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ON THE WEIGHTS OF END-PAIRS IN *n*-END CATENOIDS OF GENUS ZERO

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1. Introduction

Let $\overline{M}$ be a compact Riemann surface, and

$$X\colon M=\overline{M}\setminus\{q_1,\dots,q_n\}\longrightarrow \mathbf{R}^3$$

a conformal minimal immersion that has the ends at $q_1,\dots,q_n\in\overline{M}$. The end q_j is called a *catenoidal end* if the image of a neighborhood of q_j by X behaves asymptotic to some catenoid. When all the ends are catenoidal ends, we call X, or its image $X(M)$, an *n-end catenoid*. Choose a loop γ_j surrounding q_j from the left, and let $\vec{n}$ be a conormal such that $(\gamma_j,\vec{n})$ is positively oriented. Then the *flux vector* at the end q_j is defined by the integral

$$\varphi_j:=\int_{\gamma_j}\vec{n}\,ds,$$

where ds is the line element of $X(M)$. By the divergence formula, we get the *flux formula*

$$\sum_{j=1}^{n}\varphi_j=0.$$

When a conformal minimal immersion X has finite total curvature, the Gauss map $G\colon M\longrightarrow \mathbf{S}^2\subset\mathbf{R}^3$ of X is naturally extended to the map $G\colon\overline{M}\longrightarrow\mathbf{S}^2\subset\mathbf{R}^3$. In particular when X is an n-end catenoid, $G(q_j)$ is parallel to φ_j, and hence there exists a real number $w(q_j)$ satisfying

$$\varphi_j=4\pi w(q_j)G(q_j).$$

We call $w(q_j)$ the *weight* of the end q_j. The weight $w(q_j)$ is the size of the catenoidal end q_j relative to the standard catenoid. Note here that the weight may take a negative value. When $w(q_j)=0$, the end q_j is an *embedded planar end*.

Now, we can rewrite the flux formula as follows:

$$\sum_{j=1}^{n} w(q_j)G(q_j) = 0. \tag{1.1}$$

Conversely, we can consider an *inverse problem of the flux formula*, or a *Plateau problem at infinity*, that is a problem of finding n-end catenoids that realize the given data $G(q_j)$ and $w(q_j)$ $(j = 1, \ldots, n)$ satisfying (1.1). For this problem, Umehara, Yamada and the first author [5, Theorem 3.3], [6, Theorem 3.1] proved that, for almost all *flux data* $v_1, \ldots, v_n \in \mathbf{S}^2$ and $a_1, \ldots, a_n \in \mathbf{R} \setminus \{0\}$ satisfying $\sum_{j=1}^{n} a_j v_j = 0$, there exists an n-end catenoid of genus 0

$$X: M = \hat{\mathbf{C}} \setminus \{q_1, \ldots, q_n\} \longrightarrow \mathbf{R}^3$$

that satisfies

$$G(q_j) = v_j, \qquad w(q_j) = a_j \qquad (j = 1, \ldots, n). \tag{1.2}$$

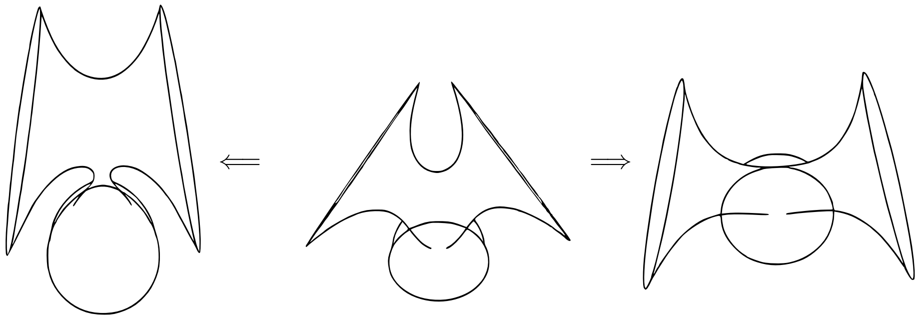
In connection with this result, we mention here that Rosenberg-Toubiana [9, Theorem 2.5] proved the general existence in the case when $\deg G = 1$ (and hence X has branch points), and that Cosín-Ros [1, Theorem 8.1] got a nessesarry and sufficient condition in the case when $\dim\langle v_1, \ldots, v_n\rangle = 2$ and X is Alexandrov embedded.

