integer $n$. Let $K$ be a natural number such that if $k \geq K$, then $d(q, h^{-k}(p))<\delta$. Then for each $k \geq K$

\[
d(p, h^{-k}(p)) \leq d(p, h^{-k}(q)) + d(h^{-k}(q), h^{-k}(p)),
\]

\[h^{-k}(p) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

But this contradicts $\lim_{k \to \infty} m[n-k] = \infty$. Thus the proof of Lemma 4 is complete.

**Lemma 5.** Let $X$ be locally compact. Suppose that $I(h)$ is a closed subset of $X$. Then for each $p \in A(h)$

\[
\lim_{n \to \pm \infty} (h^n(p)) \cap I(h) = 0.
\]

**Proof.** Let $p \in A(h)$. Then there exist open subsets $U$ and $V$ such that $U \ni p$, $V \ni I(h)$ and $\overline{U} \cap \overline{V} = 0$. Since $h$ is strongly regular at $p$, it follows from Lemma 4 that there exists $\varepsilon_0>0$ such that $n[p, U_\varepsilon(p)]$ is finite for every $\varepsilon<\varepsilon_0$. Let $\varepsilon_1>0$ be such that $\varepsilon_1<\varepsilon_0$ and that $U_{\varepsilon_1}(p) \subset U$. Since $h(I(h)) = I(h)$,

\[h(\overline{U_{\varepsilon_1}(p)}) \cap I(h) = 0
\]

for every integer $n$. Put

\[U_\varepsilon = \{x \mid x \in h^n(U_{\varepsilon_1}(p)), n = 0, 1, \ldots, n[p, U_{\varepsilon_1}(p)]-1\}.
\]

Then $U_\varepsilon$ is an open subset of $X$ and $\overline{U_\varepsilon} \cap I(h) = 0$. From the definition of $n[p, U_{\varepsilon_1}(p)]$ it follows that $h^n(p) \in U_\varepsilon$ for every integer $n$. Then

\[\lim_{n \to \pm \infty} h^n(p) \cap I(h) = 0,
\]

which completes the proof.

By Lemmas 2, 3 and 5 we have immediately the following

**Theorem 1.** Let $X$ be locally compact and $h$ a homeomorphism of $X$ onto itself. Suppose that $I(h)$ is a closed subset of $X$ and that $p \in R(h)$. Then

(1) $p \in A(h)$ if and only if $\lim_{n \to \pm \infty} h^n(p) \subset A(h)$ and $\lim_{n \to \pm \infty} h^n(p) \neq 0$,

(2) $p \in N(h)$ if and only if $\lim_{n \to \pm \infty} h^n(p) \subset I(h)$.

**Lemma 6.** Let $h$ be a homeomorphism of $X$ onto itself. Suppose that $I(h)$ is a closed subset of $X$. Then $N(h)$ is an open subset of $X$. 

PROOF. Since $I(h)$ is a closed subset of $X$, we are only to prove that if $p \in R(h) \cap A(h)$, then $p \in A(h)$. Let $\epsilon > 0$. Since $p \in R(h)$, there exists $\delta > 0$ such that if $d(p, x) < \delta$, then $d(h^n(p), h^n(x)) < \frac{\epsilon}{3}$ for every integer $n$. Since $p \in A(h)$, there exists $q \in A(h)$ such that $d(p, q) < \delta$. Since $q \in A(h)$, there exists an integer $N$ such that $d(q, h^N(q)) < \frac{\epsilon}{3}$. Then

$$d(q, h^N(p)) \leq d(p, q) + d(q, h^N(q)) + d(h^N(q), h^N(p)) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$ 

Therefore $p \in A(h)$ and the proof is complete.

**Lemma 7.** Let $X$ be locally compact. Suppose that $I(h)$ is a closed subset of $X$. Then $A(h)$ is an open subset of $X$.

**Proof.** Let $p \in A(h)$. Let $U$ be a neighborhood of $p$ such that $\overline{U} \cap I(h) = 0$. Then there exists $\epsilon > 0$ such that $U_p \subset U$. Since $p \in R(h)$, there exists $\delta > 0$ such that if $d(p, x) < \delta$, then $d(h^n(p), h^n(x)) < \epsilon$ for every integer $n$. Now we are only to prove that if $q \in U_p(p)$, then $q \in A(h)$. Since $p \in A(h)$, there exist infinitely many $n_i$ such that $d(p, h^{n_i}(p)) < \frac{\epsilon}{2}$. Then

$$d(p, h^{n_i}(q)) \leq d(p, h^{n_i}(p)) + d(h^{n_i}(p), h^{n_i}(q)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$ 

Since $U_p(p)$ is compact, $(\lim_{n \to \infty} h^n(q)) \cap U_p(p) = 0$. Then $(\lim_{n \to \infty} h^n(q)) \cap R(h) = 0$.

