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## ***On the Graphs of Knots***

By Takeshi YAJIMA and Shin'ichi KINOSHITA<sup>1)</sup>

### **1. Introduction.**

A projection of a knot or a link on a 2-dimensional plane divides the plane into several regions. It is a frequently used method to separate these regions into two classes, white regions and black regions, in the study of the knot theory. Simplifying this method, C. Bankwitz [1] introduced the notion of graph of knots in his study of alternating knots. R. J. Aumann also used the graph of knots in [3].

The present note contains a method of a graphical treatment of knots and links, and, as an application, a sufficient condition is given that a knot is amphibious. Other applications have been given in [5].

### **2. The graph of knots.**

Let  $\pi$  be a regular normed projection<sup>2)</sup> of a knot  $K$  on a 2-dimensional sphere  $S$ . If  $\pi$  has  $n$  double points  $D_1, D_2, \dots, D_n$ , then it divides  $S$  into  $n+2$  regions, each of which is homeomorphic to an open disk. Now, separate these regions into two classes  $\mathfrak{A}$  and  $\mathfrak{B}$ , in such a way that each segment of  $\pi$ , i.e. an arc from a double point to the next one, is always the common boundary of a region of  $\mathfrak{A}$  and that of  $\mathfrak{B}$ . We assume for convenience that the point at infinity is contained in a region of class  $\mathfrak{B}$ .

Let  $A_1, A_2, \dots, A_\alpha$  be the regions of class  $\mathfrak{A}$ . Take points  $P_i \in A_i$  ( $i=1, 2, \dots, \alpha$ ) and connect these points by  $n$  non-intersecting arcs  $d_1, d_2, \dots, d_n$  in such a way that each  $d_k$  corresponds to  $D_k$  ( $k=1, 2, \dots, n$ ) and  $P_i$  and  $P_j$  are connected by  $d_k$ , if and only if  $A_i$  and  $A_j$  have a common double point  $D_k$  on their boundaries.

The regions of class  $\mathfrak{A}$  in all can be considered as a projection of a surface spanning  $K$  which is twisted  $180^\circ$  at each double point of  $\pi$ .

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1) The authors express our gratitude to Professor Hidetaka Terasaka for his kind suggestions.

2) See [2], p. 6.

From this consideration, we define the sign of  $d_k$  ( $k=1, 2, \dots, n$ ) in such a way that it is  $+$  or  $-$  according as the surface is twisted at  $D_k$  as the right screw or the left one, respectively (Fig. 1).

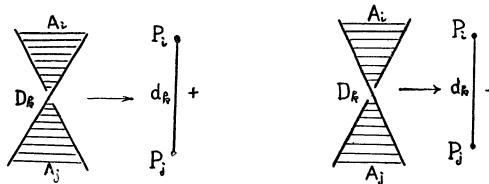


Fig. 1

Then we have a linear graph of  $\alpha$  vertices  $P_1, P_2, \dots, P_\alpha$ , and  $n$  signed arcs  $d_1, d_2, \dots, d_n$ ; each vertex of which corresponds to a region of  $\mathfrak{A}$ , and each signed arc corresponds to a normed double point. We call this linear graph the *graph* of the projection  $\pi$  and denote it by  $g(\pi)$  (c.f. 5. (1) for links).

From the same consideration about the class  $\mathfrak{B}$ , we get another graph  $g'(\pi)$ . We call this the *dual graph* of  $g(\pi)$ . We get the dual graph  $g'(\pi)$  directly from a given graph  $g(\pi)$  as follows. Let  $B'_1, B'_2, \dots, B'_\beta$  be the regions of  $S-g(\pi)$ . Take points  $Q_i \in B'_i$  ( $i=1, 2, \dots, \beta$ ) and connect these points by  $n$  non-intersecting arcs  $d'_1, d'_2, \dots, d'_n$  in such a way that each  $d'_k$  corresponds to  $d_k$  of  $g(\pi)$  ( $k=1, 2, \dots, n$ ) and  $Q_i$  and  $Q_j$  are connected by  $d'_k$ , if and only if  $d_k$  is the common boundary of  $B'_i$  and  $B'_j$ . Each arc  $d'_k$  takes the sign opposite to the sign of  $d_k$ . Thus we have a graph  $g'(\pi)$  which consists of vertices  $Q_1, Q_2, \dots, Q_\beta$  and arcs  $d'_1, d'_2, \dots, d'_n$ .

