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ON THE ASYMPTOTIC DISTRIBUTION OF EIGENVALUES OF OPERATORS ASSOCIATED WITH STRONGLY ELLIPTIC SESQUILINEAR FORMS

Dedicated to Professor Kosaku Yosida on his sixtieth birthday

KENJI MARUO AND HIROKI TANABE

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1. Introduction

Let Ω be a domain in R^n with generic point $x=(x_1, \cdots, x_n)$. We denote by $\alpha=(\alpha_1, \cdots, \alpha_n)$ a multi-index of length $|\alpha|=\alpha_1+\cdots+\alpha_n$ and use the notations

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}, D_k = -\sqrt{-1}\partial/\partial x_k .$$

For an integer $m \geqq 0$ $H_m(\Omega)$ is to be the set of all functions whose distribution derivatives of order up to m belong to $L^2(\Omega)$ and we introduce in it the usual norm

$$||u||_m = ||u||_{m,\Omega} = \Big(\int_\Omega \sum_{|\alpha| \leqq m} |D^\alpha u|^2 dx\Big)^{1/2} .$$

When $m=0$ we simply write $|| \;\; ||$ instead of $|| \;\; ||_0$, which is the norm of $L^2(\Omega)$. $\overset{\circ}{H}_m(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $H_m(\Omega)$.

Let B be a symmetric integro-differential sesquilinear form of order m with bounded coefficients

$$B[u, v] = \int_\Omega \sum_{|\alpha|,|\beta| \leqq m} a_{\alpha\beta}(x) D^\alpha u \overline{D^\beta v} dx$$

satisfying

$$B[u, u] \geqq \delta ||u||_m^2 \quad \text{for any } u \in V \qquad \text{a-(1)}$$

where δ is some positive constant and V is some closed subspace of $H_m(\Omega)$ containing $\overset{\circ}{H}_m(\Omega)$. Let A be the operator associated with this sesquilinear from: an element u of V belongs to $D(A)$ and $Au=f \in L^2(\Omega)$ if $B[u, v]=(f, v)$ is valid for any $v \in V$. As is well known A is a positive definite self-adjoint operator in $L^2(\Omega)$. In this paper assuming that Ω is a bounded domain possessing the restricted cone property (p. 11 of [1]) we investigate the asymptotic distribution of eigenvalues of the operator A, and under various smoothness assumptions on the coefficients of B deduce formulas mainly with remainder estimates which are similar to those obtained by S. Agmon [2] (see also R. Beals [5]).

In our result it is assumed that $2m>n$; however, we do not require the following inclusion relation which was assumed and essentially used by many authours[1)]:

$$D(A)\subset H_{2m}(\Omega)\,. \tag{1.1}$$

An example of an operator which does not satisfy (1.1) can be constructed with the aid of the following considerations. Let Ω be a bounded domain of R^2 with smooth boundary $\partial\Omega=\Gamma_1\cup\Gamma_2$, $\Gamma_1\cap\Gamma_2=\phi$, and denoting by (x,y) the generic point of R^2 let

$$B[u,v]=\int_\Omega\Big(\frac{\partial^2u}{\partial x^2}\overline{\frac{\partial^2v}{\partial x^2}}+2\frac{\partial^2u}{\partial x\partial y}\overline{\frac{\partial^2v}{\partial x\partial y}}+\frac{\partial^2u}{\partial y^2}\overline{\frac{\partial^2v}{\partial y^2}}\Big)dxdy$$

and V be the closure in $H_2(\Omega)$ of

$$\{u\in C^\infty(\Omega)\colon u=0 \text{ on } \Gamma_1 \text{ and } \partial u/\partial\nu=0 \text{ on } \Gamma_2\}$$

where ν is the outer normal to $\partial\Omega$. The function $u=\mathrm{Im}(x+iy)^{3/2}=r^{3/2}\sin(3\theta/2)$ satisfies $\Delta u=0$ and hence $\Delta^2u=0$ in the upper half plane $y>0$. For $x>0$, $y=0$ $u=\partial^2u/\partial y^2=0$ and for $x<0$, $y=0$ $\partial u/\partial y=\partial^3u/\partial y^3=0$. Near the origin $u\notin H_3$ although $u\in H_2$ there. Hence assuming that $\partial\Omega$ contains a part of the x-axis having the origin in its interior we can easily construct a function which belongs to $D(A)$ but not to $H_4(\Omega)$. We also note that (1.1) is not valid under boundary conditions of nicer type when the coefficients of B are not differentiable.

A great number of papers have been published on the eigenvalue distribution of elliptic operators, a survey of which is found in the introduction of [2]. In [3] S. Agmon devised an "indirect method" of estimating resolvent kernels of operators considered whereas these kernels were always estimated directly by many authors before that. This indirect method is remarkably effective in obtaining global estimates of the resolvent kernels without any complicated calculations and based upon these estimates numerous important results were derived (([1], [2], [3], [4], [5]). In this paper we follow this method; however, we need some modification since (1.1) is not necessarily satisfied as was mentioned above. To this end we extend the operator A to a mapping on V to V^* where V^* is the antidual of V(i.e. the space of continuous conjugate linear functionals on V). This extended operator which is again denoted by A is defined by

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Some related results without remainder estimates are obtained for degenerate operators. See [9] for example.

$$B[u, v] = (Au, v) \quad \text{for any } v \in V$$

where the bracket on the right stands for the duality between V^* and V in this case. Identifying $L^2(\Omega)$ with its antidual we may consider $V \subset L^2(\Omega) \subset V^*$ algebraically and topologically, and as is easily seen V is a dense subspace of V^* under this convention. The resolvent of A thus extended is a bounded linear operator on V^* to V, and in virtue of the assumption $2m > n$ we can estimate the kernels of this class of operators pointwisely by various kinds of their norms with the aid of Sobolev's inequality (Lemma 3.2 below).

In the proof of our main theorem the results of S. Agmon and Y. Kannai [4] as well as the method of S. Agmon [2] play an important role. In section 2 the main theorem is presented. In section 3 lemmas which will be used frequently in the subsequent sections are proved. In sections 3–7 the kernels of A are estimated in comparison with those of operators with smooth coefficients approximating the original ones and also with those of operators defined in a larger domain with smooth boundary. In section 8 the main theorem is proved.

It is obvious that our result remains valid when B has some boundary integrals containing derivatives of order $\leqq m-1$. It is also easy to verify that some part of the main theorem can be extended to non-symmetric cases.

2. Main theorem

For $x \in \Omega$ let $\delta(x) = \min\{1, \operatorname{dist}(x, \partial\Omega)\}$. Suppose that

$$\int_\Omega \delta(x)^{-p} dx < \infty \tag{2.1}$$

for some positive number $p < 1$ which will be specified later. Since all coefficients $a_{\alpha\beta}$ belong to $L^\infty(\Omega)$, there is a constant K such that for any $u, v \in H_m(\Omega)$

$$|B[u, v]| \leqq K \|u\|_m \|v\|_m . \qquad \text{a-(2)}$$

For an integer $k \geqq 0$ we denote by $C^k(\Omega)$ the set of all k times continuously differentiable functions in Ω. For an integer k and a positive number $h < 1$ we denote by $C^{k+h}(\Omega)$ the subclass of functions in $C^k(\Omega)$ whose derivatives of order k are Hoelder continuous of order h in Ω.

We consider the following various types of smoothness assumptions:

s-(1) For $|\alpha| = |\beta| = m$ $a_{\alpha\beta}$ is uniformly continous.

s-(2) For $|\alpha| = |\beta| = m$ $a_{\alpha\beta}$ is uniformly Hoelder continuous of order h.

s-(3) For $|\alpha| = |\beta| = m$ $a_{\alpha\beta}$ belongs to $C^{1+h}(\Omega_1)$, and for $|\alpha| + |\beta| = 2m-1$ $a_{\alpha\beta}$ is uniformly Hoelder continuous of order h. Here and in what follows Ω_1 is a domain containing $\bar{\Omega}$.

s-(4) For $|\alpha| = |\beta| = m$ $a_{\alpha\beta}$ belongs to $C^{2+h}(\Omega_1)$, for $|\alpha| + |\beta| = 2m-1$ $a_{\alpha\beta}$ belongs to $C^{1+h}(\Omega_1)$, and for $|\alpha| + |\beta| = 2m-2$ $a_{\alpha\beta}$ is uniformly Hoelder continuous of order h.

s-(5) For $|\alpha|=|\beta|=m$ $a_{\alpha\beta}$ is constant in Ω, and for $|\alpha|+|\beta|\leqq 2m-1$ $a_{\alpha\beta}$ belongs to $C^\infty(\Omega_1)$.

Throughout the paper we assume that Ω is a bounded domain having the restricted cone property (p. 11 of [1]) and that $2m>n$.

For $t>0$ let $N(t)$ be the number of eigenvalues of A which do not exceed t.

