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On Homeomorphisms which are Regular Except for a Finite Number of Points

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Introduction

All spaces considered in this paper are separable metric. Let h be a homeomorphism of a set X onto itself. Then $p \in X$ is called *regular*[1] under h, if for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $d(p, x) < \delta$, then $d(h^n(p), h^n(x)) < \varepsilon$ for every integer n. If $p \in X$ is not regular under h, then p is called *irregular*.

A set X will be called a *C^*-set* if $X - A$ is connected for any A which consists of a finite number of points of X. For example any n-manifold ($n \geqq 2$) is a C^*-set. Then one of the purpose of this paper is to prove the following

Theorem I. *Let X be a compact C^*-set and h a homeomorphism of X onto itself. If h is regular at every $x \in X$ except for a finite number of points, then the number of points which are irregular under h is at most two.*

We shall also prove the following

Theorem II.[2] *Let X be a compact C^*-set and h a homeomorphism of X onto itself such that*

(i) *h is irregular at $a, b\ (\neq) \in X$,*

(ii) *h is regular at every $x \in X - (a \cup b)$.*

Then either (1) *for each $x \in X - b$ $h^n(x)$ converges to a when $n \to \infty$ and for each $x \in X - a$ $h^n(x)$ converges to b when $n \to -\infty$, or* (2) *for each $x \in X - a$ $h^n(x)$ converges to b when $n \to \infty$ and for each $x \in X - b$ $h^n(x)$ converges to a when $n \to -\infty$.*

§ 1.

Let X be a set and h a homeomorphism of X onto itself. Let $R(h)$ be the set of all points which are regular under h and $I(h)$ the set of all points which are irregular under h. Then

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1) Introduced by B. v. Kerékjártó [5].

2) This is a converse theorem of Theorem 1 of the authors [3].

$$X = R(h) \cup I(h) \text{ and } R(h) \frown I(h) = 0 .$$

Furthermore let $A(h)$ be the set of all points which are regular and *almost periodic*[3] under h and $N(h)$ the set of all points which are regular and not almost periodic under h. Then

$$R(h) = A(h) \cup N(h) \text{ and } A(h) \frown N(h) = 0 .$$

Lemma 1. *Let $p \in R(h)$. Then $p \in A(h)$ if and only if for each $\varepsilon > 0$ there exists a natural number n such that $d(p, h^n(p)) < \varepsilon$.*

PROOF. It is clear that the condition is sufficient. We shall prove that the condition is necessary. Let $\varepsilon > 0$. Since $p \in R(h)$, there exists $\delta > 0$ such that if $d(p, x) < \delta$, then $d(h^n(p), h^n(x)) < \varepsilon$ for every integer n. Since $p \in A(h)$, there exists an integer $N(\neq 0)$ such that $d(p, h^N(p)) < \delta$. If $N > 0$, then the proof is already complete. If $N < 0$, then $d(h^{-N}(p), p) < \varepsilon$, which completes the proof.

Similarly we have the following

Lemma 1′. *Let $p \in R(h)$. Then $p \in A(h)$ if and only if for each $\varepsilon > 0$ there exists a natural number n such that $d(p, h^{-n}(p)) < \varepsilon$.*

Lemma 2. *Let $p \in R(h)$. If $(\overline{\lim_{n\to\pm\infty}} h^n(p))$[4] $\frown R(h) \neq 0$, then $p \in A(h)$.*

PROOF. Let $q \in (\overline{\lim_{n\to\pm\infty}} h^n(p)) \frown R(h)$. Given $\varepsilon > 0$, there exists $\delta > 0$ such that if $d(q, x) < \delta$, then $d(h^n(q), h^n(x)) < \frac{\varepsilon}{2}$ for every integer n. Since $q \in \overline{\lim_{n\to\pm\infty}} h^n(p)$, there exist integers m_1 and m_2 $(m_1 \neq m_2)$ such that $d(q, h^{m_1}(p)) < \delta$ and $d(q, h^{m_2}(p)) < \delta$. Then

$$d(p, h^{m_2-m_1}(p)) \leqq d(p, h^{-m_1}(q)) + d(h^{-m_1}(q), h^{m_2-m_1}(p)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon ,$$

which completes the proof.