The following examples show several graphs of the trefoil knot.

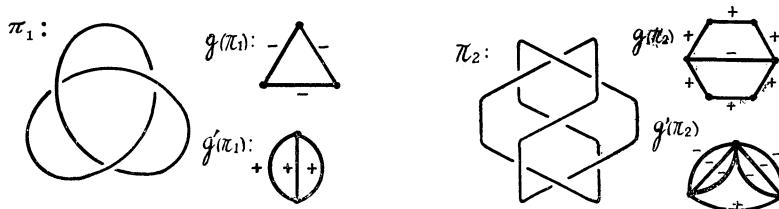


Fig. 2

**REMARK 1.** For a given projection  $\pi$  of a knot or a link, we have two graphs  $g(\pi)$  and  $g'(\pi)$ . Conversely, for a given graph, whichever  $g(\pi)$  or  $g'(\pi)$ , there is a uniquely determined projection  $\pi$ .

**REMARK 2.** For the sake of simplicity we may omit one of the signs  $+$  or  $-$  from graphs.

REMARK 3. Let  $\pi$  and  $\pi^*$  be projections of knots  $K$  and  $K^*$  respectively, which are mirror images to each other. Then  $g(\pi)$  and  $g(\pi^*)$  are of the same type and have the opposite signs. We call  $g(\pi^*)$  the *conjugate graph* of  $g(\pi)$ .

REMARK 4. R. J. Aumann [3] defined the canonical form  $K'$  from a graph  $g(\pi)$  of a knot  $K$ .  $K'$  is a knot equivalent to  $K$  situating on an orientable 2-manifold  $T$ . In general  $T - K'$  is not connected. However, we can prove that at least one of  $T - K'$  and  $T' - K''$  is connected where  $K''$  is the canonical form from  $g'(\pi)$  and  $T'$  is the orientable 2-manifold with respect to  $K''$ .

### 3. The operations on graphs.

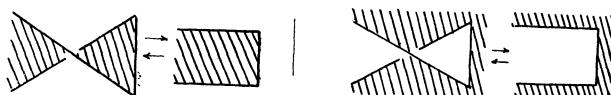
If the intersection of a connected subcomplex  $\Gamma$  of a graph  $g(\pi)$  with the remaining subcomplex  $\overline{g(\pi) - \Gamma}$  consists of at most two vertices, then we call  $\Gamma$  a *block* of  $g(\pi)$ . A single arc is the smallest block of graphs. If blocks  $\Gamma_1, \dots, \Gamma_k$  have two common vertices then we say that these blocks are *connected in parallel*. If blocks  $\Gamma_1$  and  $\Gamma_2$  have a common vertex which does not belong to any others, then we say that  $\Gamma_1$  and  $\Gamma_2$  are *connected in a series*. A common vertex of blocks will be called a *joint*.

It is well known that every deformation of a knot is equivalent to a combination of the fundamental deformation  $\Omega_1$ ,  $\Omega_2$  and  $\Omega_3$  in [2]<sup>3)</sup>. These fundamental deformations can be translated into the operations on graphs as follows. Since there are two graphs  $g(\pi)$  and  $g'(\pi)$  for a projection  $\pi$  of a knot, we have two operations on graphs  $O_i$  and  $O'_i$  for a deformation  $\Omega_i$ . Such operations will be called the *dual operations* of each other.

