

Title	Integral geometry on submanifolds of dimension one and codimension one in the product of spheres
Author(s)	Kang, Hong Jae
Citation	Osaka Journal of Mathematics. 41(1) P.107-P.117
Issue Date	2004-03
Text Version	publisher
URL	https://doi.org/10.18910/3938
DOI	10.18910/3938
rights	
Note	

Osaka University Knowledge Archive : OUKA

<https://ir.library.osaka-u.ac.jp/>

Osaka University

INTEGRAL GEOMETRY ON SUBMANIFOLDS OF DIMENSION ONE AND CODIMENSION ONE IN THE PRODUCT OF SPHERES

HONG JAE KANG

(Received December 4, 2001)

1. Introduction

Let G be a Lie group and K a closed subgroup of G . Consider two submanifolds in a Riemannian homogeneous space G/K , one fixed and the other moving under g in G . Let the fixed one be M and the moving one be gN and let μ_G be the invariant measure on G . By taking the geometric invariant $\text{vol}(M \cap gN)$, volume of the submanifold $M \cap gN$, and integrating with respect to $d\mu_G(g)$, we get so called the Poincaré formula. This can be briefly stated as follows.

Let M^p and N^q be submanifolds of dimensions p and q respectively, in a Riemannian homogeneous space G/K . Then many works in integral geometry have been concerned with computing integrals of the following form

$$\int_G \text{vol}(M \cap gN) d\mu_G(g).$$

The Poincaré formula means equalities which represent the above integral by some geometric invariants of submanifolds M and N of G/K . For example in the case that G is the group of isometries of Euclidean space \mathbf{R}^n and M and N are submanifolds of \mathbf{R}^n then the result of above integral leads to formulas of Poincaré, Crofton and other integral geometers (see [6]). Especially R. Howard [1] obtained a Poincaré formula for Riemannian homogeneous spaces as follows:

Let M and N be submanifolds of G/K with $\dim M + \dim N = \dim(G/K)$. Assume that G is unimodular. Then

$$(1.1) \quad \int_G \sharp(M \cap gN) d\mu_G(g) = \iint_{M \times N} \sigma_K(T_x^\perp M, T_y^\perp N) d\mu_{M \times N}(x, y),$$

where $\sharp(X)$ denotes the number of elements in a set X and $\sigma_K(T_x^\perp M, T_y^\perp N)$ is defined by (2.1) in Section 2.

The author was supported by grant Proj. No. R14-2002-003-01000-0 from Korea Science and Engineering Foundation.

The formula (1.1) holds under the general situation. However, it is difficult to give an explicit description through the concrete computation of $\sigma_K(T_x^\perp M, T_y^\perp N)$, and only a little is known about it ([2], [3], [7]). In the present paper, we attempt to explicitly describe this formula for submanifolds of dimension and codimension one in the product of arbitrary dimensional unit spheres. More precisely,

Theorem 1.1. *Let M be a submanifold of $S^{m+1} \times S^{n+1}$ of dimension 1 and N a submanifold of codimension 1. Assume that for almost all $g \in G$, M and gN intersect transversely. For any point $x \in M$ (resp. $y \in N$), $\sin \theta_x$ and $\cos \theta_x$ (resp. $\sin \tau_y, \cos \tau_y$) denote length of the first and second component of unit vector $u_x = (u_1, u_2)$ (resp. $v_y = (v_1, v_2)$) of $T_x M$ (resp. $T_y^\perp N$), respectively. Then we have*

$$\begin{aligned} & \int_{SO(m+2) \times SO(n+2)} \#(M \cap gN) d\mu_{SO(m+2) \times SO(n+2)}(g) \\ &= 2 \operatorname{vol}(SO(m+1) \times SO(n+1)) \iint_{M \times N} \sigma(x, y) d\mu_{M \times N}(x, y) \end{aligned}$$

where

$$\sigma(x, y) = \begin{cases} \frac{c_{xy} \operatorname{vol}(S^{n-1})}{n \operatorname{vol}(S^n)} F\left(-\frac{1}{2}, -\frac{n}{2}, \frac{1+m}{2}, \frac{s_{xy}^2}{c_{xy}^2}\right), & \text{if } s_{xy} \leq c_{xy}; \\ \frac{s_{xy} \operatorname{vol}(S^{m-1})}{m \operatorname{vol}(S^m)} F\left(-\frac{1}{2}, -\frac{m}{2}, \frac{1+n}{2}, \frac{c_{xy}^2}{s_{xy}^2}\right), & \text{if } s_{xy} \geq c_{xy}. \end{cases}$$

Here $F(a, b, c; x)$ is the Gauss hypergeometric function, and $s_{xy} = \sin \theta_x \sin \tau_y$ and $c_{xy} = \cos \theta_x \cos \tau_y$.