Lemma 3. *For each $p \in A(h)$ $(\overline{\lim_{n\to\pm\infty}} h^n(p)) \frown N(h) = 0$.*

PROOF. Given $q \in (\overline{\lim_{n\to\pm\infty}} h^n(p)) \frown R(h)$, it is easy to see that $p \in \overline{\lim_{n\to\pm\infty}} h^n(q)$. From Lemma 2 it follows that $q \in A(h)$, which completes the proof.

Now assume that $p \in A(h)$ and that U is a neighbourhood of p. Let $n(p, U)$ be the set of all integers n_i such that $h^{n_i}(p) \in U$. Furthermore assume that $n_0 = 0$ and that $n_i < n_{i+1}$. It follows from Lemmas 1 and 1′ that n_i is defined uniquely for every integer i. Put

3) Let h be a homeomorphism of X onto itself. Then $x \in X$ is called almost periodic under h, if for each $\varepsilon > 0$ there exists an integer $n \neq 0$ such that $d(x, h^n(x)) < \varepsilon$.

4) $\overline{\lim_{u\to\pm\infty}} h^n(p) = \{x \mid$ for each $\varepsilon > 0$ there exist infinitely many integers n such that $d(x, h^n(p)) < \varepsilon\}$.

$$m[n_i] = n_{i+1} - n_i$$
$$n[p, U] = \underset{n_i \in n(p, U)}{l.\ u.\ b.}\ m[n_i].$$

A homeomorphism h of X onto itself is said to be *strongly regular* at $p \in X$, if there exists a neighbourhood U of p such that h is regular for every point of U. Then we have the following

Lemma 4. *Let X be locally compact. If h is strongly regular at $p \in A(h)$, then there exists $\varepsilon_0 > 0$ such that $n[p, U_\varepsilon(p)]$*[5] *is finite for every $\varepsilon < \varepsilon_0$.*

PROOF. Since X is locally compact, there exists a neighbourhood U of p such that $\bar{U}$ is compact. From the strong regularity of h at p it follows that there exists a neighbourhood V of p such that h is regular for every point of V. Let $\varepsilon_0 > 0$ be such that

$$U_{\varepsilon_0}(p) \subset U \frown V.$$

Let $\varepsilon < \varepsilon_0$. Suppose on the contrary that $n[p, U_\varepsilon(p)]$ is not finite. Then either $\overline{\lim\limits_{i\to\infty}}\, m[n_i] = \infty$ or $\overline{\lim\limits_{i\to\infty}}\, m[n_{-i}] = \infty$.

First we suppose that $\overline{\lim\limits_{i\to\infty}}\, m[n_i] = \infty$. Then there exists a subsequence $\{n_{i_j}\}$ of $\{n_i\}$ such that $\lim\limits_{j\to\infty} m[n_{i_j}] = \infty$. Since $\overline{U_\varepsilon(p)}$ is compact, there exists a subsequence $\{n_k\}$ of $\{n_{i_j}\}$ such that $\lim\limits_{k\to\infty} h^{n_k}(p) = q$, where $q \in \overline{U_\varepsilon(p)}$. Since $q \in R(h)$, there exists $\delta > 0$ such that if $d(q, x) < \delta$, then $d(h^n(q), h^n(x)) < \frac{\varepsilon}{3}$ for every integer n. Let K be a natural number such that if $k \geqq K$, then $d(q, h^{n_k}(p)) < \delta$. Since $p \in A(h)$, there exists a natural number N such that $d(p, h^{n_K+N}(p)) < \frac{\varepsilon}{3}$. Then for each $k \geqq K$

$$d(p, h^{n_k+N}(p)) \leqq d(p, h^{n_K+N}(p)) + d(h^{n_K+N}(p), h^N(q))$$
$$+ d(h^N(q), h^{n_k+N}(p)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

But this contradicts $\lim\limits_{k\to\infty} m[n_k] = \infty$.

