| | |
|---|---|
| Title | An inverse boundary value problem for the stationary transport equation |
| Author(s) | Choulli, Mourad; Stefanov, Plamen |
| Citation | Osaka Journal of Mathematics. 1999, 36(1), p. 87-104 |
| Version Type | VoR |
| URL | https://doi.org/10.18910/3941 |
| rights | |
| Note | |

Choulli, M. and Stefanov, P.
Osaka J. Math.
**36** (1999), 87–104

# AN INVERSE BOUNDARY VALUE PROBLEM FOR THE STATIONARY TRANSPORT EQUATION

MOURAD CHOULLI and PLAMEN STEFANOV

(Received June 12, 1997)

## 1. Introduction

Let $X \subset \mathbf{R}^n$, $n \geq 2$ be an open bounded set with $C^1$ boundary $\partial X$ and let us assume also that $V \subset \mathbf{R}^n$ is open. Denote $\Gamma_\pm = \{(x,v) \in \partial X \times V;\ \pm n(x) \cdot v > 0\}$, where $n(x)$ is the outer normal to $\partial X$ at $x \in \partial X$. Let us denote by $f$ the solution (if exists) to the following boundary value problem for the stationary linear transport (Boltzmann) equation:

$$(1.1)\quad \begin{cases} -v \cdot \nabla_x f(x,v) - \sigma_a(x,v) f(x,v) + \displaystyle\int_V k(x,v',v) f(x,v')\, dv' = 0 & \text{in } X \times V, \\ \hfill f|_{\Gamma_-} = f_-. \end{cases}$$

Here $f_-$ is a given function on $\Gamma_-$. We assume that the pair $(\sigma_a, k)$ is *admissible*, i.e.

(i) $0 \leq \sigma_a \in L^\infty(X \times V)$,

(ii) $0 \leq k(x,v',\cdot) \in L^1(V)$ for a.e. $(x,v') \in X \times V$ and $\sigma_p(x,v') := \int_V k(x,v',v)\, dv$ belongs to $L^\infty(X \times V)$.

If the direct problem (1.1) is solvable, one can define the following *albedo* operator

$$\mathcal{A} : f_- \mapsto f|_{\Gamma_+},$$

that maps the incoming flux on the boundary into the outgoing one. We are interested in the following inverse problem:

**(IP)** Does the albedo operator $\mathcal{A}$ determine uniquely the coefficients $\sigma_a(x,v)$, $k(x,v',v)$?

There are a lot of papers devoted to (IP) both for the stationary transport equation (1.1) and for the time-dependent one (see e.g. [1], [6], [9], [10], [11], [12], [14]). In those papers however there are some restrictive assumptions on the coefficients $k(x,v',v)$ and $\sigma_a$, for example $k$ is assumed to be independent of some variables, small enough etc. To our best knowledge the general case has been open until the authors considered in [4] the inverse problem (IP) for the time-dependent transport equation $(\partial_t - T)u = 0$, where $T$ is the differential operator appearing in the left hand side of the first equation in (1.1). It is shown [4] that the (time-dependent) albedo operator does determine uniquely $\sigma_a$ and $k$ provided that $\sigma_a = \sigma_a(x, |v|)$. It is easy to see that some restriction on $\sigma_a$ is unavoidable, because there is a simple example showing that in the general case the uniqueness may fail (see below). In [4] it is also studied the inverse scattering problem for the time-dependent transport equation.

In the stationary case that we consider in this paper we have less data than in [4], because the time variable is missing. Next, the same objection to the uniqueness as in the time-dependent case exists. Namely, let $p = p(x, v)$ be a continuous function such that for $(x, v) \in X \times V$ we have $(x + p(x, v)v, v) \in X \times V$ as well and define $\hat{\sigma}_a(x, v) = \sigma_a(x + p(x, v)v, v)$. Then the pairs $(\sigma_a, 0)$ and $(\hat{\sigma}_a, 0)$ (i.e. $k = \hat{k} = 0$) produce the same albedo operator, while $\sigma_a \neq \hat{\sigma}_a$ in general. In order to avoid this, we will assume further that $\sigma_a = \sigma_a(x, |v|)$.

Let us introduce some notations. For $v \neq 0$, set $\tau_\pm(x, v) = \min\{t \geq 0;\ x \pm tv \in \partial X\}$, $\tau = \tau_- + \tau_+$. Consider the following two measures on $\Gamma_\pm$: $d\xi = |n(x) \cdot v| d\mu(x) dv$ and $d\tilde{\xi} = \min\{\tau(x, v), \lambda\} |n(x) \cdot v| d\mu(x) dv$, where $d\mu(x)$ is the Lebesgue measure on $\partial X$, $\lambda > 0$ is an arbitrary constant. Using some trace theorems [2], [3] we can show that $\mathcal{A} : L^1(\Gamma_-, d\xi) \to L^1(\Gamma_+, d\xi)$ is a bounded operator if (1.2) holds and $\mathcal{A} : L^1(\Gamma_-, d\tilde{\xi}) \to L^1(\Gamma_+, d\tilde{\xi})$ is bounded if (1.3) holds (see also [7]).

In general, the direct problem (1.1) may not be uniquely solvable, so we consider in this work two physically important situations where (1.1) is well posed. First we will assume that

$$\|\tau\sigma_a\|_{L^\infty} < \infty, \quad \|\tau\sigma_p\|_{L^\infty} < \infty \quad \text{and} \quad \|\tau\sigma_p\|_{L^\infty} < 1. \tag{1.2}$$

This condition in particular guarantees that the dynamics corresponding to the time-dependent Boltzmann equation is subcritical [13], i.e. the "energy" (the $L^1$-norm of the solution) is uniformly bounded for $t > 0$. Note that (1.2) holds if in particular $\||v|^{-1}\sigma_a\|_{L^\infty} < \infty$, $\mathrm{diam}(X)\||v|^{-1}\sigma_p\|_{L^\infty} < 1$. The second situation we will consider is when [7]

$$\sigma_a(x, v) - \sigma_p(x, v) \geq \nu > 0 \quad \text{for a.e. } (x, v) \in X \times V \tag{1.3}$$

with some $\nu > 0$. In other words, (1.3) says that the absorption rate is greater than the production rate. This also implies that the corresponding dynamics is subcritical.

The main result of this paper is the following.

**Theorem 1.1.** *Let $(\sigma_a, k)$, $(\hat{\sigma}_a, \hat{k})$ be two admissible pairs with $\sigma_a = \sigma_a(x, |v|)$, $\hat{\sigma}_a = \hat{\sigma}_a(x, |v|)$ and assume that they satisfy either* (1.2) *or* (1.3). *Assume that the corresponding albedo operators $\mathcal{A}$ and $\hat{\mathcal{A}}$ coincide. Then*

(a) *if $n \geq 3$, then $\sigma_a = \hat{\sigma}_a$, $k = \hat{k}$;*

(b) *if $n = 2$, then $\sigma_a = \hat{\sigma}_a$.*

Note that our proof is constructive and we obtain explicit formulas for $\sigma_a$ and $k$

in terms of the distribution kernel $\alpha$ of the albedo operator (see Proposition 4.1 and Proposition 4.2). We study the first two terms in the singular decomposition of $\alpha$ and show in Theorem 3.2 that they are delta functions supported on varieties of different dimension (if $n \geq 3$), while the remainder is a function. We show that the first singular term depends on $\sigma_a$ only and $\sigma_a$ can be recovered from it. Next, knowing $\sigma_a$, one can recover $k$ from the second term. We followed similar approach in [4] in the time-dependent case. In the two-dimensional case the second term is not a delta function, but a locally $L^1$-function and cannot be distinguished from the remainder. That is why our approach does not work for recovering $k$ in two dimensions. Notice that the idea of using singular solutions of the transport equation has been used also in [10], [14].