2. Preliminaries

In this section we shall review the Poincaré formula on Riemannian homogeneous spaces given by R. Howard [1] and recall the Gauss hypergeometric function.

Let E be a finite dimensional real vector space with an inner product. For vector subspaces V and W with orthonormal bases v_1, \dots, v_p and w_1, \dots, w_q respectively, we define $\sigma(V, W)$ by

$$\sigma(V, W) = |v_1 \wedge \dots \wedge v_p \wedge w_1 \wedge \dots \wedge w_q|.$$

This definition is independent of the choice of orthonormal bases. Furthermore, if $p + q = \dim E$ then

$$\sigma(V, W) = \sigma(V^\perp, W^\perp).$$

Let G be a Lie group and K a closed subgroup of G . We assume that G has a left invariant Riemannian metric that is also invariant under the right actions of ele-

ments of K . This metric induces a G -invariant Riemannian metric on G/K . We denote by o the origin of G/K . If $x, y \in G/K$ and V is a vector subspace of $T_x(G/K)$ and W is a vector subspace of $T_y(G/K)$ then define $\sigma_K(V, W)$ by

$$(2.1) \quad \sigma_K(V, W) = \int_K \sigma((dg_x)_o^{-1}V, dk_o^{-1}(dg_y)_o^{-1}W) d\mu_K(k)$$

where g_x and g_y are elements of G such that $g_x o = x$ and $g_y o = y$. This definition is independent of the choice of g_x and g_y in G such that $g_x o = x$ and $g_y o = y$.

We list here the basic properties of the Gauss hypergeometric function that are needed in this paper only. For further details see [4].

The Gauss hypergeometric series, convergent for $|z| < 1$, is given by the power series

$$(2.2) \quad F(a, b, c; z) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(a)} \cdot \frac{\Gamma(b+n)}{\Gamma(b)} \cdot \frac{\Gamma(c)}{\Gamma(c+n)} \cdot \frac{z^n}{n!},$$

where Γ is the gamma function. By analytic continuation $F(a, b, c; z)$ can be extended to define a function analytic and single-valued in the complex z plane cut along the positive real axis from 1 to ∞ . We remark that above series reduces to a polynomial of degree n in z when a or b is equal to $-n$, ($n = 0, 1, 2, \dots$). The series (2.2) is not defined when c is equal to $-m$, ($m = 0, 1, 2, \dots$), provided a or b is not a negative integer n with $n < m$. The hypergeometric equation

$$z(1-z)\frac{d^2u}{dz^2} + (c - (a+b+1)z)\frac{du}{dz} - abu = 0$$

has the solution $u = F(a, b, c; z)$.

The six functions $F(a \pm 1, b, c; z)$, $F(a, b \pm 1, c; z)$ and $F(a, b, c \pm 1; z)$ are called contiguous to $F(a, b, c; z)$. Relations between $F(a, b, c; z)$ and any two contiguous functions have been given by Gauss. By repeated application of these relations the function $F(a+m, b+n, c+l; z)$ with integer m, n, l can be expressed as a linear combination of $F(a, b, c; z)$ and one of its contiguous functions with coefficients which are rational functions of a, b, c, z . For examples,

$$(2.3) \quad \begin{aligned} azF(a+1, b+1, c+1; z) &= c[F(a, b+1, c; z) - F(a, b, c; z)], \\ (c-1)F(a, b, c-1; z) &= (c-a-1)F(a, b, c; z) + aF(a+1, b, c; z). \end{aligned}$$

Among the special cases are

$$(2.4) \quad (1-z)^t = F(-t, b, b; z),$$

$$(2.5) \quad \arcsin x = xF\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; z^2\right).$$

Furthermore C.F. Gauss evaluated, for $\Re(c - a - b) > 0$,

$$(2.6) \quad F(a, b, c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}.$$

In this paper, we may consider only when z is a real number.