(3.1) *If a graph  $g(\pi)$  consists of a block  $\Gamma$  and an arc  $d$  whose intersection is a single joint, then  $d$  can be omitted. If a graph  $g(\pi)$  consists of a block  $\Gamma$  and a closed arc  $d$  whose intersection is a single joint, then  $d$  can be omitted. The converse operations are also true.*

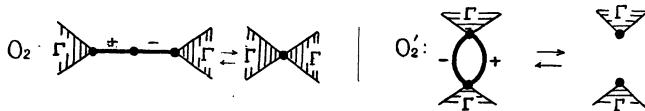


These statements are verified by the following projections.

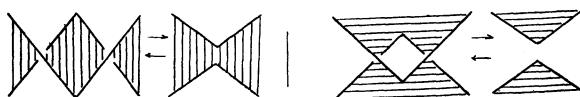


3) See [2], p. 7.

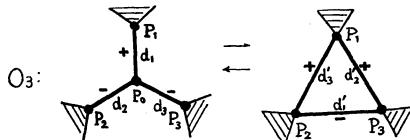
(3.2) If a graph  $g(\pi)$  consists of a block  $\Gamma$  and two arcs  $d_1, d_2$  in opposite signs which are connected in a series  $\Gamma, d_1, d_2, \Gamma$ , then the arcs can be omitted affixing the joints of  $\Gamma$ . If a graph  $g(\pi)$  consists of a block  $\Gamma$  and two arcs  $d_1, d_2$  in opposite signs which are connected in parallel, then the arcs can be omitted. The converse operations are also true.



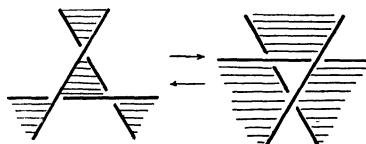
The following projections verify the statements.



(3.3) If vertices  $P_1, P_2, P_3$  are connected with a vertex  $P_0$  by arcs  $d_1, d_2, d_3$  respectively where one of the arcs is in opposite sign from the others, then  $P_0$  can be omitted and each  $d_i$  ( $i=1, 2, 3$ ) replaced by the arc  $d'_i$  connecting the vertices  $P_j, P_k$  with the opposite sign to  $d_i$ , where the suffices  $j, k$  are different from  $i$  and from each other. The converse operation is also true.



In this case  $O_3$  and  $O'_3$  coincide. The corresponding projection is as follows.

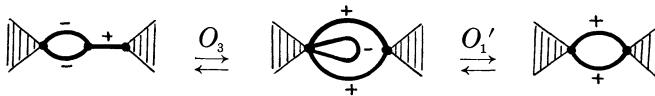


From the fundamental operations we have the following corollaries.

(3.4) If a graph  $g(\pi)$  consists of two blocks  $\Gamma_0, \Gamma_1$  and an arc  $d$  which are connected in a series  $\Gamma_0, \Gamma_1, d, \Gamma_0$  where  $\Gamma_1$  consists of two arcs of the same sign which are connected in parallel and  $d$  is in opposite sign from  $\Gamma_1$ , then  $d$  can be omitted and  $\Gamma_1$  changed the sign. The converse and their dual operations are also true.

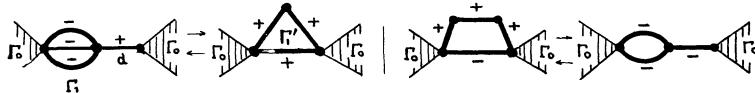


Proof.

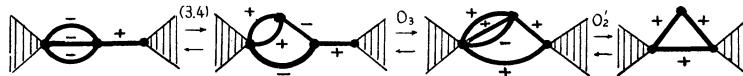


The statement of the right hand side is proved by the corresponding dual operations of the left one.