The results we obtain are not restricted to the conditions (1.2) or (1.3) only. If for example $\inf\{|v|;\ v \in V\} > 0$ and if the direct problem $Tf = g$, $f|_{\Gamma_-} = 0$ has unique solution $f \in L^1_{\text{loc}}(X \times V)$ for any $g \in L^1(X \times V)$, then Theorem 1.1 still holds. Theorem 1.1 holds in the presence of a (non-singular) source term as well, because the flux generated by such a source term is not singular and cannot affect the leading singularities of the kernel of $\mathcal{A}$. We would like to mention also that a similar result can be obtained if $V = S^{n-1}$, $n \geq 3$.

The structure of the paper is the following. In section 2. we prove some trace theorems of the type obtained in [2], [3], solve the direct problem and define the albedo operator in suitable $L^1$ spaces. In section 3. we construct a special solution $f = f(x, v, x', v')$ to (1.1) with $f_- = \delta_{\{x'\}}(x)\delta(v - v')$, where $(x', v') \in \Gamma_-$ are parameters and we study the singularities of $f$. With the aid of $f$ we obtain a singular decomposition of the distribution kernel $\alpha$ of $\mathcal{A}$. Finally, in section 4. we show how to recover explicitly $\sigma_a$, $k$ from $\alpha$.

The results of this paper have been announced in [5].

## 2. Preliminaries

We begin this section with a simple lemma.

**Lemma 2.1.** *Assume that $f \in L^1(X \times V)$. Then*

$$\int_{X\times V} f(x,v)\,dx\,dv = \int_{\Gamma_\mp}\int_0^{\tau_\pm(x',v)} f(x' \pm tv, v)\,dt\,d\xi(x',v).$$

Proof. The proof follows by performing a change of variables $X \ni x \mapsto (x', t) \in \Gamma_\mp \times (0, \tau_\pm(x, v))$ given by the formula $x' = x \mp \tau_\mp(x, v)v$, $t = \tau_\mp(x, v)$. This change is smooth except on a closed set of measure zero and $dx = |n(x')\cdot v|d\mu(x')\,dt$. □

Let us introduce some new notations. Denote

$$T_0 f = -v \cdot \nabla_x f, \qquad A_1 f = -\sigma_a f, \qquad A_2 f = \int_V k(x, v', v) f(x, v')\, dv',$$
$$T_1 = T_0 + A_1, \qquad T = T_0 + A_1 + A_2 = T_1 + A_2.$$

Notice that if $(\sigma_a, k)$ is admissible (which we will always assume), then $A_1$ and $A_2$ are bounded operators in $L^1(X \times V)$. All norms throughout this paper are in $L^1(X \times V)$ except otherwise stated.

We are going to prove next a trace theorem in the spirit of [2], [3].

**Theorem 2.1.** *We have*

$$\left\| f|_{\Gamma_\pm} \right\|_{L^1(\Gamma_\pm, d\xi)} \le \|T_0 f\| + \|\tau^{-1} f\| \tag{2.1}$$

*for any function $f(x,v)$, $(x,v) \in X \times V$ for which the right-hand side above is well-defined.*

Proof. Consider first a function $g \in L^1([0,a])$, such that $g' \in L^1([0,a])$, where $a > 0$. Then

$$g(0) = -\int_0^t g'(x)\, dx + g(t),$$

therefore

$$|g(0)| \le \int_0^a |g'(x)|\, dx + |g(t)| = \|g'\| + |g(t)| \qquad \forall t \in [0,a].$$

Here $\|\cdot\|$ is the norm in $L^1([0,a])$. After integrating that inequality we get

$$|g(0)| = \frac{1}{a} \int_0^a |g(0)|\, dt \le \|g'\| + \frac{1}{a}\|g\|. \tag{2.2}$$

Now, let $f(x,v)$ be such that $T_0 f \in L^1(X \times V)$ and $\tau^{-1} f \in L^1(X \times V)$. Set

$$g(t, x', v) = f(x' + tv, v), \quad (x', v) \in \Gamma_-, \quad 0 \le t \le \tau_+(x', v).$$

Since $T_0 f \in L^1(X \times V)$, combining Fubini's theorem and Lemma 2.1, we get that for a.e. $(x', v) \in \Gamma_-$, the function $t \mapsto \partial_t g = -(T_0 f)(x' + tv, v)$ belongs to $L^1(0, \tau_+(x', v))$ and so does $g(t, x', v)$. Next, $g(0, x', v) = f(x', v) = f|_{\Gamma_-}$. We therefore get from (2.2)

$$|f(x', v)| \le \int_0^{\tau_+(x', v)} |(T_0 f)(x' + tv, v)| dt$$

$$+\frac{1}{\tau_+(x',v)}\int_0^{\tau_+(x',v)}|f(x'+tv,v)|\,dt.$$

If we now integrate over $\Gamma_-$ with respect to $d\xi(x',v)$ and apply Lemma 2.1 to the right-hand side, we complete the proof of the theorem for $f|_{\Gamma_-}$. The proof for $f|_{\Gamma_+}$ is similar. □

Let us set

$$\mathcal{W}=\{f;\ T_0 f\in L^1(X\times V),\ \tau^{-1}f\in L^1(X\times V)\},$$
$$\|f\|_{\mathcal{W}}=\|T_0 f\|+\|\tau^{-1}f\|.$$

Then Theorem 2.1 says that taking the trace $f\mapsto f|_{\Gamma_\pm}$ is a continuous operator from $\mathcal{W}$ into $L^1(\Gamma_\pm,d\xi)$. As we will see below, the inequality (2.1) cannot be improved, for some functions it turns into equality.

Given $f_-\in L^1(\Gamma_-,d\xi)$, define $Jf_-$ as the following prolongation of $f_-$ inside $X\times V$:

$$Jf_-=e^{-\int_0^{\tau_-(x,v)}\sigma_a(x-sv,v)ds}f_-(x-\tau_-(x,v)v,v),\qquad (x,v)\in X\times V.$$

Note that $Jf_-$ is defined so that $T_1Jf_-=0$, $Jf_-|_{\Gamma_-}=f_-$, therefore $J$ is the solution operator of the problem $T_1 f=0$, $f|_{\Gamma_-}=f_-$.

**Proposition 2.1.** *Assume that* $\|\tau\sigma_a\|_{L^\infty}<\infty$. *Then*

$$\|Jf_-\|_{\mathcal{W}}\le C\|f_-\|_{L^1(\Gamma_-,d\xi)},$$

*with* $C=1+\|\tau\sigma_a\|_{L^\infty}$. *If* $\sigma_a=0$, *then we have equality above (and* $C=1$*).*

Proof. Note first that

$$\|Jf_-\|_{\mathcal{W}}\le\|\tau^{-1}Jf_-\|+\|A_1Jf_-\|\le(1+\|\tau\sigma_a\|_{L^\infty})\,\|\tau^{-1}Jf_-\|,$$

because $T_0Jf_-=-A_1Jf_-$. Next, by Lemma 2.1,

$$\begin{aligned}(2.3)\quad \int_{X\times V}\left|\tau^{-1}Jf_-\right|dx\,dv&\le\int_{X\times V}\tau^{-1}(x,v)|f_-(x-\tau_-(x,v)v,v)|\,dx\,dv\\&=\int_{\Gamma_-}\int_0^{\tau_+(x',v)}\tau_+^{-1}(x',v)|f_-(x',v)|\,dt\,d\xi\\&=\|f_-\|_{L^1(\Gamma_-,d\xi)},\end{aligned}$$

which completes the proof. □

We are going to use Theorem 2.1 and Proposition 2.1 in the case where (1.2) holds. Then we will work with functions belonging to $L^1(\Gamma_\pm, d\xi)$ on the boundary and to $\mathcal{W}$ inside $X \times V$. In the case where (1.3) holds, we will work with boundary data in $L^1(\Gamma_\pm, d\tilde{\xi})$ and then we will need the following trace theorem due to Cessenat [2], [3]. Recall that $d\tilde{\xi} = \min\{\lambda, \tau(x,v)\}d\xi$, $\lambda > 0$ is an arbitrary fixed constant.

