Hayakawa, K.
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SOME RESULTS ON THE FOURIER IMAGE OF SPECIAL CLASSES OF DISTRIBUTIONS

KANTARO HAYAKAWA

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Introduction. In this paper, we shall show that we can characterize the Fourier-Laplace image of a class of distributions with a certain decreasing condition at ∞ by a class of entire functions with a corresponding increasing condition along the imaginary axis. Many results in this concern have been obtained. Paley-Wiener [2]-Schwartz [3] shows that the Fourier transform of a distribution which vanishes identically at ∞ is equal to a function, which can be extended to an entire function of exponential type.

More generally, Schwartz [3] shows that the Fourier-Laplace image of a class of ditributions which decrease exponentially with some exponent is a class of holomorphic functions which are analytic in a corresponding strip domain along the real axis. (see Proposition 3.4. in this paper.)
Here, we use the term "the Fourier-Laplace transform" in the sense of Schwartz. Precisely, the Fourier-Laplace transform $F_u(\xi+i\eta)$ of a distribution $u(x)$ is defined by the Fourier transform of $e^{\eta x}u(x)$ when the distribution $e^{\eta x}u(x)$ belongs to $\mathcal{S}'$.

On the other hand, Gel'fand-Shilov [1] shows the following. Let $\rho(x)$ be a function with a certain increasing condition, which is similar to that in this paper. The Fourier image of a class of C^∞-functions, any element of which can be estimated by $C\exp(-\rho((1\pm\varepsilon)x)$ together with its derivatives, coincides with the class of entire functions $F(\xi+i\eta)$, which satisfies the estimate $|(\xi+i\eta)^k F(\xi+i\eta)|\leqq C_k\exp(\rho^*(\eta))$ for all k, where $\rho^*(\eta)$ is the dual function of $\rho(x)$ in the sense of Young.

In this paper, we treat this problem in $\mathcal{S}'$-category, and get some results analogous to those results in [1].
Let $\rho(x)$ be a certain function with decreasing conditions which will be mentioned in the next section.
We consider a class $\mathcal{S}_\rho'$ of distributions, which consists of all distributions of the type $e^{-\rho(x)}v$, where v is an element of $\mathcal{S}'$.

Now, our result is, roughly speaking, that the Fourier-Laplace image of $\mathcal{S}_\rho'$ is nearly equal to the class of entire functions $F(\zeta)=F(\xi+i\eta)$ which satisfies the estimate

$$|F(\xi+i\eta)|\leqq C(1+|\xi+i\eta|)^N e^{\rho^*(\eta)}$$

for an integer N (see Theorems in 4.).

In § 5 we treat the case of several variables.

And in the last section, some examples of $\rho(x)$ which satisfies conditions i)$\sim$iv) are considered.

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2. The function $\rho(x)$ and the dual function $\rho^*(\eta)$

In this paper, we denote by $\rho(x)$ a function which satisfies the next conditions i)$\sim$iv).

i) $\rho(x)$ is a C^∞-function on R.

ii) $\rho(x)$ is strictly concave.

iii) $\rho(x)/|x|$ goes to ∞ when $|x|$ tends to ∞.

iv) For any integer $k\geqq 1$, there exist some constant C_k and integer N_k with which we have

$$|\rho^{(k)}(x)|\leqq C_k(1+|x|)^{N_k} \qquad \text{for all } x,$$

where $\rho^{(k)}(x)$ denotes the k-th derivative of $\rho(x)$.

The dual function $\rho^*(\eta)$ of $\rho(x)$ in the sense of Young is defined by $\rho^*(\eta)=\mathrm{Max}(-\rho(x)+\eta x;\ -\infty<x<\infty)$.

From the assumptions and the definition of ρ^*, we have

Lemma 1.1. 1) *For any η there exists only one $x=x(\eta)$ which satisfies $\rho^*(\eta)=-\rho(x)+\eta x$.*

2) *The function $x(\eta)$ is a continuous and concave function which increases strictly in η.*

3) *We have the dual formula*

$$\rho(x)=\mathrm{Max}(-\rho^*(\eta)+x\eta;\ -\infty<\eta<\infty)$$

and for any η there exists only one $\eta=\eta(x)$ which satisfies

$$\rho(x)=-\rho^*(\eta)+x\eta\,.$$

4) *$\rho^*(\eta)/|\eta|$ goes to ∞ as $|\eta|$ tends to ∞.*

Proof. 1) Since $x(\eta)$ is the solution of the equation $-\rho'(x)+\eta=0$, the conclusions of 1) and 2) are obvious by using ii) and iii). By the definition of

$\rho^*(\eta)$, we have $\rho^*(\eta) \geqq -\rho(x)+\eta x$ for any η and x. Thus, we have $\rho(x) \geqq -\rho^*(\eta)+x\eta$. On the contrary, when we set $\eta=\rho'(x)$, which is the inverse function of $x(\eta)$, we have $\rho(x)=-\rho^*(\eta)+x\eta$. Hence we have 3).

Now, we are going to prove 4).

If we have a constant M and a sequence $\{\eta_j; j=1, 2, \cdots\}$ such that η_j goes to ∞ as j tends to ∞, and $\rho^*(\eta_j)/\eta_j < M$, then we have, by definition,

$$-\rho(x)+x\eta_j < M \qquad \text{for all } x \text{ and } j.$$

But when we set $x=x_0>M+1$, we have the estimate $M\eta_j > (M+1)\eta_j - \rho(x_0)$, which contradicts the fact that η_j goes to ∞.

3. Spaces $\mathcal{S}_\Gamma'$, $\mathcal{S}_\rho'$, $\mathcal{S}'_{\rho,\Gamma}$, and $\mathcal{S}_\rho$

We denote by $\mathcal{S}$ (or $\mathcal{S}'$) the space of rapidly decreasing C^∞-functions (or tempered distributions resp.) in the sense of Schwartz.