In our case, when $n = 3$ and $\dim\langle v_1, v_2, v_3\rangle = 2$, we can replace "almost all" by "all". Furthermore, for any flux data, a 3-end catenoid realizing the data is unique ([8], [4, Example 3.5]). On the other hand, when $n \geq 4$, such a uniquness result does not hold, and we can construct examples of n-end catenoids that have the same flux data and are not congruent to each other ([5, Example 3.7], [7, Examples 3.1, 3.2], see Example 7.1 for their Weierstrass data). In particular in the case when $n = 4$, we know that, for any data $v_1, v_2, v_3, v_4 \in \mathbf{S}^2$ and $a_1, a_2, a_3, a_4 \in \mathbf{R} \setminus \{0\}$ satisfying $\sum_{j=1}^{4} a_j v_j = 0$ and $\dim\langle v_1, v_2, v_3, v_4\rangle \geq 2$, the number of 4-end catenoids of genus 0 satisfying (1.2) is at most four. In particular, this estimate is sharp ([5, Theorems 3.3, 4.2]).

To explain our problem, let us observe an example of a family of 4-end catenoids whose limit normals v_1, v_2, v_3, v_4 are arranged in the positions of the vertices of a tetrahedron

$$\begin{aligned} &v_1 = (\cos\theta, 0, \sin\theta), && v_2 = (-\cos\theta, 0, \sin\theta), \\ &v_3 = (0, \cos\theta, -\sin\theta), && v_4 = (0, -\cos\theta, -\sin\theta) \end{aligned} \qquad \left(0 \leq \theta < \frac{\pi}{2}\right),$$

and whose weights satisfy $w(q_j) = a_j = 1$ $(j = 1, 2, 3, 4)$ ([5, Example 3.7, Figure 3.2(a), (b)]). When $\theta = \mathrm{Sin}^{-1}(1/\sqrt{3})$, X is unique and invariant under the action of the tetrahedral group, and there are two types of deformation from $\theta = \mathrm{Sin}^{-1}(1/\sqrt{3})$ to 0.



(the former type) tetrahedrally symmetric (the latter type)

Fig. 1.1. Two types of deformation of 4-end catenoids

In one type, the simple closed geodesic of $X(M)$ which separates the ends q_1, q_2 and the ends q_3, q_4 becomes shorter and shorter, and, as the limit, we get two catenoids tangent to each other.

In the other type, the closed geodesic of $X(M)$ as above does not become so short, and, as the limit, we get a Jorge-Meeks' 4-noid ([2, Examples in §5]).

What is the essence of difference of these two types of deformation? In this paper, we consider this problem.

Quite similar phenomenon as in the former type is observed also in Karcher's example ([3, Example 2.3.8, Figure 2.3.8]) whose limit normals are arranged in the positions of the vertices of a rectangle. In this example, since each X satisfies $\dim\langle v_1, v_2, v_3, v_4\rangle = 2$ and are Alexandrov embedded, we can apply the theory in [1] to explain this phenomenon by using flux poligons. However, similar phenomenon is observed also in more general case when $\dim\langle v_1, \dots, v_n\rangle = 3$ (cf. [10, Theorem 1.1], [7, Example 3.2], etc.).

When X is symmetric with respect to some plane and no ends are arranged on the plane, we can give a simple explanation. If an n-end catenoid X has such symmetry (Karcher's example has such symmetry), then a simple closed geodesic appears on the plane of the symmetry, and its length is equal to the length of the flux vector along to the closed geodesic. Therefore if the sum of the flux vectors in one side of the plane tends to 0, then the length of the closed geodesic also tends to 0. This holds also for the higher genus. However, if we do not assume such symmetry (indeed, our first example does not have such symmetry), then this explanation is not available.