**Theorem 2.2** ([2], [3]). *For any $f \in L^1(X \times V)$ such that $T_0 f \in L^1(X \times V)$ we have*

$$\|f|_{\Gamma_\pm}\|_{L^1(\Gamma_\pm, d\tilde{\xi})} \le \lambda \|T_0 f\| + \|f\|.$$

We refer to [2], [3] for a proof and for other results. The theorem follows also from Theorem 2.1 by setting $f = \min\{\lambda, \tau\}g$.

Denote

$$\tilde{\mathcal{W}} = \{f;\ f \in L^1(X \times V), T_0 f \in L^1(X \times V)\},$$
$$\|f\|_{\tilde{\mathcal{W}}} = \|T_0 f\| + \|f\|.$$

In the case where (1.3) holds, we have the following version of Proposition 2.1.

**Proposition 2.2.** *For any $f_- \in L^1(\Gamma_-, d\tilde{\xi})$*

$$\|Jf_-\|_{\tilde{\mathcal{W}}} \le C\|f_-\|_{L^1(\Gamma_-, d\tilde{\xi})},$$

*where $C = (1 + \|\sigma_a\|_{L^\infty}) \max\{1, (\nu\lambda)^{-1}\}$.*

Proof. We obtain as before that $\|Jf_-\|_{\tilde{\mathcal{W}}} \le (1 + \|\sigma_a\|_{L^\infty})\|Jf_-\|$. Next,

$$\begin{aligned}
\int_{X\times V} |Jf_-|\, dx\, dv &\le \int_{X\times V} e^{-\nu\tau_-(x,v)}|f_-(x - \tau_-(x,v)v, v)|\, dx\, dv \\
&= \int_{\Gamma_-}\int_0^{\tau_+(x',v)} e^{-\nu t}|f_-(x',v)|\, dt\, d\xi \\
&\le \int_{\Gamma_-} \frac{\min\{\tau_+(x',v), 1/\nu\}}{\min\{\tau_+(x',v), \lambda\}}|f_-(x',v)|\, d\tilde{\xi} \\
&\le \max\left\{1, \frac{1}{\nu\lambda}\right\}\|f_-\|_{L^1(\Gamma_-, d\tilde{\xi})}. \qquad \square
\end{aligned}$$

We are going now to reduce the boundary value problem (1.1) to an integral equation using standard arguments. Equation (1.1) can be rewritten as $(T_1 + A_2)f = 0$. Let us integrate the identity $\exp\{-\int_0^t \sigma_a(x - sv, v)ds\}[(T_1 + A_2)f](x - tv, v) = 0$ with respect to $t$ from 0 to $\tau_-(x,v)$ and take into account the boundary condition $f|_{\Gamma_-} = f_-$.

We thus get

$$(I+K)f = Jf_-, \tag{2.4}$$

where $I$ stands for the identity and $K$ is the following integral operator

$$Kf = -\int_0^{\tau_-(x,v)} e^{-\int_0^t \sigma_a(x-sv,v)ds}(A_2 f)(x-tv,v)\,dt. \tag{2.5}$$

Introduce the following unbounded operators:

$$\begin{aligned} \mathbf{T_1}f &= T_1 f,\ D(\mathbf{T_1}) = \{f \in L^1(X\times V);\ T_1 f \in L^1(X\times V),\ f|_{\Gamma_-} = 0\},\\ \mathbf{T}f &= Tf,\ \ D(\mathbf{T}) \ = \{f \in L^1(X\times V);\ Tf \in L^1(X\times V),\ f|_{\Gamma_-} = 0\}. \end{aligned}$$

Notice that formally $K = \mathbf{T_1^{-1}}A_2$ and for $\mathbf{T_1^{-1}}$ we have

$$\mathbf{T_1^{-1}}f = -\int_0^{\tau_-(x,v)} e^{-\int_0^t \sigma_a(x-sv,v)ds} f(x-tv,v)\,dt.$$

In the next two propositions we will show in particular that $\mathbf{T_1^{-1}}$ is well defined.

**Proposition 2.3.** *Assume* (1.2). *Then*

(a) $\tau^{-1}\mathbf{T_1^{-1}}$, $\tau^{-1}\mathbf{T}^{-1}$ *and* $A_2\tau$ *are bounded operators in* $L^1(X\times V)$ *and therefore* $K = \mathbf{T_1^{-1}}A_2$ *is a bounded operator in* $L^1(X\times V;\ \tau^{-1}dxdv)$. *Moreover, the operator norm of* $K$ *is not greater than* $\|\tau\sigma_p\|_{L^\infty} < 1$ *and therefore* $(I+K)^{-1}$ *exists in this space.*

(b) *The integral equation* (2.4) *and therefore the boundary value problem* (1.1) *are uniquely solvable for any* $f_- \in L^1(\Gamma_-, d\xi)$ *and then* $f \in \mathcal{W}$.

(c) *The albedo operator* $\mathcal{A}$ *is a bounded map* $\mathcal{A} : L^1(\Gamma_-, d\xi) \to L^1(\Gamma_+, d\xi)$.

Proof. Clearly,

$$\|\tau^{-1}\mathbf{T_1^{-1}}f\| \le \|f\|, \qquad \forall f \in L^1(X\times V).$$

Next, (1.2) implies

$$\|A_2\tau\|_{\mathcal{L}(L^1(X\times V))} \le \|\tau\sigma_p\|_{L^\infty} < 1,$$

where $\tau$ stands for the operator of multiplication by $\tau(x,v)$. Therefore, we have $K = \mathbf{T_1^{-1}}A_2$ and

$$\|\tau^{-1}Kf\| = \|\tau^{-1}\mathbf{T_1^{-1}}A_2 f\| \le \|A_2 f\| \le \|\tau\sigma_p\|_{L^\infty}\|\tau^{-1}f\|,$$

which proves (a). Since $\|\tau\sigma_p\|_{L^\infty} < 1$, the operator $I + K$ is invertible in $L^1(X \times V;\ \tau^{-1}dxdv)$ and (2.4) has unique solution

$$f = (I + K)^{-1} J f_-. \tag{2.6}$$

By (2.4) we have for $f$

$$\begin{aligned} \|\tau^{-1} f\| &\le \left(1 - \|\tau\sigma_p\|_{L^\infty}\right)^{-1} \|\tau^{-1} J f_-\| \\ &\le \left(1 - \|\tau\sigma_p\|_{L^\infty}\right)^{-1} \|f_-\|_{L^1(\Gamma_-, d\xi)}. \end{aligned} \tag{2.7}$$

Further, since $Tf = 0$, we deduce that $T_0 f = -(A_1 + A_2) f$ and therefore

$$\|T_0 f\| \le \left(\|\tau\sigma_a\|_{L^\infty} + \|\tau\sigma_p\|_{L^\infty}\right) \|\tau^{-1} f\|. \tag{2.8}$$

Then by (2.7) and (2.8) we deduce that $f \in \mathcal{W}$ and by Theorem 2.1 we conclude that $f|_{\Gamma_+} = \mathcal{A} f_- \in L^1(\Gamma_+, d\xi)$. Moreover, the operator norm of $\mathcal{A}$ does not exceed $(1 + \|\tau\sigma_a\|_{L^\infty} + \|\tau\sigma_p\|_{L^\infty})\left(1 - \|\tau\sigma_p\|_{L^\infty}\right)^{-1}$. We finally note that $\mathbf{T}$ is invertible, because we can set $\mathbf{T}^{-1} = (I + K)^{-1} T_1^{-1} : L^1(X \times V) \to L^1(X \times V;\ \tau^{-1} dxdv)$. □

**Proposition 2.4.** *Assume* (1.3). *Then*

(a) $K$, $\mathbf{T}_1^{-1}$ *and* $\mathbf{T}^{-1}$ *are bounded operators in* $L^1(X \times V)$ *and* $K = \mathbf{T}_1^{-1} A_2$. *Further,* $I + K$ *is invertible and* $(I + K)^{-1} = I - \mathbf{T}^{-1} A_2$.

