DEGENERATING FAMILIES OF FINITE BRANCHED COVERINGS

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1. Introduction

The category of finite branched coverings of a given complex projective manifold M is equivalent to the category of finite extensions $K/\mathbb{C}(M)$ of the rational function field $\mathbb{C}(M)$ of M. Hence the study of finite branched coverings of M is nothing but a geometric study of extensions of algebraic function fields. In Namba [8], we constructed and studied the moduli space of equivalence classes of finite branched coverings of the complex projective line $\mathbb{P}^1 = \mathbb{P}^1(\mathbb{C})$. If we want to compactify the moduli space, we are obliged to consider degenerations of branched coverings.

In this paper, we study degenerating families of finite branched coverings of $\mathbb{P}^1$ and $\mathbb{P}^m = \mathbb{P}^m(\mathbb{C}) (m \geq 2)$ the m-dimensional complex projective space. In order to observe the degeneration, it is useful to introduce a picture which topologically represents a finite branched covering of the complex projective line. In §3, we call such a picture a Klein picture, since we can find such pictures in Klein [5]. In §5 (resp. §7), we assert that the topological type of the central fiber of a degenerating family of finite branched coverings of $\mathbb{P}^1$ (resp. $\mathbb{P}^m$ $(m \geq 2)$) is completely determined by that of the central branch divisor and the permutation monodromy of the general fiber. In §6, we prove (Theorem 8) that the topological structure of a degenerating family of finite branched coverings of $\mathbb{P}^1$ can be determined by the permutation monodromy of the general fiber and the braid nomodromy of the family. Some results of this paper were anounced in Namba [10].

2. Terminology

For a given connected complex manifold M, a finite branched covering of M is by definition a finite proper holomorphic mapping

$$f : X \longrightarrow M$$

of an irreducible normal complex space X onto M. A ramification point of f is a point of X such that f is not biholomorphic around the point. The image by f of a ramification point is called a branched point of f. The set of all ramification points (resp. branch points) is denoted by R_f (resp. B_f). This is a hypersurface of

X (resp. M). The mapping

$$f\colon X - f^{-1}(B_f) \longrightarrow M - B_f$$

is a finite unbranched covering. Its mapping degree is denoted by $\deg(f)$ and is called the degree of f. For a hypersurface B of M, a finite branched covering f is said to branch at most at B if B_f is contained in B. Finite branched coverings $f\colon X \longrightarrow M$ and $f'\colon X' \longrightarrow M$ are said to be isomorphic if there is a biholomorphic mapping $\psi\colon X \longrightarrow X'$ such that $f = f' \cdot \psi$. In this case, we denote $f \simeq f'$. Finite branched coverings $f\colon X \longrightarrow M$ and $f'\colon X' \longrightarrow M'$ are said to be equivalent (resp. topologically equivalent) if there are biholomorphic mappings (resp. orientation preserving hemeomorphisms) $\psi\colon X \longrightarrow X'$ and $\varphi\colon M \longrightarrow M'$ such that $\varphi \cdot f = f' \cdot \psi$. In this case, we denote $f \sim f'$ (resp. $f \sim f'$ (top.)).

Theorem 1 (Grauert-Remmert [4]). *Let B be a hypersurface of a connected complex manifold M and $f'\colon X' \longrightarrow M - B$ be a finite unbranched covering. Then there exists a unique (up to isomorphisms) finite covering $f\colon X \longrightarrow M$ which branches at most at B and is an extension of f'.*

A topological version of Theorem 1 is given in Fox [3]. Theorem 1 asserts that the correspondence $f \longleftrightarrow f'$ gives a categorical equivalence between finite unbranched coverings of $M - B$ and finite coverings of M branching at most at B. Thus we can apply terminology of finite unbranched coverings of $M - B$ to finite coverings of M branching at most at B; for example, covering transformations, Galois coverings, abelian coverings, cyclic coverings, etc.

Corollary 1. *There is a one-to-one correspondence between the set of all isomorphism classes of finite coverings of M branching at most at B and the set of all conjugacy classes of subgroups of finite index of the fundamental group $\pi_1(M - B, q_0)$ of $M - B$.*

3.   Monodromy representations and Klein pictures

Let $f\colon X \longrightarrow M$ be a finite branched covering of a connected complex manifold M of degree d branching at most at a hypersurface B of M. Take a reference point q_0 of $M - B$ and put $f^{-1}(q_0) = \{p_1, \ldots, p_d\}$. The homotopy class $[\gamma]$ of a loop γ in $M - B$ starting from q_0 gives the homotopy class of the pull-back over f of γ starting from every point p_j, $(j = 1, \ldots, d)$. Hence its end point $p_{j'}$ is determined. Thus we obtain a mapping

$$\Phi_f\colon \pi_1(M - B, q_0) \longrightarrow S_d,$$

which maps $[\gamma]$ to the permutation $j \to j'$, where S_d is the d-th symmetric group. We define the product of pathes α and β as $\alpha\beta$, where the end point of α is the initial point of β. We also define the product of permutations as in the following example:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.$$

The mapping Φ_f is then a homomorphism and is called the (permutation) monodromy representation of the covering f. Note that the representation class $[\Phi_f]$ of Φ_f does not depend on the choice of the arrangement of the points $p_1, \ldots, p_d$, nor the choice of the reference point q_0. That is, if one changes the arrangement of the points $p_1, \ldots, p_d$ or one chooses another reference point, then Φ_f is changed to $A^{-1}\Phi_f A$ for a fixed permutation A. Note also that the image of Φ_f is a transitive subgroup of S_d, for $X - f^{-1}(B)$ is connected. The image is called the monodromy group of the covering f. Monodromy groups of finite branched coverings correspond to Galois groups of algebraic equations. By the theorem of Grauert-Remmert and its corollary, we easily have the following 2 theorems:

Theorem 2. (1) *Finite branched coverings f and f' of M are isomorphic if and only if $B_f = B_{f'}$ and $[\Phi_f] = [\Phi_{f'}]$. ($[\Phi_f]$ is the representation class of Φ_f.)*
(2) *Finite branched coverings f of M and f' of M' are equivalent (resp. topologically equivalent) if and only if there is a biholomorphic mapping (resp. orientation preserving homeomorphism) $\varphi \colon M \longrightarrow M'$ such that $\varphi(B_f) = B_{f'}$ and $[\Phi_{f'} \cdot \varphi_*] = [\Phi_f]$.*

Theorem 3. *For a given homomorphism $\Phi \colon \pi_1(M - B, q_0) \longrightarrow S_d$ whose image is transitive, there exists a unique (up to isomorphisms) covering $f \colon X \longrightarrow M$ of degree d branching at most at B such that $\Phi_f = \Phi$.*

However it is a difficult problem in general to construct covering $f \colon X \longrightarrow M$ in the theorem from a given Φ concretely (analytically or algebraically). The problem for the case $M = \mathbb{P}^1$ the complex projective line and $B = \{0, 1, \infty\}$ is studied in number theorey (see Schneps [11]).

We construct branched coverings of the complex projective line $\mathbb{P}^1$ topologically for any given Φ, by drawing a picture which we call a Klein picture, the idea of which comes from Klein [5]. Let $B = \{q_1, \ldots, q_n\}$ be a set of n distinct points of $\mathbb{P}^1$. Let $f \colon X \longrightarrow \mathbb{P}^1$ be a covering of degree d branching at most at B. We draw a simple loop in $\mathbb{P}^1$ passing through all points q_j, $j = 1, \ldots, n$, oriented in this order which bounds a domain (the inside area) clockwisely (see Fig. 1). We regard the inside area of the loop as a continent and the outside area as an ocean. We assume that the reference point q_0 is contained in the continent. We then pull them back over the covering f. Then we get a checked pattern of d continents and d oceans on X. We call such a
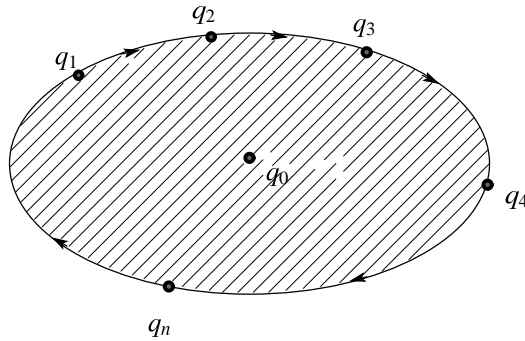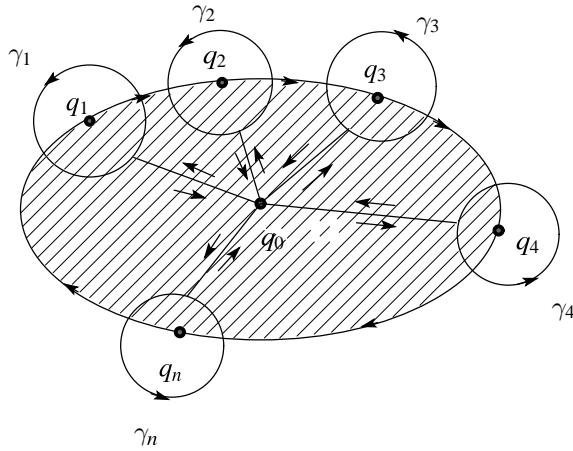
Fig. 1.

Fig. 2.

pattern the Klein picture of the covering f. The Klein picture represents the branched covering f topologically. Starting from a homomorphism $\Phi\colon \pi_1(\mathbb{P}^1 - B,\, q_0) \longrightarrow S_d$ such that $\operatorname{Im}\Phi$ is transitive, we construct the branched covering f in Theorem 2 topologically by drawing its Klein picture as follows: Put

$$A_j = \Phi(\gamma_j) \in S_d,\ j = 1,\, \ldots,\, n,$$

where γ_j are lassos surrounding the points q_j as in Fig. 2. Note that

$$\pi_1(\mathbb{P}^1 - B,\, q_0)\ = \langle \gamma_1,\, \ldots,\, \gamma_n \mid \gamma_n \cdots \gamma_1 = 1 \rangle,$$
$$A_n \cdots A_1 = 1 \in S_d.$$

Thus the representation Φ is determined by the permutations A_j. Decompose each A_j into mutually prime cyclic permutations A_{jk} whose length are e_{jk}. Put (by Riemann-
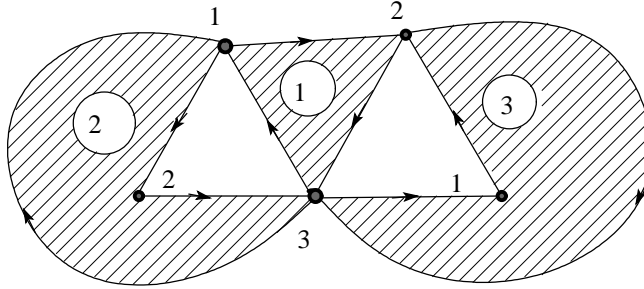
Fig. 3.