DEFINITION 3.1. Let Γ be an open interval in R and $\rho(x)$ be a function as in §2. We define spaces $\mathcal{S}_\rho$, $\mathcal{S}_\Gamma'$, $\mathcal{S}_\rho'$ and $\mathcal{S}'_{\rho,\Gamma}$ as follows:

$$\begin{aligned}
\mathcal{S}_\rho &= \{u(x) \in C^\infty(R);\ e^{\rho(x)}u(x) \in \mathcal{S}\} \\
\mathcal{S}_\Gamma' &= \{u(x) \in \mathcal{D}'(R);\ e^{\lambda x}u(x) \in \mathcal{S}' \quad \text{for all } \lambda \text{ in } \Gamma\} \\
\mathcal{S}_\rho' &= \{u(x) \in \mathcal{D}'(R);\ e^{\rho(x)}u(x) \in \mathcal{S}'\} \\
\mathcal{S}'_{\rho,\Gamma} &= \{u(x) \in \mathcal{D}'(R);\ e^{\lambda x+\rho(x)}u(x) \in \mathcal{S}' \quad \text{for all } \lambda \in \Gamma\}
\end{aligned}$$

DEFINITION 3.2. (Schwartz [3]) Let $u(x)$ be in $\mathcal{S}_\Gamma'$, then we can define the Fourier transform $U_\eta(\xi)$ of $e^{\eta x}u(x)$ for all η in Γ. We shall call $U_\eta(\xi)$ the Fourier-Laplace transform of u.

Proposition 3.3. (Schwartz [3]) *A distribution $u(x) \in \mathcal{D}'(R)$ is in $\mathcal{S}'$ if and only if it is a derivative of a continuous function which increases in at most algebraic order, that is, we can find a bounded continuous function $f(x)$ and integers p, k, with which we have $u(x)=D_x^p((1+x^2)^k f(x))$.*

Proposition 3.4. (Schwartz [3]) *If $u(x)$ is in $\mathcal{S}_\Gamma'$, the Fourier-Laplace transform of u is a holomorphic function $F_u(\xi+i\eta)$ in the tubular domain $\Xi+i\Gamma=\{\xi+i\eta: \eta\in\Gamma\}$ which satisfies;*

$$(3, 1) \qquad |F_u(\xi+i\eta)| \leqq C_\eta(1+|\xi+i\eta|)^{N_\eta} \qquad \textit{for any } \eta \textit{ in } \Gamma.$$

In (3,1), *C_η is a constant and N_η is an η integer depending only on which can be taken uniformly on any compact subset of Γ. Conversely, if a holomorphic function $F(\xi+i\eta)$ in $\Xi+i\Gamma$ satisfies the estimate* (3,1), *we can find a distribution $u(x)$ in $\mathcal{S}_\Gamma'$, whose Fourier-Laplace transform coincides with $F(\xi+i\eta)$.*

4. Main results

By the definition, the C^∞–function $e^{\eta x-\rho(x)}$ is in $\mathcal{S}$, hence we have $\mathcal{S}_\rho'\subset\mathcal{S}_R'$. And we can consider the Fourier-Laplace transform for an element of $\mathcal{S}_\rho'$. Moreover, we have the following theorems.

Theorem 4.1. *If u is in $\mathcal{S}_\rho'$, the Fourier-Laplace transform of u is an entire function $F_u(\xi+i\eta)$ which satisfies the following condition* T–1).

T–1) *There exists an integer N, and for any positive real ε, there exists a constant C_ε, such that we have the following estimate* (4,1) *with these N and C_ε.*

$$|F_u(\xi+i\eta)|\leqq C_\varepsilon(1+|\xi+i\eta|)^N e^{\rho^*(\eta\pm\varepsilon)} \tag{4, 1}$$

where we take $+\varepsilon$ (or $-\varepsilon$) if $\eta>0$ (or $\eta<0$ resp.).

Theorem 4.2. *Let $F(\zeta)=F(\xi+i\eta)$ be an entire function which satisfies the following condition* T–2).

T–2) *There exist an integer N and a constant C with which we have*

$$|F(\zeta)|\leqq C(1+|\zeta|)^N e^{\rho^*(\eta)}. \tag{4, 2}$$

Then, we can find a distribution $u(x)$ in $\mathcal{S}_\rho'$ uniquely, so that the Fourier-Laplace transform of u is equal to $F(\zeta)$.

In general, we cannot take $\varepsilon=0$ in T–1), (see a counter example in the last section), however we have

Theorem 4.3. *If $\rho(x)$ satisfies the following condition* v) *in addition to* i)∼iv):

v) *$\rho(x)/|x|^p$ goes to ∞ as $|x|$ tends to ∞ for some $p>1$,*
then we can take $\varepsilon=0$ in T–1).

Combining the consequences of Theorem 4.1. and 4.2, we have

Theorem 4.4. *If u is in $\mathcal{S}'_{\rho,\Gamma}$, then the Fourier-Laplace transform of u is an entire function $F_u(\xi+i\eta)$ which satisfies the following condition* T–3).

T–3) *There exists an integer N_λ for any λ in Γ, and there exists a constant C_λ, such that, with these N_λ and C_λ, we have*

$$|F_u(\xi+i\eta)|\leqq C_\lambda(1+|\xi+i\eta|)^{N_\lambda} e^{\rho^*(\eta-\lambda)}. \tag{4, 3}$$

Conversely, if an entire function $F(\zeta)$ satisfies the above condition T–3), we can find a distribution u in $\mathcal{S}'_{\rho,\Gamma}$ whose Fourier-Laplace transform coincides with $F(\zeta)$.

Above theorems are all concerned with the $\mathcal{S}'$-category, on the other hand for the space $\mathcal{S}_\rho$, we have the following

Theorem 4.5. *If u is in $\mathcal{S}_\rho$, the Fourier-Laplace transform of u is an entire function $F_u(\xi+i\eta)$ having the following property* T–4).