(b) *The integral equation* (2.4) *and therefore the boundary value problem* (1.1) *are uniquely solvable for any* $f_- \in L^1(\Gamma_-, d\tilde{\xi})$ *and then* $f \in \tilde{\mathcal{W}}$.

(c) $\mathcal{A}$ *is a bounded map* $\mathcal{A} : L^1(\Gamma_-, d\tilde{\xi}) \to L^1(\Gamma_+, d\tilde{\xi})$.

Proof. Since $\exp\{-\int_0^t \sigma_a(x - sv, v) ds\} \le e^{-\nu t}$, $\|\mathbf{T}_1^{-1}\|_{\mathcal{L}(L^1(X \times V))} \le \nu^{-1}$. The boundedness of $\mathbf{T}^{-1}$ is proven in [7] under the assumption (1.3). Next, it is easy to check directly that $(I + K)(I - \mathbf{T}^{-1} A_2) = (I - \mathbf{T}^{-1} A_2)(I + K) = I$, therefore $(I + K)^{-1} = I - \mathbf{T}^{-1} A_2$. This proves (a). Observe next that if $f_- \in L^1(\Gamma_-, d\tilde{\xi})$, then by Proposition 2.4 $J f_- \in \tilde{\mathcal{W}}$ and the solution $f$ to (2.4) is given by $f = (I - \mathbf{T}^{-1} A_2) J f_- \in L^1(X \times V)$. Moreover, since $Tf = 0$, we have $T_0 f = -(A_1 + A_2) f \in L^1(X \times V)$. Therefore $T_0 f \in \tilde{\mathcal{W}}$ and by Theorem 2.2, $\mathcal{A} f_- = f|_{\Gamma_+} \in L^1(\Gamma_+, d\tilde{\xi})$. □

## 3. The special solution

In this section we study the distribution kernel of $\mathcal{A}$. To this end first we solve (1.1) with

$$f_- = \delta_{\{x'\}}(x) \delta(v - v'),$$

where $(x', v') \in \Gamma_-$ are regarded as parameters and $\delta_{\{x'\}}$ is a distribution on $\partial X$ defined by $(\delta_{\{x'\}}, \varphi) = \int \delta_{\{x'\}}(x)\varphi(x)d\mu(x) = \varphi(x')$. On the other hand, we will denote by $\delta$ the ordinary Dirac delta function in $\mathbf{R}^n$. The integral above is to be considered in distribution sense. Let us denote by $f(x, v, x', v')$ the solution (in distribution sense) of

$$\begin{cases} Tf = 0 & \text{in } X \times V, \\ f|_{\Gamma_-} = f_-. \end{cases} \tag{3.1}$$

Since we assume that the boundary is $C^1$-smooth and not necessarily $C^\infty$-smooth, we have to work in fact with distributions which are linear functionals on $C_0^1$. All singular distributions appearing in the analysis of $f$ and the kernel of $\mathcal{A}$ are in fact delta-type distributions supported on some hypersurfaces and can be regarded also as measures.

To solve (3.1), pick $\varphi_- \in C_0^1(\Gamma_-)$ and denote by $\varphi$ the solution of

$$\begin{cases} T\varphi = 0 & \text{in } X \times V, \\ \varphi|_{\Gamma_-} = \varphi_-. \end{cases} \tag{3.2}$$

Assume that either (1.2) or (1.3) holds. Then, according to Proposition 2.3 and Proposition 2.4, equation (3.2) has unique solution $\varphi = (I + K)^{-1}J\varphi_-$, which also admits the representation

$$\varphi = J\varphi_- - KJ\varphi_- + (I + K)^{-1}K^2 J\varphi_-.$$

Since $(I + K)^{-1}\mathbf{T}_1^{-1} = \mathbf{T}^{-1}$, $K = \mathbf{T}_1^{-1}A_2$, we get $(I + K)^{-1}K^2J\varphi_- = (I + K)^{-1}\mathbf{T}_1^{-1}A_2KJ\varphi_- = \mathbf{T}^{-1}A_2KJ\varphi_-$. Therefore,

$$\varphi = J\varphi_- - KJ\varphi_- + \mathbf{T}^{-1}A_2KJ\varphi_-. \tag{3.3}$$

All terms in (3.3) belong to $L^1(X \times V;\ \tau^{-1}dxdv)$ or $L^1(X \times V)$, respectively. We proceed with analysis of each term in the right-hand side of (3.3).

For $J\varphi_-$ we have

$$\begin{aligned} J\varphi_- &= E(x, v)\varphi_-(x - \tau_-(x, v)v, v), \\ E(x, v) &:= \exp\left\{ -\int_0^{\tau_-(x,v)} \sigma_a(x - sv, v)ds \right\}. \end{aligned}$$

Choose $\psi \in C_0^\infty(X \times V)$ and consider

$$\begin{aligned} (J\varphi_-, \psi) &= \int_{X\times V} (J\varphi_-)\psi \, dx \, dv \\ &= \int_{X\times V} E(x, v)\varphi_-(x - \tau_-(x, v)v, v)\psi(x, v) \, dx \, dv. \end{aligned}$$

By Lemma 2.1,

$$
\begin{aligned}
(J\varphi_-,\psi) &= \int_{\Gamma_-}\int_0^{\tau_+(x',v)} E(x'+tv,v)\varphi_-(x',v)\psi(x'+tv,v)\,dt\,d\xi(x',v)\\
&= \int\!\!\dots\!\!\int E(x,v)\varphi_-(x',v')\psi(x,v)\delta(x-x'-tv)\delta(v-v')\,dt\,d\xi(x',v')\,dx\,dv,
\end{aligned}
$$

where the last integral is to be considered in the sense of distributions. Therefore,

$$
\begin{aligned}
J\varphi_- = \int_{\Gamma_-}&\left(\int_0^{\tau_+(x',v')} E(x,v)\delta(x-x'-tv)\delta(v-v')\,dt\right)\\
&\times|v'\cdot n(x')|\varphi_-(x',v')\,d\mu(x')\,dv'.
\end{aligned}
$$

Thus,

$$
J\varphi_- = \int_{\Gamma_-} f_1(x,v,x',v')\varphi_-(x',v')\,d\mu(x')\,dv' \tag{3.4}
$$

with

$$
\begin{aligned}
f_1(x,v,x',v') = |v'\cdot n(x')|&\int_0^{\tau_+(x',v')} e^{-\int_0^{\tau_-(x,v)}\sigma_a(x-sv,v)ds}\\
&\times\delta(x-x'-tv)\delta(v-v')\,dt.
\end{aligned} \tag{3.5}
$$

In other words, $f_1 = Jf_-$ is given by (3.5).