Now we suppose that $\overline{\lim\limits_{i\to\infty}}\, m[n_{-i}] = \infty$. Then there exists a subsequence $\{n_{i_j}\}$ of $\{n_i\}$ such that $\lim\limits_{j\to\infty} m[n_{-i_j}] = \infty$. Since $\overline{U_\varepsilon(p)}$ is compact, there exists a subsequence $\{n_k\}$ of $\{n_{i_j}\}$ such that $\lim\limits_{k\to\infty} h^{n_{-k}}(p) = q$, where $q \in \overline{U_\varepsilon(p)}$. Since $q \in R(h)$, there exists $\delta > 0$ such that if $d(q, x)$

5) $U_\varepsilon(p) = \{x \mid d(p, x) < \varepsilon\}$.

$<\delta$, then $d(h^n(q), h^n(x))<\frac{\varepsilon}{2}$ for every integer n. Let K be a natural number such that if $k\geqq K$, then $d(q, h^{n_{-k}}(p))<\delta$. Then for each $k\geqq K$

$$d(p, h^{n_{-k}-n_{-K}}(p))\leqq d(p, h^{-n_{-K}}(q))+d(h^{-n_{-K}}(q),$$
$$h^{n_{-k}-n_{-K}}(p))<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon\,.$$

But this contradicts $\lim_{k\to\infty} m[n_{-k}]=\infty$. Thus the proof of Lemma 4 is complete.

Lemma 5. *Let X be locally compact. Suppose that $I(h)$ is a closed subset of X. Then for each $p\in A(h)$*

$$\overline{\lim_{n\to\pm\infty}}\,(h^n(p))\frown I(h)=0\,.$$

PROOF. Let $p\in A(h)$. Then there exist open subsets U and V such that $U\ni p$, $V\supset I(h)$ and $\bar{U}\frown\bar{V}=0$. Since h is strongly regular at p, it follows from Lemma 4 that there exists $\varepsilon_0>0$ such that $n[p, U_\varepsilon(p)]$ is finite for every $\varepsilon<\varepsilon_0$. Let $\varepsilon_1>0$ be such that $\varepsilon_1<\varepsilon_0$ and that $U_{\varepsilon_1}(p)\subset U$. Since $h(I(h))=I(h)$,

$$\overline{h^n(U_{\varepsilon_1}(p))}\frown I(h)=0$$

for every integer n. Put

$$U_0=\{x\,|\,x\in h^n(U_{\varepsilon_1}(p)),\ n=0\ 1,\cdots, n[p, U_{\varepsilon_1}(p)]-1\}\,.$$

Then U_0 is an open subset of X and $\bar{U}_0\frown I(h)=0$. From the definition of $n[p, U_{\varepsilon_1}(p)]$ it follows that $h^n(p)\in U_0$ for every integer n. Then

$$\overline{\lim_{n\to\pm\infty}}\,h^n(p)\frown I(h)=0\,,$$

which completes the proof.

By Lemmas 2, 3 and 5 we have immediately the following

Theorem 1. *Let X be locally compact and h a homeomorphism of X onto itself. Suppose that $I(h)$ is a closed subset of X and that $p\in R(h)$. Then*

(1) $p\in A(h)$ *if and only if* $\overline{\lim_{n\to\pm\infty}}\,h^n(p)\subset A(h)$ *and* $\overline{\lim_{n\to\pm\infty}}\,h^n(p)\neq 0$,

(2) $p\in N(h)$ *if and only if* $\overline{\lim_{n\to\pm\infty}}\,h^n(p)\subset I(h)$.