T–4) *For any integers N and m, we can take a constant* $C_{N,m}$, *with which we have*

$$|\partial_\xi^m F_u(\xi+i\eta)| \leqq C_{N,m}(1+|\xi+i\eta|)^{-N} e^{\rho^*(\eta)}. \tag{4, 4}$$

Conversely, if $F(\zeta)$ *is entire and satisfies* T–4), *we can find a function u in* $\mathcal{S}_\rho$ *whose Fourier-Laplace transform equals F.*

Proof of Theorem 4.1. From Proposition 3.4, $F_u(\zeta)$ is an entire function. So, we prove the estimate for F_u. Since $e^{\eta x-\rho(x)}$ is an element of $\mathcal{S}$ for any η, we have

$$F_u(\xi+i\eta) = {}_{\mathcal{S}'}\langle e^{\rho(x)}u(x),\ e^{-\rho(x)-i(\xi+i\eta)x}\rangle_{\mathcal{S}}. \tag{4. 5}$$

By the definition, $e^{\rho(x)}u(x)$ is in $\mathcal{S}'$, hence we have $e^{\rho(x)}u=D_x^p((1+x^2)^k f(x))$ where f is a bounded continuous function. Thus we have

$$\begin{aligned} F_u(\xi+i\eta) &= {}_{\mathcal{S}'}\langle(1+x^2)^k f(x),\ (-D_x)^p e^{-\rho(x)-i\zeta x}\rangle_{\mathcal{S}} \\ &= \int (1+x^2)^k f(x)(-D_x)^p e^{-\rho(x)-i(\xi+i\eta)x}\,dx\,. \end{aligned} \tag{4. 6}$$

Using the Leibniz formula

$$(-D_x)^p e^{-\rho(x)} e^{-i\zeta x} = \sum_{l=0}^{p} p_{p,l}(\rho',\cdots,\rho^{(l)})\zeta^{p-l} e^{-\rho(x)-i\zeta x} \tag{4. 7}$$

where $P_{p,l}\,(r_1,\cdots,r_l)$ is a polynomial in $r=(r_1,\cdots,r_l)$.
Now we substitute this into (4,6) and we have for $\eta>1$

$$\begin{aligned} |F_u(\xi+i\eta)| &< \int_{-\infty}^{\infty}\sum_{l=0}^{p} C_{p,l}(1+x^2)^k \sup|f(x)|\,|\zeta|^l(1+x^2)^{n_{p,l}} e^{-\rho(x)+\eta x}\,dx \\ &\leqq \sum_l C_{l,\varepsilon}\int_0^{\infty}(1+x^2)^{k+n_{p,l}} \sup|f(x)|\,|\zeta|^l e^{-\varepsilon x}\,dx\, e^{-\rho^*(\eta+\varepsilon)} + \\ &\quad +\sum_l C_{l,\varepsilon}\int_{-\infty}^{0}(1+x^2)^{k+n_{p,l}} \sup|f(x)|\,|\zeta|^l e^{-\rho(x)}\,dx \\ &\leqq \sum_{l=0}^{p}|\zeta|^l C_{l,\varepsilon} e^{-\rho^*(\eta+\varepsilon)} \\ &\leqq C_\varepsilon(1+|\zeta|)^p e^{-\rho^*(\eta+\varepsilon)}\,. \end{aligned} \tag{4. 8}$$

Similarly we have for $\eta<-1$

$$|F_u(\xi+i\eta)| \leqq C_\varepsilon(1+|\zeta|)^p e^{-\rho^*(\eta-\varepsilon)}\,.$$

Before going to the proof of Theorem 4.2 we prepare the following lemma.

Lemma 4.6. *Let* $F(\zeta)$ *be an entire function. For any integer* ν *we can find a function* $F_0(\zeta)=\sum_{k=0}^{\nu} a_k e^{i\alpha_k\zeta}$ *so that it satisfies*

$$F(0)=F_0(0),\ \partial_\zeta F(0)=\partial_\zeta F_0(\zeta),\ \cdots,\ \partial_\zeta^\nu F(0)=\partial_\zeta^\nu F_0(0)$$

where α_k's are real numbers and a_k's are complex constants.

Proof. We take real numbers α_k so that we have $\alpha_j \neq \alpha_k$ for $j \neq k$. Then we can find a_k ($k=1, 2, \cdots, \nu$) which satisfies

$$\begin{pmatrix} 1, & 1, & \cdots, & 1 \\ i\alpha_0, & i\alpha_1, & \cdots, & i\alpha_\nu \\ \vdots & \vdots & & \\ (i\alpha_0)^\nu, & (i\alpha_1)^\nu, & \cdots, & (i\alpha_\nu)^\nu \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_\nu \end{pmatrix} = \begin{pmatrix} F(0) \\ \partial_\zeta F(0) \\ \vdots \\ \partial_\zeta^\nu F(0) \end{pmatrix}$$

since the above matrix is nonsingular. With these α_k and a_k we have

$$\partial_\zeta^l F_0(0)=\sum_{k=0}^{\nu}(i\alpha_k)^l a_k=\partial_\zeta^l F(0).$$

Proof of Theorem 4.2.

For the entire function $F(\zeta)$ we take $F_0(\zeta)$ as in Lemma 4.5 where we set $\nu=N+1$. We write the function $F_0(\zeta)$ as the sum of two functions $F_0(\zeta)$ and $F_1(\zeta)=F(\zeta)-F_0(\zeta)$. $F_0(\zeta)$ is the Fourier-Laplace transform of $u_0(x)=\sum_{k=0}^{N+1} a_k\delta(x-\alpha_k)$, where $\delta(x)$ is the Dirac distribution.