Main Theorem. *In the situation stated above the following asymptotic formulas for $N(t)$ hold as $t\to\infty$:*

$$N(t) = c_0 t^{n/2m}+o(t^{n/2m}) \qquad \textit{under } \text{s-(1)},$$
$$N(t) = c_0 t^{n/2m}+O(t^{(n-\theta)/2m})$$

for any number θ satisfying

$$0<\theta<h/(h+3) \qquad \textit{under } \text{s-(2)},$$
$$0<\theta<(h+1)/(h+4) \qquad \textit{under } \text{s-(3)},$$
$$0<\theta<(h+2)/(h+5) \qquad \textit{under } \text{s-(4)},$$
$$0<\theta<1 \qquad \textit{under } \text{s-(5)}$$

where

$$c_0 = \frac{\sin(n\pi/2m)}{n\pi/2m}\int_\Omega c_0(x)dx\,,$$
$$c_0(x) = (2\pi)^{-n}\int_{R^n}\{\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)\xi^{\alpha+\beta}+1\}^{-1}d\xi\,. \tag{2.2}$$

REMARK. If s-(4) is satisfied for $h=1$ the above formula coincides with the one of Agmon [2]. Under the assumption s-(2) the formula is the same as that of Beals [5].

3. Some lemmas

To begin with we shall prove four lemmas of which we make frequent use in the subsequent sections. Let λ be a complex number which is not on the positive real axis. According to Lax-Milgram theorem $A-\lambda$ has a bounded inverse defined in the whole of V^*. Let $d(\lambda)$ be the distance from the point λ to the positive real axis. For a bounded operator S on V^* to V we use the notations $||S||_{V^*\to V}$, $||S||_{V^*\to L^2}$, etc. to denote the norms of S considered as an operator on V^* to V, V^* to $L^2(\Omega)$., etc.

Lemma 3.1. *There exists a constant C such that*

$$\text{(i)} \qquad ||(A-\lambda)^{-1}||_{L^2\to L^2}\leqq 1/d(\lambda),$$

(ii) $\|(A-\lambda)^{-1}\|_{L^2\to V} \leqq C|\lambda|^{1/2}/d(\lambda)$,

(iii) $\|(A-\lambda)^{-1}\|_{V^*\to V} \leqq C|\lambda|/d(\lambda)$,

(iv) $\|(A-\lambda)^{-1}\|_{V^*\to L^2} \leqq C|\lambda|^{1/2}/d(\lambda)$.

Proof. The statement (i) is clear since A is a positive definite self-adjoint operator in $L^2(\Omega)$. If $u=(A-\lambda)^{-1}f$, $f\in L^2(\Omega)$, then by the definition of A

$$B[u, u] = (f, u)+\lambda(u, u). \tag{3.1}$$

In view of a-(1) and (i) we get

$$\begin{aligned}\delta\|u\|_m^2 &\leqq \|f\|\,\|u\| + |\lambda|\,\|u\|^2 \\ &\leqq \|f\|^2/d(\lambda) + |\lambda|(\|f\|/d(\lambda))^2 \\ &\leqq 2|\lambda|(\|f\|/d(\lambda))^2\end{aligned}$$

from which (ii) follows. Since $B[u, u]$ is real we get from (3.1)

$$\operatorname{Im}\lambda\|u\|^2 = -\operatorname{Im}(f, u),$$

whence it follows that

$$|\operatorname{Im}\lambda|\,\|u\|^2 \leqq \|f\|_{V^*}\|u\|_m \tag{3.2}$$

which implies

$$d(\lambda)\|u\|^2 \leqq \|f\|_{V^*}\|u\|_m$$

if $\operatorname{Re}\lambda\geqq 0$. From (3.1) we get also

$$0\leqq B[u, u] = \operatorname{Re}(f, u)+\operatorname{Re}\lambda\|u\|^2,$$

and hence if $\operatorname{Re}\lambda<0$

$$|\operatorname{Re}\lambda|\,\|u\|^2 = -\operatorname{Re}\lambda\|u\|^2 \leqq \operatorname{Re}(f, u) \leqq \|f\|_{V^*}\|u\|_m.$$

Combining this inequality with (3.2) we obtain

$$d(\lambda)\|u\|^2 \leqq \sqrt{2}\,\|f\|_{V^*}\|u\|_m. \tag{3.3}$$

It follows from (3.1) and (3.3) that

$$\begin{aligned}\delta\|u\|_m^2 &\leqq \|f\|_{V^*}\|u\|_m + |\lambda|\,\|u\|^2 \\ &\leqq \|f\|_{V^*}\|u\|_m + \sqrt{2}\,|\lambda|\,\|f\|_{V^*}\|u\|_m/d(\lambda) \\ &\leqq (1+\sqrt{2})|\lambda|\,\|f\|_{V^*}\|u\|_m/d(\lambda)\end{aligned}$$

from which (iii) follows immediately. Finally with the aid of (iii) and the following inequality

$$|\lambda|\,||u||^2 \leqq K||u||_m^2 + ||f||_{V^*}||u||_m$$

which is a simple consequence of (3.1) we can easily show (iv).

Lemma 3.2. *Let S be a bounded operator on V^* to V. Then S has a kernel M in the following sense:*

$$(Sf)(x) = \int_\Omega M(x, y) f(y) dy \quad \text{for} \quad f \in L^2(\Omega).$$

$M(x, y)$ is continuous in $\bar{\Omega}\times\bar{\Omega}$ and there exists a constant C such that for any $x, y \in \Omega$

$$|M(x, y)|$$

$$\leqq C||S||_{V^*\to V}^{n^2/4m^2}\,||S||_{V^*\to L^2}^{n/2m-n^2/4m^2}\,||S||_{L^2\to V}^{n/2m-n^2/4m^2}\,||S||_{L^2\to L^2}^{(1-n/2m)^2}.$$

Proof. That S has a kernel of the type stated in the lemma is a consequence of the general theory of Hilbert-Schmidt operators (see p. 211 of [1] where these operators are called operators of finite double norm). Applying Sobolev's inequality to $M(x, y)$ considered as a function of y we get

$$|M(x, y)| \leqq \gamma ||M(x, \cdot)||_m^{n/2m} ||M(x, \cdot)||^{1-n/2m} \tag{3.4}$$

Next applying the same inequality to Sf

$$|(Sf)(x)| \leqq \gamma ||Sf||_m^{n/2m} ||Sf||^{1-n/2m}$$

$$\leqq \gamma ||S||_{V^*\to V}^{n/2m} ||S||_{V^*\to L^2}^{1-n/2m} ||f||_{V^*}.$$

Hence noting that V is reflexive we find that $M(x, \cdot) \in V$ for any fixed $x \in \Omega$ and

$$||M(x, \cdot)||_m = ||M(x, \cdot)||_V \leqq \gamma ||S||_{V^*\to V}^{n/2m} ||S||_{V^*\to L^2}^{1-n/2m}. \tag{3.5}$$

In a similar manner we obtain

$$||M(x, \cdot)|| \leqq \gamma ||S||_{L^2\to V}^{n/2m} ||S||_{L^2\to L^2}^{1-n/2m}. \tag{3.6}$$

Combining (3.4), (3.5) and (3.6) we complete the proof of the lemma.

Lemma 3.3. *There exists a constant C such that for any integer $0 \leqq k \leqq m$*

$$||(A-\lambda)^{-1}f||_k \leqq C|\lambda|^{1/2+k/2m}/d(\lambda)||f||_{V^*} \quad \text{for } f \in V^*, \tag{3.7}$$

$$||(A-\lambda)^{-1}f||_k \leqq C|\lambda|^{k/2m}/d(\lambda)||f|| \quad \text{for } f \in L^2(\Omega). \tag{3.8}$$

Proof. Under our assumption on Ω the following interpolation inequality is true:

$$||u||_k \leqq c||u||_m^{k/m}||u||^{1-k/m}. \tag{3.9}$$

Applying this inequality to $(A-\lambda)^{-1}f$ and then using Lemma 3.1 we easily obtain (3.7) and (3.8).

Lemma 3.4. *There is a constant C such that for any integer $0 \leqq k \leqq m$*

$$||v||_k \leqq C|\lambda|^{-(m-k)/2m}(||v||_m + |\lambda|^{1/2}||v||) \qquad \textit{for } v \in V.$$

Proof. In view of the interpolation inequality (3.9) and Young's inequality we conclude

$$\begin{aligned} ||v||_k &\leqq c(|\lambda|^{-(m-k)/2m}||v||_m)^{k/m}(|\lambda|^{k/2m}||v||)^{1-k/m} \\ &\leqq c|\lambda|^{-(m-k)/2m}(||v||_m + |\lambda|^{1/2}||v||). \end{aligned}$$

4. Estimates of resolvent kernels - 1

In this section we shall estimate the difference between the resolvent kernels of A and those of the operator A_0 associated with B under the Dirichlet boundary conditions. By definition for any $u, v \in \mathring{H}_m(\Omega)$ we have

$$B[u, v] = (A_0 u, v)$$

where the bracket on the right denotes the pairing between the antidual $H_{-m}(\Omega)$ of $\mathring{H}_m(\Omega)$ and $\mathring{H}_m(\Omega)$ this case. Under our convention of identifying $L^2(\Omega)$ with its antidual there is no fear of confusion if we use the notation (,) standing for the L^2-inner product to denote also the pairing between V^* and V as well as that between $H_{-m}(\Omega)$ and $\mathring{H}_m(\Omega)$. Obviously for the operator A_0 the analogues of Lemmas 3.1–3.4 hold. We denote by Λ a class of functions in $C_0^\infty(R^n)$ the supports of which are contained in the set $\{x \in R^n : |x| < 1\}$ and which take the value 1 at the origin. We fix a point $x_0 \in \Omega$. For the sake of simplicity we put $\varepsilon = \delta(x_0)$ for the time being and $\xi_\varepsilon(x) = \xi((x-x_0)/\varepsilon)$ for $\xi \in \Lambda$. Let $S_{\lambda\varepsilon}$ be the operator defined by

$$S_{\lambda\varepsilon} f = \xi_\varepsilon\{(A-\lambda)^{-1}f - (A_0-\lambda)^{-1}(rf)\}$$

for any $f \in V^*$ where rf is the restriction of $f \in V^*$ to $\mathring{H}_m(\Omega)$. Obviously $S_{\lambda\varepsilon}$ is a bounded operator on V^* to $\mathring{H}_m(\Omega)$ and hence *a fortiori* to V.