3. Proof of Theorem 1.1

Let S^m be the unit sphere of dimension m . The special orthogonal group $SO(m + 1)$ acts transitively on S^m . The isotropy subgroup of $SO(m + 1)$ at a point in S^m is $SO(m)$. Thus $S^{m+1} \times S^{n+1}$ can be realized as a homogeneous space $(SO(m+2) \times SO(m+2))/(SO(m+1) \times SO(m+1))$. We have set, to simplify notation,

$$G = SO(m+2) \times SO(m+2), \quad K = SO(m+1) \times SO(m+1).$$

Let $\mathfrak{g} = \mathfrak{so}(m+2) \times \mathfrak{so}(m+2)$ be the Lie algebra of G . Define an inner product on \mathfrak{g} by

$$(X, Y) = -\frac{1}{2} \text{Trace}(XY) \quad (X, Y \in \mathfrak{g}).$$

We extend this inner product (\cdot, \cdot) on \mathfrak{g} to the left invariant Riemannian metric on G . Then we obtain a bi-invariant Riemannian metric on G . This bi-invariant Riemannian metric on G induces a G -invariant Riemannian metric on $G/K = S^{m+1} \times S^{n+1}$.

Let M be a submanifold of $S^{m+1} \times S^{n+1}$ of dimension one and N a submanifold of codimension one. By the formula (1.1), we have

$$(3.1) \quad \int_G \#(M \cap gN) d\mu_G(g) = \int \int_{M \times N} \sigma_K(T_x M, T_y N) d\mu_{M \times N}(x, y).$$

Let $u_x = (u_1, u_2)$ and $v_y = (v_1, v_2)$ be unit vectors of $T_x M$ and $T_y^\perp N$ respectively. By the action of K , we can transport u_x to $((\sin \theta_x, 0, \dots, 0), (\cos \theta_x, 0, \dots, 0))$ and v_y to $((\sin \tau_y, 0, \dots, 0), (\cos \tau_y, 0, \dots, 0))$ respectively. Let e_1, \dots, e_{m+n+2} be the standard orthonormal basis of \mathbf{R}^{m+n+2} . Thus we can take

$$(-\cos \tau_y e_1 + \sin \tau_y e_{m+2}), e_2, \dots, e_{m+1}, e_{m+3}, \dots, e_{m+n+2}$$

as an orthonormal basis of $T_y N$. We can simply write

$$\sigma(\theta_x, \tau_y) = \sigma_K(T_x M, T_y N),$$

since $\sigma_K(T_x M, T_y N)$ is dependent only on θ_x and τ_y . Then we have

$$\sigma(k^{-1} T_x M, T_y N)$$

$$\begin{aligned}
 &= |(\sin \theta_x e_1 + \cos \theta_x e_{m+2})k \wedge (-\cos \tau_y e_1 + \sin \tau_y e_{m+2}) \wedge \hat{e}_{1,m+2}| \\
 &= |\sin \theta_x \sin \tau_x k_{11} + \cos \theta_y \cos \tau_y k_{m+2m+2}|,
 \end{aligned}$$

where

$$\hat{e}_{1,m+2} = e_2 \wedge \cdots \wedge e_{m+1} \wedge e_{m+3} \wedge \cdots \wedge e_{m+n+2},$$

and

$$k = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \in SO(m+1) \times SO(n+1).$$

For simplicity of writing we put $\sin \theta \sin \tau = s$ and $\cos \theta \cos \tau = c$. Then we get

$$\sigma(\theta, \tau) = \iint_K |sk_{11} + ck_{m+2m+2}| d\mu_K(k)$$

We now have to compute following:

$$(3.2) \quad \int_{SO(m+1)} \int_{SO(n+1)} |tk_{11} + g_{11}| d\mu_{SO(n+1)}(g) d\mu_{SO(m+1)}(k)$$

We here give the following lemma to compute the above integral.