(3.5) If a graph  $g(\pi)$  consists of two blocks  $\Gamma_0$ ,  $\Gamma_1$  and an arc  $d$  which are connected in a series  $\Gamma_0$ ,  $\Gamma_1$ ,  $d$ ,  $\Gamma_0$  where  $\Gamma_1$  consists of three arcs of the same sign connected in parallel and where  $d$  is in opposite sign to  $\Gamma_1$ , then  $d$  can be omitted and  $\Gamma_1$  replaced by the triangular block  $\Gamma_1'$  of the opposite sign from  $\Gamma_1$ . The converse and their dual operations are also true.



Proof.



The statement of the right hand side is proved by the corresponding dual operations of the left one.

The above-mentioned operations on graphs correspond to the local deformations of knots. We have further operations on graphs, which correspond to the deformations in the global. They are the consequence of the fundamental operations on graphs.

(3.6) Every graph  $g(\pi)$  of a knot is equivalent to its dual graph  $g'(\pi)$  by the fundamental operations.

Proof<sup>4)</sup>. Suppose  $g(\pi)$  is a graph of the class  $\mathfrak{A}$ , and  $B_\infty$  is the region of  $S - \pi$  containing the point at infinity. Let  $D_1$  be a double point of  $\pi$  on the boundary of  $B_\infty$ , and  $D_2$  be a consecutive double point of  $\pi$  along the boundary of  $B_\infty$ . Let  $s$  be a bounding segment of  $B_\infty$  whose boundaries are  $D_1$  and  $D_2$ . Let  $D'_1$ ,  $D'_2$  be two points on  $s$ , and

4) We can also prove (3, 6) by a pure graphical method.

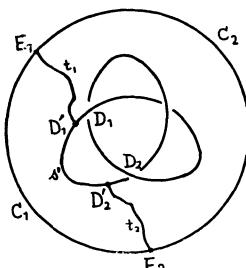


Fig. 3

$s'$  be a sub-segment of  $s$  whose boundaries are  $D_1'$  and  $D_2'$ .

We can suppose that  $\pi$  is contained in a circle  $C$ . If we take two points  $E_1, E_2$  on the circle  $C$ , then  $C$  is divided into two arcs  $C_1$  and  $C_2$ . We can connect  $E_1$  with  $D_1'$  and  $E_2$  with  $D_2'$  by disjoint arcs  $t_1$  and  $t_2$  respectively where  $t_1$  and  $t_2$  do not intersect  $\pi$  and  $C$  except their boundaries. Put  $\pi_i = \pi - s' + (C_i + t_1 + t_2)$ , ( $i=1, 2$ ). Either  $C_1 + t_1 + t_2$  or  $C_2 + t_1 + t_2$  can be deformed

into  $s'$  in  $S - \pi + s'$ . Assume the former. Then we have  $g(\pi_i) = g(\pi)$  and  $g(\pi_2) = g'(\pi)$ . Since  $\pi_1$  is equivalent to  $\pi_2$  by the deformation of  $C_1$  into  $C_2$ ,  $g(\pi_1)$  is equivalent to  $g(\pi_2)$  by the fundamental operations. Hence  $g(\pi)$  and  $g'(\pi)$  are equivalent.

(3.7) *If a graph  $g(\pi)$  consists of blocks  $\Gamma_1, \dots, \Gamma_k$  which are connected in parallel, then we can deform a block in a cyclic order.*

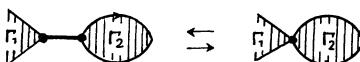
Since  $g(\pi)$  is situated on a sphere, the statement is evident.

(3.8) *If a graph  $g(\pi)$  consists of two blocks  $\Gamma_1, \Gamma_2$  and two arcs  $d_1, d_2$  in opposite signs such that they are connected in a series  $\Gamma_1, d_1, \Gamma_2, d_2, \Gamma_1$ , then the arcs can be omitted with a rotation of  $\Gamma_2$  in  $180^\circ$  around the axis through the joints of  $\Gamma_2$ . The converse operation is also true.*



This statement corresponds to the deformation  $\Omega_5$  in [2]<sup>5)</sup>.