Lemma 4.1. *If $\varepsilon^{-1}|\lambda|^{-1/2m} \leqq 1$, then for any positive integer j there is a constant K_j such that*

$$||S_{\lambda\varepsilon}||_{V^* \to V} \leqq K_j(\varepsilon^{-1}|\lambda|^{1-1/2m}d(\lambda)^{-1})^j |\lambda|/d(\lambda),$$

$$\left.\begin{array}{l}||S_{\lambda\varepsilon}||_{V^*\to L^2}\\ ||S_{\lambda\varepsilon}||_{L^2\to V}\end{array}\right\}\leqq K_j(\varepsilon^{-1}|\lambda|^{1-1/2m}d(\lambda)^{-1})^j|\lambda|^{1/2}/d(\lambda)\,,$$

$$||S_{\lambda\varepsilon}||_{L^2\to L^2}\leqq K_j(\varepsilon^{-1}|\lambda|^{1-1/2m}d(\lambda)^{-1})^j/d(\lambda)\,.$$

Proof. Let $u=(A-\lambda)^{-1}f-(A_0-\lambda)^{-1}(rf)$ and $v=\xi_\varepsilon u=S_{\lambda\varepsilon}f$. Now

$$\begin{aligned}&B[v,v]-\lambda(v,v)\\ &=B[v,v]-B[u,\xi_\varepsilon v]+B[u,\xi_\varepsilon v]-\lambda(u,\xi_\varepsilon v)\\ &=\{B[v,v]-B[u,\xi_\varepsilon v]\}+B[(A-\lambda)^{-1}f,\xi_\varepsilon v]-\lambda((A-\lambda)^{-1}f,\xi_\varepsilon v)\\ &-B[(A_0-\lambda)^{-1}(rf),\xi_\varepsilon v]+\lambda((A_0-\lambda)^{-1}(rf),\xi_\varepsilon v)\\ &=B[v,v]-B[u,\xi_\varepsilon v]\,.\end{aligned}$$

Noting that $B[u,u]\geqq 0$ and $d(\lambda)/|\lambda|=|\sin(\arg\lambda)|$ we get for some constant C

$$\begin{aligned}|B[v,v]-\lambda(v,v)|&\geqq\max\{B[v,v],\,|\lambda|(v,v)\}d(\lambda)/|\lambda|\\ &\geqq C(||v||_m+|\lambda|^{1/2}||v||)^2d(\lambda)/|\lambda|\,.\end{aligned}\tag{4.2}$$

Next from (4.1) it follows that

$$\begin{aligned}&|B[v,v]-\lambda(v,v)|=|B[v,v]-B[u,\xi_\varepsilon v]|\\ &=\Big|\int_\Omega\sum_{|\alpha|,|\beta|\leqq m}a_{\alpha\beta}(x)(D^\alpha(\xi_\varepsilon u)\overline{D^\beta v}-D^\alpha u\overline{D^\beta(\xi_\varepsilon v)})dx\Big|\\ &\leqq\Big|\int_\Omega\sum_{|\alpha|,|\alpha|\leqq m}a_{\alpha\beta}(x)\sum_{\alpha>\gamma}\binom{\alpha}{\gamma}D^{\alpha-\gamma}\xi_\varepsilon D^\gamma u\overline{D^\beta v}dx\Big|\\ &+\Big|\int_\Omega\sum_{|\alpha|,|\alpha|\leqq m}a_{\alpha\beta}(x)\sum_{\beta>\gamma}\binom{\alpha}{\gamma}D^\alpha u D^{\beta-\gamma}\xi_\varepsilon\overline{D^\gamma v}dx\Big|=I_1+I_2\,.\end{aligned}\tag{4.3}$$

We shall proceed inductively and at first consider the case $j=1$. Noting that $||rf||_{-m}\leqq||f||_{V^*}$ we get from Lemma 3, 3

$$||u||_k\leqq C|\lambda|^{1/2+k/2m}d(\lambda)^{-1}||f||_{V^*}\quad\text{for } f\in V^*,\tag{4.4}$$

$$||u||_k\leqq C|\lambda|^{k/2m}d(\lambda)^{-1}||f||\qquad\text{for } f\in L^2(\Omega)\tag{4.5}$$

if $0\leqq k\leqq m$. Clearly for some constant C independent of x and x_0 we have

$$|D^\gamma\xi_\varepsilon(x)|\leqq C\varepsilon^{-|\gamma|}\,.\tag{4.6}$$

From (4.4), (4.5) and (4.6) it follows that

$$\begin{aligned}I_1&\leqq C\sum_{k=0}^{m-1}\varepsilon^{k-m}||u||_k||v||_m\\ &\leqq C\sum_{k=0}^{m-1}\varepsilon^{k-m}|\lambda|^{(k-m)/2m}|\lambda|d(\lambda)^{-1}||f||_{V^*}||v||_m\end{aligned}$$

for any $f \in V^*$ and

$$I_1 \leqq C \sum_{k=0}^{m-1} \varepsilon^{k-m} |\lambda|^{(k-m)/2m} |\lambda|^{1/2} d(\lambda)^{-1} ||f|| \, ||v||_m$$

for any $f \in L^2(\Omega)$. Hence using the assumption $\varepsilon^{-1}|\lambda|^{-1/2m} \leqq 1$ we get

$$I_1 \leqq C|\lambda|^{1-1/2m} \varepsilon^{-1} d(\lambda)^{-1} ||f||_{V^*} ||v||_m \qquad \text{for } f \in V^*, \qquad (4.7)$$

$$I_1 \leqq C|\lambda|^{1-1/2m} \varepsilon^{-1} d(\lambda)^{-1} |\lambda|^{-1/2} ||f|| \, ||v||_m \quad \text{for } f \in L^2(\Omega). \quad (4.8)$$

Next in view of (4.6) and Lemma 3.1

$$I_2 \leqq C|\lambda| d(\lambda)^{-1} ||f||_{V^*} \sum_{k=0}^{m-1} \varepsilon^{k-m} ||v||_k \qquad \text{for } f \in V^*,$$

$$I_2 \leqq C|\lambda|^{1/2} d(\lambda)^{-1} ||f|| \sum_{k=0}^{m-1} \varepsilon^{k-m} ||v||_k \qquad \text{for } f \in L^2(\Omega).$$

Using here Lemma 3.4 and the assumption $\varepsilon^{-1}|\lambda|^{-1/2m} \leqq 1$ we easily obtain

$$I_2 \leqq C|\lambda|^{1-1/2m} \varepsilon^{-1} d(\lambda)^{-1} ||f||_{V^*} (||v||_m + |\lambda|^{1/2} ||v||), \qquad (4.9)$$

$$I_2 \leqq C|\lambda|^{1-1/2m} \varepsilon^{-1} d(\lambda)^{-1} |\lambda|^{1/2} ||f|| (||v||_m + |\lambda|^{1/2} ||v||) \qquad (4.10)$$

for any $f \in V^*$, $f \in L^2(\Omega)$ respectively. Combining (4.2), (4.3), (4.7) and (4.9) or combining (4.2), (4.3), (4.8) and (4.10) we get

$$||v||_m + |\lambda|^{1/2} ||v|| \leqq K_1 (|\lambda|^{1-1/2m} \varepsilon^{-1} d(\lambda)^{-1}) (|\lambda| / d(\lambda)) ||f||_{V^*}$$

or

$$||v||_m + |\lambda|^{1/2} ||v|| \leqq K_1 (|\lambda|^{1-1/2m} \varepsilon^{-1} d(\lambda)^{-1}) (|\lambda|^{1/2} / d(\lambda)) ||f||$$

according as $f \in V^*$ or $f \in L^2(\Omega)$ where K_1 is some constant independent of x, x_0 and λ. Recalling the definition of v we can easliy establish the desired estimates for $j=1$ with the aid of these two inequalities. Assume that the lemma has been proved for some k. We pick another function $\eta \in \Lambda$ such that $\eta(x)=1$ for any $x \in \operatorname{supp} \xi$ and write $\eta_\varepsilon(x) = \eta((x-x_0)/\varepsilon)$. Now, by the easily verified inequality

$$I_1 = \left| \int_\Omega \sum_{|\alpha|, |\beta| \leqq m} a_{\alpha\beta}(x) \sum_{\alpha > \gamma} \binom{\alpha}{\gamma} D^{\alpha-\gamma} \xi_\varepsilon D^\gamma (\eta_\varepsilon u) \overline{D^\beta v} dx \right|$$

$$\leqq C \sum_{k=0}^{m-1} \varepsilon^{k-m} ||\eta_\varepsilon u||_k ||v||_m$$

we get using Lemma 3.4 and the assumption $\varepsilon^{-1}|\lambda|^{-1/2m} \leqq 1$

$$I_1 \leqq C \varepsilon^{-1} |\lambda|^{-1/2m} (||\eta_\varepsilon u||_m + |\lambda|^{1/2} ||\eta_\varepsilon u||) ||v||_m. \qquad (4.11)$$