Consider further the second term $-KJ\varphi_-$ in the right-hand side of (3.3).

$$
\begin{aligned}
-KJ\varphi_- &= \int_0^{\tau_-(x,v)}\int_V e^{-\int_0^s\sigma_a(x-pv,v)dp}\\
&\quad\times k(x-sv,v',v)(J\varphi_-)(x-sv,v')\,dv'\,ds\\
&= \int_0^{\tau_-(x,v)}\int_V e^{-\int_0^s\sigma_a(x-pv,v)dp}\\
&\quad\times k(x-sv,v',v)E(x-sv,v')\\
&\quad\times\varphi_-(x-sv-\tau_-(x-sv,v')v',v')\,dv'\,ds.
\end{aligned} \tag{3.6}
$$

Arguing as above, we get

$$
-KJ\varphi_- = \int_{\Gamma_-} f_2(x,v,x',v')\varphi_-(x',v')\,d\mu(x')\,dv', \tag{3.7}
$$

where

$$
f_2 = |n(x')\cdot v'|\int_0^{\tau_-(x,v)}\int_0^{\tau_+(x',v')} e^{-\int_0^s\sigma_a(x-pv,v)dp} \tag{3.8}
$$

$$\times e^{-\int_0^{\tau_-(x-sv,v')}\sigma_a(x-sv-pv',v')dp}$$
$$\times k(x-sv,v',v)\delta(x-x'-sv-tv')\,dt\,ds.$$

And finally, let us consider the last term in (3.3). We first start with $A_2KJ\varphi_-$.

$$A_2KJ\varphi_- = \int_V\int_V\int_0^{\tau_-(x,v'')} e^{-\int_0^s \sigma_a(x-pv'',v'')dp} \times E(x-sv'',v'')k(x,v'',v)k(x-sv'',v',v'') \times\varphi_-(x-sv''-\tau_-(x-sv'',v')v',v')\,ds\,dv'\,dv''. \tag{3.9}$$

We claim that (3.9) is absolutely convergent integral provided that (1.2) or (1.3) holds. Indeed, assume that (1.2) holds. Then by Proposition 2.3 and (2.4),

$$\begin{aligned}\int_{X\times V}|A_2KJ\varphi_-|\,dx\,dv &= \|A_2KJ\varphi_-\|_{L^1(X\times V)}\\ &\le \|A_2\tau\|_{\mathcal{L}(L^1(X\times V))}\|\tau^{-1}K\tau\|_{\mathcal{L}(L^1(X\times V))}\|\tau^{-1}J\varphi_-\|\\ &\le \|\varphi_-\|_{L^1(\Gamma_-,d\xi)}.\end{aligned}$$

Next, in the case where (1.3) holds, by Proposition 2.2 and Proposition 2.4,

$$\begin{aligned}\int_{X\times V}|A_2KJ\varphi_-|\,dx\,dv &= \|A_2KJ\varphi_-\|_{L^1(X\times V)}\\ &\le \|A_2\|_{\mathcal{L}(L^1(X\times V))}\|K\|_{\mathcal{L}(L^1(X\times V))}\|J\varphi_-\|\\ &\le C\|\varphi_-\|_{L^1(\Gamma_-,d\tilde{\xi})}.\end{aligned}$$

In conclusion, we showed that $(A_2KJ\varphi_-)(x,v)$ is absolutely integrable in $X\times V$ and therefore by the Fubini theorem the integral (3.9) is absolutely convergent for a.e. $(x,v)\in X\times V$. This proves our claim.

Let us extend all functions depending on $v$, $v'$, $v''$ by zero outside $V$. Change the order of integration in (3.9) from $ds\,dv'\,dv''$ to $dv''\,dv'\,ds$ and perform first the change of variables $y = x - sv''$, $dv'' = s^{-n}dy$ in (3.9) and next, using Lemma 2.1 let us change the variables again $x' = y - \tau_-(y,v')v'$, $t = \tau_-(y,v')$. Then $(x',v')\in\Gamma_-$ and $dy\,dv' = dt\,d\xi(x',v')$. We thus get

$$A_2KJ\varphi_- = \int_0^\infty\int_{\Gamma_-}\int_0^{\tau_+(x',v')} e^{-\int_0^s\sigma_a(x-pv'',v'')dp}E(x'+tv',v'')k(x,v'',v) \times k(x'+tv',v',v'')\chi\varphi_-(x',v')s^{-n}\,dt\,d\xi(x',v')\,ds,$$

with $v'' = (x-x'-tv')/s$. Here $\chi = 1$ if the line segment $[x,x'+tv']$ belongs to $X$ and $\chi = 0$ otherwise. The obtained integral is also absolutely convergent despite the singularity $s^{-n}$ appearing in it. By the Fubini theorem we can change the order of integration $dt\, d\xi(x',v')\, ds \to dt\, ds\, d\xi(x',v')$ and after the first two integrations with respect to $t$ and $s$ we obtain a function of $(x',v')$ which is integrable with values in $L^1(\mathbf{R}^n_x \times V_v)$. In other words,

$$A_2KJ\varphi_- = \int_{\Gamma_-} \tilde f_3(x,v,x',v')\varphi_-(x',v')\, d\mu(x')\, dv', \tag{3.10}$$

where

$$\begin{aligned}\tilde f_3 = |n(x')\cdot v'| \int_0^\infty \int_0^{\tau_+(x',v')} & e^{-\int_0^s \sigma_a(x-pv'',v'')dp} E(x'+tv',v'') \\ & \times k(x,v'',v)k(x'+tv',v',v'')\chi s^{-n}\, dt\, ds,\end{aligned}$$

where $v'' = (x-x'-tv')/s$. Moreover, the arguments above imply that the function $\Gamma_- \ni (x',v') \to |n(x')\cdot v'|^{-1}\tilde f_3(x,v,x',v')\varphi_-(x',v') \in L^1(X\times V)$ belongs to $L^1(\Gamma_-,d\xi)$ for any $\varphi_- \in L^1(\Gamma_-,d\xi)$ provided that (1.2) holds and, respectively, $(\min\{\tau,\lambda\})^{-1}|n(x')\cdot v'|^{-1}\tilde f_3\varphi_-$ belongs to $L^1(\Gamma_-,d\tilde\xi)$ for any $\varphi_- \in L^1(\Gamma_-,d\tilde\xi)$ provided that (1.3) holds. Therefore,

$$\begin{aligned} |n(x')\cdot v'|^{-1}\tilde f_3 &\in L^\infty(\Gamma_-;\ L^1(X_x\times V_v)), \text{ if (1.2) holds,}\\ (\min\{\tau,\lambda\})^{-1}|n(x')\cdot v'|^{-1}\tilde f_3 &\in L^\infty(\Gamma_-;\ L^1(X_x\times V_v)), \text{ if (1.3) holds.}\end{aligned} \tag{3.11}$$

We are ready now to estimate the term $\mathbf{T}^{-1}A_2KJ\varphi_-$ appearing in (3.3). By (3.10),

$$\mathbf{T}^{-1}A_2KJ\varphi_- = \int_{\Gamma_-} f_3(x,v,x',v')\varphi_-(x',v')\, d\mu(x')\, dv', \tag{3.12}$$

where

$$f_3 := \mathbf{T}^{-1}\tilde f_3. \tag{3.13}$$

Here $f_3 = f_3(x,v,x',v')$ and in the formula above $\mathbf{T}^{-1}$ acts with respect to $(x,v)$ and $(x',v') \in \Gamma_-$ are considered as parameters. Assume first that (1.2) holds. Then by Proposition 2.3 $\mathbf{T}^{-1} : L^1(X\times V) \to L^1(X\times V;\ \tau^{-1}dxdv)$ is bounded and we get immediately from (3.11) that

$$|n(x')\cdot v'|^{-1} f_3 \in L^\infty(\Gamma_-;\ L^1(X_x\times V_v;\ \tau^{-1}dxdv)) \quad \text{when (1.2) holds.} \tag{3.14}$$