To explain the above phenomenon in more general case, we define, in §2, the *relative weights* w_{jk} of end-pairs (q_j, q_k) $(j, k = 1, \dots, n\,;\ \ j \neq k)$, which are conformal invariants satisfying

$$\sum_{k=1;k\neq j}^{n} w_{jk} = \sum_{k=1;k\neq j}^{n} w_{kj} = w(q_j),$$

and, in §3–5, prove the following result:

Theorem 1.1. *Let* $X\colon M = \hat{\mathbf{C}} \setminus \{q_1, \dots, q_n\} \longrightarrow \mathbf{R}^3$ *be an* n*-end catenoid of genus* 0 *satisfying* (1.2) ($n \geq 4$), *and* w_{jk} *the relative weight of the end-pair* (q_j, q_k) $(j, k = 1, \dots, n\ ;\ \ j \neq k)$. *Assume that there exist positive numbers* C_1, C_2, C_3, *and* ϵ_1, ϵ_2 *small enough satisfying*

$$\begin{cases} C_1 \leq |w_{jk}| \leq C_2 & (j, k = 1, \dots, m \text{ or } j, k = m+1, \dots, n\ ;\ \ j \neq k) \\ \epsilon_1 \leq |w_{jk}| \leq \epsilon_2 & (j = 1, \dots, m\ ;\ \ k = m+1, \dots, n) \end{cases} \tag{1.3}$$

$(2 \leq m \leq n-2)$ *and*

$$\angle(v_j, v_k) \geq C_3 \qquad (j, k = 1, \dots, n\ ;\ \ j \neq k). \tag{1.4}$$

Then there exists a positive number $C = C(C_1, C_2, C_3, \epsilon_2/\epsilon_1, n)$ *such that the length* l *of the minimal closed geodesic that separates the surface* $X(M)$ *to the side of the ends* $q_1, \dots, q_m$ *and the side of the ends* $q_{m+1}, \dots, q_n$ *satisfies*

$$l \leq C\epsilon_2.$$

As for the lower estimate by flux, it is clear that the length l of any (simple) closed geodesic satisfies

$$l \geq \left|\sum_{j=1}^{m} \varphi_j\right| \left(= \left|\sum_{k=m+1}^{n} \varphi_k\right| \right),$$

where $q_1, \dots, q_m$ are the ends in one side of the geodesic. Unfortunately, this estimate does not make sense if the right-hand side is equal to 0. But we can show that if all the ratios w_{jl}/w_{kl} take values close to a common nonzero complex number w_k^j independent of l $(j,\ k = 1, \dots, n)$, then any closed geodesic is not short. In the most typical case when the common complex numbers are equal to 1, our assertion is stated as follows:

Theorem 1.2. *Let* $X\colon M = \hat{\mathbf{C}} \setminus \{q_1, \dots, q_n\} \longrightarrow \mathbf{R}^3$ *be an* n*-end catenoid of genus* 0 *satisfying* (1.2) ($n \geq 4$), *and* w_{jk} *the relative weight. Assume that there exist a nonzero complex number* w, *a positive number* ϵ *small enough, and a positive number* C_3 *satisfying*

$$\left|\frac{w_{jk}}{w} - 1\right| \leq \epsilon \qquad (j, k = 1, \dots, n\ ;\ \ j \neq k) \tag{1.5}$$

and (1.4)*. Then there exists a positive number* $C' = C'(C_3, n)$ *such that the length* l *of the minimal closed geodesic of the surface* $X(M)$ *satisfies*

$$l \geq \min\{n^2 C_3 |w|(1 - C'\epsilon), 4\pi \min |a_j|\}.$$

In §6, we give a proof for this assertion under a more general assumption.

Theorem 1.1 (resp. 1.2) describes the phenomenon observed in the former (resp. latter) type of deformation in our first example. We see this in §7.