From Lemma 2 it follows that $q \in A(h)$ and the proof is complete.

By Lemmas 6 and 7 we have immediately the following

**Theorem 2.** Let $X$ be locally compact and $h$ a homeomorphism of $X$ onto itself. Suppose that $I(h)$ is a closed subset of $X$. If $R(h)$ is connected, then $A(h) = 0$ or $N(h) = 0$.

By the definition of the regularity we have clearly that if $p \in R(h)$, then $p \in R(h^m)$ for every integer $m$. Conversely we have the following

**Lemma 8.** Let $X$ be compact. If $p \in R(h^m)$ for some integer $m(>0)$, then $p \in R(h)$.

**Proof.** Without loss of generality we may assume that $m > 1$. Let $\epsilon > 0$. Since $X$ is compact, $h$ is uniformly continuous on $X$. Then there exists $\delta_0 > 0$ such that if $d(x, y) < \delta_0$, then $d(h^k(x), h^k(y)) < \epsilon$ for $k = 0, 1, \ldots, m-1$. From the regularity of $h^m$ it follows that there exists $\delta > 0$ such that if $d(p, x) < \delta$, then $d(h^{m^m}(p), h^{m^m}(x)) < \delta$ for every integer $n$. Then it is easy to see that if $d(p, x) < \delta$, then $d(h^n(p), h^n(x)) < \epsilon$ for every integer $n$, and the proof is complete.
Let $X$ be compact. From the above Lemma and the definition of $A(h)$ it follows clearly that if $p \in A(h^m)$ for some integer $m(\neq 0)$, then $p \in A(h)$. Conversely we have the following

**Lemma 9.** If $p \in A(h)$, then $p \in A(h^m)$ for every integer $m(\neq 0)$.

**Proof.** This follows immediately from the theorem of P. Erdös and A. H. Stone [2].

By Lemma 8 and 9 we have immediately the following

**Theorem 3.** Let $X$ be compact and $h$ a homeomorphism of $X$ onto itself. Then $I(h) = I(h^m)$, $A(h) = A(h^m)$ and $N(h) = N(h^m)$ for every integer $m(\neq 0)$.

§ 2.

Let $h$ be a homeomorphism of $X$ onto itself. An isolated point of the set $I(h)$ is said to be an isolated irregular point of $h$ and furthermore if $h(p) = p$, then $p$ is said to be an isolated irregular fixed point.

**Lemma 10.** Let $h$ be a homeomorphism of $X$ onto itself. Suppose that there exists an isolated irregular fixed point $p$ of $h$ and that $X$ is locally compact at $p$. Then there exists a point $q \in R(h)$ such that $\lim_{n \to +\infty} h^n(q) = p$.

**Proof.** Since $p$ is an isolated irregular point of $h$, there exists a neighbourhood $U$ of $p$ such that $h$ is regular for every point of $U - p$. Since $h$ is irregular at $p$ and $h(p) = p$, there exists $\varepsilon_0 > 0$ which satisfies the following condition: Given $\varepsilon < \varepsilon_0$, for each $\delta < 0$ there exists a point $x$ with $d(p, x) < \delta$ such that there exists an integer $n(\delta)$ with $d(p, h_{n(\delta)}(x)) \geq \varepsilon$. Since $X$ is locally compact at $p$, there exists a neighbourhood $V$ of $p$ such that $V$ is compact. Then there exists $\varepsilon_1 > 0$ such that

$$U_{\varepsilon_1}(p) \subseteq U \cap U_{\varepsilon_1}(p) \cap V.$$ 

Since $h(p) = p$, there exists $\varepsilon_2(\leq \varepsilon_1)$ such that

$$h(U_{\varepsilon_2}(p)) \cup h^{-1}(U_{\varepsilon_1}(p)) \subseteq U_{\varepsilon_1}(p).$$

From this it follows that if $x \in U_{\varepsilon_2}(p)$ and $h^n(x) \cap U_{\varepsilon_1}(p) = 0$ for some integer $n$, then there exists an integer $n'$ such that $h^{n'}(x) \in U_{\varepsilon_1}(p) - U_{\varepsilon_2}(p)$.

Let $\delta_i (\geq 0)$ be a sequence such that $\delta_0 = \varepsilon_2, \delta_1 > \delta_2 > \delta_3 > \cdots$ and $\lim \delta_n = 0$. Then for each $\delta_n$ there exists $x_n$ with $d(p, x_n) < \delta_n$ such that $d(p, h^{m(n)}(x_n)) \geq \varepsilon_1$ for some integer $m(n)$. Therefore there exists an integer $m'(n)$ such that $h^{m'(n)}(x_n) \in U_{\varepsilon_1}(p) - U_{\varepsilon_2}(p)$ Then there exist a $q \in U_{\varepsilon_1}(p) - U_{\varepsilon_2}(p)$ and a subsequence $\{n_i\}$ such that $\lim_{i \to +\infty} h^{m'(n_i)}(x_{n_i}) = q$. 