Moreover, since $T_0 f_3 = -(A_1+A_2)f_3$ and $A_1+A_2 : L^1(X\times V;\ \tau^{-1}dxdv) \to L^1(X\times V)$ is bounded, we also get

$$|n(x')\cdot v'|^{-1} T_0 f_3 \in L^\infty(\Gamma_-;\ L^1(X_x\times V_v)) \quad \text{when (1.2) holds.} \tag{3.15}$$

Let us assume next that (1.3) holds. Then by Proposition 2.4, $\mathbf{T}^{-1}$ is bounded in $L^1(X \times V)$ and hence by (3.11),

$$(\min\{\tau,\lambda\})^{-1}|n(x')\cdot v'|^{-1} f_3 \in L^\infty(\Gamma_-;\ L^1(X_x \times V_v)) \quad \text{when (1.3) holds.} \tag{3.16}$$

We also get as above that

$$(\min\{\tau,\lambda\})^{-1}|n(x')\cdot v'|^{-1} T_0 f_3 \in L^\infty(\Gamma_-;\ L^1(X_x \times V_v)) \quad \text{when (1.3) holds.} \tag{3.17}$$

Combining (3.3), (3.4), (3.7) and (3.12), we see that the solution to (3.2) is given by

$$\varphi(x,v) = \int_{\Gamma_-} f(x,v,x',v')\varphi_-(x',v')\, d\mu(x')\, dv', \tag{3.18}$$

where the integral is to be considered in distribution sense and $f$ is given by

$$f = f_1 + f_2 + f_3,$$

with $f_1$, $f_2$ defined by (3.5), (3.8) and $f_3$ satisfying (3.14), (3.15) (respectively (3.16), (3.17)). It is also clear that $f$ solves (3.1) in distribution sense. Let us formulate this in the following theorem.

**Theorem 3.1.** *Assume that $(\sigma_a, k)$ is admissible and either* (1.2) *or* (1.3) *holds. Then for the solution $f(x,v,x',v')$ of* (3.1) *we have $f = f_1 + f_2 + f_3$, where*

$$\begin{aligned}
f_1 &= |n(x')\cdot v'| \int_0^{\tau_+(x',v')} e^{-\int_0^{\tau_-(x,v)} \sigma_a(x-pv,v)dp}\delta(x-x'-tv)\delta(v-v')\,dt \\
f_2 &= |n(x')\cdot v'| \int_0^{\tau_-(x,v)} \int_0^{\tau_+(x',v')} e^{-\int_0^{s} \sigma_a(x-pv,v)dp} \\
&\quad \times e^{-\int_0^{\tau_-(x-sv,v')} \sigma_a(x-sv-pv',v')dp} k(x-sv,v',v)\delta(x-x'-sv-tv')\,dt\,ds
\end{aligned}$$

$|n(x')\cdot v'|^{-1} f_3 \in L^\infty(\Gamma_-;\ \mathcal{W})$, *if* (1.2) *holds,*
$(\min\{\tau,\lambda\})^{-1}|n(x')\cdot v'|^{-1} f_3 \in L^\infty(\Gamma_-;\ \tilde{\mathcal{W}})$, *if* (1.3) *holds.*

By (3.18) the so constructed solution $f(x,v,x',v')$ is the distribution kernel of the solution operator $\varphi_- \mapsto \varphi$ of (3.2). In order to find the distribution kernel $\alpha(x,v,x',v')$ $((x,v) \in \Gamma_+,\ (x',v') \in \Gamma_-)$ of the albedo operator $\mathcal{A}$, it is enough to set

$$\alpha(x,v,x',v') := f(x,v,x',v')|_{(x,v)\in\Gamma_+}, \quad (x',v') \in \Gamma_-.$$

Then Theorem 3.1 yields the following.

**Theorem 3.2.** *Assume that* $(\sigma_a, k)$ *is an admissible pair and that either* (1.2) *or* (1.3) *holds. Then the distribution kernel* $\alpha(x,v,x',v')$ *of* $\mathcal{A}$ *satisfies* $\alpha = \alpha_1 + \alpha_2 + \alpha_3$ *with*

$$
\begin{aligned}
\alpha_1 &= e^{-\int_0^{\tau_-(x,v)} \sigma_a(x-pv,v)dp}\delta_{\{x-\tau_-(x,v)v\}}(x')\delta(v-v') \\
\alpha_2 &= \int_0^{\tau_-(x,v)} e^{-\int_0^s \sigma_a(x-pv,v)dp} e^{-\int_0^{\tau_-(x-sv,v')} \sigma_a(x-sv-pv',v')dp} \\
&\quad \times k(x-sv,v',v)\delta_{\{x-sv-\tau_-(x-sv,v')v'\}}(x')\, ds \\
&|n(x')\cdot v'|^{-1}\alpha_3 \in L^\infty(\Gamma_-;\, L^1(\Gamma_+, d\xi)), \quad \textit{if } (1.2) \textit{ holds and} \\
&\min\{\tau(x',v'),\lambda\}^{-1}|n(x')\cdot v'|^{-1}\alpha_3 \in L^\infty(\Gamma_-;\, L^1(\Gamma_+, d\tilde{\xi})), \quad \textit{if } (1.3) \textit{ holds.}
\end{aligned}
$$

Proof. Formally $\alpha_j = f_j|_{(x,v)\in\Gamma_+}$, $j = 1,2,3$. To show that $\alpha_j$ are well defined, pick $\varphi \in C_0^1(\Gamma_-)$ as before and consider $\varphi_j = \int_{\Gamma_-} f_j\varphi_- d\mu(x')dv'$, $j = 1,2,3$. By (3.4), $\varphi_1 = J\varphi_- = E(x,v)\varphi_-(x-\tau_-(x,v)v,v)$. This proves the formula for $\alpha_1$ above. Next, from (3.6) we get similarly $\varphi_2 = \int \alpha_2\varphi_- d\mu(x')dv'$ with $\alpha_2$ as stated above. Finally, combining Theorem 3.1 with Theorem 2.1 and Theorem 2.2, we obtain the properties stated for the trace $\alpha_3$ of $f_3(\cdot,\cdot,x',v')$ on $\Gamma_+$. □

REMARK. The first two terms $\alpha_1$, $\alpha_2$ above are written as distributions with respect to the variables $(x',v') \in \Gamma_-$ with $(x,v) \in \Gamma_+$ considered as parameters (more precisely as linear functionals on $C^1(\Gamma_-;\ L^1(\Gamma_+, d\xi))$). One can also write them down as distributions with respect to $(x,v) \in \Gamma_+$ with $(x',v') \in \Gamma_-$ considered as parameters:

$$
\begin{aligned}
\alpha_1 &= \frac{|n(x')\cdot v'|}{n(x)\cdot v} e^{-\int_0^{\tau_-(x,v)} \sigma_a(x-pv,v)dp}\delta_{\{x'+\tau_+(x',v')v'\}}(x)\delta(v-v'), \\
\alpha_2 &= \frac{|n(x')\cdot v'|}{n(x)\cdot v}\int_0^{\tau_+(x',v')} e^{-\int_0^{\tau_+(x'+tv',v)} \sigma_a(x-pv,v)dp} e^{-\int_0^t \sigma_a(x+pv',v')dp} \\
&\quad \times k(x+tv',v',v)\delta_{\{x'+tv'+\tau_+(x'+tv',v)\}}(x)\, dt.
\end{aligned}
$$