Lemma 3.1. *If $|\alpha| \leq 1$ then*

$$\int_{S^n} |\alpha + x_1| d\mu_{S^n}(x) = 2\text{vol}(S^{n-1}) \left\{ \frac{1}{n} \sqrt{1 - \alpha^2}^n + \alpha^2 F\left(\frac{1}{2}, 1 - \frac{n}{2}, \frac{3}{2}; \alpha^2\right) \right\}.$$

Proof. Define a mapping $\phi: (-1, 1) \times S^{n-1} \rightarrow S^n$ by

$$(t, x = (x_1, x_2, \dots, x_n)) \mapsto (t, \sqrt{1 - t^2} x).$$

Using ϕ as a variable transformation we have

$$\begin{aligned}
 \int_{S^n} |\alpha - x_1| d\mu_{S^n}(x) &= \int_{-1}^1 \int_{S^{n-1}} |\alpha - t| \sqrt{1 - t^2}^{n-2} d\mu_{S^{n-1}}(x) dt \\
 &= \text{vol}(S^{n-1}) \int_{-1}^1 |\alpha - t| \sqrt{1 - t^2}^{n-2} dt.
 \end{aligned}$$

Here $\text{vol}(S^n)$ is the surface area of the n -dimensional unit sphere and its value is

$$\text{vol}(S^n) = \frac{2\pi^{(n+1)/2}}{\Gamma((n+1)/2)},$$

where Γ denotes the gamma function. By a simple calculation, we have

$$\int \sqrt{1-x^2}^n dx = xF\left(\frac{1}{2}, -\frac{n}{2}, \frac{3}{2}; x^2\right).$$

Hence we obtain

$$\int_{-1}^1 |\alpha - t| \sqrt{1-t^2}^{n-2} dt = \frac{2}{n} \sqrt{1-\alpha^2}^n + 2\alpha^2 F\left(\frac{1}{2}, 1 - \frac{n}{2}, \frac{3}{2}; \alpha^2\right),$$

which implies Lemma 3.1. □

We first consider the case where $0 \leq t \leq 1$ in (3.2). Then we have

$$\begin{aligned} & \int_{SO(n+1)} |tk_{11} + g_{11}| d\mu_{SO(n+1)}(g) \\ (3.3) \quad &= \text{vol}(SO(n)) \int_{S^n} |tk_{11} + x_1| d\mu_{S^n}(x) \\ &= 2\text{vol}(SO(n))\text{vol}(S^{n-1}) \\ (3.4) \quad & \times \left\{ \frac{1}{n} \sqrt{1-t^2(k_{11})^2}^n + t^2(k_{11})^2 F\left(\frac{1}{2}, 1 - \frac{n}{2}, \frac{3}{2}, t^2(k_{11})^2\right) \right\}. \end{aligned}$$

Equality (3.3) follows from the fibering of $SO(n+1)$ over S^n with the fiber $SO(n)$, and (3.4) follows from Lemma 3.1.

Notice

$$(3.5) \quad \int_0^\pi \sin x dx \int_0^\pi \sin^2 x dx \cdots \int_0^\pi \sin^m x dx = \frac{\text{vol}(S^{m+1})}{2\pi} = \frac{\text{vol}(S^{m-1})}{m},$$

then, using spherical coordinate transformation, the integral of the first term in (3.4) over $SO(m+1)$ is as follows:

$$\begin{aligned} & \int_{SO(m+1)} \sqrt{1-t^2(k_{11})^2}^n d\mu_{SO(m+1)}(k) \\ &= \text{vol}(SO(m)) \int_{S^m} \sqrt{1-t^2(x_1)^2}^n d\mu_{S^m}(x) \\ &= 2 \text{vol}(SO(m)) \text{vol}(S^{m-1}) \int_0^{\pi/2} \sqrt{1-t^2 \cos^2 \theta_1}^n \cdot \sin^{m-1} \theta_1 d\theta_1 \\ &= 2 \text{vol}(SO(m)) \text{vol}(S^{m-1}) \int_0^1 \sqrt{1-t^2 x^2}^n \cdot \sqrt{1-x^2}^{m-2} dx. \end{aligned}$$