Since $u_0(x)$ has a compact support, it belongs to $\mathcal{S}_\rho'$. So, for the proof of this theorem, it is enough to show that $F_1(\zeta)$ is the Fourier-Laplace transform of an element of $\mathcal{S}_\rho'$. From the form of $F_0(\zeta)$ we have $|F_0(\zeta)| \leqq C\exp \mathrm{Max}|\alpha_k||\eta|$ for some constant C, so the function $F_1(\zeta)=F(\zeta)-F_0(\zeta)$ satisfies the same estimate as (4,2)

$$|F_1(\xi+i\eta)| \leqq C_1(1+|\zeta|)^N e^{\rho^*(\eta)} \tag{4. 9}$$

for some constant C_1 and the same integer N.
And from the definition we have $\partial_\zeta^l F_1(0)=0$ for $l=0, 1, 2, \cdots, N+1$. Hence if we set $G(\zeta)=F_1(\zeta)/\zeta^{N+2}$, $G(\zeta)$ is an entire function having the following estimate

$$|G(\xi+i\eta)| \leqq C_2(1+|\zeta|)^{-2} e^{\rho^*(\eta)} \tag{4. 10}$$

Now by Proposition 3.4 we can find a distribution $u_1(x)$ in $\mathcal{S}_R'$ whose Fourier-Laplace transform is equal to $F_1(\zeta)$. We must show that this distribution u_1 belongs to $\mathcal{S}_\rho$. To prove this we have only to show that the distribution $e^{\rho(x)} u_1(x)$ can be extended continuously and linearly on $\mathcal{S}$.
Let $\varphi(x)$ be an arbitrary test function in $\mathcal{D}(R)$.

$$\begin{aligned} &{}_{\mathcal{D}'}\langle e^{\rho(x)}u_1(x),\ \varphi(x)\rangle_{\mathcal{D}} \\ &= {}_{\mathcal{D}'}\langle u_1(x),\ e^{\rho(x)}\varphi(x)\rangle_{\mathcal{D}} \\ &= \frac{1}{2\pi}\int F_1(\xi)d\xi\int e^{-i\xi x}e^{\rho(-x)}\varphi(-x)dx \end{aligned}$$

$$= \frac{1}{2\pi}\int \xi^{N+2} G(\xi) d\xi \int e^{-i\xi x} e^{\rho(-x)} \varphi(-x) dx$$

since we have $\xi^{N+2} e^{-i\xi x} = (-D_x)^{N+2} e^{-i\xi x}$ and $D_x^{N+2} e^{\rho(-x)} \varphi(-x) = \sum_{k=0}^{N+2} P_{N+2,k}$ $(\rho', \cdots, \rho^{(N+2)}) e^{\rho(-x)} D_x^k \varphi(-x)$ the above equation is equal to

$$\frac{1}{2\pi}\int G(\xi) d\xi \int \sum_{k=9}^{N+2} P_{N+2,k} e^{-i\xi x} e^{\rho(-x)} D_x^k \varphi(-x) dx .$$

where $P_{N+2,k}(r_1, \cdots, r_{N+2})$ are polynomials of $(r_1, \cdots, r_{N+2})$. The functions $P_{N+2,k} e^{\rho(-x)} D_x^k \varphi(-x)$ are C^∞-functions with compact support, and for $G(\zeta)$ we have the estimate (4,10). Hence, the integrand in the previous integral by $dx\, d\xi$ is absolutely summable and we can exchange the order of integral. Then the above integral equals

$$\frac{1}{2\pi}\int \sum_{k=0}^{N+2} P_{N+2,k} D_x^k \varphi(-x) dx \int G(\xi) e^{-i\xi x + \rho(-x)} d\xi .$$

Now the function $G(\zeta)$ is entire and satisfies the estimate (4,10), we can change the path of integral in $d\xi$ from the real line to an arbitrary line, parallel to the real axis. So, we take the path on the line $\{\xi + i\eta(-x);\ -\infty < \xi < \infty\}$, where $\eta(x)$ is the function in Lemma 1.1. Then the previous integral coincides with the next integral

$$\frac{1}{2\pi}\sum_k \int_{-\infty}^{\infty} P_{N+1,k} D_x^k \varphi(-x) \int_{-\infty}^{\infty} G(\xi + i\eta(-x)) e^{x\eta(-x)+\rho(-x)-i\xi x} d\xi\, dx .$$

By the definition of $\eta(-x)$, we have $x\eta(-x) + \rho(-x) = -\rho^*(\eta(-x))$. So, using the estimate (4,10), we have

$$\begin{aligned} &|G(\xi + i\eta(-x))\, e^{x\eta(-x)+\rho(-x)-i\xi x}| \\ &= |G(\xi + i\eta(-x))\, e^{-\rho^*(\eta(-x))}| \leqq C(1+|\xi|)^{-2} . \end{aligned}$$

Therefore, we have

$$\begin{aligned} &|\,{}_{\mathscr{D}'}\langle e^{\rho(x)} u_1(x), \varphi(x)\rangle_{\mathscr{D}}| \\ &\leqq \sum_k \int_{-\infty}^{\infty} |P_{N+1,k}(\rho', \cdots, \rho^{(N+1)}) D_x^k \varphi(-x)|\, dx \int_{-\infty}^{\infty} C(1+|\xi|)^{-2} d\xi . \end{aligned}$$

Since $P_{N+1,k}$ is the polynomial and $\rho^{(l)}(x)$ is estimated by a polynomial, we have an integer N_0 and a constant C_0 such that

$$|P_{N+1,k}(\rho', \cdots, \rho^{(N+1)})| \leqq C_0(1+x^2)^{N_0} \qquad \text{for all } k.$$

Finally we have

$$|{}_{\mathcal{D}'}\langle e^{\rho(x)}u_1(x),\ \varphi(x)\rangle_{\mathcal{D}}| \leqq C \sup_{\substack{-\infty<x<\infty \\ 0\leqq k\leqq N+2}} |(1+x^2)^{N_0+2} D_x^k \varphi(x)|$$

where C is a constant independent of φ. So, $e^{\rho(x)}u_1(x)$ belongs to $\mathcal{S}'$, i.e. u_1 is in $\mathcal{S}_\rho'$. As we have proved before that u_0 belongs to $\mathcal{S}_\rho'$, $u(x)=u_0(x)+u_1(x)$ belongs also to $\mathcal{S}_\rho'$. And the Fourier-Laplace transform of $u(x)$ is equal to $F_0(\zeta)+F_1(\zeta)=F(\zeta)$.
This proves Theorem 4.2.