Now we shall prove that $\lim_{n \to +\infty} h^n(q) \ni p$. Given $\varepsilon' > 0$, there exists a natural number $n_0$ such that $\delta < \varepsilon'$. Since $h$ is regular at $q$, there exists $\delta' > 0$ such that if $d(q, x) < \delta'$, then $d(h^n(q), h^n(x)) < \varepsilon' - \delta$. For every integer $n$. Since $\lim_{n \to \infty} h^n(x_n) = q$, there exists an integer $N(> n_0)$ such that $d(h^n(x_n), q) < \delta'$. Then

$$d(p, h^{-n(N)}(q)) < d(p, x_N) + d(x_N, h^{-n(N)}(q)) < \delta_N + (\varepsilon' - \delta) < \varepsilon'. $$

This proves that $\lim_{n \to +\infty} h^n(q) \ni p$ and the proof of Lemma 10 is complete.

**Lemma 11.** Let $X$ be locally compact and $h$ a homeomorphism of $X$ onto itself. Suppose that $I(h)$ is a closed subset of $X$ and that there exists an isolated irregular fixed point $p$ of $h$. Let $q \in R(h)$. If $\lim_{n \to +\infty} h^n(q) \ni p$, then $p = \lim_{n \to +\infty} h^n(q)$.

**Proof.** Suppose on the contrary that $h^n(q)$ does not converge to $p$ when $n \to +\infty$. Then there exists $\varepsilon_i > 0$ such that for infinitely many natural numbers $n_i$, $d(p, h^n(q)) > \varepsilon_i$. Let $\varepsilon(\leq \varepsilon_i)$ be such that $\bar{U}_\varepsilon(p)$ is compact and that $U_{\delta} \cap I(h) = \emptyset$. Since $h(p) = p$, there exists $\delta(\leq \varepsilon)$ such that $h(U_{\delta}(p)) \subset U_{\delta}(p)$. Then it is easy to see that there exist infinitely many natural numbers $n_i$ such that $h^{n_i}(q) \in U_{\varepsilon}(p) - U_{\delta}(p)$.

Since $U_{\varepsilon}(p) - U_{\delta}(p)$ is compact, $\lim_{n \to +\infty} h^n(q) \cap U_{\varepsilon}(p) - U_{\delta}(p) \ni 0$. Since $U_{\varepsilon}(p) - U_{\delta}(p) \cap I(h) = 0$, $\lim_{n \to +\infty} h^n(q) \cap I(h) = 0$. From Lemma 2 it follows that $q \in A(h)$ and therefore $\lim_{n \to +\infty} h^n(q) \cap I(h) = 0$ by Theorem 1. This contradiction completes the proof.

**Lemma 12.** Let $X$ be locally compact. Suppose that $I(h)$ is a closed subset of $X$ and that there exists an isolated irregular fixed point $p$ of $h$. Put

$$P = \{ x \mid \lim_{n \to +\infty} h^n(x) = p, \ x \in R(h) \}.$$

Then $P$ is an open and closed subset of $R(h)$.

**Proof.** To prove that $P$ is an open subset of $R(h)$: There exists $\varepsilon < 0$ such that $\bar{U}_\varepsilon(p)$ is compact and that $\bar{U}_\varepsilon(p) \cap I(h) = \emptyset$. Let $x \in P$. Then $x \in N(h)$ by Theorem 1. Since $N(h)$ is an open subset of $X$ by Theorem 2, there exists a neighbourhood $U(x)$ of $x$ such that $U(x) \subset N(h)$. Then $U(x) \cap \{ x \mid \lim_{n \to +\infty} h^n(x) = p \} = \emptyset$. This proves that $P$ is an open subset of $R(h)$.
Then there exists $\delta > 0$ such that if $d(x, y) < \delta$, then $d(h^n(x), h^n(y)) < \frac{\varepsilon}{2}$ for every integer $n$ and that $U_\delta(x) \subset U(x)$. Since $\lim h^n(x) = p$, there exists a natural number $N$ such that for each $n > N$ $d(p, h^n(x)) < \frac{\varepsilon}{2}$. Then for each $n > N$, if $d(x, y) < \delta$,
\[ d(p, h^n(y)) \leq d(p, h^n(x)) + d(h^n(x), h^n(y)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \]

Since $U_\delta(p)$ is compact, $\lim h^n(y) \cap U_\varepsilon(p) \neq 0$. Since $y \in N(h)$, $\lim h^n(y) = p$ by Theorem 2 and Lemma 11. Therefore $P$ is an open subset of $R(h)$.