Lemma 6. *Let h be a homeomorphism of X onto itself. Suppose that $I(h)$ is a closed subset of X. Then $N(h)$ is an open subset of X.*

PROOF. Since $I(h)$ is a closed subset of X, we are only to prove that if $p\in R(h)\cap\overline{A(h)}$, then $p\in A(h)$. Let $\varepsilon>0$. Since $p\in R(h)$, there exists $\delta>0$ such that if $d(p, x)<\delta$, then $d(h^n(p), h^n(x))<\frac{\varepsilon}{3}$ for every integer n. Since $p\in\overline{A(h)}$, there exists $q\in A(h)$ such that $d(p, q)<\delta$. Since $q\in A(h)$, there exists an integer N such that $d(q, h^N(q))<\frac{\varepsilon}{3}$. Then

$$d(q, h^N(p))\leqq d(p, q)+d(q, h^N(q))+d(h^N(q), h^N(p))<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon\,.$$

Therefore $p\in A(h)$ and the proof is complete.

Lemma 7. *Let X be locally compact. Suppose that $I(h)$ is a closed subset of X. Then $A(h)$ is an open subset of X.*

PROOF. Let $p\in A(h)$. Let U be a neighbourhood of p such that $\overline{U}$ is compact and that $\overline{U}\cap I(h)=0$. Then there exists $\varepsilon>0$ such that $U_\varepsilon(p)\subset U$. Since $p\in R(h)$, there exists $\delta>0$ such that if $d(p, x)<\delta$, then $d(h^n(p), h^n(x))<\varepsilon$ for every integer n. Now we are only to prove that if $q\in U_\delta(p)$, then $q\in A(h)$. Since $p\in A(h)$, there exist infinitely many n_i such that $d(p, h^{n_i}(p))<\frac{\varepsilon}{2}$. Then

$$d(p, h^{n_i}(q))\leqq d(p, h^{n_i}(p))+d(h^{n_i}(p), h^{n_i}(q))<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon\,.$$

Since $\overline{U_\varepsilon(p)}$ is compact, $(\overline{\lim_{n\to\pm\infty}}\, h^n(q))\cap\overline{U_\varepsilon(p)}\neq 0$. Then $(\overline{\lim_{n\to\pm\infty}}\, h^n(q))\cap R(h)\neq 0$. From Lemma 2 it follows that $q\in A(h)$ and the proof is complete.

By Lemmas 6 and 7 we have immediately the following

Theorem 2. *Let X be locally compact and h a homeomorphism of X onto itself. Suppose that $I(h)$ is a closed subset of X. If $R(h)$ is connected, then $A(h)=0$ or $N(h)=0$.*

By the definition of the regularity we have clearly that if $p\in R(h)$, then $p\in R(h^m)$ for every integer m. Conversely we have the following

Lemma 8. *Let X be compact. If $p\in R(h^m)$ for some integer $m(\neq 0)$, then $p\in R(h)$.*

PROOF. Without loss of generality we may assume that $m>1$. Let $\varepsilon>0$. Since X is compact, h is uniformly continuous on X. Then there exists $\delta_0>0$ such that if $d(x, y)<\delta_0$, then $d(h^k(x), h^k(y))<\varepsilon$ for $k=0, 1, \cdots, m-1$. From the regularity of h^m it follows that there exists $\delta>0$ such that if $d(p, x)<\delta$, then $d(h^{mn}(p), h^{mn}(x))<\delta_0$ for every integer n. Then it is easy to see that if $d(p, x)<\delta$, then $d(h^n(p), h^n(x))<\varepsilon$ for every integer n, and the proof is complete.

Let X be compact. From the above Lemma and the definition of $A(h)$ it follows clearly that if $p \in A(h^m)$ for some integer $m(\neq 0)$, then $p \in A(h)$. Conversely we have the following

Lemma 9. *If $p \in A(h)$, then $p \in A(h^m)$ for every integer $m(\neq 0)$.*

PROOF. This follows immediately from the theorem of P. Erdös and A. H. Stone [2].

By Lemma 8 and 9 we have immediately the following

Theorem 3. *Let X be compact and h a homeomorphism of X onto itself. Then $I(h)=I(h^m)$, $A(h)=A(h^m)$ and $N(h)=N(h^m)$ for every integer $m(\neq 0)$.*

§ 2.