Proof of Theorem 4.3. To prove this theorem, it is enough to show that $\sup\{(1+x^2)^k e^{-\rho(x)+\eta x-\rho^*(\eta)};\ -\infty<x<\infty\}$ can be estimated by a polynomial in η.

First we shall prove this for $\eta>0$.
We denote by $q_k(x, \eta)$ the function $(1+x^2)^k e^{-\rho(x)+\eta x-\rho^*(\eta)}$.
When $x<0$, we have $0<q_k(x, \eta)<(1+x^2)^k e^{-\rho(x)}\leqq M$ for some constant M. And when $0\leqq x\leqq 1$, we have $0<q_k(x, \eta)\leqq(1+x^2)^k\leqq 2^k$. Therefore, we must prove the estimate only when $x>1$.
By iii) we have a constant c_0 satisfying that $\rho(x)>c_0x^p$ for all $x>1$. From this, we have

$$-\rho(x)+\eta x-\rho^*(\eta)\leqq -x \text{ if } x\geqq((\eta+1)/c_0)^{1/(p-1)}=L_\eta\,.$$

Thus,

$$\begin{aligned}
&\sup_{x>1} q_k(x, \eta)\\
&=\operatorname{Max}\{\sup_{1<x<L_\eta} q_k(x, \eta),\ \sup_{L_\eta\leqq x} q_k(x, \eta)\}\\
&\leqq \operatorname{Max}\{\sup_{1<x<L_\eta}(1+x^2)^k,\ \sup_{L_\eta\leqq x}(1+x)^k e^{-x}\}\\
&\leqq C_k(1+\eta)^{2k/(p-1)}\,.
\end{aligned}$$

So, we have a constant C_k by which

$$\sup|q_k(x, \eta)|\leqq C_k(1+\eta)^{2k/(p-1)} \qquad \text{when } \eta>0.$$

For $\eta<0$, it goes very similarly to the case $\eta>0$. Putting this into (4,8), we have

$$\begin{aligned}
|F_u(\xi+i\eta)| &\leqq \sum_l \int (1+x^2)^{k+n_{p,l}} \sup|f(x)|\,|\zeta|^l e^{-\rho(x)+\eta x} c_l\,dx\\
&=\sum_l \int c_l \sup|f|\,|\zeta|^l q_{k+n_{p,l}+1}(x, \eta)(1+x^2)^{-1} e^{\rho^*(\eta)}dx\\
&\leqq \sum_{l=0}^{p} c_l C_{k+n_{p,l}+1} \sup|f|(1+|\zeta|^2)^{k/(p-1)} e^{\rho^*(\eta)}.
\end{aligned}$$

Proof of Theorem 4.5 is very simple. For, if $u(x)$ is in $\mathcal{S}_\rho$, we have

$$F_u(\xi+i\eta)=\int e^{\rho(x)}u(x)e^{-\rho(x)-i(\xi+i\eta)x}dx\,.$$

Thus, we have, for any nonnegative integers N and m

$$(\xi+i\eta)^N \partial_\zeta^m F_u(\xi+i\eta) = \int D_x^N(e^{\rho(x)}u(x)e^{-\rho(x)})(-ix)^m e^{-i(\xi+i\eta)x}dx.$$

Using the Leibniz formula, we have

$$D_x^N((e^{\rho(x)}u(x))e^{-\rho(x)}) = \sum_{l=0}^{N} \bar{P}_{N,l}(x)D_x^l(e^{\rho(x)}u(x))e^{-\rho(x)},$$

where $\bar{P}_{N,l}$ is a polynomial of $(\rho'(x), \cdots, \rho^{(N)}(x))$. By iii), $\bar{P}_{N,l}(x)$ can be estimated by a polynomial of x. Hence, from the fact that $(1+x^2)^k D_x^l(e^{\rho(x)}u(x))$ is absolutely summable for any k and l, and the fact that $-\rho(x)+\eta x \leqq \rho^*(\eta)$, we get the estimate (4,4). Conversely if $F(\xi+i\eta)$ is an entire function satisfying the estimate (4,4), we can define a function $u(x)$ by

$$u(x) = \int e^{i(\xi+i\eta)x}F(\xi+i\eta)d\xi$$

where we can take real η arbitrarily.

Now, we must show that this function u belongs to $\mathcal{S}_\rho$. But this is true, because we have

$$\begin{aligned}(1+x^2)^k D_x^N(e^{\rho(x)}u(x)) &= (1+x^2)^k \int D_x^N\{e^{i\xi x-\eta x}e^{\rho(x)}\}F(\xi+i\eta)d\xi \\ &= (1+x^2)^k \int \sum_l P_{N,l}\zeta^{N-l}e^{i\xi x-\eta x+\rho(x)}F(\xi+i\eta)d\xi\end{aligned}$$

where $P_{N,l}$ is a polynomial of $(\rho', \cdots, \rho^{(N)})$ that appeared in (4,7). By the assumption iii), we can take an integer N_1 and a constant M_1, with which we have

$$|P_{N,l}(x)| \leqq M_1(1+x^2)^{N_1} \qquad \text{for all } x \text{ and } l=1, 2, \cdots, N,$$