Hurwitz formula)

$$g = \frac{1}{2}\left[\sum_{j,\,k}(e_{j_k} - 1) - 2d\right] + 1.$$

We prepare an oriented compact surface X of genus g. We then draw the Klein picture, that is, a checked pattern of d continents and d oceans on X which is compatible with Φ. Here, the compatibility means that, for the point p_{j_k} of $f^{-1}(q_j)$ which corresponds to A_{j_k}, e_{j_k} continents and oceans are arranged alternately and counterclockwisely around p_{j_k}.

EXAMPLE 1. Put $n = 3$, $d = 3$ and

$$A_1 = \Phi(\gamma_1) = (1\ 2),\ \ A_2 = \Phi(\gamma_2) = (1\ 3),\ \ A_3 = \Phi(\gamma_3) = (1\ 2\ 3).$$

The genus of X is 0. The Klein picture in this case is as in Fig. 3, in which the points j denote the points in $f^{-1}(q_j)$ and the circled number $\textcircled{i}$ denotes the i-th continent. Observe that the points 1, 2 and 3 are seaside cities (vertices) of every continents arranged colckwisely in this order, while for example the continents $\textcircled{1}$, $\textcircled{2}$ and $\textcircled{3}$ are arranged counterclockwisely in this order around the city 3, which means $A_3 = (1\ 2\ 3)$. (Conversely, we can read the monodromy from the Klein picture.) Put

$$f: X \longrightarrow \mathbb{P}^1,\ (z,\ w) \longmapsto z,$$

where X is the Riemann surface of the algebraic function $w = w(z)$ given by the equation $w^3 - 3w - z = 0$. Then $q_1 = -2$, $q_2 = 2$, $q_3 = \infty$ and $\Phi_f = \Phi$.

EXAMPLE 2. Put $n = 3$, $d = 3$ and

$$A_j = \Phi(\gamma_j) = (1\ 2\ 3),\quad j = 1,\ 2,\ 3.$$

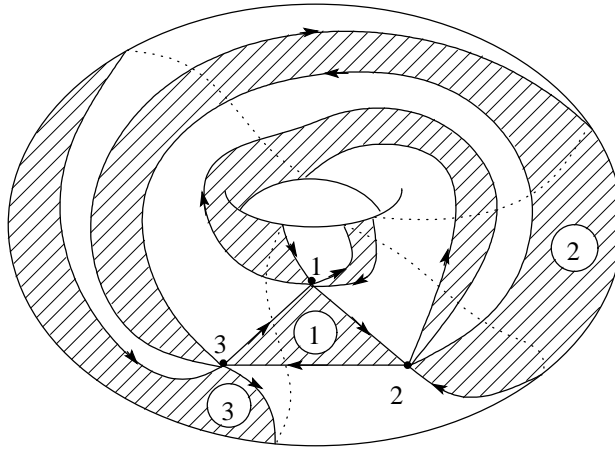The genus of X is 1. The Klein picture in this case is as in Fig. 4. Put

Fig. 4.

$$f\colon X \longrightarrow \mathbb{P}^1,\ (z,\,w) \longmapsto z,$$

where X is the Riemann surface of the algebraic function $w = w(z)$ given by the equation $w^3 - z^3 + 1 = 0$. Then f is a cyclic covering such that $\Phi_f = \Phi$.

EXAMPLE 3. Put $n = 4$, $d = 3$ and

$$A_1 = \Phi(\gamma_1) = (1\ 3\ 2), \quad A_2 = \Phi(\gamma_2) = (1\ 3\ 2),$$
$$A_3 = \Phi(\gamma_3) = (1\ 2\ 3), \quad A_4 = \Phi(\gamma_4) = (1\ 2\ 3).$$

The genus of X is 2. The Klein picture in this case is as in Fig. 5. Put

$$f\colon X \longrightarrow \mathbb{P}^1,\ (z,\,w) \longmapsto z,$$

where X is the Riemann surface of the algebraic function $w = w(z)$ given by the equation $w^3 - z^2(z-1)^2(z-2) = 0$. Then f is a cyclic covering such that $q_1 = 0$, $q_2 = 1$, $q_3 = 2$, $q_4 = \infty$ and $\Phi_f = \Phi$.

4. Families of finite branched coverings

Let T be a connected complex manifold. A family of connected complex manifolds with the parameter space T is by definition a smooth holomorphic mapping

$$\pi\colon M \longrightarrow T$$

of a connected complex manifold M onto a connected complex manifold T such that every fiber is connected. Here the smoothness means that the Jacobian matrix $d\pi$ is of
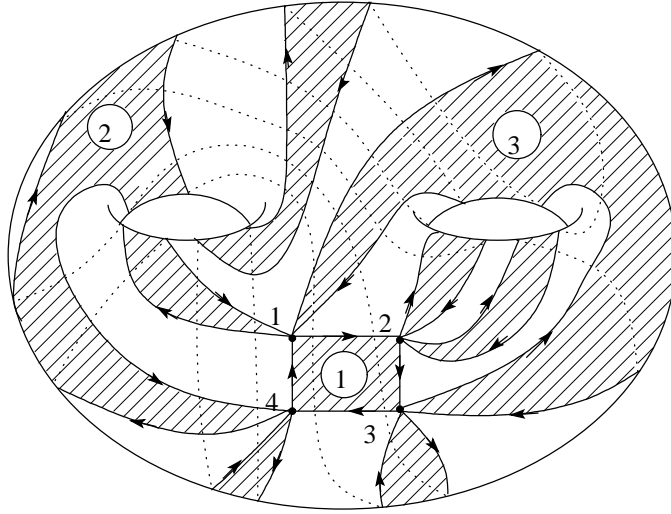
Fig. 5.

maximal rank at every point of M. Every fiber $M_t = f^{-1}(t)$ of $t \in M$ is a connected complex manifold. We denote

$$M = \{M_t\}_{t \in T} .$$

Let $M = \{M_t\}_{t \in T}$ be a family of connected complex manifolds. A family of finite branched coverings of $M = \{M_t\}_{t \in T}$ is by definition a finite branched covering

$$f \colon X \longrightarrow M$$

such that
(i) $M_t \not\subset B_f$ for every $t \in T$,
(ii) there is a hypersurface V of T such that

$$f_t = f \colon X_t = f^{-1}(M_t) \longrightarrow M_t$$

is a finite branched covering of M_t for every $t \in T - V$.
(iii) For any t and t' in $T - V$, f_t and $f_{t'}$ are topologically equivalent.

We denote $f = \{f_t\}$. In particular if $\pi \colon M \longrightarrow T$ is a holomorphic $\mathbb{P}^m$-bundle, then we call $f \colon X \longrightarrow M$ a family of finite branched coverings of $\mathbb{P}^m$.

REMARK. X and $X_t\,(t \in T - V)$ have only normal singularity, while $X_t\,(t \in V)$, the degenerated coverings, may not be normal. In this sense, our definition of degenerations is different from the usual one.

We are interested in X_t for $t \in V$, that is, degenerated coverings. In the subse-

quent sections, we restrict our consideration to degenerating families of finite branched coverings of $\mathbb{P}^m$ and a disc in $\mathbb{C}$.

EXAMPLE 4.   Put $Y = \big\{ ((a_0 : a_1 : a_2 : a_3), (x_0 : x_1)) \in \mathbb{P}^3 \times \mathbb{P}^1 \mid a_0 x_1^3 + a_1 x_1^2 x_0 + a_2 x_1 x_0^2 + a_3 x_0^3 = 0 \}$, $g: ((a_0 : a_1 : a_2 : a_3), (x_0 : x_1)) \in Y \longmapsto (a_0 : a_1 : a_2 : a_3) \in \mathbb{P}^3$, where $(a_0 : a_1 : a_2 : a_3)$ and $(x_0 : x_1)$ are homogeneous coordinate systems of $\mathbb{P}^3$ and $\mathbb{P}^1$, respectively. Then Y is non-singular and g is a branched covering of degree 3 whose branch locus is the discriminant locus

$$B_f = \big\{ (a_0 : a_1 : a_2 : a_3) \in \mathbb{P}^3 \mid a_1^2 a_2^2 - 4a_0 a_2^3 + 18a_0 a_1 a_2 a_3 - 4a_1^3 a_3 - 27a_0^2 a_3^2 = 0 \big\}.$$

Let $\mathbb{P}^{3*}$ be the dual projective space of $\mathbb{P}^3$ and put

$$M = \big\{ ((t_0 : t_1 : t_2 : t_3), (a_0 : a_1 : a_2 : a_3)) \in \mathbb{P}^{3*} \times \mathbb{P}^3 \mid t_0 a_0 + t_1 a_1 + t_2 a_2 + t_3 a_3 = 0 \big\},$$
$$\pi: ((t_0 : t_1 : t_2 : t_3), (a_0 : a_1 : a_2 : a_3)) \in M \longmapsto (t_0 : t_1 : t_2 : t_3) \in \mathbb{P}^{3*},$$
$$\pi': (t_0 : t_1 : t_2 : t_3), (a_0 : a_1 : a_2 : a_3)) \in M \longmapsto (a_0 : a_1 : a_2 : a_3) \in \mathbb{P}^3,$$

where $(t_0 : t_1 : t_2 : t_3)$ is a homogeneous coordinate system of $\mathbb{P}^{3*}$. Then π is a $\mathbb{P}^2$-bundle on $\mathbb{P}^{3*}$. Let X be the normalization of the fiber product $M \times_{\mathbb{P}^3} Y$ of $\pi': M \longrightarrow \mathbb{P}^3$ and $g: Y \longrightarrow \mathbb{P}^3$. Let

$$f: X \longrightarrow M$$

be the composition of the normalization

$$X \longrightarrow M \times_{\mathbb{P}^3} Y$$

and the projection

$$M \times_{\mathbb{P}^3} Y \longrightarrow M.$$

Then $f = \{f_t\}_{t \in \mathbb{P}^{3*}}$ is a family of branched coverings of $\mathbb{P}^2$. ($f_t: X_t \longrightarrow \pi^{-1}(t)$ is the restriction to the plane $\pi^{-1}(t)$ in $\mathbb{P}^3$ of g.)