From the induction assumption with η in place of ξ it follows that

$$\begin{aligned}&\|\eta_\varepsilon u\|_m+|\lambda|^{1/2}\|\eta_\varepsilon u\|\\&\leqq\begin{cases}K_k(\varepsilon^{-1}|\lambda|^{1-1/2m}d(\lambda)^{-1})^k|\lambda|d(\lambda)^{-1}\|f\|_{V^*}, & (4.12)\\ K_k(\varepsilon^{-1}|\lambda|^{1-1/2m}d(\lambda)^{-1})^k|\lambda|^{1/2}d(\lambda)^{-1}\|f\|. & (4.13)\end{cases}\end{aligned}$$

Therefore, in virtue of (4.11), (4.12) and (4.13) we get

$$I_1\leqq C(\varepsilon^{-1}|\lambda|^{1-1/2m}d(\lambda)^{-1})^{k+1}\|f\|_{V^*}\|v\|_m, \tag{4.14}$$

$$I_1\leqq C(\varepsilon^{-1}|\lambda|^{1-1/2m}d(\lambda)^{-1})^{k+1}|\lambda|^{-1/2}\|f\|\,\|v\|_m. \tag{4.15}$$

On the other hand

$$\begin{aligned}I_2&=\left|\int_\Omega\sum_{|\alpha|,|\beta|\leqq m}a_{\alpha\beta}(x)\sum_{\beta>\gamma}\binom{\beta}{\gamma}D^\alpha(\eta_\varepsilon u)D^{\beta-\gamma}\xi_\varepsilon\overline{D^\gamma v}dx\right|\\&\leqq C\|\eta_\varepsilon u\|_m\sum_{k=0}^{m-1}\varepsilon^{k-m}\|v\|_k.\end{aligned} \tag{4.16}$$

Combining (4.12), (4.13) and (4.16), and again using Lemma 3.4 and the assumption $\varepsilon^{-1}|\lambda|^{-1/2m}\leqq 1$, we find that I_2 is dominated by the right member of (4.14) or (4.15) with $\|v\|_m+|\lambda|^{1/2}\|v\|$ in place of $\|v\|_m$ according as $f\in V^*$ or $f\in L^2(\Omega)$. Hence combining (4.2), (4.3) and the estimates for I_1 and I_2 just obtained we conclude there is a constant K_{k+1} such that

$$\begin{aligned}&\|v\|_m+|\lambda|^{1/2}\|v\|\\&\leqq K_{k+1}(\varepsilon^{-1}|\lambda|^{1-1/2m}d(\lambda)^{-1})^{k+1}|\lambda|d(\lambda)^{-1}\|f\|_{V^*}\end{aligned}$$

or

$$\leqq K_{k+1}(\varepsilon^{-1}|\lambda|^{1-1/2m}d(\lambda)^{-1})^{k+1}|\lambda|^{1/2}d(\lambda)^{-1}\|f\|$$

according as $f\in V^*$ or $f\in L^2(\Omega)$. Thus recalling the definition of v again we finish the proof of the present lemma.

Let $M_{\lambda\varepsilon}$, K_λ and K^0_λ be the kernels of the operators $S_{\lambda\varepsilon}$, $(A-\lambda)^{-1}$ and $(A_0-\lambda)^{-1}$ respectively. Then clearly we have the relation

$$M_{\lambda\varepsilon}(x,y)=\xi_\varepsilon(x)\{K_\lambda(x,y)-K^0_\lambda(x,y)\}. \tag{4.17}$$

Lemma 4.2. *For any $p>0$ the following inequality holds:*

$$|K_\lambda(x_0,x_0)-K^0_\lambda(x_0,x_0)|\leqq C_p\frac{|\lambda|^{n/2m}}{d(\lambda)}\left(\frac{|\lambda|^{1-1/2m}}{\delta(x_0)d(\lambda)}\right)^p,\quad |\lambda|\geqq 1, \tag{4.18}$$

where C_p is a constant depending on p but not on x_0 and λ.

Proof. First let us assume that p is an integer. If $\delta(x_0)|\lambda|^{1/2m}\geqq 1$, then in view of Lemmas 3.2 and 4.1 we know

$$|M_{\lambda\varepsilon}(x, y)| \leqq C_p \frac{|\lambda|^{n/2m}}{d(\lambda)} \left(\frac{|\lambda|^{1-1/2m}}{\delta(x_0)d(\lambda)}\right)^p \tag{4.19}$$

and therefore recalling (4.17) we obtain (4.18). In general for any λ we have

$$\begin{aligned} &|K_\lambda(x_0, x_0) - K^0_\lambda(x_0, x_0)| \\ &\quad \leqq |K_\lambda(x_0, x_0)| + |K^0_\lambda(x_0, x_0)| \leqq C|\lambda|^{n/2m} d(\lambda)^{-1} \end{aligned} \tag{4.20}$$

by Lemmas 3.1 and 3.2. So if $\delta(x_0)|\lambda|^{1/2m} \leqq 1$, (4.18) trivially follows from (4.20). Thus (4.18) is proved when p is an integer. That (4.18) also holds for non-integral vlaues of p follows by interpolation.

5. Approximation of coefficients by smooth functions

We shall approximate the coefficients $a_{\alpha\beta}$ by functions in $C^\infty(R^n)$ so that we may apply the results of S. Agmon [2].

Let $\tilde{\rho}$ be the real valued even function in $C_0(R^1)$ the support of which is contained in the set $\{x: |x| \leqq n^{-1/2}\}$. We write for $x = (x_1, \cdots, x_n) \in R^n$

$$\rho(x) = \tilde{\rho}(x_1) \cdots \tilde{\rho}(x_n),\ \rho_\varepsilon(x) = \varepsilon^{-n} \rho(x/\varepsilon)$$

and

$$\rho_\varepsilon * f(x) = \int_{R^n} \rho_\varepsilon(x-y) f(y) dy\,.$$

Here ε is an arbitrary positive number. Moreover we use the notation $\partial^\alpha_x = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$ for $\alpha = (\alpha_1, \cdots, \alpha_n)$. First we shall prove the following

Lemma 5.1. *Let $f \in C^2(\Omega)$, x_0 be a point of Ω, and δ be any positive number. We set*

$$f_0(x) = \begin{cases} \displaystyle\sum_{|\alpha| \leqq 2} \frac{1}{\alpha!} (x-x_0)^\alpha \partial^\alpha_x f(x_0), & \text{if } |x-x_0| \leqq \delta \\ \displaystyle\sum_{|\alpha| \leqq 2} \frac{1}{\alpha!} (x_1-x_0)^\alpha \partial^\alpha_x f(x_0) & \text{if } |x-x_0| > \delta \end{cases} \tag{5.1}$$

where x_1 is the point of intersection of the sphere $|x-x_0| = \delta$ and the line segment connecting x_0 and x. Then

(i) *$\rho_\varepsilon * f_0$ is a function in $C^\infty(R^n)$;*

(ii) *when $\varepsilon < \delta$, we have $\rho_\varepsilon * f_0(x) = f_0(x) + C_\varepsilon(x_0)$ in the set $\{x \in R^n: |x-x_0| < \delta - \varepsilon\}$ where $C_\varepsilon(x_0)$ which is independent of x satisfies*

$$|C_\varepsilon(x_0)| \leqq \varepsilon^2 \sum_{|\alpha|=2} |\partial^\alpha_x f(x_0)|\,; \tag{5.2}$$

(iii) *for any $x \in R^n$*

$$|\rho_\varepsilon * f_0(x) - f(x_0)|$$
$$\leqq \delta \sum_{|\alpha|=1} |\partial^\alpha f(x_0)| + \delta^2 \sum_{|\alpha|=2} |\partial_x^\alpha f(x_0)| . \tag{5.3}$$

Proof. (i) is obvious.

(ii) Noting that $|x+z-x_0|<\delta$ if $|z|<\varepsilon$ and $|x-x_0|<\delta-\varepsilon$ we know by the change of variables

$$\begin{aligned}
&\rho_\varepsilon * f_0(x) - f_0(x) \\
&= \int_{R^n} \rho_\varepsilon(z) f_0(x+z)dz - f_0(x) \\
&= \int_{R^n} \rho_\varepsilon(z) \sum_{0<|\alpha|\leqq 2} \frac{1}{\alpha!}\{(x+z-x_0)^\alpha - (x-x_0)^\alpha\} \partial_x^\alpha f(x_0) dz \\
&= \sum_{|\alpha|=1} \partial_x^\alpha f(x_0) \int_{R^n} \rho_\varepsilon(z) z^\alpha dz + \sum_{|\alpha|=2} \frac{1}{\alpha!} \partial_x^\alpha f(x_0) \int_{R^n} z^\alpha \rho_\varepsilon(z) dz \\
&+ \sum_{|\alpha|=2} \frac{1}{\alpha!} \partial_x^\alpha f(x_0) \sum_{0<\gamma<\alpha} \binom{\alpha}{\gamma} (x-x_0)^{\alpha-\gamma} \int_{R^n} z^\gamma \rho_\varepsilon(z) dz \\
&= I_1 + I_2 + I_3 .
\end{aligned}$$

It is easy to see that $I_1=0$ and $I_3=0$ in virtue of the eavenness of the function ρ. Thus by a suitable change of variables in the integral of I_2 we get the following relation:

$$\rho_\varepsilon * f_0(x) - f_0(x) = C_\varepsilon(x_0)$$

where

$$C_\varepsilon(x_0) = \varepsilon^2 \sum_{|\alpha|=2} \frac{1}{\alpha!} \partial_x^\alpha f(x_0) \int z^\alpha \rho(z) dz . \tag{5.4}$$

Clearly $C_\varepsilon(x_0)$ dose not depend on x and we easily find that (5.2) is valid.