In the last integral,

$$\int_0^1 \sqrt{1-x^2}^{m-2} dx$$

is an Euler beta function. Hence it can be evaluated in terms of gamma function, and we find that

$$(3.6) \quad \int_0^1 \sqrt{1-x^2}^{m-2} dx = \frac{\text{vol}(S^m)}{2 \text{vol}(S^{m-1})}.$$

By (3.6) and integration by part, for any even number l we obtain

$$\int_0^1 x^l \sqrt{1-x^2}^{m-2} dx = \frac{(l-1)!(m-1)!!}{(m+l-1)!!} \cdot \frac{\text{vol}(S^m)}{2 \text{vol}(S^{m-1})},$$

where

$$m!! = \begin{cases} m(m-2) \cdots 4 \cdot 2, & m: \text{even}; \\ m(m-2) \cdots 3 \cdot 1, & m: \text{odd}. \end{cases}$$

And by (2.4) we have

$$\sqrt{1-t^2x^2}^n = F\left(-\frac{n}{2}, 1, 1; t^2x^2\right).$$

So we obtain

$$\int_0^1 \sqrt{1-t^2x^2}^n \sqrt{1-x^2}^{m-2} dx = \frac{\text{vol}(S^m)}{2 \text{vol}(S^{m-1})} \cdot F\left(\frac{1}{2}, -\frac{n}{2}, \frac{1+m}{2}; t^2\right).$$

Hence we have

$$(3.7) \quad \begin{aligned} & \int_{SO(m+1)} \sqrt{1-t^2(k_{11})^2}^n d\mu_{SO(m+1)}(k) \\ &= \text{vol}(SO(m)) \text{vol}(S^m) F\left(\frac{1}{2}, -\frac{n}{2}, \frac{1+m}{2}; t^2\right). \end{aligned}$$

On the other hand, the integral of the second term in (3.4) on $SO(m+1)$ is as follows:

$$\begin{aligned} & \int_{SO(m+1)} (k_{11})^2 F\left(\frac{1}{2}, 1 - \frac{n}{2}, \frac{3}{2}; t^2(k_{11})^2\right) d\mu_{SO(m+1)}(k) \\ &= \text{vol}(SO(m)) \int_{S^m} (x_1)^2 F\left(\frac{1}{2}, 1 - \frac{n}{2}, \frac{3}{2}; t^2(x_1)^2\right) d\mu_{S^m}(x). \end{aligned}$$

Using again spherical coordinate transformation we get

$$\begin{aligned} & \int_{S^m} (x_1)^2 F\left(\frac{1}{2}, 1 - \frac{n}{2}, \frac{3}{2}; t^2(x_1)^2\right) d\mu_{S^m}(x) \\ &= 2 \text{vol}(S^{m-1}) \int_0^1 x^2 \sqrt{1-x^2}^{m-2} F\left(\frac{1}{2}, 1 - \frac{n}{2}, \frac{3}{2}; t^2x^2\right) dx. \end{aligned}$$

Let $a_n x^{2n}$ be a general term of the series $F(1/2, 1 - n/2, 3/2; t^2 x^2)$. Then we arrive at the relation

$$F\left(\frac{1}{2}, 1 - \frac{n}{2}, \frac{3}{2}; t^2 x^2\right) = 1 + b_1 \cdot \frac{t^2 x^2}{1!} + b_2 \cdot \frac{t^4 x^4}{2!} + \cdots + b_n \cdot \frac{t^{2n} x^{2n}}{n!} + \cdots,$$

where $b_n = n! a_n$. Since

$$\int_0^1 x^2 \sqrt{1-x^2}^{m-2} dx = \frac{1}{m+1} \cdot \frac{\text{vol}(S^m)}{2 \text{vol}(S^{m-1})},$$

we have

$$\begin{aligned} & \int_0^1 x^2 \sqrt{1-x^2}^{m-2} F\left(\frac{1}{2}, 1 - \frac{n}{2}, \frac{3}{2}; t^2 x^2\right) dx \\ &= \frac{\text{vol}(S^m)}{2(m+1) \text{vol}(S^{m-1})} \cdot F\left(\frac{1}{2}, 1 - \frac{n}{2}, \frac{3+m}{2}; t^2\right). \end{aligned}$$