Let h be a homeomorphism of X onto itself. An isolated point of the set $I(h)$ is said to be an *isolated irregular point* of h and furthermore if $h(p)=p$, then p is said to be an *isolated irregular fixed point.*

Lemma 10. *Let h be a homeomorphism of X onto itself. Suppose that there exists an isolated irregular fixed point p of h and that X is locally compact at p. Then there exists a point $q \in R(h)$ such that $\overline{\lim_{n\to\pm\infty}} h^n(q) \ni p$.*

PROOF. Since p is an isolated irregular point of h, there exists a neighbourhood U of p such that h is regular for every point of $U-p$. Since h is irregular at p and $h(p)=p$, there exists $\varepsilon_0>0$ which satisfies the following condition: Given $\varepsilon<\varepsilon_0$, for each $\delta<0$ there exists a point x with $d(p, x)<\delta$ such that there exists an integer $n(\delta)$ with $d(p, h^{n(\delta)}(x)) \geqq \varepsilon$. Since X is locally compact at p, there exists a neighbourhood V of p such that $\bar{V}$ is compact. Then there exists $\varepsilon_1>0$ such that

$$\overline{U_{\varepsilon_1}(p)} \subset U \frown U_{\varepsilon_0}(p) \frown V.$$

Since $h(p)=p$, there exists $\varepsilon_2(<\varepsilon_1)$ such that

$$h(U_{\varepsilon_2}(p)) \smile h^{-1}(U_{\varepsilon_2}(p)) \subset U_{\varepsilon_1}(p).$$

From this it follows that if $x \in U_{\varepsilon_2}(p)$ and $h^n(x) \frown U_{\varepsilon_1}(p)=0$ for some integer n, then there exists an integer n' such that $h^{u'}(x) \in U_{\varepsilon_1}(p)-U_{\varepsilon_2}(p)$.

Let $\delta_n(>0)$ be a sequence such that $\delta_1=\varepsilon_2$, $\delta_1>\delta_2>\delta_3>\cdots$ and $\lim_{n\to\infty}\delta_n=0$. Then for each δ_n there exists x_n with $d(p, x_n)<\delta_n$ such that $d(p, h^{m(n)}(x_n)) \geqq \varepsilon_1$ for some integer $m(n)$. Therefore there exists an integer $m'(n)$ such that $h^{m'(n)}(x_n) \in U_{\varepsilon_1}(p)-U_{\varepsilon_2}(p)$ Then there exist a $q \in \overline{U_{\varepsilon_1}(p)-U_{\varepsilon_2}(p)}$ and a subsequence $\{n_i\}$ such that $\lim_{i\to\infty} h^{m'(n_i)}(x_{n_i})=q$.

Now we shall prove that $\overline{\lim_{n\to\pm\infty}} h^n(q) \ni p$. Given $\varepsilon'>0$, there exists a natural number n_0 such that $\delta_{n_0}<\varepsilon'$. Since h is regular at q, there exists $\delta'>0$ such that if $d(q, x)<\delta'$, then $d(h^n(q), h^n(x))<\varepsilon'-\delta_{n_0}$ for every integer n. Since $\lim_{i\to\infty} h^{m'(n_i)}(x_{n_i})=q$, there exists an integer $N(>n_0)$ such that $d(h^{m'(N)}(x_N), q)<\delta'$. Then

$$d(p, h^{-m'(N)}(q))<d(p, x_N)+d(x_N, h^{-m'(N)}(q))<\delta_N+(\varepsilon'-\delta_{n_0})<\varepsilon'.$$

This proves that $\overline{\lim_{n\to\pm\infty}} h^n(q) \ni p$ and the proof of Lemma 10 is complete.