Further, using the fact that $(1+x^2)e^{i\zeta x}=(1-\partial_\zeta^2)e^{i\zeta x}$, and integrating by parts, we get that the previous integral equals

$$\begin{aligned}&\sum_l \frac{P_{N,l}(x)}{(1+x^2)^{N_1}} \int e^{i\zeta x+\rho(x)}(1-\partial_\zeta^2)^{k+N_1}(\zeta^{N-l}F(\zeta))d\xi \\ &= \sum_l \sum_m \frac{P_{N,l}(x)}{(1+x^2)^{N_1}} \int e^{i(\xi+i\eta)x+\rho(x)}P_m(\zeta)(\partial_\zeta^m F)(\xi+i\eta)d\xi.\end{aligned}$$

After putting $\eta=\eta(x)$, which is the function defined in Lemma 1.1, we use the estimate (4,3). Because of the fact that $\rho(x)+\rho^*(\eta(x))+\eta(x)x=0$, we have, for any integers k and N,

$$|(1+x^2)^k D_x^N(e^{\rho(x)}u(x))| \leqq \sum_{m,l} \frac{|P_{N,l}(x)|}{(1+x^2)^{N_1}} \int C_{N+2,m}(1+|\xi|)^{-2}d\xi.$$

This shows that $u(x)$ belongs to $\mathcal{S}_\rho$.

5. The case of several variables

In the above sections, we restrected our consideration to the case of one variable. In this section, we treat the several variable case.

Generic points in the Euclidean space R^n and in the complex dual space $\Xi^n+i\Xi^n$ will be denoted by $x=(x_1, x_2, \cdots, x_n)$ and $\zeta=(\zeta_1, \zeta_2, \cdots, \zeta_n)$ respectively. And the form of duality $x_1\zeta_1+\cdots+x_n\zeta_n$ will be denoted by $x\zeta$. If $\rho_1(x_1), \cdots, \rho_n(x_n)$ are functions satisfying i)$\sim$iv) in §2, we denote by $\rho(x)$ the function $\rho_1(x_1)+\cdots+\rho_n(x_n)$. And we also denote by $\rho^*(\eta)$ the function $\rho_1^*(\eta_1)+\cdots+\rho_n^*(\eta_n)$. By Γ we denote an open convex set in R^n.

Under these notations, results in the n-variable case are all the same as in the one variable case. Proofs of Theorem 4.1, 4.3, 4.4 and 4.5 hold without any alteration in the n-variable case. But, as to Theorem 4.2, we must prepare a few lemmas related to the technique which was used in the proof of 1-variable case. We shall prove in the case of $n=2$.

Lemma 5.1. *If $F(\zeta_1, \zeta_2)$ is an entire function of 2 variables which has the next estimate*

$$|F(\zeta_1, \zeta_2)|\leqq C(1+|\zeta_1|)^{N_1}(1+|\zeta_2|)^{N_2}e^{\rho^*(\eta)}, \tag{5, 1}$$

then, for non-negative integer m, we have

$$|\partial_{\zeta_1}^m F(0, \zeta_2)|\leqq C_m(1+|\zeta_2|)^{N_2}e^{\rho_2^*(\eta_2)}. \tag{5, 2}$$

Lemma 5.2. *If an entire function $G(\zeta_1, \zeta_2)$ of 2 variables satisfies the following estimate*

$$|\zeta_1^N G(\zeta_1, \zeta_2)|\leqq C(1+|\zeta_1|)^{N_1}|(1+|\zeta_2|)^{N_2}e^{\rho^*(\eta)}. \tag{5, 3}$$

then we have

$$|G(\zeta_1, \zeta_2)|\leqq C(1+|\zeta_1|)^{N_1-N}(1+|\zeta_2|)^{N_2}e^{\rho^*(\eta)}. \tag{5, 4}$$

By using these lemmas, we can prove Theorem 4.2 in the case of 2 variables. Let $F(\zeta)=F(\xi_1+i\eta_1, \xi_2+i\eta_2)$ be an entire function of 2 variables (ζ_1, ζ_2), and satisfy the next estimate.

$$|F(\zeta_1, \zeta_2)|\leqq C(1+|\zeta_1|)^{N_1}(1+|\zeta_2|)^{N_2}e^{\rho^*(\eta)}. \tag{5, 5}$$

As in Lemma 4.6, we can find a function $F_0(\zeta_1, \zeta_2)=\sum_{k=0}^{N_1+1} e^{i\alpha_k\zeta_1}f_k(\zeta_2)$, so that it satisfies the following;

$$F_0(0, \zeta_2) = F(0, \zeta_2), \cdots, \partial_{\zeta_1}^{N_1+1}F_0(0, \zeta_2) = \partial_{\zeta_1}^{N_1+1}F(0, \zeta_2).$$

By Lemma 5.1, $\partial_{\zeta_1}^{\nu}F(0, \zeta_2)$ satisfies (5,2) for all $\nu=0, \cdots, N_1+1$. Since $f_k(\zeta_2)$ is a linear combination of $F(0, \zeta_2), \cdots, \partial_{\zeta_1}^{N_1+1}F(0, \zeta_2)$, it also satisfies (5,2). And, from the consequences for 1 variable case, $f_k(\zeta_2)$ is the Fourier-Laplace transform of a distribution $u_k(x_2)$ in $\mathcal{S}'_{\rho_2}$. Thus, if we set $u_0(x_1, x_2)=\sum_{k=1}^{N_1+1}\delta(x_j-\alpha_k)u_k(x_2)$, we have that $u_0(x_1, x_2)$ is in $\mathcal{S}_\rho'$ and its Fourier-Laplace transform coincides with $F_0(\zeta_1, \zeta_2)$. We shall treat $F_1(\zeta_1, \zeta_2)=F(\zeta_1, \zeta_2)-F_0(\zeta_1, \zeta_2)$. Now, we have $\partial^\nu_{\zeta_1}F(0, \zeta_2)=0$, for $\nu=0, 1, \cdots, N_1+1$, hence we conclude that $G(\zeta_1, \zeta_2)=F_1(\zeta_1, \zeta_2)/\zeta_1^{N_1+2}$ is entire. From the form of $F_0(\zeta)$, we have $|F_0(\zeta)|\leqq C(1+|\zeta_2|)^{N_2}e^{a|\zeta_1|}e^{\rho_2*(\eta_2)}$. So, the function satisfies the estimate (5,5), where C might be changed, but N is not. By using Lemma 5.2, we have $|G(\zeta_1, \zeta_2)|\leqq C'(1+|\zeta_1|)^{-2}(1+|\zeta_2|)^{N_2}e^{\rho*(\eta)}$. Again, we shall carry out the same consideration about $F_1(\zeta)$ as we have done about $F(\zeta)$, and we decompose the function $F(\zeta_1, \zeta_2)$ into $F_0(\zeta_1, \zeta_2)+\zeta_1^{N_1+2}G_0(\zeta_1, \zeta_2)+\zeta_1^{N_1+2}\zeta_2^{N_2+2}G_1(\zeta_1, \zeta_2)$.