So we obtain

$$\begin{aligned} (3.8) \quad & \int_{SO(m+1)} (k_{11})^2 F\left(\frac{1}{2}, 1 - \frac{n}{2}, \frac{3}{2}; t^2 (k_{11})^2\right) d\mu_{SO(m+1)}(k) \\ &= \text{vol}(SO(m)) \text{vol}(S^m) \cdot \frac{1}{m+1} \cdot F\left(\frac{1}{2}, 1 - \frac{n}{2}, \frac{3+m}{2}; t^2\right). \end{aligned}$$

A simple calculation shows that

$$\begin{aligned} & F\left(\frac{1}{2}, -\frac{n}{2}, \frac{1+m}{2}; t^2\right) + \frac{nt^2}{m+1} F\left(\frac{1}{2}, 1 - \frac{n}{2}, \frac{3+m}{2}; t^2\right) \\ &= F\left(-\frac{1}{2}, -\frac{n}{2}, \frac{1+m}{2}; t^2\right). \end{aligned}$$

As the result, from (3.7), (3.8) and the last equality, we have

$$\begin{aligned} & \int_{SO(m+1)} \int_{SO(n+1)} |tk_{11} + g_{11}| d\mu_{SO(n+1)}(g) d\mu_{SO(m+1)}(k) \\ &= \frac{2 \text{vol}(K) \text{vol}(S^{n-1})}{n \text{vol}(S^n)} F\left(-\frac{1}{2}, -\frac{n}{2}, \frac{1+m}{2}; t^2\right). \end{aligned}$$

It remains to compute the case where $t \geq 1$ of (3.2). In this case we may compute the following:

$$t \int_{SO(m+1)} \int_{SO(n+1)} \left| k_{11} + \frac{1}{t} g_{11} \right| d\mu_{SO(n+1)}(g) d\mu_{SO(m+1)}(k).$$

This integration is nothing but (3.2) which replace m with n . Hence we immediately obtain

$$t \cdot \frac{2 \operatorname{vol}(K) \operatorname{vol}(S^{m-1})}{m \operatorname{vol}(S^m)} F\left(-\frac{1}{2}, -\frac{m}{2}, \frac{1+n}{2}; \frac{1}{t^2}\right).$$

Therefore we have

$$\sigma(\theta, \tau) = \begin{cases} 2 \operatorname{vol}(K)c \cdot \frac{\operatorname{vol}(S^{n-1})}{n \operatorname{vol}(S^n)} F\left(-\frac{1}{2}, -\frac{n}{2}, \frac{1+m}{2}, \frac{s^2}{c^2}\right), & \text{if } s \leq c; \\ 2 \operatorname{vol}(K)s \cdot \frac{\operatorname{vol}(S^{m-1})}{m \operatorname{vol}(S^m)} F\left(-\frac{1}{2}, -\frac{m}{2}, \frac{1+n}{2}, \frac{c^2}{s^2}\right), & \text{if } s \geq c, \end{cases}$$

which implies Theorem 1.1.

Up to this point, we unrestrainedly used the notation $\operatorname{vol}(SO(n+1))$ the volume of the special orthogonal group $SO(n+1)$. This value is given by

$$\operatorname{vol}(SO(n+1)) = \operatorname{vol}(SO(n)) \cdot \operatorname{vol}(S^n).$$

EXAMPLE. The case where $m = n = 1$ in (3.2). It is well known that

$$K(k) = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}, 1; k^2\right), \quad E(k) = \frac{\pi}{2} F\left(-\frac{1}{2}, \frac{1}{2}, 1; k^2\right),$$

where $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kind respectively. By a simple calculation we have

$$K(k) - E(k) = \frac{\pi}{4} k^2 F\left(\frac{1}{2}, \frac{3}{2}, 2; k^2\right).$$

Hence, for $0 < t < 1$, we have

$$\begin{aligned} \sigma(\theta, \tau) &= 2c \operatorname{vol}(SO(2))^2 \cdot \frac{\operatorname{vol}(S^0)}{\operatorname{vol}(S^1)} \cdot F\left(-\frac{1}{2}, -\frac{1}{2}, 1; t^2\right) \\ &= 16c(2E(t) - (1 - t^2)K(t)). \end{aligned}$$

REMARK. Let $M = S^1$ and $N = S^m \times S^{n+1}$ in Theorem 1.1. Then, for almost all $g \in G = SO(m+2) \times SO(n+2)$, we have $\sharp(M \cap gN) = 2$. Thus we have

$$\int_G \sharp(M \cap gN) d\mu_G(g) = 2 \operatorname{vol}(G).$$

Finally we can give the following inequalities as an application of the integral formula in Theorem 1.1.