Lemma 11. *Let X be locally compact and h a homeomorphism of X onto itself. Suppose that $I(h)$ is a closed subset of X and that there exists an isolated irregular fixed point p of h. Let $q \in R(h)$. If $\overline{\lim_{n\to\infty}} h^n(q) \ni p$, then $p = \lim_{n\to\infty} h^n(q)$.*

Proof. Suppose on the contrary that $h^n(q)$ does not converge to p when $n \to \infty$. Then there exists $\varepsilon_1>0$ such that for infinitely many natural numbers n_i $d(p, h^{n_i}(q))\geqq\varepsilon_1$. Let $\varepsilon(\leqq\varepsilon_1)$ be such that $\overline{U_\varepsilon(p)}$ is compact and that $U_\varepsilon(p)\frown I(h)=p$. Since $h(p)=p$, there exists $\delta(<\varepsilon)$ such that $h(U_\delta(p))\subset U_\varepsilon(p)$. Then it is easy to see that there exist infinitely many natural numbers n_i' such that

$$h^{n_i'}(q) \in U_\varepsilon(p)-U_\delta(p)\,.$$

Since $\overline{U_\varepsilon(p)-U_\delta(p)}$ is compact, $\overline{\lim_{n\to\infty}} h^n(q) \frown \overline{U_\varepsilon(p)-U_\delta(p)} \neq 0$. Since $\overline{U_\varepsilon(p)-U_\delta(p)} \frown I(h)=0$, $\overline{\lim_{n\to\infty}} h^n(q) \frown R(h) \neq 0$. From Lemma 2 it follows that $q \in A(h)$ and therefore $\overline{\lim_{n\to\infty}} h^n(q)\frown I(h)=0$ by Theorem 1. This contradiction completes the proof.

Lemma 12. *Let X be locally compact. Suppose that $I(h)$ is a closed subset of X and that there exists an isolated irregular fixed point p of h. Put*

$$P=\{x \mid \lim_{n\to\infty} h^n(x)=p,\ x\in R(h)\}.$$

Then P is an open and closed subset of $R(h)$.

Proof. To prove that P is an open subset of $R(h)$: There exists $\varepsilon<0$ such that $\overline{U_\varepsilon(p)}$ is compact and that $\overline{U_\varepsilon(p)}\frown I(h)=p$. Let $x\in P$. Then $x \in N(h)$ by Theorem 1. Since $N(h)$ is an open subset of X by Theorem 2, there exists a neighbourhood $U(x)$ of x such that $U(x)\subset N(h)$.

Then there exists $\delta>0$ such that if $d(x,y)<\delta$, then $d(h^n(x), h^n(y))<\frac{\varepsilon}{2}$ for every integer n and that $U_\delta(x)\subset U(x)$, Since $\lim_{n\to\infty} h^n(x)=p$, there exists a natural number N such that for each $n>N$ $d(p, h^n(x))<\frac{\varepsilon}{2}$. Then for each $n>N$, if $d(x,y)<\delta$,

$$d(p, h^n(y))\leqq d(p, h^n(x))+d(h^n(x), h^n(y))<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

Since $\overline{U_\varepsilon(p)}$ is compact, $\overline{\lim_{n\to\infty}} h^n(y)\frown\overline{U_\varepsilon(p)}\neq 0$. Since $y\in N(h)$, $\lim_{n\to\infty} h^n(y)=p$ by Theorem 2 and Lemma 11. Therefore P is an open subset of $R(h)$.

To prove that P is a closed subset of $R(h)$: Suppose $x\in R(h)-P$. From Lemma 11 it follows that $\overline{\lim_{n\to\infty}} h^n(x)\frown p = 0$. Then $\overline{\bigcup_{n=0}^\infty h^n(x)}\frown p=0$. Put

$$a = d(p, \overline{\bigcup_{n=0}^\infty h^n(x)})\,.$$

Since $x\in R(h)$, there exists $\delta>0$ such that if $d(x,y)<\delta$, then $d(h^n(x), h^n(y))<\frac{a}{2}$ for every integer n. Then

$$\overline{\bigcup_{n=0}^\infty h^n(y)}\subset U_{\frac{a}{2}}(\overline{\bigcup_{n=0}^\infty h^n(x)})\,.$$

Therefore $\overline{\lim_{n\to\infty}} h^n(y)\frown p=0$. Hence $R(h)-P$ is an open subset of $R(h)$, and the proof is complete.