(iii) follows from the fact that $|f_0(y)-f(x_0)|$ is dominated by the right hand side of (5.3) throughout R^n.

Lemma 5.2. *Let f be a function in $C^1(\Omega)$. We set*

$$f_0(x) = \begin{cases} \sum_{|\alpha|\leqq 1} (x-x_0)^\alpha \partial_x^\alpha f(x_0) & \text{for } |x-x_0| \leqq \delta , \\ \sum_{|\alpha|\leqq 1} (x_1-x_0)^\alpha \partial_x^\alpha f(x_0) & \text{for } |x-x_0| > \delta \end{cases} \tag{5.5}$$

where x_1 is the point defined in Lemma 5.1. Then

(i) *$\rho_\varepsilon * f_0$ is a fuction in $C^\infty(R^n)$;*

(ii) *when $\varepsilon<\delta$ $\rho_\varepsilon * f_0(x) = f_0(x)$ in the set $\{x\in R^n: |x-x_0|<\delta-\varepsilon\}$,*

(iii) *for any $x \in R^n$*

$$|\rho_\varepsilon * f_0(x) - f(x_0)| \leqq \delta \sum_{|\alpha|=1} |\partial^\alpha f(x_0)|.$$

Proof is similar to that of the preceding lemma.

It is known that for some constant $c_0 > 0$ we have

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)\xi^{\alpha+\beta} \geqq c_0 |\xi|^{2m} \tag{5.6}$$

for any $x \in \Omega$ and $\xi \in R^n$ under the condition $a-(1)$ (see [1]).

Now suppose that the coefficients of B satisfy the smoothenss condition $s-(4)$. Let x_0 be an arbitrary fixed point of Ω and $0<\varepsilon'<\delta$. We shall apply Lemma 5.1 to $a_{\alpha\beta}$, $|\alpha|=|\beta|=m$, and Lemma 5.2 to $a_{\alpha\beta}$, $|\alpha|+|\beta|=2m-1$. For $|\alpha|=|\beta|=m$ let $a^0_{\alpha\beta}$ and $C^{\alpha\beta}_{\varepsilon'}(x_0)$ be the function and the constant defined by (5.1) and (5.4) with f and ε replaced by $a_{\alpha\beta}$ and ε' and set $a^1_{\alpha\beta} = \rho_{\varepsilon'} * a^0_{\alpha\beta} - C^{\alpha\beta}_{\varepsilon'}(x_0)$. For $|\alpha|+|\beta|=2m-1$ letting $a^0_{\alpha\beta}$ be the function defined by (5.5) with $a_{\alpha\beta}$ in place of f we set $a^1_{\alpha\beta} = \rho_{\varepsilon'} * a^0_{\alpha\beta}$. For $|\alpha|+|\beta|=2m-2$ or $|\alpha|+|\beta| \leqq 2m-3$ we put $a^1_{\alpha\beta}(x) = a_{\alpha\beta}(x_0)$ or $a^1_{\alpha\beta}(x)=0$ respectively. We shall consider the following symmetric sesquilinear form:

$$B_1[u, v] = \sum_{|\alpha|,|\beta| \leqq m} \int_\Omega a^1_{\alpha\beta}(x) D^\alpha u \overline{D^\beta v} dx .$$

Lemma 5.3. *There are two positive constants c_0' and C such that*

$$B_1[u, u] \geqq c_0' \|u\|_m^2 - C\|u\|^2$$

for any $u \in \overset{\circ}{H}_m(\Omega)$ provided that δ and ε' are sufficiently small independently of x_0.

Proof. We write

$$\begin{aligned} &\sum_{|\alpha|=|\beta|=m} a^1_{\alpha\beta}(x)\xi^{\alpha+\beta} \\ &= \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x_0)\xi^{\alpha+\beta} + \sum_{|\alpha|=|\beta|=m} \{a^1_{\alpha\beta}(x) - a_{\alpha\beta}(x_0)\}\xi^{\alpha+\beta} \end{aligned}$$

and then use (5.6) and Lemma 5.1 in order to estimate both sums in the right hand side. It then follows immediately from the assumption s-(4) that if δ and ε' are sufficiently small independently of x_0 then for any $x \in \Omega$ and $\xi \in R^n$

$$\sum_{|\alpha|=|\beta|=m} a^1_{\alpha\beta}(x)\xi^{\alpha+\beta} \geqq (c_0/2)|\xi|^{2m} .$$

Since clearly the coefficients of B_1 are all uniformlly bounded it is a well known fact that the assertion of the lemma is true. q.e.d.

Next consider the case where the coefficients $a_{\alpha\beta}$ satisfy s-(3). For $|\alpha|=|\beta|=m$ letting $a^0_{\alpha\beta}$ be the function defined by (5.5) with $a_{\alpha\beta}$ in place of f we put $a^2_{\alpha\beta}=\rho_{\varepsilon'}*a^0_{\alpha\beta}$. According as $|\alpha|+|\beta|=2m-1$ or $|\alpha|+|\beta|\leqq 2m-2$ we put $a^2_{\alpha\beta}(x)=a_{\alpha\beta}(x_0)$ or $a^2_{\alpha\beta}(x)=0$. Difining the coefficients $a^2_{\alpha\beta}$ in this way we set

$$B_2[u, v] = \sum_{|\alpha|,|\beta|\leqq m}\int_\Omega a_{\alpha\beta}(x)D^\alpha u\overline{D^\beta v}dx .$$

In case where the coefficients of B satisfy s-(2) let B_3 be the sesquilinear form defined by

$$B_3[u, v] = \sum_{|\alpha|=|\beta|=m}\int_\Omega a_{\alpha\beta}(x_0)D^\alpha u\overline{D^\beta v}dx .$$

Then the following lemma can be proved analogously to Lemma 5.3.

Lemma 5.4. *There exist positive constants c_0' and C such that for $i=2$ and 3*

$$B_i[u, u]\geqq c_0'||u||_m^2-C||u||^2$$

for any $u\in \mathring{H}_m(\Omega)$ provided that δ and ε' are sufficiently small independently of x_0.

6. Estimates of resolvent kernels - 2

From now on we fix constants δ and ε' in the range specified in Lemmas 5.3 and 5.4. Let $\varepsilon_0=\mathrm{dist}(\Omega, \partial\Omega_1)$. Let A_1 be the operator associated with B_1 restricted to $\mathring{H}_m(\Omega)\times\mathring{H}_m(\Omega)$, that is,

$$B_1[u, v] = (A_1u, v) \qquad \text{for any } u, v\in\mathring{H}_m(\Omega) .$$

We intend to estimate the difference between the resolvent kernels of A_0 and those of A_1. To this end we define

$$S^1_{\lambda\varepsilon}f = \xi_\varepsilon\{(A_0-\lambda)^{-1}-(A_1-\lambda)^{-1}\}f$$

for $f\in H_{-m}(\Omega)$ where ξ is again a function in Λ and ε is an arbitrary positive number. We note here that the range of $S^1_{\lambda\varepsilon}$ is contained in $\mathring{H}_m(\Omega)$ whether the support of ξ_ε is contained in Ω or not. For an operator S on $H_{-m}(\Omega)$ to $\mathring{H}_m(\Omega)$ we denote by $||S||_{(-m,m)}$, $||S||_{(-m,0)}$, $||S||_{(0,m)}$, $||S||_{(0,0)}$ the norms of S considered as an operator on $H_{-m}(\Omega)$ to $\mathring{H}_m(\Omega)$, on $H_{-m}(\Omega)$ to $L^2(\Omega)$, on $L^2(\Omega)$ to $\mathring{H}_m(\Omega)$, on $L^2(\Omega)$ to $L^2(\Omega)$ respectively.