2. Relative weights of end-pairs

In this paper, we use the *Weierstrass representation formula* of the following type:

$$X(z) = \mathrm{Re} \int_{z_0}^{z} \bigl(1 - g^2, \sqrt{-1}\,(1 + g^2), 2g\bigr)\eta,$$

where g is a meromorphic function on $\overline{M}$ defined by the composition of $\sigma\colon \mathbf{S}^2 \longrightarrow \hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$, the stereograhic projection from the north pole, and the Gauss map G extended to $\overline{M}$, i.e. $g := \sigma \circ G\colon \overline{M} \longrightarrow \hat{\mathbf{C}}$, and η is a melomorphic 1-form on $\overline{M}$ which is holomorphic on M. We call (g, η) the *Weierstrass data* of X.

For n-end catenoids of genus 0, Umehara, Yamada and the first author proved the following result:

Theorem 2.1 ([5, Theorem 2.4]). *Let* $X\colon M = \hat{\mathbf{C}} \setminus \{q_1, \ldots, q_n\} \longrightarrow \mathbf{R}^3$ *be an* n*-end catenoid of genus* 0 *satisfying* (1.2)*. Assume* $v_j \neq (0, 0, 1)$*, and set* $p_j := \sigma(v_j)$ $(j = 1, \ldots, n)$*. Then its Weierstrass data is given by*

$$g(z) = \sum_{j=1}^{n} \frac{p_j b_j}{z - q_j} \Bigg/ \sum_{j=1}^{n} \frac{b_j}{z - q_j}, \qquad \eta = -\left(\sum_{j=1}^{n} \frac{b_j}{z - q_j}\right)^2 dz, \tag{2.1}$$

where $b_1, \ldots, b_n$ *are nonzero complex numbers satisfying the following equations*:

$$\begin{cases} b_j \displaystyle\sum_{k=1;k\neq j}^{n} b_k \frac{p_k - p_j}{q_k - q_j} = a_j \\ b_j \displaystyle\sum_{k=1;k\neq j}^{n} b_k \frac{\overline{p}_j p_k + 1}{q_k - q_j} = 0 \end{cases} \qquad (j = 1, \ldots, n). \tag{2.2}$$

Conversely, for any given data $p_1, \ldots, p_n \in \mathbf{C}$*, and* $a_1, \ldots, a_n \in \mathbf{R} \setminus \{0\}$ *satisfying* $\sum_{j=1}^{n} a_j \sigma^{-1}(p_j) = 0$*, if there exist* $q_1, \ldots, q_n \in \mathbf{C} = \hat{\mathbf{C}} \setminus \{\infty\}$ *and* $b_1, \ldots, b_n \in \mathbf{C} \setminus \{0\}$ *satisfying* (2.2)*, and if the degree of* g *given by* (2.1) *is equal to* $n - 1$*, then the conformal minimal immersion given by the Weierstrass data* (2.1) *is an* n*-end catenoid satisfying* (1.2) *with* $v_j = \sigma^{-1}(p_j)$ $(j = 1, \ldots, n)$*.*

Now, let us define the relative weights.

DEFINITION 2.2. Let $X\colon M = \hat{\mathbf{C}} \setminus \{q_1, \dots, q_n\} \longrightarrow \mathbf{R}^3$ be an n-end catenoid of genus 0 given by the Weierstrass data (2.1) with (2.2). We call

$$w_{jk} := b_j b_k \frac{p_k - p_j}{q_k - q_j}$$

the *relative weight* of the end-pair (q_j, q_k) $(j, k = 1, \dots, n\ ;\ \ j \neq k)$.

While the weight $w(q_j)$ always takes a real value, the relative weight w_{jk} may take an imaginary value. The value of w_{jk} is independent of the parametrization of the surface $X(M)$ up to multiplying ± 1. Indeed, for w_{jk}, we have the following:

Proposition 2.3. *w_{jk} is invariant under the conformal transformations of $\hat{\mathbf{C}}$ and the orientation preserving congruent transformations of $\mathbf{R}^3$.*

To show this proposition, we prepare the transformation rules for the Weierstrass data of n-end catenoids.