To prove that $P$ is a closed subset of $R(h)$: Suppose $x \in R(h) - P$. From Lemma 11 it follows that $\lim h^n(x) \cap \bar{p} = 0$. Then $\bigcup_{n=0}^{\infty} h^n(x) \cap \bar{p} = 0$. Put
\[ a = d(p, \bigcup_{n=0}^{\infty} h^n(x)). \]
Since $x \in R(h)$, there exists $\delta > 0$ such that if $d(x, y) < \delta$, then $d(h^n(x), h^n(y)) < \frac{a}{2}$ for every integer $n$. Then
\[ \bigcup_{n=0}^{\infty} h^n(y) \subset U_\frac{a}{2}(\bigcup_{n=0}^{\infty} h^n(x)). \]
Therefore $\lim h^n(y) \cap \bar{p} = 0$. Hence $R(h) - P$ is an open subset of $R(h)$, and the proof is complete.

By Lemmas 10, 11 and 12 we have immediately the following

**Theorem 4.** Let $X$ be locally compact and $h$ a homeomorphism of $X$ onto itself. Suppose that $I(h)$ is a closed subset of $X$ and that $R(h)$ is connected. If there exists an isolated irregular fixed point $p$ of $h$, then either for each $x \in R(h) \lim h^n(x) = p$ or for each $x \in R(h) \lim h^n(x) = p$.

**Theorem 5.** Let $X$ be compact and $h$ a homeomorphism of $X$ onto itself. Suppose that $I(h)$ is a closed subset of $X$ and that $R(h)$ is connected. Let $p$ be an isolated irregular point of $h$. If $h^m(p) = p$ for some natural number $m$, then $p$ is an isolated irregular fixed point of $h$.

**Proof.** We are only to prove that $h(p) = p$. Suppose on the contrary that $h(p) \neq p$. It follows from Theorem 3 that $I(h) = I(h^m)$. Therefore $p$ is an isolated irregular fixed point of $h^m$. Then it follows from Theorem 4 that either for each $x \in R(h) \lim h^{mn}(x) = p$ or for each $x \in R(h) \lim h^{mn}(x) = p$. Without loss of generality we may assume that $\lim h^{mn}(x) = p$ for each $x \in R(h)$. Then we have that
\[ \lim h^{mn+1}(x) = \lim h(h^{mn}(x)) = h(\lim h^{mn}(x)) = h(p) = p. \]
On the other hand we have that
\[ \lim_{n \to \infty} h^{m+1}(x) = \lim_{n \to \infty} h^m(h(x)) = p. \]
This is a contradiction and the proof is complete.

**Proof of Theorem I.** Let \( X \) be a compact \( C^* \)-set and \( h \) a homeomorphism of \( X \) onto itself which is regular for every \( x \in X \) except for a finite number of points. Put \( I(h) = \{ p_0, p_1, \ldots, p_m \} \). Then \( I(h) \) is a closed subset of \( X \) and all \( p_i (1 \leq i \leq m) \) are isolated irregular points of \( h \). Furthermore \( R(h) \) is connected. It is easy to see that for each \( p_i \) there exists a natural number \( n_i \) such that \( h^{n_i}(p_i) = p_i \). It follows from Theorem 5 that all \( p_i \) are isolated irregular fixed points of \( h \). Then by Theorem 4 either for each \( x \in R(h) \lim h^n(x) = p_i \) or for each \( x \in R(h) \lim h^n(x) = p_i (1 \leq i \leq m) \). Therefore the number of points which are irregular under \( h \) is at most two and the proof is complete.

**Proof of Theorem II.** This is clear from the proof of Theorem I.

By the theorem of the authors [4] we have the following

**Theorem 6.** If \( h \) is a homeomorphism of \( S^3 \) onto itself such that (i) \( h \) is irregular at \( a, b \) \((=) \in S^3 \), (ii) \( h \) is regular at every \( x \in S^3 - (a \cup b) \), then \( h \) is topologically equivalent to the dilatation in \( S^3 \).

**Remark 1.** B. v. Kerékjártó [5] proved that if \( h \) is a homeomorphism of \( S^3 \) onto itself which is regular for every \( x \in S^3 \) except for a finite number of points, then \( h \) is topologically equivalent to a linear transformation of complex numbers.

**Remark 2.** For the case where \( h \) is a homeomorphism of \( S^n \) onto itself which is regular except for only one point see H. Terasaka [7].

**Remark 3.** It is proved by R. H. Bing [1] and D. Montgomery and L. Zippin [6] respectively that there exist a sense-reversing and a sense-preserving homeomorphisms \( h_1 \) and \( h_2 \) of \( S^3 \) onto itself with period 2 (then they are regular for every \( x \in S^3 \)) such that \( h_1 \) is not topologically equivalent to the reflexion and \( h_2 \) is not topologically equivalent to the rotation in \( S^3 \).

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References