Here, $G_0(\zeta_1, \zeta_2)$ has the form $\sum_{k=0}^{N_1+1}e^{i\beta_k\zeta_2}g_k(\zeta_1)$, and $G_1(\zeta_1, \zeta_2)$ is an entire function which satisfies

$$|G_1(\zeta_1, \zeta_2)|\leqq D(1+|\zeta_1|)^{-2}(1+|\zeta_2|)^{-2}e^{-\rho_1*(\eta_1)}e^{\rho_2*(\eta_2)}.$$

Using integration by parts in the same way as in the proof for 1 variable case, we can prove that $\zeta_1^{N_1+2}\zeta_2^{N_2+2}G_1(\zeta_1, \zeta_2)$ is the Fourier-Laplace transform of a distribution $u_2(x_1, x_2)$ in $\mathcal{S}_\rho'$. On the other hand, the second term $\zeta_1^{N_1+2}G_0(\zeta_1, \zeta_2)$ is the Fourier-Laplace transform of a distribution $u_1(x_1, x_2)=\sum_{k=0}^{N_2+1}\delta(x_2-\beta_k)v_k(x_1)$, where $v_k(x_1)$ is in $\mathcal{S}'_{\rho_1}$. This proves that $F(\zeta)$ is the Fourier-Laplace transform of a distribution $u(x)=u_0(x)+u_1(x)+u_2(x)$, which is in $\mathcal{S}_\rho'$.

Proof of Lemma 5,1. From Cauchy's integral formula, $\partial^m_{\zeta_1}F(0, \zeta_2)$ is represented by the following form

$$\partial^m_{\zeta_1}F(0, \zeta_2)=\frac{m!}{2\pi i}\int_{|\zeta_1|=1}\frac{F(\zeta_1, \zeta_2)}{\zeta_1^{m+1}}d\zeta_1.$$

Thus, (5,1) leads (5,2).

Proof of Lemma 5.2. When $|\zeta_1|\geqq 1$, (5,4) is an obvious consequence of (5,3). Else, if $|\zeta_1|<1$, we have

$$G(\zeta_1, \zeta_2)=\frac{1}{2\pi i}\int_{|z_1|=2}\frac{z_1G(z_1, \zeta_2)}{(z_1-\zeta_1)z_1}dz_1.$$

From this we have

$$|G(\zeta_1, \zeta_2)|\leqq C(1+|\zeta_2|)^{N_2}e^{\rho_2*(\eta_2)} \qquad \text{for } |\zeta_1|<1.$$

Therefore, we have (5,4).

6. Examples and a counter example

In this paragraph, we give some examples of $\rho(x)$ satisfying the conditions i)~iv) and the counter example announced in §4.

EXAMPLE 1. $\rho(x)=\frac{1}{p}|x|^p$ for $x>1$ where $p>1$. We can extend this function suitably on the whole line R so that it satisfies the conditions i)~iv). In this case $\rho(x)$ also satisfies the condition v) in Theorem 4.3. The dual function $\rho^*(\eta)$ is $\frac{1}{p^*}|\eta|^{p^*}$ for $|\eta|>1$, where $\frac{1}{p^*}+\frac{1}{p}=1$.

EXAMPLE 2. $\rho(x)=|x|\log|x|$ for $|x|>e$. Extended suitably on R, this function satisfies i)~iv), and the dual function is $e^{|\eta|-1}$ for $|\eta|>2$.

Now, we give another example of $\rho(x)$. This will be available in considering a convolution equation

$$\int_{-\infty}^{\infty}\frac{e^{\lambda-\mu}}{e^{e^{\lambda-\mu}}-1}\Phi(\mu)d\mu=\Psi(\lambda)$$

which concerns the Bose-Einstein equation in statistical mechanics

$$\int_0^{\infty}\frac{1}{e^{t/s}-1}\varphi(s)ds=\psi(t) \qquad t>0\,.$$

EXAMPLE 3. We set

$$m(x)=\int_{-\infty}^{\infty}\frac{e^{\eta}}{e^{e^{\eta}}-1}e^{x\eta}\,d\eta \qquad \text{for } x<0\,.$$

Then we have

Lemma 6.1. 1) *When $x>0$, $m(x)$ is a strictly positive C^{∞}-function.*
2) *$m(x)/\Gamma(x+1)$ goes to 1 as x tends to ∞, where $\Gamma(x)$ is the gamma function.*
3) *for any integer $k\geqq 1$, $m^{(k)}(x)/\Gamma(x+2)$ remains bounded when x tends to ∞.*

From this lemma $\rho(x)=\log m(x)$ satisfies the conditions i)~iv) for $x>1$.

Lemma 6.2. 1) *$\rho(x)$ is a C^{∞}-function on $x>1$.*
2) *$\rho(x)$ is strictly concave on $x>1$.*
3) *$\rho(x)/x$ goes to ∞ when x tends to ∞.*
4) *for any integer k, $\rho^{(k)}(x)$ is estimated by a polynomial of x when $x>1$.*

Proof of Lemma 6.1. 1) is ovbious.