(3.9) *If a graph  $g(\pi)$  consists of two blocks  $\Gamma_1, \Gamma_2$  and an arc  $d$  in any signs such that they are connected in a series  $\Gamma_1, d, \Gamma_2$ , then  $d$  can be omitted with a rotation of  $\Gamma_2$  in  $180^\circ$ . The converse operation is also true.*



This is the operation corresponding to the deformation  $\Omega_4$  in [2]<sup>6)</sup>, and the proof may be omitted. However, from  $O_1, O_2$  and (3.8) it is easy to prove the statement by the graphical method.

5) See [2], p. 11.

6) See [2], p. 10.

#### 4. Applications.

To trace the process of deformations of knots is a main purpose of the graphical representation of knots. For example, the equivalence of  $\pi_2$  and  $\pi_1$  (Fig. 2) is proved as follows.

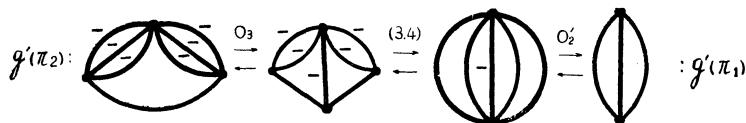


Fig. 4

Thus the deformation of knots, therefore the equivalence of knots, is explained schematically by the graphical representation of knots.

The graphical representation of knots also make a contribution to the study of construction of knots. For example, as the union<sup>7)</sup> of knots  $3_1$  and  $4_1$  in the table of Alexander-Briggs<sup>8)</sup>, we have the following possible cases.

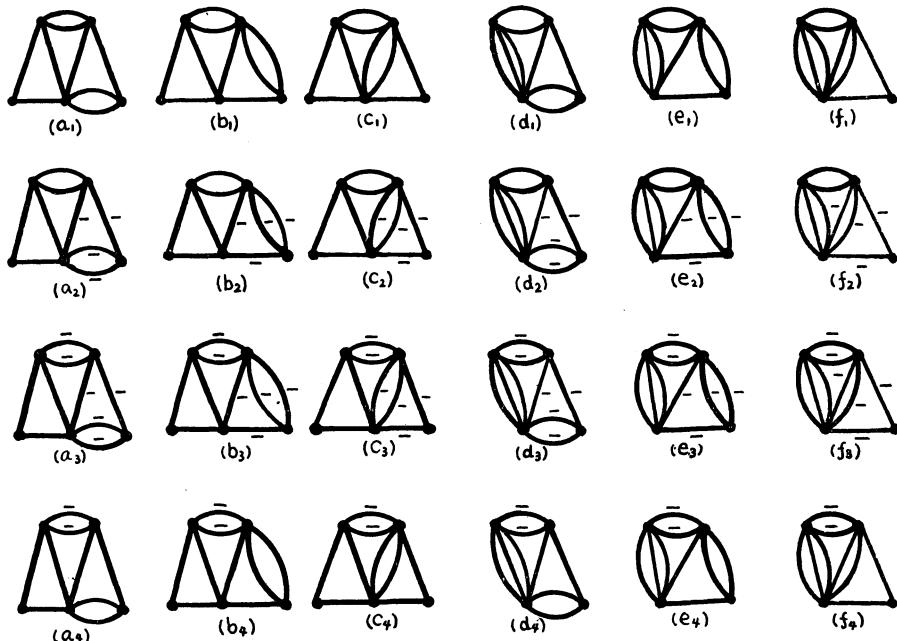


Fig. 5

7) See [5], p. 134.

8) See [2], p. 70.

$(a_i)$  and  $(b_i)$  are equivalent by the following operation.

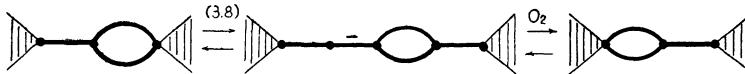


Fig. 6

$(d_i)$  and  $(e_i)$  are also equivalent by the same operation.