By Lemmas 10, 11 and 12 we have immediately the following

Theorem 4. *Let X be locally compact and h a homeomorphism of X onto itself. Suppose that $I(h)$ is a closed subset of X and that $R(h)$ is connected. If there exists an isolated irregular fixed point p of h, then either for each $x\in R(h)$ $\lim_{n\to\infty} h^n(x)=p$ or for each $x\in R(h)$ $\lim_{n\to-\infty} h^n(x)=p$.*

Theorem 5. *Let X be compact and h a homeomorphism of X onto itself. Suppose that $I(h)$ is a closed subset of X and that $R(h)$ is connected. Let p be an isolated irregular point of h. If $h^m(p)=p$ for some natural number m, then p is an isolated irregular fixed point of h.*

PROOF. We are only to prove that $h(p)=p$. Suppose on the contrary that $h(p)\neq p$. It follows from Theorem 3 that $I(h)=I(h^m)$. Therefore p is an isolated irregular fixed point of h^m. Then it follows from Theorem 4 that either for each $x\in R(h)$ $\lim_{n\to\infty} h^{mn}(x)=p$ or for each $x\in R(h)$ $\lim_{n\to-\infty} h^{mn}(x)=p$. Without loss of generality we may assume that $\lim_{n\to\infty} h^{mn}(x)=p$ for each $x\in R(h)$. Then we have that

$$\lim_{n\to\infty} h^{mn+1}(x) = \lim_{n\to\infty} h(h^{mn}(x)) = h(\lim_{n\to\infty} h^{mn}(x)) = h(p)\neq p.$$

On the other hand we have that

$$\lim_{n\to\infty} h^{mn+1}(x) = \lim_{n\to\infty} h^{mn}(h(x)) = p\ .$$

This is a contradiction and the proof is complete.

PROOF OF THEOREM I. Let X be a compact C^*-set and h a homeomorphism of X onto itself which is regular for every $x \in X$ except for a finite number of points. Put $I(h) = \{p_0, p_1, \cdots, p_m\}$. Then $I(h)$ is a closed subset of X and all $p_i (1 \leqq i \leqq m)$ are isolated irregular points of h. Furthermore $R(h)$ is connected. It is easy to see that for each p_i there exists a natural number n_i such that $h^{n_i}(p_i) = p_i$. It follows from Theorem 5 that all p_i are isolated irregular fixed points of h. Then by Theorem 4 either for each $x \in R(h)$ $\lim_{n\to\infty} h^n(x) = p_i$ or for each $x \in R(h)$ $\lim_{n\to-\infty} h^n(x) = p_i (1 \leqq i \leqq m)$. Therefore the number of points which are irregular under h is at most two and the proof is complete.

PROOF OF THEOREM II. This is clear from the proof of Theorem I.

By the theorem of the authors [4] we have the following

Theorem 6. *If h is a homeomorphism of S^3 onto itself such that (i) h is irregular at a, $b\ (\neq) \in S^3$, (ii) h is regular at every $x \in S^3 - (a \cup b)$, then h is topologically equivalent to the dilatation in S^3.*

Remark 1. B. v. Kerékjártó [5] proved that if h is a homeomorphism of S^2 onto itself which is regular for every $x \in S^2$ except for a finite number of points, then h is topologically equivelent to a linear transformation of complex numbers.

Remark 2. For the case where h is a homeomorphism of S^n onto itself which is regular except for only one point see H. Terasaka [7].

Remark 3. It is proved by R. H. Bing [1] and D. Montgomery and L. Zippin [6] respectively that there exist a sense-reversing and a sense-preserving homeomorphisms h_1 and h_2 of S^3 onto itself with period 2 (then they are regular for every $x \in S^3$) such that h_1 is not topologically equivalent to the reflexion and h_2 is not topologically equivalent to the rotation in S^3.

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