We explain this as follows: Let

$$C_3 = \big\{ (1 : u : u^2 : u^3) \in \mathbb{P}^{3*} \mid u \in \mathbb{P}^1 \big\}$$

be the rational normal curve, which is the image curve of the holomorphic imbedding

$$\Phi_{|D|}: \mathbb{P}^1 \longrightarrow \mathbb{P}^{3*}$$

of the unique complete linear system $|D|$ of degree 3 (D is a divisor on $\mathbb{P}^1$ of degree 3). B_f is then the dual variety of C_3. That is, B_f is the set of all planes in $\mathbb{P}^{3*}$

which contain tangent lines to C_3. Every divisor in $|D|$ is the intersection of C_3 with a (unique) plane in $\mathbb{P}^{3*}$. In this sense, $|D|$ is identified with $\mathbb{P}^3 = (\mathbb{P}^{3*})^*$. By the uniqueness of the complete linear system $|D|$ of degree 3, every automorphism of $\mathbb{P}^1$ acts on $|D| = \mathbb{P}^3$ (resp. on $\mathbb{P}^{3*}$) as a projective transformation, which maps B_f to B_f (resp. C_3 to C_3). Let V be the ruled surface in $\mathbb{P}^{3*}$ consisting of tangent lines to C_3.

For any two points t and t' in $\mathbb{P}^{3*} - V$, there is an automorphism φ of $\mathbb{P}^1$ such that $\varphi(t) = t'$. In fact, there are just 3 points p_1, p_2 and p_3 in C_3 (resp. p_1', p_2' and p_3' in C_3) such that the osculating plane at p_j (resp. at p_j') to C_3 passes through t (resp. t') for $j = 1, 2, 3$. Then $\varphi \in \mathrm{Aut}(\mathbb{P}^1)$ such that $\varphi(p_j) = p_j'$ $(j = 1, 2, 3)$ maps t to t'. Thus $\mathrm{Aut}(\mathbb{P}^1)$ acts on $\mathbb{P}^{3*} - V$ transitively. (The orbits of the group action of $\mathrm{Aut}(\mathbb{P}^1)$ on $\mathbb{P}^{3*}$ are $\mathbb{P}^{3*} - V$, V and C_3.) The projection π_t with the center t maps C_3 onto a rational plane cubic curve C_t with a node and 3 flexes $\pi_t(p_1)$, $\pi_t(p_2)$ and $\pi_t(p_3)$. The plane projective transformation induced by φ maps C_t to $C_{t'}$. The branch locus B_t (resp. $B_{t'}$) of f_t (resp. $f_{t'}$) is the dual curve of C_t (resp. $C_{t'}$) which is a rational plane quartic curve with 3 simple cusps. Hence the plane projective transformation induced by φ maps B_t to $B_{t'}$. Now, φ induces an automorphism of the projective manifold Y which, by the above discussion, induces an equivalence of f_t and $f_{t'}$.

A similar discussion shows that if $t \in V - C_3$ (say $t = (0 : 1 : 0 : 0)$), then B_t is the union of a rational plane cubic curve with 1 simple cusp and a line passing through a flex of the curve. For any points t and t' in $V - C_3$, f_t and $f_{t'}$ are equivalent.

If $t \in C_3$ (say $t = (0 : 0 : 0 : 1)$), then B_t is the union of an irreducible conic and a double tangent line to the conic. In this case, X_t is not irreducible.

5. Degenerating families of finite branched coverings of $\mathbb{P}^1$

Let

$$\Delta = \Delta(0, \epsilon) = \{t \in \mathbb{C} \mid |t| < \epsilon\}$$

be a disc and $\Delta^* = \Delta - \{0\}$ be the punctured disc. A finite branched covering

$$f : X \longrightarrow \Delta \times \mathbb{P}^1$$

is called a degenerating family of finite branched coverings of $\mathbb{P}^1$ and is denoted by $f = \{f_t\}$, if the following three conditions are satisfied:
(1) $t \times \mathbb{P}^1 \nsubseteq B_f$ for every $t \in \Delta$.
(2) For every $t \in \Delta^*$, $t \times \mathbb{P}^1$ meets at n points transversally with B_f. (n is constant for $t \in \Delta^*$.)
(3) For every $t \in \Delta^*$,

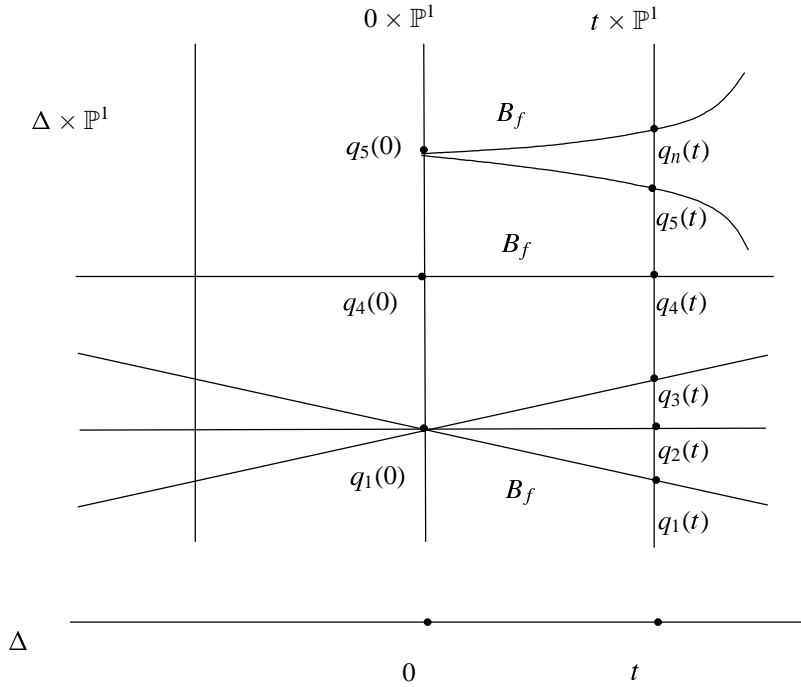$$f_t = f : X_t = f^{-1}(t \times \mathbb{P}^1) \longrightarrow t \times \mathbb{P}^1$$

Fig. 6.

is a covering of $\mathbb{P}^1$ of degree $d = \deg(f)$ branching at $B_t = B_f \cap (t \times \mathbb{P}^1) = \{q_1(t), \ldots, q_n(t)\}$ (see Fig. 6).

The central fiber $X_0 = f^{-1}(0 \times \mathbb{P}^1)$ is a degeneration of a general fiber X_t for $t \neq 0$. The Klein picture of f_t degenerates to a picture on X_0, which we call the Klein picture of f_0. This represents (X_0, f_0) topologically.

EXAMPLE 5.   Let X_t be the Riemann surface of the algebraic function $w = w(z)$ given by the equation $w^3 - 3tw - z = 0$. Put

$$f_t \colon X_t \longrightarrow \mathbb{P}^1, \ (z, w) \longmapsto z.$$

Then $f = \{f_t\}$ is a degenerating family of branched coverings of $\mathbb{P}^1$. For a fixed non-zero t, the monodromy representation Φ_t and the Klein picture of f_t are given as same as in Example 1. Note that $q_1(t) = -2t^{3/2}$, $q_2(t) = 2t^{3/2}$ and $q_3(t) = \infty$. As $t \longrightarrow 0$, both branch points $q_1(t)$ and $q_2(t)$ converge to $q_1(0) = q_2(0) = 0$, so the pathes connecting the points 1 and 2 in Fig. 3 converge to the point $1 = 2$, and we get the Klein picture of f_0 as in Fig. 7. In fact $X_0 \colon w^3 - z = 0$.
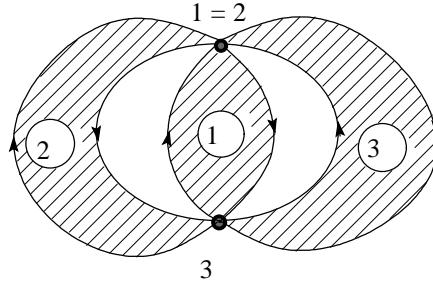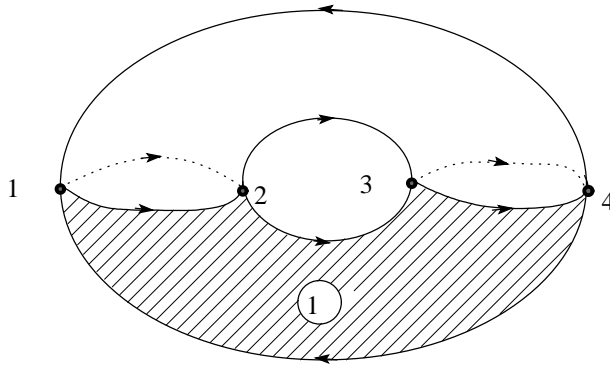
Fig. 7.

Fig. 8.

EXAMPLE 6.   Let X_t be the Riemann surface of the algebraic function $w = w(z)$ given by the equation $w^2 - z(z - t)(z - 1) = 0$. Put

$$f_t \colon X_t \longrightarrow \mathbb{P}^1, \ (z, w) \longmapsto z.$$

Then $f = \{f_t\}$ is a degenerating family of branched double coverings of $\mathbb{P}^1$. Note that $q_1(t) = 0$, $q_2(t) = t$, $q_3(t) = 1$ and $q_4(t) = \infty$. For a fixed non-zero t, the monodromy representation Φ_t is given by $A_j = \Phi_t(\gamma_j) = (1\ 2)$ for $j = 1, 2, 3, 4$. The Klein picture of f_t is as in Fig. 8 in which the continent ② is the upper backside of the torus. As $t \longrightarrow 0$, both $q_1(t)$ and $q_2(t)$ converge to $q_1(0) = q_2(0) = 0$, so the pathes connecting the points 1 and 2 in Fig. 8 converge to the point $1 = 2$, and we get the Klein picture of f_0 as in Fig. 9 in which the continent ② is also the upper backside. In fact, $X_0 : w^2 - z^2(z - 1) = 0$.

EXAMPLE 7.   Let X_t be the Riemann surface of genus 1 of the algebraic function $w = w(z)$ given by the equation $w^2 - z(z - t)(z - 1)(z - 1 - t) = 0$. Put

$$f_t \colon X_t \longrightarrow \mathbb{P}^1, \quad (z, w) \longmapsto z.$$

$\boxed{2}$ is the upper
backside.

Fig. 9.

$\boxed{2}$ is the upper backside.
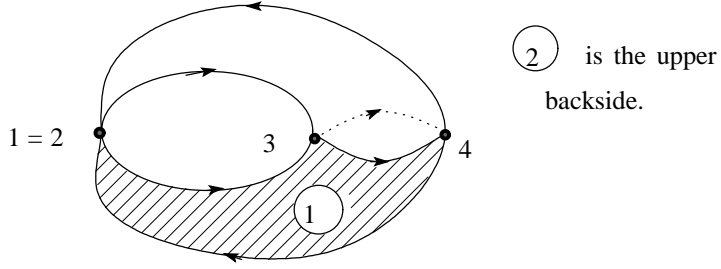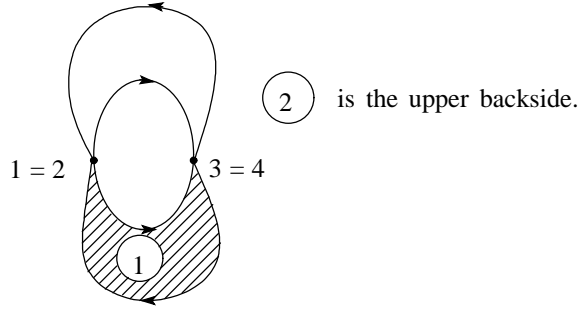
Fig. 10.