2)
$$\begin{aligned} m(x) &= \int_{-\infty}^{\infty}\frac{e^{\eta}}{e^{e^{\eta}}-1}e^{x\eta}\,d\eta \\ &= \int_{-\infty}^{0}\frac{e^{\eta}}{e^{e^{\eta}}-1}e^{x\eta}\,d\eta+\int_0^{\infty}\frac{e^{\eta}}{e^{e^{\eta}}-1}e^{x\eta}\,d\eta \\ &= I_1+I_2 \end{aligned}$$

The first term I_1 is bounded for $x>1$. On the other hand, as to the second term I_2, we have

$$\begin{aligned} I_2(x) &= \int_0^\infty \frac{e^\eta}{e^{e^\eta}-1} e^{x\eta}\, d\eta \\ &= \int_0^\infty \frac{e^{e^\eta}}{e^{e^\eta}-1} \frac{e^\eta}{e^{e^\eta}} e^{x\eta}\, d\eta \\ &= \int_{-\infty}^\infty \frac{e^\eta}{e^{e^\eta}} e^{x\eta}\, d\eta - \int_{-\infty}^0 \frac{e^\eta}{e^{e^\eta}} e^{x\eta}\, d\eta + \int_0^\infty \frac{1}{e^{e^\eta}-1} \frac{e^\eta}{e^{e^\eta}} e^{x\eta}\, d\eta \\ &= I_{21}+I_{22}+I_{23}\,. \end{aligned}$$

Here, $I_{21}=\Gamma(x+1)$, I_{22} is bounded and I_{23}/I_{21} goes to 0 when x tends to ∞. Thus, we have 2).

3) For any integer k, we have

$$\begin{aligned} m^{(k)}(x) &= \int_{-\infty}^\infty \frac{\eta^k e^\eta}{e^{e^\eta}-1} e^{x\eta}\, d\eta \\ &= \int_0^\infty \frac{\eta^k e^\eta}{e^{e^\eta}-1} e^{x\eta}\, d\eta + \int_{-\infty}^0 \frac{\eta^k e^\eta}{e^{e^\eta}-1} e^{x\eta}\, d\eta\,. \end{aligned}$$

As to the first term, we have

$$0<\int_0^\infty \frac{\eta^k e^\eta}{e^{e^\eta}-1} e^{x\eta}\, d\eta < C_k \int_{-\infty}^\infty \frac{e^\eta}{e^{e^\eta}} e^{(x+1)\eta}\, d\eta\,.$$

And the second term is a bounded function of x. This proves 3).

Proof of Lemma 6.2. 1) is obvious.

2) For any a; $0<a<1$, and any x_0, $x_1>1$, the Holder's inequality shows

$$\begin{aligned} &\int_{-\infty}^\infty e^{(1-a)x_0\eta} e^{ax_1\eta} \frac{e^\eta}{e^{e^\eta}-1}\, d\eta \\ &\leqq \left\{\int_{-\infty}^\infty e^{x_0\eta} \frac{e^\eta}{e^{e^\eta}-1}\, d\eta\right\}^{1-a} \left\{\int_{-\infty}^\infty e^{x_1\eta} \frac{e^\eta}{e^{e^\eta}-1}\, d\eta\right\}^{a}. \end{aligned}$$

Hence, we have $m((1-a)x_0+ax_1)\leqq m(x_0)^{1-a}m(x_1)^a$, where the equality is realized only when $x_0=x_1$. This shows 2).

3) Since the support of the integrand in the definition of $m(x)$ is not bounded above, $m(x)$ cannot be estimated by a exponential function e^{Mx}, however large M may be. This shows that the function $\rho(x)/x$ is not bounded above. From 2) of this Lemma, this function $\rho(x)/x$ increases monotonically for sufficiently large x. Thus, $\rho(x)/x$ goes to ∞ as x tends to ∞.

4) By the successive calculation we have

$$\rho^{(k)}(x)=\frac{P_k(m, m', \cdots, m^{(k)})}{(m(x))^k}$$

where $P(m, \cdots, m^{(k)})$ is a polynomial of $(m, \cdots, m^{(k)})$ of degree k. So, by using the facts 2) and 3) of the previous lemma, we get 4).
Now, we give the counter example announced in §4.

COUNTER-EXAMPLE. Take $\rho(x)=x \log x$ for $x>e$, and extended it to $x\leqq e$ suitably, so that it satisfies the condition i)~iv). Then we have $\rho^*(\eta)=e^{\eta-1}$ for $\eta>2$. Now, the function $u(x)=e^{-\rho(x)}$ is in $\mathcal{S}_\rho'$, and

$$F_u(i\eta)=\int_{-\infty}^{\infty} e^{-\rho(x)+\eta x}\,dx\,.$$

Since the integrand is positive, we have

$$F_u(i\eta)>\int_e^{\infty} e^{-x\log x+\eta x}\,dx\,.$$

Changing the variable from x to $e^{1-\eta}y$, we have

$$\int_e^{\infty} e^{-x\log x+\eta x}\,dx=\int_{e^{2-\eta}}^{\infty} e^{-e^{\eta-1}\{y\log y-y+1\}}\,e^{\eta-1}\,dy\,e^{e^{\eta-1}}\,.$$

By the fact that $y\log y-y+1<(y-1)^2$ for $1/2<y<3/2$ we have

$$\begin{aligned}F_u(i\eta)&>\int_{1/2}^{3/2} e^{\eta-1}e^{-e^{\eta-1}(y-1)^2}\,dy\,e^{e^{\eta-1}}\\&=\int_{|z|<(1/2)e^{(\eta-1)/2}} e^{-z^2}\,dz\,e^{(\eta-1)/2}\,e^{e^{\eta-1}}\,.\end{aligned}$$

So, it is impossible to get an estimate of the following type:

$$|F_u(i\eta)|\leqq C(1+|\eta|)^N e^{\rho^*(\eta)}\qquad \text{for } \eta>3.$$

OSAKA UNIVERSITY

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