Lemma 6.1. *If $\varepsilon^{-1}|\lambda|^{1-1/2m}d(\lambda)^{-1}\leqq 1$, then for any positive integer j there is a constant K_j independent of x_0, ε and λ such that*

$$||S^1_{\lambda\varepsilon}||_{(-m,m)}\leqq K_jR^j_{\lambda\varepsilon}, \quad ||S^1_{\lambda\varepsilon}||_{(-m,0)}\leqq K_jR^j_{\lambda\varepsilon}|\lambda|^{-1/2m},$$
$$||S^1_{\lambda\varepsilon}||_{(0,m)}\leqq K_jR^j_{\lambda\varepsilon}|\lambda|^{-1/2}, \quad ||S^1_{\lambda\varepsilon}||_{(0,0)}\leqq K_jR^j_{\lambda\varepsilon}|\lambda|^{-1}$$

where

$$R^j_{\lambda\varepsilon} = \varepsilon^{2+h}\Big(\frac{|\lambda|}{d(\lambda)}\Big)^2 + \Big(\frac{|\lambda|^{1-1/2m}}{\varepsilon d(\lambda)}\Big)^j \frac{|\lambda|}{d(\lambda)}.$$

Proof. First consider $a_{\alpha\beta}$ with $|\alpha|=|\beta|=m$. Let $0<\varepsilon\leqq\min(\varepsilon_0, \delta-\varepsilon')$. Then in view of Lemma 5.1 we get for $|x-x_0|<\varepsilon$

$$a^1_{\alpha\beta}(x) = a^0_{\alpha\beta}(x) = \sum_{|\gamma|\leqq 2}\frac{1}{\gamma!}(x-x_0)^\gamma\partial_x^\gamma a_{\alpha\beta}(x_0)$$

whence it follows from the Taylor expansion of $a_{\alpha\beta}$ at x_0 that

$$|a_{\alpha\beta}(x)-a^1_{\alpha\beta}(x)|\leqq C\varepsilon^{2+h}. \tag{6. 1}$$

Replacing C by another constant if necessary we find that (6.1) is true without any restriction on $\varepsilon>0$. Similarly for $|\alpha|+|\beta|=2m-1$ we get

$$|a_{\alpha\beta}(x)-a^1_{\alpha\beta}(x)|\leqq C\varepsilon^{1+h} \tag{6. 2}$$

and for $|\alpha|+|\beta|=2m-2$

$$|a_{\alpha\beta}(x)-a^1_{\alpha\beta}(x)|\leqq C\varepsilon^{h} \tag{6. 3}$$

if $|x-x_0|<\varepsilon$.

Now let us write $u=\{(A_0-\lambda)^{-1}-(A_1-\lambda)^{-1}\}f$ and $v=\xi_\varepsilon u=S^1_{\lambda\varepsilon}f$. Then we find

$$\begin{aligned}
&B[v, v]-\lambda(v, v)\\
&\quad= B[u, \xi_\varepsilon v]-\lambda(u, \xi_\varepsilon v)+B[v, v]-B[u, \xi_\varepsilon v]\\
&\quad= B[(A_0-\lambda)^{-1}f, \xi_\varepsilon v]-B[(A_1-\lambda)^{-1}f, \xi_\varepsilon v]\\
&\quad- \lambda((A_0-\lambda)^{-1}f, \xi_\varepsilon v)+\lambda((A_1-\lambda)^{-1}f, \xi_\varepsilon v)\\
&\quad+ B[v, v]-B[u, \xi_\varepsilon v]\\
&\quad= (B_1-B)[(A_1-\lambda)^{-1}f, \xi_\varepsilon v]+B[v, v]-B[u, \xi_\varepsilon v].
\end{aligned}\tag{6. 4}$$

We shall proceed by induction. When $j=1$, using (6.1), (6.2) and (6.3) we get

$$\begin{aligned}
&|(B_1-B)[(A_1-\lambda)^{-1}f, \xi_\varepsilon v]|\\
&\quad\leqq \Big|\int_\Omega \sum_{|\alpha|+|\beta|\geqq 2m-2}(a_{\alpha\beta}(x)-a^1_{\alpha\beta}(x))D^\alpha((A_1-\lambda)^{-1}f)\overline{D^\beta(\xi_\varepsilon v)}dx\Big|\\
&\quad+\Big|\int_\Omega \sum_{|\alpha|+|\beta|\leqq 2m-3}a_{\alpha\beta}(x)D^\alpha((A_1-\lambda)^{-1}f)D^\beta(\xi_\varepsilon v)dx\Big|\\
&\quad\leqq C\varepsilon^{2+h}\|(A_1-\lambda)^{-1}f\|_m\|\xi_\varepsilon v\|_m\\
&\quad+ C\varepsilon^{1+h}\sum_{k=0}^{1}\|(A_1-\lambda)^{-1}f\|_{m-k}\|\xi_\varepsilon v\|_{m-1+k}+C\varepsilon^h\sum_{k=0}^{2}\|(A_1-\lambda)^{-1}f\|_{m-k}\|\xi_\varepsilon v\|_{m-2+k}
\end{aligned}$$

$$+C\sum_{k=0}^{3}||(A_1-\lambda)^{-1}f||_{m-k}||\xi_\varepsilon v||_{m-3+k}=I_1+I_2+I_3+I_4\,.$$

Using Lemma 3.4 and writing

$$Q=(||(A_1-\lambda)^{-1}f||_m+|\lambda|^{1/2}||(A_1-\lambda)^{-1}f||)(||\xi_\varepsilon v||_m+|\lambda|^{1/2}||\xi_\varepsilon v||)$$

we find

$$I_1\leqq C\varepsilon^{2+h}Q,\qquad I_2\leqq C\varepsilon^{1+h}|\lambda|^{-1/2m}Q$$
$$I_3\leqq C\varepsilon^{h}|\lambda|^{-2/2m}Q,\quad I_4\leqq C|\lambda|^{-3/2m}Q\,.$$

Applying Leibniz's formula and using the interpolation inequality (3.9) we easily verify

$$||\xi_\varepsilon v||_m+|\lambda|^{1/2}||\xi_\varepsilon v||\leqq C(||v||_m+|\lambda|^{1/2}||v||)\,. \tag{6. 5}$$

Hence from Lemma 3.1, (6.5) and the hypothesis of the present lemma it follows that for $f\in H_{-m}(\Omega)$

$$\begin{aligned}&|(B_1-B)[(A_1-\lambda)^{-1}f,\ \xi_\varepsilon v]|\\ &\quad\leqq C\varepsilon^{2+h}|\lambda|/d(\lambda)||f||_{-m}(||v||_m+|\lambda|^{1/2}||v||)\,.\end{aligned} \tag{6. 6}$$

Next by the same argument as in Lemma 4.1 we find that if $f\in H_{-m}(\Omega)$ $|B[u,\ \xi_\varepsilon v]-B[v,\ v]|$ is dominated by the value of the same form as the right member of (4.9) except for the replacement of $||f||_{V^*}$ by $||f||_{-m}$. From (4.2), (6.4), (6.6), and the estimate for $B[u,\ \xi_\varepsilon v]-B[v,\ v]$ just mentioned we easily conclude that for some constant K_1

$$||v||_m+|\lambda|^{1/2}||v||\leqq K_1R^1_{\lambda\varepsilon}||f||_{-m}\qquad\text{for } f\in H_{-m}(\Omega)\,.$$

Similarly if $f\in L^2(\Omega)$ we get

$$||v||_m+|\lambda|^{1/2}||v||\leqq K_1R^1_{\lambda\varepsilon}|\lambda|^{1/2}||f||\,.$$

Thus the lemma has been proved when $j=1$. Assume now the lemma has been proved for some k. Let η be a function in Λ such that $\eta(x)=1$ for any $x\in\operatorname{supp}\xi$, and set $\eta_\varepsilon(x)=\eta((x-x_0)/\varepsilon)$. Then we find

$$|B[u,\ \xi_\varepsilon v]-B[v,\ v]|\leqq II_1+II_2$$

where II_1 and II_2 have the same from as I_1 and I_2 which appeared in the proof of Lemma 4.1 (see the relation just before (4.11) as well as (4.16)). With the aid of Lemma 3.4 and the hypothesis of the present lemma we get

$$II_1\leqq C\varepsilon^{-1}|\lambda|^{-1/2m}(||\eta_\varepsilon u||_m+|\lambda|^{1/2}||\eta_\varepsilon u||)||v||_m\,,$$
$$II_2\leqq C\varepsilon^{-1}|\lambda|^{-1/2m}||\eta_\varepsilon u||_m(||v||_m+|\lambda|^{1/2}||v||)\,,$$

and hence using the induction hypothesis applied to $\|\eta_\varepsilon u\|_m + |\lambda|^{1/2}\|\eta_\varepsilon u\|$ for any $f \in H_{-m}(\Omega)$

$$II_1 \leqq C\varepsilon^{-1}|\lambda|^{-1/2m} R^k_{\lambda\varepsilon}\|f\|_{-m}\|v\|_m, \tag{6.7}$$

$$II_2 \leqq C\varepsilon^{-1}|\lambda|^{-1/2m} R^k_{\lambda\varepsilon}\|f\|_{-m}(\|v\|_m + |\lambda|^{1/2}\|v\|). \tag{6.8}$$

Combining (4.2), (6.4), (6.6), (6.7) and (6.8) we get for some constant K_{k+1}

$$\|v\|_m + |\lambda|^{1/2}\|v\| \leqq K_{k+1} R^{k+1}_{\lambda\varepsilon}\|f\|_{-m}.$$

In an analogous manner if $f \in L^2(\Omega)$ we obtain

$$\|v\|_m + |\lambda|^{1/2}\|v\| \leqq K'_{k+1} R^{k+1}_{\lambda\varepsilon}|\lambda|^{-1/2}\|f\|$$

for some constant K'_{k+1} which may be assumed to coincide with K_{k+1}. Thus the lemma has been completely proved.

Let A_i be the operator associated with B_i considered as a sesquilinear form defined on $\overset{\circ}{H}_m(\Omega) \times \overset{\circ}{H}_m(\Omega)$ for $i=2$ and 3, i.e.

$$B_i[u, v] = (A_i u, v) \qquad \text{for any } u, v \in \overset{\circ}{H}_m(\Omega). \tag{6.9}$$

We denote by $K^i_\lambda(x, y)$ the resolvent kernel of A_i for $i=1$, 2 and 3.