Then $f = \{f_t\}$ is a degenerating family of branched double covering of $\mathbb{P}^1$. Note that

$$q_1(t) = 0, \quad q_2(t) = t, \quad q_3(t) = 1, \quad q_4(t) = 1 + t.$$

For fixed t with $0 < |t| < 1$, the monodromy representation Φ_t is a given by $A_j = \Phi_t(\gamma_j) = (1\,2)$ for $j = 1, 2, 3, 4$. The Klein picture of f_t is as same as that in Fig. 8 for Example 6. As $t \longrightarrow 0$, $q_2(t)$ and $q_4(t)$ converge to $q_1(0) = 0$, $q_3(0) = 1$, respectively, so the pathes connecting the points 1 to 2 and 3 to 4 in Fig. 8 converge to $1 = 2$ and $3 = 4$, respectively. Hence we get the Klein picture of f_0 as in Fig. 10 in which the continent $\textcircled{2}$ is also the upper backside. In fact

$$X_0 : w^2 - z^2(z - 1)^2 = 0,$$

which is not globally irreducible.

EXAMPLE 8. Let X_t be the Riemann surface of genus 1 of the algebraic function $w = w(z)$ given by the equation $w^2 - z(z - t)(z - 2t) = 0$. Put

$$f_t : X_t \longrightarrow \mathbb{P}^1, \quad (z, w) \longmapsto z.$$

As $t \longrightarrow 0$, the Klein picture of f_t converges to that of f_0 as in Fig. 11, in which $\textcircled{2}$
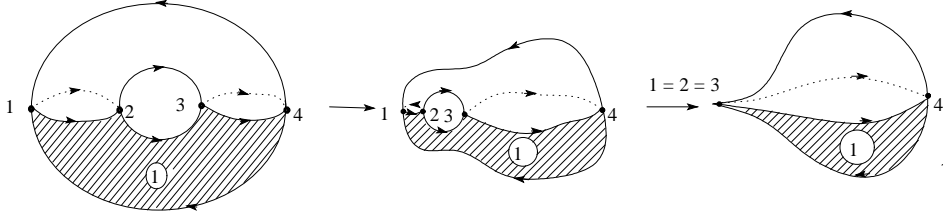
Fig. 11.

are the upper backside.

X_0 has a cusp singularity at the point $1 = 2 = 3$. In fact

$$X_0 : w^2 - z^3 = 0.$$

Now we assume and put

$$q_1(0) = \cdots = q_{k_1}(0) = q_1^0,$$
$$q_{k_1+1}(0) = \cdots = q_{k_1+k_2}(0) = q_2^0,$$
$$\cdots \cdots$$
$$q_{k_1+\cdots+k_{r-1}+1}(0) = \cdots = q_{k_1+\cdots+k_r}(0) = q_r^0$$

where $k_\rho \geq 1$ ($\rho = 1, \ldots, r$), $k_1 + \cdots + k_r = n$ and $q_1^0, q_2^0, \ldots, q_r^0$ are mutually distinct. We regard

$$B_0 = B_f \cap (0 \times \mathbb{P}^1) = \{ q_1^0, q_2^0, \ldots, q_r^0 \}$$

not as a point set but as a divisor on $\mathbb{P}^1$:

$$B_0 = k_1 q_1^0 + k_2 q_2^0 + \cdots + k_r q_r^0.$$

We draw a simple loop in $\mathbb{P}^1$ passing through all points $q_1^0, \ldots, q_r^0$ oriented in this order which bounds a domain clockwisely as in Fig. 1. We call the Klein picture of f_0 for the checked patern on X_0 which is the pull-back of the picture over f_0.

Now, we show that topologically, the degenerating curve $X_0 = f^{-1}(0 \times \mathbb{P}^1)$ can be described by the divisor B_0 and the monodromy $\Phi_t = \Phi_{f_t}$, where $t \in \Delta^*$ is a fixed point.

Let $\gamma_j(t)\,(1 \leq j \leq n)$ be the lasso around $q_j(t)$ as in Fig. 2 and put

$$A_1 = \Phi_t(\gamma_1), \ldots, A_n = \Phi_t(\gamma_n).$$

Let H_ρ ($1 \leq \rho \leq r$) be the subgroup of S_d generated by

$$A_{k_1+\cdots+k_{\rho-1}+1}, \ldots, A_{k_1+\cdots+k_{\rho-1}+k_\rho}.$$

H_ρ may not be a transitive subgroup of S_d. We denote

$$\mathfrak{A}_1^\rho, \ldots, \mathfrak{A}_{v_\rho}^\rho$$

the orbits of H_ρ on $\{1, 2, \ldots, d\}$. v_ρ is the number of orbits. Put

$$\begin{aligned}
A_1^0 &= A_{k_1} A_{k_1-1} \cdots A_1, \\
A_2^0 &= A_{k_1+k_2} A_{k_1+k_2-1} \cdots A_{k_1+1}, \\
&\quad\cdots\cdots \\
A_r^0 &= A_{k_1+\cdots+k_r} A_{k_1+\cdots+k_r-1} \cdots A_{k_1+\cdots+k_{r-1}+1}
\end{aligned}$$

and

$$K = \left\langle A_1^0, \ldots, A_r^0 \right\rangle$$

the subgroup of S_d generated by $A_1^0, \ldots, A_r^0$. Let

$$\gamma_1^0, \ldots, \gamma_r^0$$

be lassos around the points

$$q_1^0, \ldots, q_r^0,$$

respectively in $0 \times \mathbb{P}^1$ as in Fig. 2. Put

$$\Phi_0(\gamma_\rho^0) = A_\rho^0 \quad (1 \le \rho \le r).$$

Then

$$\Phi_0 : \pi_1(0 \times \mathbb{P}^1 - \{q_1^0, \ldots, q_r^0\}, q_0) \longrightarrow S_d$$

is a homomorphism.

DEFINITION 1. For a permutation $A \in S_d$, if A is written as $A = A_1 \cdots A_w$, the product of mutually prime cyclic permutations, then we call the number $w = w(A)$ the weight of A. ($w(A)$ depends also on d. For example, if $d = 4$ and $A = (1\ 2\ 3)$, then $w(A) = w((1\ 2\ 3)(4)) = 2$.)

Let $\chi(X_t)$ denote the Euler characteristic of X_t.

Theorem 4. *Let $t \ne 0$. Then the following* (1)–(4) *hold*:
(1) $\chi(X_t) = 2 - 2g = 2d - nd + \sum_{j=1}^{n} w(A_j)$.
(2) $\chi(X_0) = 2d - \{nd - \sum_{\rho=1}^{r}(k_\rho - 1)d\} + \sum_{\rho=1}^{r} v_\rho$.
(3) $\chi(X_0) - \chi(X_t) = \sum_{\rho=1}^{r}(k_\rho - 1)d + \sum_{\rho=1}^{r} v_\rho - \sum_{j=1}^{n} w(A_j)$.

(4) $\chi(X_0) \geq \chi(X_t)$.

Proof.   (1) The Klein picture of the covering $f_t \colon X_t \longrightarrow \mathbb{P}^1$ gives a cellular decomposition of X_t. The number of vertices is $\sum_{j=1}^{n} w(A_j)$, the number of sides is nd and the number of faces is $2d$. Hence

$$\chi(X_t) = 2d - nd + \sum_{j=1}^{n} w(A_j).$$

(2) Let $\hat{G}$ be the (oriented) graph on X_t of the pull-back by f_t of the cycle

$$q_1(t) \longrightarrow q_2(t) \longrightarrow \cdots \longrightarrow q_n(t) \longrightarrow q_1(t).$$

Then $\hat{G}$ is the graph whose points and lines are vertices and sides, respectively, of the Klein picture of f_t. Every point of $\hat{G}$ has been numbered as a vertix of the Klein picture. We put the circled number $\textcircled{j}$ on every sides of j-th continent. Thus we get a graph $\hat{G}$ with numbered points and circle numbered lines.

Let $G_\rho \, (1 \leq \rho \leq r)$ be the (oriented) graph on X_t of the pull-back by f_t of the tree

$$q_{k_1+\cdots+k_{\rho-1}+1}(t) \longrightarrow q_{k_1+\cdots+k_{\rho-1}+2}(t) \longrightarrow \cdots \longrightarrow q_{k_1+\cdots+k_{\rho-1}+k_\rho}(t).$$

Then every G_ρ is a subgraph of $\hat{G}$. Let

$$k_1 + \cdots + k_{\rho-1} + 1 \leq i < i + 1 \leq k_1 + \cdots + k_{\rho-1} + k_\rho.$$

If the permutaion A_i and A_{i+1} are witten as, say,

$$A_i = \begin{pmatrix} \cdots & a & \cdots \\ \cdots & b & \cdots \end{pmatrix}, \qquad A_{i+1} = \begin{pmatrix} \cdots & a & \cdots \\ \cdots & c & \cdots \end{pmatrix},$$

then there are lines $\textcircled{a}$ and $\textcircled{b}$ in G_ρ which have the starting point i (a point of $G_\rho \cap f_t^{-1}(q_i)$), and so are connected at the point i. Moreover there is a line $\textcircled{c}$ in G_ρ such that the lines $\textcircled{a}$ and $\textcircled{c}$ have the same end point $i+1$, and so are connected at the point $i+1$. Hence the lines $\textcircled{a}$, $\textcircled{b}$ and $\textcircled{c}$ are connected in G_ρ.

Now the Klein picture of f_0 gives a cellular decomposition of X_0 which can be obtained from that of X_t by converging every connected component of the graphs $G_\rho \, (1 \leq \rho \leq r)$ to a point. Hence the number of vertices is $\sum_{\rho=1}^{r} v_\rho$, the number of sides is $nd - \sum_{\rho=1}^{r}(k_\rho - 1)d$ and the number of faces is $2d$. Hence

$$\chi(X_0) = 2d - \left\{ nd - \sum_{\rho=1}^{r}(k_\rho - 1)d \right\} + \sum_{\rho=1}^{r} v_\rho.$$

(3) This follows from (1) and (2).

(4) For a graph G, the following inequality holds:

$$b \geq a - c,$$

where a is the number of points of G, b is the number of lines of G and c is the number of connected components of G. Here the equality holds if and only if every component is a tree, that is, a graph without cycles.