From the corollary (3.5) we have

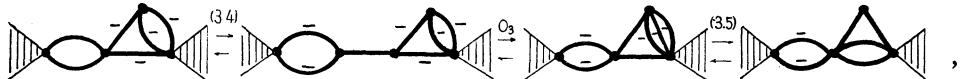


Fig. 7

which shows that  $(a_2)$  and  $(c_4)$ ,  $(a_4)$  and  $(c_2)$ ,  $(d_2)$  and  $(f_4)$ ,  $(d_4)$  and  $(f_2)$  are equivalent to each other respectively.

From the same corollary we have



Fig. 8

which shows that  $(a_4)$  and  $(d_3)$ ,  $(f_4)$  and  $(c_3)$ ,  $(d_4)$  and  $(a_3)$ ,  $(c_4)$  and  $(f_3)$  are equivalent to each conjugate graph of the other, respectively.

In the consequence we have

$(a_1) \sim 9_{30}$	of Alexander-Briggs's table,
$(c_1) \sim 9_{25}$	" ,
$(d_1) \sim 9_{22}$	" ,
$(f_1) \sim 9_{36}$	" ,
$(a_2) \sim (c_4) \sim (f_3) \sim 9_{44}$	" ,
$(c_2) \sim (a_4) \sim (d_3) \sim 9_{45}$	" ,
$(d_2) \sim (f_4) \sim (c_3) \sim 9_{42}$	" ,
$(f_2) \sim (d_4) \sim (a_3) \sim 9_{43}$	" .

Further application of graphs to the knot-theory is the following

**Theorem.** *If  $g(\pi)$  and  $g'(\pi)$  are of the same type and have the opposite signs, then the original knot  $K$  of  $\pi$  is amphibious, that is to say,  $K$  is equivalent to its mirror image  $K^*$ <sup>9)</sup>.*

9) A necessary condition for the amphibious knot is given in [2], p. 31. But this is not a sufficient condition. Recently H. Schubert [4] has given a necessary and sufficient condition for the amphibious knot which have the two-bridge representation.

Proof. Since  $g(\pi)$  and  $g'(\pi)$  are of the same type and have the opposite signs,  $g(\pi^*)$  and  $g'(\pi)$  coincide. Therefore, from (3.6)  $\pi$  and  $\pi^*$  are equivalent. Hence  $K$  and  $K^*$  are equivalent.

From this theorem we can verify that the knots  $4_1$ ,  $6_3$ ,  $8_3$ ,  $8_9$ ,  $8_{12}$ ,  $8_{17}$ , and  $8_{18}$  of Alexander-Briggs's table are amphibious.

### 5. Remarks for links.

(1) In the case of links, the construction of the graph must be modified as follows: For a connected projection  $\pi$  of a link  $L$ , we construct  $g(\pi)$  by the same manner as the graph of knots. If  $\pi$  is not connected, then there exist some regions which are not simply connected. Suppose  $A_i$  is such a region and there exist  $r$  components  $\pi^1, \pi^2, \dots, \pi^r$  of  $\pi$ , which intersect the boundary of  $A_i$ . Instead of the point  $P_i$  in the construction of the graph of knots, we take  $r$  points  $P_{i1}, P_{i2}, \dots, P_{ir}$  and presume that  $P_{ij}$  is the representative of  $A_i$  with respect to  $\pi^j$  ( $j=1, 2, \dots, r$ ). Then we have the graph in the same manner as in the case of knots.

(2) In the operation  $O_2$ , let  $P_1$  and  $P_2$  be the joints of  $\Gamma$  with the arcs  $d_1$  and  $d_2$  respectively, and  $P_3$  be the joint of  $d_1$  with  $d_2$ .  $P_1$  does not coincide with  $P_2$  in the case of knots. However  $P_1$  and  $P_2$  may coincide in the case of links. In this case,  $d_1$  and  $d_2$  can be omitted and  $P_3$  remained without cancelling.

The other operations are also valid in the case of links.

(Received September 1, 1957)

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