Lemma 6.2. *There exists a constant C depending on j but not on x_0, ε and λ such that if* $|\lambda|^{1-1/2m}\varepsilon^{-1}d(\lambda)^{-1} \leqq 1$

$$|K^0_\lambda(x_0, x_0) - K^1_\lambda(x_0, x_0)| \leqq C|\lambda|^{n/2m-1} R^j_{\lambda\varepsilon} \quad \textit{under } \text{s-(4)}, \tag{6.10}$$

$$|K^0_\lambda(x_0, x_0) - K^2_\lambda(x_0, x_0)| \leqq C|\lambda|^{n/2m-1}\, {}^1R^j_{\lambda\varepsilon} \quad \textit{under } \text{s-(3)}, \tag{6.11}$$

$$|K^0_\lambda(x_0, x_0) - K^3_\lambda(x_0, x_0)| \leqq C|\lambda|^{n/2m-1}\, {}^2R^j_{\lambda\varepsilon} \quad \textit{under } \text{s-(2)} \tag{6.12}$$

where $R^j_{\lambda\varepsilon}$ was defined in Lemma 6.1 *and for $i=1$ and* 2

$${}^iR^j_{\lambda\varepsilon} = \varepsilon^{2-i+h}\left(\frac{|\lambda|}{d(\lambda)}\right)^2 + \left(\frac{|\lambda|^{1-1/2m}}{\varepsilon d(\lambda)}\right)^j.$$

Proof. The inequality (6.10) follows from Lemmas 3.2 and 6.1. The remaining inequalities can be proved analogously and we omit the proof.

7. Estimates for resolvent kernels - 3

Let Ω_R be the spherical domain $\{|x|<R\}$ containing $\bar{\Omega}$. It is clear by the argument of the preceding sections that for $i=1, 2, 3$

$$\sum_{|\alpha|=|\beta|=m} a^i_{\alpha\beta}\, \xi^{\alpha+\beta} \geqq c|\xi|^{2m}$$

holds in Ω_R for some positive constant c independent of $x_0 \in \Omega$. Let B_i, $i=4, 5, 6$, be the symmetric sesquilinear form

$$B_i[u, v] = \int_{\Omega_R} \sum_{|\alpha|, |\beta| \leq m} a_{\alpha\beta}^{i-3}(x) D^\alpha u \overline{D^\beta v} dx$$

and A_i, $i=4, 5, 6$, be the operator associated with B_i considered to be defined on $\mathring{H}_m(\Omega_R) \times \mathring{H}_m(\Omega_R)$. For the sake of simplicity we write $\varepsilon = \delta(x_0) = \min \{\text{dist}(x_0, \partial\Omega), 1\}$ in the following two lemmas. For a function $u \in \mathring{H}_m(\Omega)$ let $\tilde{u} = u$ in Ω and $\tilde{u} = 0$ in $\Omega_R - \Omega$. Then $\tilde{u}$ belongs to $\mathring{H}_m(\Omega_R)$ and by this correspondence $\mathring{H}_m(\Omega)$ may be considered as a closed subspace of $\mathring{H}_m(\Omega_R)$. For $i=4, 5, 6$ let

$$S_{\lambda\varepsilon}^i f = \xi_\varepsilon \{((A_{i-3} - \lambda)^{-1}(rf))^\sim - (A_i - \lambda)^{-1} f\}$$

for $f \in H_{-m}(\Omega_R)$ where rf is the restriction of f to $\mathring{H}_m(\Omega)$. Evidently $S_{\lambda\varepsilon}^i$ is a bounded linear operator on $H_{-m}(\Omega_R)$ to $\mathring{H}_m(\Omega_R)$.

Lemma 7.1. *If $\varepsilon^{-1} |\lambda|^{-1/2m} \leq 1$, then for any integer $j \geq 0$ there exists a constant K_j independent of x_0 and λ such that for $i=4, 5, 6$*

$$\|S_{\lambda\varepsilon}^i\|_{(-m, m)} \leq K_j (|\lambda|^{1-1/2m} \varepsilon^{-1} d(\lambda)^{-1})^j |\lambda| / d(\lambda),$$

$$\left.\begin{array}{l} \|S_{\lambda\varepsilon}^i\|_{(0, m)} \\ \|S_{\lambda\varepsilon}^i\|_{(-m, 0)} \end{array}\right\} \leq K_j (|\lambda|^{1-1/2m} \varepsilon^{-1} d(\lambda)^{-1})^j |\lambda|^{1/2} / d(\lambda),$$

$$\|S_{\lambda\varepsilon}^i\|_{(0, 0)} \leq K_j (|\lambda|^{1-1/2m} \varepsilon^{-1} d(\lambda)^{-1})^j / d(\lambda).$$

Proof. Let $v = \xi_\varepsilon u = S_{\lambda\varepsilon}^i f$. Noting that the support of v is contained in Ω we find

$$B_i[v, v] - \lambda(v, v) = B_i[v, v] - B_i[u, \xi_i v].$$

The present lemma can be proved just as Lemma 4.1 based upon this equality.

Let K_λ^i be the resolvent kernel of A_i, $i=4, 5, 6$.

Lemma 7.2. *For any $p \geq 0$ there exists a constant C_p depending on p but not on x_0 and λ such that*

$$|K_\lambda^{i-3}(x_0, x_0) - K_\lambda^i(x_0, x_0)| \leq C_p \frac{|\lambda|^{n/2m}}{d(\lambda)} \left(\frac{|\lambda|^{1-1/2m}}{\delta(x_0) d(\lambda)} \right)^p$$

for $i=4, 5, 6$.

The lemma can be proved analogously to Lemma 4.2 applying Lemma 3.2 to $S_{\lambda\varepsilon}^i$.

The following lemma is a consequence of Theorem 3.1 of S. Agmon and Y. Kannai [4] on the asymptotic expansion of the resolvent kernels in the interior of the domain considered.

Lemma 7.3. *For any positive number ε there is a constant C depending on ε but not on $x_0\in\Omega$ such that the following inequality holds for $d(\lambda)\geqq|\lambda|^{1-1/4m+\varepsilon}$, $|\lambda|\geqq 1$ and $x\in\Omega$ and for $i=4, 5, 6$:*

$$|K_\lambda^i(x, x)-c_0^i(x)(-\lambda)^{n/2m-1}|\leqq C|\lambda|^{(n-1)/2m-1}$$

where $c_0^i(x)$ is a function defined by (2.2) with $a_{\alpha\beta}$ replaced by $a_{\alpha\beta}^{i-3}$.

REMARK. $c_0^i(x_0)=c_0(x_0)$ for $i=4, 5, 6$.

Using the results proved up to now we get the following theorem.

Theorem 7.1. *Suppose that s-(4) is satisfied. Then for any number $p\geqq 0$ and $\theta\in(0, (h+2)/(h+5))$ we have*

$$|K_\lambda(x, x)-c_0(x)(-\lambda)^{n/2m-1}|$$
$$\leqq C_p\left\{|\lambda|^{(n-\theta)/2m-1}+\frac{|\lambda|^{n/2m}}{d(\lambda)}\left(\frac{|\lambda|^{1-1/2m}}{\delta(x)d(\lambda)}\right)^p\right\}$$

for any $x\in\Omega$ and λ satisfying $d(\lambda)\geqq|\lambda|^{1-\theta/2m}$, $|\lambda|\geqq L$ where C_p is a constant depending on p and θ but not on x and λ, and L is some constant depending only on B, Ω and Ω_1. Under the assumption s-(3) or s-(2) the same conclusion remains valid for $0<\theta<(h+1)/(h+4)$ or $0<\theta<h/(h+3)$ respectively.

Proof. First suppose s-(4) is satisfied. If we take $\varepsilon=|\lambda|^{-3\theta/2m(h+2)}$, then we can apply Lemma 6.2 in the region $d(\lambda)\geqq|\lambda|^{1-\theta/2m}$, $|\lambda|\geqq L$ to otbain

$$|K_\lambda^0(x_0, x_0)-K_\lambda^0(x_0, x_0)|\leqq C|\lambda|^{m/2m-1}R_{\lambda\varepsilon}^j\leqq C|\lambda|^{(n-\theta)/2m-1}$$

choosing j sufficiently large depending on θ. Combining this with Lemmas 4.2, 7.2 and 7.3 we get the assertion of the theorem. The remaining part of the theorem can be proved in parallel taking $\varepsilon=|\lambda|^{-3\theta/2m(h+1)}$ in case of s-(3) and $\varepsilon=|\lambda|^{-3\theta/2mh}$ in case of s-(2).

8. Proof of the main theorem

In this section we shall prove the main theorem essentially following the method of S. Agmon [2]. The resolvent kernel $K_\lambda(x, y)$ of A is continuous in $\bar{\Omega}\times\bar{\Omega}$ and since A is selfadjoint it is symmetric: $K_\lambda(x, y)=\overline{K_{\bar{\lambda}}(y, x)}$. Let $\{\lambda_j\}$ and $\{\phi_j\}$ be the sequence of eigenvalues and the corresponding sequence of orthonormal eigenfunctions of A respectively. Evidently the proof of Mercer's theorem applies to our case and we have

$$K_\lambda(x, y)=\sum_{j=0}^{\infty}\frac{\phi_j(x)\overline{\phi_j(y)}}{\lambda_j-\lambda}, \tag{8. 1}$$

the series on the right being convergent absolutely and uniformly in $\Omega\times\Omega$ for any fixed λ in the resolvent set of A. Let $\sigma_x(t)=\sum_{\lambda_j\leqq t}|\phi_j(x)|^2$. Then

$$\sum_{j=0}^{\infty}\frac{|\phi_j(x)|^2}{\lambda_j-\lambda}=\int_0^\infty\frac{d\sigma_x(t)}{t-\lambda}. \tag{8.2}$$

We introduce here Pleijel's formula which was used in his simple proof of Malliavin's tauberian theorem.