We apply this to every graph G_ρ. Then

$$a = a_\rho = w(A_{k_1+\cdots+k_{\rho-1}+1}) + \cdots + w(A_{k_1+\cdots+k_{\rho-1}+k_\rho}),$$
$$b = b_\rho = (k_\rho - 1)d,$$
$$c = c_\rho = v_\rho.$$

Note that

$$\sum_{\rho=1}^{r} a_\rho = \sum_{j=1}^{n} w(A_j).$$

Hence by (3)

$$\chi(X_0) - \chi(X_t) = \sum_{\rho=1}^{r} b_\rho + \sum_{\rho=1}^{r} c_\rho - \sum_{\rho=1}^{r} a_\rho = \sum_{\rho=1}^{r}(b_\rho + c_\rho - a_\rho) \geq 0. \qquad \square$$

Theorem 5. (1) $f_0^{-1}(q_\rho^0)$ consists of v_ρ points, which can be identified with $\mathfrak{A}_1^\rho, \ldots, \mathfrak{A}_{v_\rho}^\rho$.

(2) Every $A_\rho^0 (1 \leq \rho \leq r)$ induces a permutation $A_{\rho j}^0 \colon \mathfrak{A}_j^\rho \longrightarrow \mathfrak{A}_j^\rho$. X_0 has local $w(A_{\rho j}^0)$ irreducible components at the point corresponding to $\mathfrak{A}_j^\rho$.

(3) There is a natural one-to-one correspondence between the set of global irreducible components of X_0 and the set of orbits of $K = \langle A_1^0, \ldots, A_r^0 \rangle$ on $\{1, \ldots, d\}$. Φ_0, regarded as the representation to permutations on an orbit of K on $\{1, \ldots, d\}$, gives the monodromy representation of the branched covering

$$f_0 \cdot \eta \colon \hat{X}_0' \longrightarrow 0 \times \mathbb{P}^1,$$

where $\eta \colon \hat{X}_0' \longrightarrow X_0'$ is the normalization of the global irreducible component X_0' of X_0 corresponding to the orbit of K.

Proof. (1) follows from the proof (2) of Theorem 4. For a sufficiently small $|t|$,

$$q_{k_1+\cdots+k_{\rho-1}+1}(t), \ \ldots, \ q_{k_1+\cdots+k_{\rho-1}+k_\rho}(t)$$

are in a small neighborhood of q_ρ^0. Hence the lasso γ_ρ^0 is homotopic to the product

$$\gamma_{k_1+\cdots+k_{\rho-1}+k_\rho}(t) \cdots \gamma_{k_1+\cdots+k_{\rho-1}+1}(t).$$

Let

$$i\colon N = 0 \times \mathbb{P}^1 \longrightarrow M = \Delta \times \mathbb{P}^1$$

be the inclusion mapping. Then the fiber product $N \times_M X$ can be identified with X_0. Now, (2) and (3) of Theorem 5 follow from the following lemma, whose proof is straightforward and is omitted. $\square$

Lemma 1. *Let M and N be connected complex manifolds and $h\colon N \longrightarrow M$ be a holomorphic mapping. Let $f\colon X \longrightarrow M$ be a finite unbranched covering of M of degree d and Φ_f be its monodromy representation. Let $f'\colon N \times_M X \longrightarrow N$ be the projection of the fiber product $N \times_M X$ onto N. Then the followings hold*:
(1) *There is a one-to-one correspondence between the set of orbits of $\Phi_f \cdot h_*(\pi_1(N, p_0))$ on $\{1, \ldots, d\}$ and the set of connected components of $N \times_M X$.*
(2) *For a connected component Y of $N \times_M X$, $f'\colon Y \longrightarrow N$ is a finite unbranched covering of N whose monodromy representation is equal to $\Phi_f \cdot h_*$ regarded as the representation to permutations on the orbit of $\Phi_f \cdot h_*(\pi_1(N, p_0))$ on $\{1, \ldots, d\}$ corresponding to Y.*

Theorem 6. *The following four conditions are mutually equivalent*:
(1) *X_0 is homeomorphic to X_t for $t \neq 0$.*
(2) *$\chi(X_0) = \chi(X_t)$ for $t \neq 0$.*
(3) *$\sum_{\rho=1}^{r}(k_\rho - 1)d = \sum_{j=1}^{n} w(A_j) - \sum_{\rho=1}^{r} v_\rho$.*
(4) *$\sum_{\rho=1}^{r}(k_\rho - 1)d = \sum_{j=1}^{h} w(A_j) - \sum_{\rho=1}^{r} w(A_\rho^0)$.*

Proof. If X_0 is homeomorphic to X_t, then $\chi(X_0) = \chi(X_t)$. If $\chi(X_0) = \chi(X_t)$, then every connected component of the graphs $G_\rho (1 \leq \rho \leq r)$ is a tree as is shown in the proof of Theorem 4. When t converges to 0, every connected component of the graphs $G_\rho (1 \leq \rho \leq r)$ converges to a point. This means that X_0 is homeomorphic to X_t. Next, note that

$$A_\rho^0 = A_{\rho 1}^0 \cdots A_{\rho v_\rho}^0,$$

where $A_{\rho j}^0$ is the permutation on the orbit $\mathfrak{A}_j^\rho$ induced by A_ρ^0. Hence

$$w(A_\rho^0) = w(A_{\rho 1}^0) + \cdots + w(A_{\rho v_\rho}^0).$$

In particular

$$w(A_\rho^0) \geq v_\rho.$$

Here the equality holds if and only if every $A_{\rho j}^0$ is a cyclic permutation. Hence, by (2) of Theorem 5, the equality holds if and only if X_0 is locally irreducible at ev-

ery point $\mathfrak{A}_j^\rho$ ($1 \leq j \leq v_\rho$). Now, by (4) of Theorem 4, the following inequality holds:

$$\sum_{\rho=1}^{r}(k_\rho - 1)d \geq \sum_{j=1}^{n} w(A_j) - \sum_{\rho=1}^{r} v_\rho \geq \sum_{j=1}^{n} w(A_j) - \sum_{\rho=1}^{r} w(A_\rho^0).$$

If

$$\sum_{\rho=1}^{r}(k_\rho - 1)d = \sum_{j=1}^{k} w(A_j) - \sum_{\rho=1}^{r} w(A_\rho^0),$$

then

$$\sum_{\rho=1}^{r}(k_\rho - 1)d = \sum_{j=1}^{n} w(A_j) - \sum_{\rho=1}^{r} v_\rho.$$

Hence $\chi(X_0) = \chi(X_t)$ by Theorem 4.

Conversely, if $\chi(X_0) = \chi(X_t)$, then X_0 is homeomorphic to X_t. In particular, X_0 is locally irreducible at every point $\mathfrak{A}_j^\rho$ ($1 \leq j \leq v_\rho$, $1 \leq \rho \leq r$). Thus

$$\sum_{\rho=1}^{r}(k_\rho - 1)d = \sum_{j=1}^{n} w(A_j) - \sum_{\rho=1}^{r} v_\rho = \sum_{j=1}^{n} w(A_j) - \sum_{\rho=1}^{r} w(A_\rho^0). \qquad \square$$

REMARK. If one of the conditions of Theorem 6 is satisfied, then X_0 is non-singular. In fact, if one of the conditions of Theorem 6 is satisfied, then every connected component of every graph G_ρ ($1 \leq \rho \leq r$) is a tree. X_0 is obtained from X_t converging every tree to a point. Hence X_0 is still a manifold. (The total space X is also non-singular.)

This can be also shown in the following way: The arithmetic genus is constant. In particular, the arithmetic genus of X_0 is equal to the geometric genus of X_t ($t \neq 0$). If X_0 is singular, then the geometric genus of X_0 is less than the arithmetic genus, a contradiction to the assumption $\chi(X_0) = \chi(X_t)$. On the other hand, if $\chi(X_0) > \chi(X_t)$, then the graph G contains a cycle. As $t \longrightarrow 0$, such a cycle Γ converges to a point p, while, for a connected open neighborhood U of Γ, $U - \Gamma$, which has two connected components, moves homeomorphically. Hence X_0 is locally a cone with the vertex p. Thus X_0 can not be a manifold, so X_0 is singular.

6. Topological equivalence of families

In this section we show that the topologial structure of the degenerating family $f = \{f_t\}$ of finite branched coverings of $\mathbb{P}^1$ is not determined by Φ_t alone, but depends also on the braid monodromy $\theta(\delta)$. Here

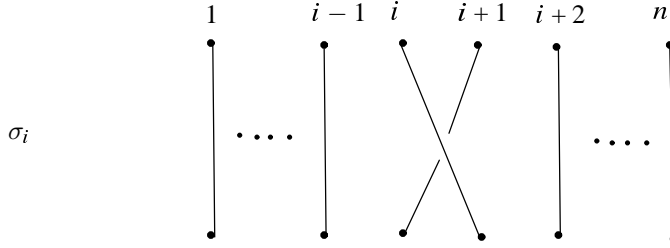$$\delta: u \longmapsto t = t_0 e^{iu}, \ (0 \leq u \leq 2\pi)$$

Fig. 12.

is the loop around $t = 0$. ($t_0 \in \Delta^*$ is a fixed point.) In this section we assume for simplicity that $q_j(t) \neq \infty$ for every $t \in \Delta$ and $1 \leq j \leq n$. Then

$$\{q_1(t_0 e^{iu}), \ldots, q_n(t_0 e^{iu})\}_{0 \leq u \leq 2\pi}$$

gives an (Artin) braid of n strings, which is called the braid monodromy of the curve B_f around $t = 0$ and is denoted by $\theta(\delta)$. The braid $\theta = \theta(\delta)$ can not be arbitrary. It is given by a complex analytic curve B_f. So such a braid we call a complex analytic braid. We fix a reference point $t_0 \in \Delta^*$ and put

$$q_j = q_j(t_0) \quad \text{for} \quad 1 \leq j \leq n.$$

Then the Artin braid group B_n naturally acts on the fundamental group $\pi_1(\mathbb{P}^1 - \{q_1, \ldots, q_n\}, q_0)$ as follows:

$$\sigma_i(\gamma_i) = \gamma_i^{-1}\gamma_{i+1}\gamma_i,$$
$$\sigma_i(\gamma_{i+1}) = \gamma_i,$$
$$\sigma_i(\gamma_j) = \gamma_j \quad (j \neq i, \ i+1),$$

where γ_j ($j = 1, \ldots, n$) are the lassos as in Fig. 2 and σ_i ($i = 1, \ldots, n-1$) are the generators of B_n defined as in Fig. 12.