## 4. The inverse problem

Theorem 3.2 suggests the following way for solving (IP). Assume that we are given the albedo operator $\mathcal{A}$, corresponding to some admissible pair $(\sigma_a, k)$, satisfying either (1.2) or (1.3). Then we also know the distribution $\alpha(x,v,x',v')$. By Theorem 3.2, $\alpha = \alpha_1 + \alpha_2 + \alpha_3$. Here $\alpha_1$ is a delta-type distribution supported on a $(2n-1)$-dimensional variety in $\Gamma_+ \times \Gamma_-$. Next, $\alpha_2$ is also a delta-type distribution (provided that $n \geq 3$) supported on a $3n$-dimensional variety in $\Gamma_+ \times \Gamma_-$, while $\alpha_3$ is a (locally $L^1$) function on the $(4n-2)$-dimensional $\Gamma_+ \times \Gamma_-$. Notice that if $n = 2$, then $\alpha_2$ is a function as well. Therefore, if $n \geq 3$, one can distinguish between $\alpha_1 + \alpha_2$ and $\alpha_3$. Moreover, since $\alpha_1$ and $\alpha_2$ have different degrees of singularities, one can recover $\alpha_1$ and $\alpha_2$. Now, if $\sigma_a = \sigma_a(x, |v|)$, then $\alpha_1$ determines the X-ray transform $\int \sigma_a(x + s\omega, |v|)ds$ of $\sigma_a$ for all $x$, $|v|$ and $\omega$ in an open subset of $S^2$ (for all $\omega \in S^2$ if $V$ is spherically symmetric). This determines uniquely $\sigma_a$ (see e.g. [8]). Next, once we know $\sigma_a$, from $\alpha_2$ we can recover $k$. If $n = 2$, then we can recover $\alpha_1$ and therefore $\sigma_a$, but we cannot (at least using those arguments) distinguish between $\alpha_2$ and $\alpha_3$ which are both functions and therefore our approach does not work for reconstructing $k$ in two dimensions. Below we make those arguments precise and moreover we find explicit formulas (see also [4] for the time-dependent case) for $\sigma_a$, $k$ in terms of $\alpha$.

Assume that $\varphi \in C_0^\infty(\mathbf{R}^n)$, $0 \leq \varphi \leq 1$, $\varphi(0) = 1$, $\int \varphi(x)dx = 1$. Given $\varepsilon > 0$, set

$$\phi_\varepsilon(x, v, x', v') = \varphi\left(\frac{x - x' - \tau_-(x, v)}{\varepsilon}\right)\varphi\left(\frac{v - v'}{\varepsilon}\right).$$

**Proposition 4.1.** *If either* (1.2) *or* (1.3) *holds, then*

$$\lim_{\varepsilon \to 0} \int_{\Gamma_-} \alpha(x, v, x', v')\phi_\varepsilon(x, v, x', v')\, d\mu(x')\, dv' = e^{-\int_0^{\tau_-(x,v)} \sigma_a(x - pv, v)dp}, \tag{4.1}$$

*where the integral is to be considered in distribution sense and the limit holds in* $L^1_{\text{loc}}(\Gamma_+, d\xi)$.

Proof. It should be noted first that $\tau_-(x, v)$ is smooth except on a closed subset of $\Gamma_+$ of measure zero, where it may have jumps. Nevertheless, the formal integral above is well-defined as will become clear from the proof.

It is easy to see that the limit (4.1) is trivially satisfied with $\alpha$ replaced by $\alpha_1$. We will show below that if we replace $\alpha$ by $\alpha_2$ and $\alpha_3$, respectively, then the limit in (4.1) vanishes considered in $L^1(\Gamma_+ \cap \{|v| \leq M\}, d\xi)$ for any $M > 0$. To this end, choose $0 \leq \chi \in C_0^\infty(V)$. Then

$$\begin{aligned}
0 &\leq \int_{\Gamma_+}\int_{\Gamma_-} \alpha_2(x, v, x', v')\phi_\varepsilon(x, v, x', v')\chi(v)\, d\mu(x')dv'\, d\xi(x, v) \\
&\leq \int_{\Gamma_+}\int_V \int_0^{\tau_-(x,v)} \varphi\left(\frac{v - v'}{\varepsilon}\right) k(x - sv, v', v)\chi(v)\, ds\, dv'\, d\xi(x, v) \\
&\leq \int_{X \times V}\int_V \varphi\left(\frac{v - v'}{\varepsilon}\right) k(x, v', v)\chi(v)\, dv'\, dx\, dv \\
&\leq \int_{W_\varepsilon} \chi(v)k(x, v', v)\, dx\, dv'\, dv \\
&\to 0, \quad \text{as } \varepsilon \to 0,
\end{aligned} \tag{4.2}$$

Here $W_\varepsilon = \{(x, v', v) \in X \times V \times V;\ v \in \operatorname{supp}\chi,\ |v - v'| < c\varepsilon\}$ with $c > 0$ depending on $\varphi$. Since $\chi(v)k(x, v', v)$ belongs to $L^1(X \times V \times V)$ and $\operatorname{meas}(W_\varepsilon) \to 0$, as $\varepsilon \to 0$, we get that the limit in (4.2) is zero, as stated.

Finally, assume that (1.2) holds. Then

$$
\begin{aligned}
(4.3)\qquad & \int_{\Gamma_+} \left| \int_{\Gamma_-} \alpha_3(x, v, x', v') \phi_\varepsilon(x, v, x', v') \chi(v)\, d\mu(x') dv' \right| d\xi(x, v) \\
& \leq \int_{E_\varepsilon} |n(x') \cdot v'|^{-1} |\alpha_3(x, v, x', v')| \chi(v)\, d\xi(x', v')\, d\xi(x, v) \\
& \to 0, \quad \text{as } \varepsilon \to 0,
\end{aligned}
$$

where $E_\varepsilon = \{(x, v, x', v') \in \Gamma_+ \times \Gamma_-;\ v \in \operatorname{supp}\chi,\ |v - v'| < c\varepsilon\}$. Since every bounded function on $\Gamma_+$, vanishing for large $|v|$, belongs to $L^1(\Gamma_+, d\xi)$, by Theorem 3.2 we conclude that the integrand above belongs to $L^1(\Gamma_+ \times \Gamma_-; d\xi(x, v) d\xi(x', v'))$. The limit in (4.3) is zero as stated, because $\operatorname{meas}(E_\varepsilon) \to 0$, as $\varepsilon \to 0$. Combining (4.2) and (4.3) we complete the proof in the case where (1.2) holds. The proof in the case (1.3) is similar. □

Next, denote by $\pi_{v,v'}(x)$ the projection of $x$ onto the plane spanned by $v$, $v'$ provided that $v$ and $v'$ are linearly independent. Pick a vector $m(v, v') \neq 0$ in $\operatorname{span}\{v, v'\}$, such that $m \cdot v' = 0$, for example, $m(v, v') = (v \cdot v'/|v'|^2)v' - v$. Choose $\varphi_1 \in C_0^\infty(\mathbf{R}^1)$ with $0 \leq \varphi_1 \leq 1$, $\int \varphi_1(s) ds = 1$. Consider the function

$$
\phi_{\varepsilon_1,\varepsilon_2}(x', v, v') = \frac{1}{\varepsilon_1} \varphi_1\left( \frac{x' \cdot m(v, v')}{\varepsilon_1 v \cdot m(v, v')} \right) \varphi\left( \frac{x' - \pi_{v,v'}(x')}{\varepsilon_2} \right).
$$

Denote by $D \subset V^2 := V \times V$ the variety $D = \{(v, v') \in V^2;\ v \text{ and } v' \text{ are linearly dependent}\}$.