Corollary 3.2. *Put $G = SO(m+2) \times SO(n+2)$. Under the hypothesis of Theorem 1.1:*

(1) *If $N = S^m \times S^{n+1}$ then we have*

$$\frac{1}{\text{vol}(G)} \int_G \#(M \cap gN) d\mu_G(g) \leq 2 \cdot \frac{\text{vol}(M)}{\text{vol}(S^1)}.$$

The inequality becomes an equality if and only if M is a curve in S^{m+1} .

(2) *If $M = S^1 \subset S^{m+1}$ then we have*

$$\frac{1}{\text{vol}(G)} \int_G \#(M \cap gN) d\mu_G(g) \leq 2 \cdot \frac{\text{vol}(N)}{\text{vol}(S^m \times S^{n+1})}.$$

The equality holds if and only if N is a submanifold of $L \times S^{n+1}$. Here L is a submanifold in S^{m+1} .

Proof. (1) In this case we can take $\sin \theta_x e_1 + \cos \theta_x e_{m+2}$ and e_2, \dots, e_{m+n+2} as an orthonormal basis of $T_x M$ and $T_y N$ respectively. Here e_1, \dots, e_{m+n+2} is the standard orthonormal basis of \mathbf{R}^{m+n+2} . Hence we obtain

$$\sigma(x, y) = \frac{\text{vol}(S^{m-1})}{m \text{vol}(S^m)} \sin \theta_x.$$

We therefore have

$$\begin{aligned} \int_G \#(M \cap gN) d\mu_G(g) &= 2 \text{vol}(K) \frac{\text{vol}(S^{m-1})}{m \text{vol}(S^m)} \text{vol}(N) \int_M \sin \theta_x d\mu_M(x) \\ &= \text{vol}(G) \cdot \frac{1}{\pi} \int_M \sin \theta_x d\mu_M(x). \end{aligned}$$

(2) In this case we can obtain

$$\sigma(x, y) = \frac{\text{vol}(S^{m-1})}{m \text{vol}(S^m)} \sin \tau_y.$$

This, by a computation similar to that in (1), completes the proof. □

References

- [1] R. Howard: *The kinematic formula in Riemannian homogeneous spaces*, Mem. Amer. Math. Soc. **106** (1993), vi+69.
- [2] H.J. Kang and H. Tasaki: *Integral geometry of real surfaces in complex projective spaces*, Tsukuba J. Math. **25** (2001), 155–164.

- [3] H.J. Kang and H. Tasaki: *Integral geometry of real surfaces in the complex projective plane*, Geom. Dedicata **90** (2002), 99–106.
- [4] W. Miller, Jr.: *Lie Theory and Special Functions*, Academic Press, New York, 1968.
- [5] L.A. Santaló: *Integral geometry in Hermitian spaces*, Amer. J. Math. **74** (1952), 423–434.
- [6] L.A. Santaló: *Integral Geometry and Geometric Probability*, Addison-Wesley, London, 1977.
- [7] H. Tasaki: *Generalization of Kähler angle and integral geometry in complex projective spaces*, Steps in Differential Geometry (Debrecen, 2000), 349–361, Inst. Math. Inform., 2001.
- [8] H. Tasaki: *Integral Geometry under the action of the first symplectic group*, Archiv der Mathematik **80** (2003), 106–112.
- [9] H. Umemura: *Theory of Elliptic Functions: Analysis of Elliptic Curves* (in Japanese), University of Tokyo Press, 2000.

Institute of Mathematics
University of Tsukuba
Tsukuba, Ibaraki
305-8571, Japan
e-mail: kang@math.tsukuba.ac.jp

Current address:
Major in Mathematics
Dong-A University
Saha-ku, Busan
604-714, Korea