A theorem of Zariski-van Kampen (see e.g. Dimca [2]) asserts

Theorem 7 (Zariski-van Kampen).

$$\pi_1(\Delta \times \mathbb{P}^1 - B_f, q_0)$$
$$= \langle \gamma_1, \ldots, \gamma_n \mid \gamma_n \cdots \gamma_1 = 1, \ \theta(\delta)\gamma_j = \gamma_j, \ (1 \leq j \leq n)\rangle,$$

where γ_j are lassos as in Fig. 2 *for $f_{t_0}: X_{t_0} \longrightarrow \mathbb{P}^1$.*

The monodromy representation Φ_f of $f: X \longrightarrow \Delta \times \mathbb{P}^1$ is equal to $\Phi_{t_0} = \Phi_{f_{t_0}}$.

By Theorem 7, Φ_{t_0} satisfies

$$\Phi_{t_0} \cdot \theta(\delta) = \Phi_{t_0}.$$

DEFINITION 2. $f = \{f_t\}$ and $f' = \{f'_{t'}\}$ are said to be topologically equivalent if there are orientation preserving homeomorphisms ψ, φ and η which make the following diagram commutative:

$$
\begin{array}{ccc}
X & \xrightarrow{\ \psi\ } & X' \\
f\big\downarrow & & \big\downarrow f' \\
\Delta \times \mathbb{P}^1 & \xrightarrow{\ \varphi\ } & \Delta' \times \mathbb{P}^1 \\
\big\downarrow & & \big\downarrow \\
\Delta & \xrightarrow{\ \eta\ } & \Delta'
\end{array}
$$

Using fundamental results in the theory of fiber bundles (see Steenrod [12]), we get the following theorem, which can be regarded as a branched covering version of a theorem in Matsumoto-Montesinos [6]:

Theorem 8. *There exists a one to one correspondence between $\{$topological equivalence class of $f = \{f_t\}$, where f_{t_0} ($t_0 \neq 0$) has the degree d and n branched points $q_1, \ldots, q_n\}$ and $\{([\Phi], \theta) \mid [\Phi]$ is the representation class of $\Phi\colon \pi_1(\mathbb{P}^1 - \{q_1, \ldots, q_n\}, q_0) \longrightarrow S_d$ such that $\mathrm{Im}\,\Phi$ is transitive, and $\theta \in B_n$ is a complex analytic braid such that $\Phi \cdot \theta = \Phi\}/B_n$. Here $\sigma \in B_n$ acts on $([\Phi], \theta)$ as follows*:

$$\sigma([\Phi], \theta) = ([\Phi \cdot \sigma^{-1}], \sigma\theta\sigma^{-1}).$$

Proof.   For two families

(1) $f = \{f_t\}\colon X \longrightarrow \Delta \times \mathbb{P}^1, \quad f' = \{f'_t\}\colon X' \longrightarrow \Delta' \times \mathbb{P}^1,$

with the assumption

$$(\Delta \times \{\infty\}) \cap B_f = \emptyset, \quad (\Delta' \times \{\infty\}) \cap B_{f'} = \emptyset,$$

we may assume that there are $q_0 \in \mathbb{C}$ and $q'_0 \in \mathbb{C}$ such that

$$(\Delta \times \{q_0\}) \cap B_f = \emptyset, \quad (\Delta' \times \{q'_0\}) \cap B_{f'} = \emptyset.$$

(For example, take q_0 and q'_0 such that $|q_0|$ and $|q'_0|$ are sufficiently large.)

Take reference points $t_0 \in \Delta^*$ and $t_0' \in \Delta'^*$. Put

$$(t_0 \times \mathbb{C}) \cap B_f = \{q_1 = q_1(t_0), \ldots, q_n = q_n(t_0)\},$$
$$(t_0' \times \mathbb{C}) \cap B_{f'} = \{q_1' = q_1'(t_0'), \ldots, q_n' = q_n'(t_0')\}.$$

There is an orientation preserving homeomorphism

(2) $$\xi \colon t_0 \times \mathbb{C} = \mathbb{C} \longrightarrow t_0' \times \mathbb{C} = \mathbb{C}$$

such that

$$\xi(q_j) = q_j' \quad (j = 0, 1, \ldots, n).$$

We identify q_j' with q_j ($j = 0, 1, \ldots, n$) through ξ.

Now $\Delta^* \times \mathbb{C} - B_f$ is a topological fiber bundle with the base space Δ^* and the standard fiber $\mathbb{C} - \{n \text{ points}\}$ (see Dimca [2] and Matsuno [7]). Put

$$G = \{\alpha \colon \mathbb{C} \longrightarrow \mathbb{C} \mid \alpha \text{ is an orientation preserving homoemorphism such that}$$
$$\alpha(q_0) = q_0, \ \alpha(\{q_1, \ldots, q_n\}) = \{q_1, \ldots, q_n\}\}.$$

G is then a topological group with compact-open topology. Let G_e be its connected component of the identity. Put

$$\pi_0(G) = G/G_e.$$

Then $\pi_0(G)$ can be naturally identified with the Artin braid group B_n of n strings (see Birman [1, p. 165]).

Now assume that the above two families

$$f = \{f_t\} \colon X \longrightarrow \Delta \times \mathbb{P}^1,$$
$$f' = \{f_t'\} \colon X' \longrightarrow \Delta' \times \mathbb{P}^1$$

are topologically equivalent. We may assume that

$$\eta(t_0) = t_0',$$
$$\varphi \colon t_0 \times \mathbb{C} = \mathbb{C} \longrightarrow t_0' \times \mathbb{C} = \mathbb{C},$$
$$\varphi(q_0) = q_0'.$$

Let

$$\chi \colon \pi_1(S^1) \longrightarrow \pi_0(G)$$
$$(\text{resp. } \chi' \colon \pi_1(S^1) \longrightarrow \pi_0(G))$$

be the characteristic homomorphism of the bundle

$$\Delta^* \times \mathbb{C} - B_f \longrightarrow \Delta^*$$
$$(\text{resp. } \Delta^{'*} \times \mathbb{C} - B_f \longrightarrow \Delta^{'*})$$

(see Steenrod [12, p. 96]). Let δ (resp. $\delta^{'}$) be the loop around $t = 0$ as before.

Two bundles

$$\Delta^* \times \mathbb{C} - B_f \quad \text{and} \quad \Delta^{'*} \times \mathbb{C} - B_f$$

over the base space Δ^* (which is homeomorphic to $(0, 1) \times S^1$) and $\Delta^{'*}$ are weakly equivalent in the sence of Steenrod [12, p. 99]. Hence by Steenrod [12, p. 100], the characteristic $\chi(\delta)$ and $\chi^{'}(\delta^{'})$ of these bundles satisfy either

$$\chi(\delta) = \chi^{'}(\delta^{'}) \quad \text{or} \quad \chi(\delta) = \chi^{'}(\delta^{'})^{-1}$$

in $\pi_0(G)$. The equality here is up to conjugacy in $\pi_0(G)$. But the last equality does not occur by Steenrod [12, p. 100], for η is orientation preserving. Hence

(3) $\chi(\delta) = \chi^{'}(\delta^{'})$ (up to conjugacy).

But $\pi_0(G)$ can be identified with B_n as noted above, Under the identification, $\chi(\delta)$ (resp. $\chi^{'}(\delta^{'})$) is equal to $\theta(\delta)$ (resp. $\theta^{'}(\delta^{'})$), the braid monodromy. Hence by (3), there is $\sigma \in B_n$ such that

(4) $\theta^{'}(\delta^{'}) = \sigma \theta(\delta) \sigma^{-1}.$

Now the restriction

$$\varphi \colon t_0 \times \mathbb{C} = \mathbb{C} \longrightarrow t_0^{'} \times \mathbb{C} = \mathbb{C}$$

of φ is an orientation preserving homeomorphism. By the assumption of topological equivalence,

(5) $[\Phi_{f_{t_0}} \cdot \varphi_*^{-1}] = [\Phi_{f_{t_0'}^{'}}].$

Consider an isotopy φ_t ($0 \leq t \leq 1$) on $\mathbb{C}$ such that $\varphi_0 = $ the identity and $\varphi_1 = \varphi$. This gives a braid σ. We may write

$$\varphi = \sigma.$$

Then by (5)

(6) $[\Phi_{f_{t_0}} \cdot \sigma^{-1}] = [\Phi_{f_{t_0'}^{'}}].$

Now the braid σ in (4) and σ in (6) are the same. In fact, the braid σ in the relation

$$\theta^{'}(\delta^{'}) = \sigma\theta(\delta)\sigma^{-1}$$

is nothing but

$$\sigma = \varphi \colon t_0 \times \mathbb{C} \longrightarrow t_0^{'} \times \mathbb{C}$$

if we regard $\theta^{'}(\delta^{'})$ and $\theta(\delta)$ as elements of $\pi_0(G)$ (see Steenrod [12, p. 97–p. 98, p. 9–p. 12]). On the other hand, σ in (6) is also

$$\sigma = \varphi \colon t_0 \times \mathbb{C} \longrightarrow t_0^{'} \times \mathbb{C}.$$

Hence the braid σ in (4) and σ in (6) are the same. Thus there is $\sigma \in B_n$ such that

$$([\Phi_{f'}], \theta^{'}(\delta^{'})) = ([\Phi_f \cdot \sigma^{-1}], \sigma\theta(\delta)\sigma^{-1}).$$

Conversely, for two families in (1), we identify q_j' with q_j ($j = 0, 1, \ldots, n$) through ξ in (2) and suppose that there is $\sigma \in B_n$ such that

$$([\Phi_{f'}], \theta^{'}(\delta^{'})) = ([\Phi_f \cdot \sigma^{-1}], \sigma\theta(\delta)\sigma^{-1}).$$

Since $\theta^{'}(\delta^{'}) = \sigma\theta(\delta)\sigma^{-1}$, the above discussion shows that two bundles

$$\Delta^* \times \mathbb{C} - B_f \quad \text{and} \quad \Delta^{'*} \times \mathbb{C} - B_{f'}$$

over Δ^* and $\Delta^{'*}$ respectively are weakly equivalent. That is, there are orientation preserving homeomorphism φ and η such that (i) the following diagram commutes:

$$
\begin{array}{ccc}
\Delta^* \times \mathbb{C} - B_f & \xrightarrow{\ \varphi\ } & \Delta^{'*} \times \mathbb{C} - B_{f'} \\
\downarrow & & \downarrow \\
\Delta^* & \xrightarrow{\ \eta\ } & \Delta^{'*}
\end{array}
$$

(ii) $\eta(t_0) = t_0^{'}$ and
(iii) $\varphi = \sigma \colon t_0 \times \mathbb{C} = \mathbb{C} \longrightarrow t_0^{'} \times \mathbb{C} = \mathbb{C}$.