**Proposition 4.2.** *Assume that $n \geq 3$ and either (1.2) or (1.3) holds. Then for $x \in X$ we have*

$$
\begin{aligned}
(4.4)\qquad & \lim_{\varepsilon_1 \to 0} \lim_{\varepsilon_2 \to 0} \int_{\partial X} \alpha(x + \tau_+(x, v)v, v, x', v') \\
& \quad \times \phi_{\varepsilon_1,\varepsilon_2}(x' - x + \tau_-(x, v')v', v, v')\, d\mu(x') \\
& = e^{-\int_0^{\tau_-(x,v')} \sigma_a(x - pv, v')dp} e^{-\int_0^{\tau_+(x,v)} \sigma_a(x + pv, v)dp} k(x, v', v),
\end{aligned}
$$

*where the limit holds in $L^1_{\text{loc}}(X \times (V^2 \setminus D))$.*

Proof. Denote

$$
E(s, x, v, v') = e^{-\int_0^s \sigma_a(x - pv, v)dp} e^{-\int_0^{\tau_-(x - sv, v')} \sigma_a(x - sv - pv', v')dp}
$$

(see the formula for $\alpha_2$ in Theorem 3.2). Since $\alpha_1 = 0$ for $v \neq v'$, we get that (4.4) vanishes with $\alpha$ replaced by $\alpha_1$. Next, for $\alpha_2$ we get

$$\begin{aligned}
&\int_{\partial X} \alpha_2(x + \tau_+(x,v)v, v, x', v')\phi_{\varepsilon_1,\varepsilon_2}(x' - x + \tau_-(x,v')v', v, v')\, d\mu(x') \\
&= \int_0^{\tau(x,v)} E(s, x + \tau_+(x,v)v, v, v')k(x + (\tau_+(x,v) - s)v, v', v) \\
&\quad \times \phi_{\varepsilon_1,\varepsilon_2}\Big([\tau_+(x,v) - s]v - [\tau_-(x + \tau_+(x,v)v - sv, v') - \tau_-(x,v')]v', v, v'\Big)\, ds \\
&= \int_0^{\tau(x,v)} \frac{1}{\varepsilon_1}\varphi_1\left(\frac{\tau_+(x,v) - s}{\varepsilon_1}\right) E(s, x + \tau_+(x,v)v, v, v') \\
&\quad \times k(x + (\tau_+(x,v) - s)v, v', v)\, ds.
\end{aligned}$$

Since the function $s \to E(s, x + \tau_+(x,v)v, v, v')k(x + (\tau_+(x,v) - s)v, v', v) \in L^1_{\mathrm{loc}}(X \times V \times V)$ is continuous, we get that the limit above as $\varepsilon_1 \to 0$ exists and equals $E(\tau_+(x,v), x + \tau_+(x,v)v, v, v')k(x, v', v)$ which is exactly the right-hand side of (4.4).

In order to complete the proof, we have to show that (4.4) vanishes for $\alpha = \alpha_3$. Fix $\chi \in C_0^\infty(X \times (V^2 \setminus D))$.

$$\begin{aligned}
(4.5)\qquad &\int_{X\times V}\int_{\Gamma_-} \frac{1}{\varepsilon_1}\varphi_1\left(\frac{(x'-x)\cdot m}{\varepsilon_1 v\cdot m}\right)\varphi\left(\frac{x' - x - \pi_{v,v'}(x'-x)}{\varepsilon_2}\right) \\
&\quad \times \frac{\alpha_3(x + \tau_+(x,v)v, v, x', v')}{|n(x')\cdot v'|}\chi(x,v,v')\, d\xi(x',v')\, dx\, dv \\
&\le \frac{1}{\varepsilon_1}\int_{F_{\varepsilon_2}} \frac{\alpha_3(x + \tau_+(x,v)v, v, x', v')}{|n(x')\cdot v'|}\chi(x,v,v')\, dx\, dv\, d\xi(x',v'),
\end{aligned}$$

where $F_{\varepsilon_2} = \{(x,v,x',v') \in X \times V \times \Gamma_-;\ (x,v,v') \in \operatorname{supp}\chi,\ |x - x' - \pi_{v,v'}(x - x')| < c\varepsilon_2\}$. By Theorem 3.2 and Lemma 2.1,

$$\frac{\alpha_3(x + \tau_+(x,v)v, v, x', v')}{|n(x')\cdot v'|\tau(x,v)} \in L^\infty(\Gamma_-;\ L^1(X_x \times V_v))$$

and clearly, $0 \le \tau(x,v) \le C < \infty$ for $(x,v,v') \in \operatorname{supp}\chi$. Therefore, the integrand in (4.5) is an $L^1$-function. On the other hand, meas$(F_{\varepsilon_2}) \to 0$, as $\varepsilon_2 \to 0$, because $F_{\varepsilon_2}$ is an $\varepsilon_2$-small neighborhood of a variety of dimension $3n + 1$ in the $4n - 1$ dimensional $X \times V \times \Gamma_-$. Consequently, (4.5) tends to zero, as $\varepsilon_2 \to 0$. □

## References


[1] Yu.E. Anikonov and B.A. Bubnov: *Inverse problems of transport theory*, Soviet Math. Doklady. **37** (1988), 497–499.

[2] M. Cessenat: *Théorèmes de trace $L^p$ pour des espaces de fonctions de la neutronique*, C.R. Acad. Sci. Paris, Série I. **299** (1984), 831–834.

[3] M. Cessenat: *Théorèmes de trace pour des espaces de fonctions de la neutronique*, C.R. Acad. Sci. Paris, Série I. **300** (1985), 89–92.

[4] M. Choulli and P. Stefanov: *Inverse scattering and inverse boundary value problems for the linear Boltzmann equation*, Comm. P.D.E. **21** (1996), 763–785.

[5] M. Choulli and P. Stefanov: *Reconstruction of the coefficients of the stationary transport equation from boundary measurements*, Inverse Prob. **12** (1996), 19–23.

[6] M. Choulli and A. Zeghal: *Laplace transform approach for an inverse problem*, Transport Theory and Stat. Phys. (9) **24** (1995), 1353–1367.

[7] R. Dautray and J.L. Lions: *Analyse Mathématique et Calcul Numérique pour les Sciences et les Techniques*, Masson, Paris. **9** (1988).

[8] S. Helgason: The Radon Transform, Birkhäuser, Boston, Basel, 1980.

[9] E.W. Larsen: *Solution of multidimensional inverse transport problems*, J. Math. Phys. (1) **25** (1984), 131–135.

[10] E. Larsen: *Solution of three-dimensional inverse transport problems*, Transport Theory Statist. Phys. (2-3) **17** (1988), 147–167.

[11] N.J. McCormick: *Recent developments in inverse scattering transport methods*, Trans. Theory and Stat. Phys. **13** (1984), 15–28.

[12] A.I. Prilepko and N.P. Volkov: *Inverse problems for determining the parameters of nonstationary kinetic transport equation from additional information on the traces of the unknown function*, Differentsialnye Uravneniya **24** (1988), 136–146.

[13] M. Reed and B. Simon: *Methods of Modern Mathematical Physics*, Academic Press, New York. **3** (1979).

[14] V. Romanov: *Stability estimates in problems of recovering the attenuation coefficient and the scattering indicatrix for the transport equation*, J. Inverse Ill-Posed Probl. **4** (1996), 297–305.



M. Choulli
Departement de Mathématiques
Université de Metz
Ile Du Saulcy, 57045 Metz cedex 1, France

P. Stefanov
Department of Mathematics
East Carolina University
Greenville, NC 27858, USA