Lemma 8.1. *Suppose $\sigma(t)$ is a non-decreasing function defined on* $[0, \infty)$. *Let*

$$f(z)=\int_0^\infty\frac{d\sigma(t)}{t-z}, \tag{8.3}$$

and

$$I(\xi)=\frac{1}{2\pi i}\int_{L(\xi)}f(z)dz \tag{8.4}$$

where $L(\xi)$ is an oriented curve in the complex plane from $\bar{\xi}$ to $\xi=t+i\tau$ not intersecting $[0, \infty)$. *Then for $t>0$, $\tau>0$*

$$|I(\xi)-(\tau/\pi)\operatorname{Re}f(\xi)-\sigma(t)+\sigma(0)|\leqq\tau\operatorname{Im}f(\xi).$$

Proof is given in [6].

Lemma 8.2. *There exists a constant C such that for any $t>0$ and $x\in\Omega$*

$$\sigma_x(t)\leqq Ct^{n/2m}.$$

Proof. The assertion follows from

$$\frac{1}{2t}\sum_{\lambda_j\leqq t}|\phi_j(x)|^2\leqq\sum_{\lambda_j\leqq t}\frac{|\phi_j(x)|^2}{\lambda_j+t}\leqq K_{-t}(x, x)\leqq Ct^{n/2m-1}.$$

Lemma 8.3. *Suppose that the assumption s-(2), s-(3) or s-(4) is satisfied. Then there exists a constant C independent of x and t such that*

$$\left|\sigma_x(t)-(2\pi)^{-n}\frac{\sin(n\pi/2m)}{n\pi/2m}c_0(x)t^{n/2m}\right|\leqq Ct^{(n-\theta)/2m}\delta(x)^{-\theta} \tag{8.5}$$

for any sufficiently large $t>0$ and any $x\in\Omega$ where θ is the same number as the one defined in the main theorem.

Proof. 1st case: $t^{1/2m}\delta(x)\leqq1$. The above estimate is trivial in virtue of Lemma 8.2.

2nd case: $t^{1/2m}\delta(x)>1$. We shall prove the assertion with the aid of Lemma 8.1. Let $f_x(z)$ and $I_x(\xi)$ be the functions defined by (8.3) and (8.4) with σ_x in place of σ. Then in virtue of (8.1) and (8.2) we have

$$f_x(z)=K_z(x, x)\,. \tag{8.6}$$

Using Lemma 8.1 and noting (8.6) and $\sigma_x(0)=0$ we get

$$|I_x(\xi)-(\tau/\pi)\operatorname{Re}K_\xi(x, x)-\sigma_x(t)|\leqq\tau\operatorname{Im}K_\xi(x, x)\,. \tag{8.7}$$

From (8.7) it follows that

$$|\sigma_x(t)-\frac{1}{2\pi i}\int_{L(\xi)}c_0(x)(-z)^{n/2m-1}dz|\leqq I_1+I_2 \tag{8.8}$$

where

$$\begin{aligned} I_1 &= \frac{1}{2\pi}\left|\int_{L(\xi)}\{K_z(x, x)-c_0(x)(-z)^{n/2m-1}\}dz\right|,\\ I_2 &= (\tau/\pi)|\operatorname{Re}K_\xi(x, x)|+\tau\operatorname{Im}K_\xi(x, x)\\ &\leqq(1+\pi^{-2})^{1/2}|K_\xi(x, x)|. \end{aligned}$$

We define $\tau_x(t)=ct(t_x{}^{1/2m}\delta(x))^{-\theta}$ where c is a constant satisfying $c\geqq(1+c^2)^{(1-\theta/2m)/2}$. In the above inequality we take $\tau=\tau_x(t)$ and

$$\begin{aligned} L(\xi)=&\{z=t+iu:\ \tau_x(t)\leqq|u|\leqq ct\}\\ &\cup\{z:\ |z|=(1+c^2)^{1/2}t,\ \operatorname{Re}z\leqq t\}\,. \end{aligned}$$

Then on account of the present assumption $t^{1/2m}\delta(x)>1$ and the choice of c we get $L(\xi)\subset\{\lambda:\ d(\lambda)\geqq|\lambda|^{1-\theta/2m}\}$ and hence in view of Theorem 7.1

$$\begin{aligned} I_2&\leqq C\tau_x(t)\{|K_\xi(x, x)-c_0(x)(-\xi)^{n/2m-1}|+|c_0(x)|\ |\xi|^{n/2m-1}\}\\ &\leqq C_p\tau_x(t)\left\{|\xi|^{(n-\theta)/2m-1}+\frac{|\xi|^{n/2m}}{d(\xi)}\left(\frac{|\xi|^{1-1/2m}}{\delta(x)d(\xi)}\right)^p+|\xi|^{n/2m-1}\right\}\\ &\leqq C_p\left\{|\xi|^{n/2m}\left(\frac{|\xi|^{1-1/2m}}{\delta(x)\tau_x(t)}\right)^p+\tau_x(t)|\xi|^{n/2m-1}\right\}. \end{aligned}$$

Noting that $t\leqq|\xi|\leqq(1+c^2)^{1/2}t$ we get

$$I_2\leqq C_p\{t^{n/2m}(t^{1/2m}\delta(x))^{-(1-\theta)p}+t^{(n-\theta)/2m}\delta(x)^{-\theta}\}$$

from which we obtain

$$I_2\leqq Ct^{(n-\theta)/2m}\delta(x)^{-\theta} \tag{8.9}$$

taking $p=\theta(1-\theta)$. On the other hand again by Theorem 7.1

$$
\begin{aligned}
I_1 &\leqq C\int_{L(\xi)}\left\{|z|^{(n-\theta)/2m-1}+\frac{|z|^{n/2m}}{d(z)}\left(\frac{|z|^{1-1/2m}}{\delta(x)d(z)}\right)^p\right\}|dz| \\
&\leqq C\left[\int_{\tau_x(t)}^{ct} du\cdot t^{(n-\theta)/2m-1}+\int_{\tau_x(t)}^{ct} u^{-p-1}du\cdot t^{n/2m+p}(t^{1/2m}\delta(x))^{-p}\right. \\
&\left.+\int_{|z|=(1+c^2)^{1/2}t}|dz|\{t^{(n-\theta)/2m-1}+t^{n/2m-1}(t^{1/2m}\delta(x))^{-p}\}\right] \\
&\leqq C\{t^{(n-\theta)/2m}+t^{n/2m}(t^{1/2m}\delta(x))^{-p(1-\theta)}+t^{n/2m}(t^{1/2m}\delta(x))^{-p}\} \\
&\leqq Ct^{(n-\theta)/2m}\delta(x)^{-\theta} \qquad (8.10)
\end{aligned}
$$

where we again used the choice of $p=\theta/(1-\theta)(\geqq\theta)$. Combining (8.8), (8.9) and (8.10) we get

$$
|\sigma_x(t)-\frac{1}{2\pi i}\int_{L(\xi)}c_0(x)(-z)^{n/2m-1}dz|\leqq Ct^{(n-\theta)/2m}\delta(x)^{-\theta}\,.
$$

Finally noting that

$$
\left|\frac{1}{2\pi i}\int_{L(\xi)}(-z)^{n/2m-1}dz-t^{n/2m}\frac{\sin(n\pi/2m)}{n\pi/2m}\right| \leqq Ct^{(n-\theta)/2m}\delta(x)^{-\theta}
$$

we obtain the desired estimate.

If (2.1) is satisfied for θ in Lemma 8.3, then integrating (8.5) over Ω we immediately obtain the asymptotic formula for $N(t)$ described in the main theorem in case of the condition s-(2), s-(3) or s-(4).

If the condition s-(5) is satisfied, then we can make use of the part of Theorem 3.1 of S. Agmon and Y. Kannai [4] for the case of constant coefficients in the principal part. Considering the sesquilinear from having the following functions as coefficients:

$$
\begin{aligned}
&a^4_{\alpha\beta}(x)=a_{\alpha\beta}(x)(=\text{constant}) \text{ for } |\alpha|=|\beta|=m\,, \\
&a^4_{\alpha\beta}\in C_0^\infty(\Omega_R) \text{ and } a^4_{\alpha\beta}(x)=a_{\alpha\beta}(x) \text{ in } \Omega \text{ for} \\
&|\alpha|+|\beta|\leqq 2m-1\,,
\end{aligned}
$$

we can verify that (8.5) is true for any positive number $\theta<1$. Therefore if (2.1) is satisfied for any $p<1$, then we obtain the desired formula for $N(t)$. Finally the assertion of the main theorem under s-(1) can be proved in a similar, but a simpler, manner. We need only investigate the asymptotic behaviour of $K_\lambda(x, x)$ for λ real and $\rightarrow-\infty$ and apply the tauberian theorem of Hardy and Littlewood.

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