Now the fiber bundle structures on $\Delta^* \times \mathbb{C} - B_f$ and $\Delta^{'*} \times \mathbb{C} - B_{f'}$ can be naturally extended to those on $\Delta^* \times \mathbb{C}$ and $\Delta^{'*} \times \mathbb{C}$ respectively (see Lemma 2 in Matsuno [7]). Hence φ can be extended to an orientation preserving homeomorphism

$$\varphi \colon \Delta^* \times \mathbb{P}^1 \longrightarrow \Delta^{'*} \times \mathbb{P}^1$$

such that the following diagram commutes:

$$
\begin{array}{ccc}
\Delta^* \times \mathbb{P}^1 & \xrightarrow{\ \varphi\ } & \Delta'^* \times \mathbb{P}^1 \\
\downarrow & & \downarrow \\
\Delta^* & \xrightarrow{\ \eta\ } & \Delta'^*
\end{array}
$$

We show that φ and η can be extended so that the following diagram commutes:

$$
\begin{array}{ccc}
\Delta \times \mathbb{P}^1 & \xrightarrow{\ \varphi\ } & \Delta' \times \mathbb{P}^1 \\
\downarrow & & \downarrow \\
\Delta & \xrightarrow{\ \eta\ } & \Delta'
\end{array}
$$

We assume and put as in §5

$$
q_1(0) = \cdots = q_{k_1}(0) = q_1^0,
$$
$$
(\text{resp. } q_1'(0) = \cdots = q_{k_1}'(0) = q_1'^0),
$$
$$
q_{k_1+1}(0) = \cdots = q_{k_1+k_2}(0) = q_2^0,
$$
$$
(\text{resp. } q_{k_1+1}'(0) = \cdots = q_{k_1+k_2}'(0) = q_2'^0),
$$
$$
\cdots \cdots
$$
$$
q_{k_1+\cdots+k_{r-1}+1}(0) = \cdots = q_{k_1+\cdots+k_{r-1}+k_r}(0) = q_r^0,
$$
$$
(\text{resp. } q_{k_1+\cdots+k_{r-1}+1}'(0) = \cdots = q_{k_1+\cdots+k_{r-1}+k_r}'(0) = q_r'^0),
$$

where $k_\nu \geq 1$ ($\nu = 1, \ldots, r$), $k_1 + \cdots + k_r = n$ and $q_1^0, \ldots, q_r^0$ (resp. $q_1'^0, \ldots, q_r'^0$) are mutually distinct.

We may assume that there is a continuous function $\rho(|t|)$ of $|t|$ such that
(i) $\rho(|t|) > 0$ for $|t| > 0$,
(ii) $\rho(0) = 0$,
(iii) $\Delta(q_\nu^0, \rho(|t|))$ (resp. $\Delta(q_\nu'^0, \rho(|t|))$) ($\nu = 1, \ldots, r$) are mutually disjoint,
(iv) each $\Delta(q_\nu^0, \rho(|t|))$ (resp. $\Delta(q_\nu'^0, \rho(|t|))$) ($\nu = 1, \ldots, r$) contains

$$
q_{k_1+\cdots+k_{\nu-1}+1}(t), \cdots, q_{k_1+\cdots+k_{\nu-1}+k_\nu}(t)
$$
$$
(\text{resp. } q_{k_1+\cdots+k_{\nu-1}+1}'(t), \cdots, q_{k_1+\cdots+k_{\nu-1}+k_\nu}'(t)).
$$

Now Lemma 2 in Matsuno [7] implies that the bundle structure on $\Delta^* \times \mathbb{C} - B_f$ coincides with that of the product bundle $\Delta^* \times \mathbb{C}$ outside

$$
T = \bigcup_{0<|t|<\epsilon} \bigcup_{\nu=1}^{r} \Delta(q_\nu^0, \rho(|t|)).
$$

Similar assertion holds for the bundle structure on $\Delta'^* \times \mathbb{C} - B_{f'}$. Hence we may assume that φ does not depend on t outside T. Thus φ can be extended to an orientation

preserving homeomorphism

$$\varphi\colon \Delta \times \mathbb{C} - \{q_1^0, \ldots, q_r^0\} \longrightarrow \Delta' \times \mathbb{C} - \{q_1'^0, \ldots, q_r'^0\}.$$

Moreover if we define

$$\varphi(q_\nu^0) = q_\nu'^0 \quad (\nu = 1, \ldots, r),$$

then φ is extended to an orientation preserving homeomorphism

$$\varphi\colon \Delta \times \mathbb{P}^1 \longrightarrow \Delta' \times \mathbb{P}^1.$$

Put also $\eta(0) = 0$. Then η is extended to an orientation preserving homeomorphism

$$\eta\colon \Delta \longrightarrow \Delta'$$

and the following diagram commutes:

$$
\begin{array}{ccc}
\Delta \times \mathbb{P}^1 & \xrightarrow{\ \varphi\ } & \Delta' \times \mathbb{P}^1 \\
\downarrow & & \downarrow \\
\Delta & \xrightarrow{\ \eta\ } & \Delta'
\end{array}
$$

Next, note that

$$\varphi(B_f) = B_{f'},$$
$$\varphi = \sigma\colon t_0 \times \mathbb{C} \longrightarrow t_0' \times \mathbb{C}.$$

Note also that

$$\varphi \cdot f\colon X \longrightarrow \Delta' \times \mathbb{P}^1$$

is unbranched on $\Delta' \times \mathbb{P}^1 - B_{f'}$. By Theorem 1, $\varphi \cdot f$ can be extended to a branched covering

$$f''\colon X'' \longrightarrow \Delta' \times \mathbb{P}^1.$$

$\varphi \cdot f$ and f'' coincides on $\Delta' \times \mathbb{C} - B_{f'}$ and both are Fox completions of the same unbranched coverings of $\Delta' \times \mathbb{C} - B_{f'}$. Hence by the uniqueness of the Fox completion (see Fox [3]), there is a homeomorphism

$$\psi'\colon X \longrightarrow X''$$

such that the following diagram commutes:

$$
\begin{array}{ccc}
X & \xrightarrow{\psi'} & X'' \\
{\scriptstyle \varphi \cdot f}\downarrow & & \downarrow{\scriptstyle f''} \\
\Delta' \times \mathbb{P}^1 & \xrightarrow{id} & \Delta' \times \mathbb{P}^1
\end{array}
$$

Note that ψ' is orientation preserving. Now, the representaion class of the monodromy of f'' is equal to that of $\varphi \cdot f$, which is cleary equal to $[\Phi_f \cdot \varphi_*^{-1}]$. By the assumption

$$[\Phi_f \cdot \varphi_*^{-1}] = [\Phi_f \cdot \sigma^{-1}] = [\Phi_{f'}],$$

we have

$$[\Phi_{f''}] = [\Phi_{f'}].$$

Hence there is a biholomorphic mapping

$$\psi'' : X'' \longrightarrow X'$$

which makes the following diagram commutative:

$$
\begin{array}{ccc}
X'' & \xrightarrow{\psi''} & X' \\
{\scriptstyle f''}\downarrow & & \downarrow{\scriptstyle f'} \\
\Delta' \times \mathbb{P}^1 & \xrightarrow{id} & \Delta' \times \mathbb{P}^1
\end{array}
$$

Now put $\psi = \psi'' \cdot \psi'$. Then

$$\psi : X \longrightarrow X'$$

is an orientation preserving homeomorphism which makes the following diagram commutative:

$$
\begin{array}{ccc}
X & \xrightarrow{\psi} & X' \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\
\Delta \times \mathbb{P}^1 & \xrightarrow{\varphi} & \Delta' \times \mathbb{P}^1
\end{array}
$$

Hence $f = \{f_t\}$ and $f' = \{f_t'\}$ are topologically equivalent. $\square$

Considering a trivial family, we get the following corollary, which can be also derived directly from Theorem 2.

Corollary 2 (cf. Wajnryb [13]). *There exists a one to one correspondence between* {*topological equivalence class of* $f: X \longrightarrow \mathbb{P}^1$ *of degree* d *with* n *branched points* $q_1, \ldots, q_n$} *and* {$[\Phi]$ | $[\Phi]$ *is the representation class of* $\Phi: \pi_1(\mathbb{P}^1 - \{q_1, \ldots, q_n\}, q_0) \longrightarrow S_d$ *such that* $\mathrm{Im}\,\Phi$ *is transitive*}/B_n.

REMARK. As one can see in the proof of Theorem 8, we do not need to mention the points $\{q_1, \ldots, q_n\}$ in the statement of Theorem 8 and its corollary, if we replace $\pi_1(\mathbb{P}^1 - \{q_1, \ldots, q_n\}, q_0)$ by the abstract group

$$\langle \gamma_1, \ldots, \gamma_n \mid \gamma_n \cdots \gamma_1 = 1 \rangle.$$

7. Degenerating families of finite branched coverings of $\mathbb{P}^m$

Let $\Delta = \Delta(0, \epsilon)$ be a disc and

$$f: X \longrightarrow \Delta \times \mathbb{P}^m$$

be a finite branched covering. As in the case of $\mathbb{P}^1$, f is called a degenerating family of finite branched covering of $\mathbb{P}^m$ and is denoted by $f = \{f_t\}$ if the following 4 conditions are satisfied

(1) $t \times \mathbb{P}^m \not\subset B_f$ for every $t \in \Delta$.

(2) For every $t \in \Delta^*$, $t \times \mathbb{P}^m$ meets transversally with B_f and putting $(t \times \mathbb{P}^m) \cap B_f = t \times B_t$, B_t is a hypersurface of $\mathbb{P}^m$ of degree n. (n is constant for $t \in \Delta^*$.)

(3) For every $t \in \Delta^*$,

$$f_t = f: X_t = f^{-1}(t \times \mathbb{P}^m) \longrightarrow t \times \mathbb{P}^m$$

is a covering of $\mathbb{P}^m$ of degree $d = \deg(f)$ branching at B_t.

(4) For any points t and t' in Δ^*, f_t and $f_{t'}$ are topologically equivalent.

The central fiber $X_0 = f^{-1}(0 \times \mathbb{P}^m)$ is a degeneration of a general fiber X_t for $t \neq 0$.

We show that, topologically, the central fiber X_0 can be described by the central branch divisor B_0, where $0 \times B_0 = (0 \times \mathbb{P}^m) \cap B_f$ and by the monodromy $\Phi_t = \Phi_{f_t}$, where $t \in \Delta^*$ is a fixed point. We explain this as follows:

Let L be a general line in $\mathbb{P}^m$. We may assume that L meets transversally with every B_t for $t \in \Delta$. Consider the restriction

$$f^L: X^L = f^{-1}(\Delta \times L) \longrightarrow \Delta \times L$$

of f to $X^L = f^{-1}(\Delta \times L)$. Then

Lemma 2. (1) *Every point of* $(X - f^{-1}(B_f)) \cap X^L$ *is a non-singular point of* X^L.

(2) *For $t \neq 0$, every point of $f^{-1}(\mathrm{Reg}(B_f) \cap (t \times L))$ is non-singular point of X^L.*
($\mathrm{Reg}(B_f)$ is the set of non-singular points of the branch locus B_f.)
(3) *For $t \neq 0$, the restriction*

$$f_t^L : X_t^L = f^{-1}(t \times L) \longrightarrow t \times L$$

of f^L is a branched covering of degree $d = \deg(f)$

Proof. (1) Let $p \in (X - f^{-1}(B_f)) \cap X^L$. Then there are local coordinate systems $(t, x_1, \ldots, x_m)$ and $(t, y_1, \ldots, y_m)$ around p in X and $q = f(p)$ in $\Delta \times \mathbb{P}^m$ such that (i) t is a local coordinate system in Δ and $(y_1, \ldots, y_m)$ is that in $\mathbb{P}^m$, (ii) L is locally given by the equation $y_2 = \cdots = y_m = 0$ and (iii) f is locally given by

$$f : (t, x_1, \ldots, x_m) \longmapsto (t, y_1, \ldots, y_m) = (t, x_1, \ldots, x_m).$$

Then f^L is locally given by

$$f^L : (t, x_1) \longmapsto (t, y_1) = (t, x_1).$$

In particular, p is a non-singular point of X^L.
(2) Let $t_0 \neq 0$ and $p \in f^{-1}(\mathrm{Reg}(B_f) \cap (t_0 \times L))$. Then there are local coordinate systems $(t, x_1, \ldots, x_m)$ and $(t, y_1, \ldots, y_m)$ around p in X and $q = f(p)$ in $\Delta \times \mathbb{P}^m$ such that (i) t is a local coordinate system in Δ around t_0 and $(y_1, \ldots, y_m)$ is that in $\mathbb{P}^m$, (ii) L is locally given by the equation $y_2 = \cdots = y_m = 0$ and (iii) f is locally given by

$$f : (t, x_1, x_2, \ldots, x_m) \longmapsto (t, y_1, y_2, \ldots, y_m) = (t, x_1^e, x_2, \ldots, x_m).$$

Then f^L is locally given by

$$f^L : (t, x_1) \longmapsto (t, y_1) = (t, x_1^e).$$

In particular, p is a non-singular point of X^L. Moreover p is a ramification point of $f_{t_0}^L$ with the ramification index e.
(3) For $t \neq 0$, the branched covering

$$f_t : X_t \longrightarrow \mathbb{P}^m$$

gives a linear system on X_t. By Bertini's theorem, X_t^L is non-singular and globally irreducible. Hence, by the proof of (2),

$$f_t^L : X_t^L \longrightarrow t \times L$$

is a branched covering of degree $d = \deg(f)$. $\qquad\square$

This lemma shows that the singular locus $\mathrm{Sing}(X^L)$ of X^L is contained in $f^{-1}(0 \times (B_0 \cap L))$, which is a finite set. X^L is globally irreducible. Let

$$\mu \colon \tilde{X}^L \longrightarrow X^L$$

be the normalization of X^L. Since $\mathrm{Sing}(X^L)$ is a finite set, μ is a bijective holomorphic mapping. In fact, suppose that there are distinct points p_1 and p_2 in $\tilde{X}^L$ such that

$$p = \mu(p_1) = \mu(p_2) \in f^{-1}(0 \times (B_0 \cap L)).$$

Then there are disjoint connected open neighborhoods W_1 and W_2 of p_1 and p_2 respectively such that

$$\mu(W_1) = \mu(W_2) = W$$

and W is connected open neighborhood of p in X^L. Since $f^{-1}(0 \times (B_0 \cap L))$ is a finite set, we may assume that

$$f^{-1}(0 \times (B_0 \cap L)) \cap W = \{\, p \,\}.$$

We may assume that $X_t^L \cap W$ is connected for non-zero t with $|t|$ sufficiently small. Hence $W - \{\, p \,\}$ is a connected 2-dimensional complex manifold. Since μ is the normalization of X^L,

$$W_1 - \{\, p_1 \,\} = W_2 - \{\, p_2 \,\}$$

and

$$\mu \colon W_1 - \{\, p_1 \,\} = W_2 - \{\, p_2 \,\} \longrightarrow W - \{\, p \,\}$$

is biholomorphic, a contradiction. Thus μ is bijective. The composition

$$f^L \cdot \mu \colon \tilde{X}^L \longrightarrow \Delta \times L$$

is a degenerating family of finite branched coverings of $L = \mathbb{P}^1$, which we denote

$$f^L \cdot \mu = \{f_t^L\}$$

by abuse of notation.

Lemma 3. (1) *Let $X_0 = X_{01} \cup \cdots \cup X_{0u}$ be the global irreducible decomposition of X_0. Then*

$$X_0^L = (X_{01} \cap X_0^L) \cup \cdots \cup (X_{0u} \cap X_0^L)$$

is the global irreducible decomposition of X_0^L.

(2) Let $\mathrm{Sing}^{m-1}(X_0)$ be the union of global irreducible components of $\mathrm{Sing}(X_0)$ which are hypersurfaces of X_0. Then (i) $\mathrm{Sing}^{m-1}(X_0) \subset f_0^{-1}(B_0)$ and (ii) $(\mathrm{Sing}^{m-1}(X_0)) \cap X_0^L = \mathrm{Sing}(X_0^L)$.

(3) For a point $p \in \mathrm{Sing}(X_0^L)$, let

$$(X_0)_p = Z_1 \cup \cdots \cup Z_v$$

be the local irreducible decompoition of X_0 at p. Then the local irreducible decomposition of X_0^L at p is given by

$$(X_0^L)_p = (Z_1 \cap X_0^L) \cup \cdots \cup (Z_v \cap X_0^L).$$

Proof.   (1)   Let

$$\mu_j \colon \hat{X}_{0j} \longrightarrow X_{0j}$$

$(1 \leq j \leq u)$ be the normalization of X_{0j}. By the proof of (1) of Lemma 2,

$$f_{0j} \cdot \mu_j \colon \hat{X}_{0j} \longrightarrow 0 \times \mathbb{P}^m \quad (f_{0j} = f_0 \mid X_{0j})$$

is a finite branched covering. By Bertini's theorem, $(f_{0j} \cdot \mu_j)^{-1}(0 \times L)$ is a non-singular connected curve of $\hat{X}_{0j}$. Hence $f_{0j}^{-1}(0 \times L)$ is a global irreducible component of X_0^L and

$$X_0^L = f_0^{-1}(0 \times L) = \bigcup_{j=1}^{u} f_{0j}^{-1}(0 \times L) = \bigcup_{j=1}^{u} (X_{0j} \cap X_0^L)$$

is the irreducible decomposition of X_0^L.

(2) By (2) of Lemma 2, every component of $\mathrm{Sing}^{m-1}(X_0)$ is a global irreducible component R_0 of $f_0^{-1}(B_{01})$, where B_{01} is a global irreducible component of B_0. Let p be a point of $X_0^L \cap R_0$. Then p is clearly a singular point of X_0^L. Conversely, if p is a singular point of X_0^L, then $f_0(p) = q$ is on $L \cap B_{01}$ for an irreducible component B_{01} of B_0. Since L is a general line, every point on a global irreducible component R_0 with $p \in R_0$ of $f_0^{-1}(B_{01})$ is a singular point of X_0. Hence R_0 is a component of $\mathrm{Sing}^{m-1}(X_0)$. This shows (i) and (ii) of (2).

(3) We use the same notation as in the proof of (1). Every Z_k is an open set of some X_{0j}. Hence

$$\mu_{jk} \colon \hat{Z}_k = \mu_j^{-1}(Z_k) \longrightarrow Z_k \quad (\mu_{jk} = \mu_j \mid \hat{Z}_k)$$

is the normalization of Z_k. $(f_{0jk} \cdot \mu_{jk})^{-1}(0 \times L)$ is a non-singular connected curve of $\hat{Z}_k$, where $f_{0jk} = f_0 \mid Z_k$. Hence $f_{0jk}^{-1}(0 \times L) = Z_k \cap X_0^L$ is a local irreducible component

of X_0^L at p and

$$(X_0^L)_p = (Z_1 \cap X_0^L) \cup \cdots \cup (Z_v \cap X_0^L)$$

is the local irreducible decomposition of X_0^L. at p.   □

Now we refer a theorem of Zariski-van Kampen. Let B be a hypersurface of degree n in $\mathbb{P}^m$. Take a general point q_0 in $\mathbb{P}^m - B$ and let

$$\pi: \mathbb{P}^m - \{q_0\} \longrightarrow \mathbb{P}^{m-1}$$

be the projection with the center q_0. Put

$$\hat{\pi} = \pi \mid B: B \longrightarrow \mathbb{P}^{m-1}$$

be the restriction. Let D be the branch locus of $\hat{\pi}$. A theorem of Zariski-van Kampen in this case can be described as follows (cf. Matsuno [7]).

Theorem 9 (Zariski-van Kampen).

$$\pi_1(\mathbb{P}^m - B, q_0)$$
$$= \langle \gamma_1, \ldots, \gamma_n \mid \gamma_n \cdots \gamma_1 = 1, \theta(\delta_k)\gamma_j = \gamma_j \ (1 \leq j \leq n, \ 1 \leq k \leq s) \rangle,$$

where γ_j are lassos as in Fig. 2 *on $\pi^{-1}(q_0)$, the line deleted the point $\{q_0\}$, δ_k are the generators of $\pi_1(\mathbb{P}^{m-1} - D, r_0)$ for a reference point $r_0 \in \mathbb{P}^{m-1} - D$, and $\theta(\delta_k)$ are the braid monodromy along δ_k.*

This theorem shows in particular that the monodromy Φ_{f_t} of f_t is equal to the monodromy $\Phi_{f_t}^L$ of f_t^L for a general line L passing through q_0. Hence, we conclude by Theorems 4, 5, 6 and Lemma 3 that topologically, the central fiber X_0 can be determined by the central branched divisor B_0 and by the monodromy $\Phi_t = \Phi_{f_t}$, where $t \in \Delta^*$ is a fixed point.

REMARK.   If $\deg B_0 = \deg B_t$ $(t \neq 0)$, then there is a surjective homomorphism

$$\pi_1(\mathbb{P}^m - B_0, q_0) \longrightarrow \pi_1(\mathbb{P}^m - B_t, q_0) \longrightarrow 0$$

(see Zariski [14]). In this case, X_0 is irreducible and

$$\dim \mathrm{Sing}(X_0) \leq m - 2.$$

Hence degenerations such that

$$\dim \mathrm{Sing}(X_0) = m - 1$$

happen only if $\deg B_0 < \deg B_t$ $(t \neq 0)$, that is, only if B_0 has a multiple component as a divisor.

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