Lemma 2.4. *Let X be an n-end catenoid of genus* 0 *given by* (2.1) *with* (2.2). *For any conformal transformation*

$$\psi(z) = \frac{az + b}{cz + d} \qquad (ad - bc \neq 0)$$

on $\hat{\mathbf{C}}$, the Weierstrass data of $\tilde{X} = X \circ \psi^{-1}$ is given by

$$\tilde{q}_j = \psi(q_j), \qquad \tilde{b}_j = \pm\sqrt{\psi'(q_j)}\, b_j \left(= \frac{\pm\sqrt{ad - bc}}{cq_j + d} b_j \right) \qquad (j = 1, \dots, n).$$

Proof. Since $\tilde{X} = X \circ \psi^{-1}$, we have $\tilde{q}_j = \psi(q_j)$ $(j = 1, \dots, n)$. On the other hand,

$$\begin{aligned}
(\psi^{-1})^* \eta &= -\left(\sum_{j=1}^{n} \frac{b_j}{\psi^{-1}(\tilde{z}) - q_j} \right)^2 (\psi^{-1})'(\tilde{z})\, d\tilde{z} \\
&= -\left(\sum_{j=1}^{n} \frac{b_j}{(d\tilde{z} - b)/(-c\tilde{z} + a) - q_j} \right)^2 \frac{ad - bc}{(-c\tilde{z} + a)^2}\, d\tilde{z} \\
&= -\left(\sum_{j=1}^{n} \frac{\sqrt{ad - bc}\, b_j}{(cq_j + d)\tilde{z} - (aq_j + b)} \right)^2 d\tilde{z} \\
&= -\left(\sum_{j=1}^{n} \frac{\{\sqrt{ad - bc}/(cq_j + d)\} b_j}{\tilde{z} - \psi(q_j)} \right)^2 d\tilde{z}
\end{aligned}$$

$$= -\left(\sum_{j=1}^{n} \frac{\sqrt{\psi'(q_j)}\, b_j}{\tilde{z} - \psi(q_j)}\right)^2 d\tilde{z},$$

from which it follows that $\tilde{b}_j = \pm\sqrt{\psi'(q_j)}\, b_j$ $(j = 1, \dots, n)$. □

Lemma 2.5. *Let X be an n-end catenoid of genus* 0 *given by* (2.1) *with* (2.2). *For any orthogonal transformation P of $\mathbf{R}^3$ such that*

$$F(\zeta) = \sigma \circ P|_{\mathbf{S}^2} \circ \sigma^{-1}(\zeta) = \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta} \qquad (\alpha\delta - \beta\gamma \neq 0),$$

the Weierstrass data of $\tilde{X} = P \circ X$ is given by

$$\tilde{p}_j = F(p_j), \qquad \tilde{b}_j = \pm\frac{1}{\sqrt{F'(p_j)}} b_j \left(= \frac{\gamma p_j + \delta}{\pm\sqrt{\alpha\delta - \beta\gamma}}\, b_j\right) \qquad (j = 1, \dots, n).$$

Proof. Since $\tilde{g} = F \circ g$, we have $\tilde{p}_j = F(p_j)$ $(j = 1, \dots, n)$. On the other hand, since the Hopf differential $\eta \cdot dg$ is invariant under the action of $SO(3)$,

$$\begin{aligned}
\tilde{\eta} &= \frac{\eta \cdot dg}{d\tilde{g}} = \frac{\eta \cdot dg}{d(F \circ g)} = \frac{\eta \cdot dg}{F' \circ g\, dg} = \frac{1}{F' \circ g}\eta \\
&= -\frac{(\gamma g + \delta)^2}{\alpha\delta - \beta\gamma}\left(\sum_{j=1}^{n} \frac{b_j}{z - q_j}\right)^2 dz = -\frac{1}{\alpha\delta - \beta\gamma}\left(\sum_{j=1}^{n} \frac{(\gamma p_j + \delta) b_j}{z - q_j}\right)^2 dz \\
&= -\left(\sum_{j=1}^{n} \frac{\left(1/\sqrt{F'(p_j)}\right) b_j}{z - q_j}\right)^2 dz,
\end{aligned}$$

from which it follows that $\tilde{b}_j = \pm b_j/\sqrt{F'(p_j)}$ $(j = 1, \dots, n)$. □

Proof of Proposition 2.3. In Lemma 2.4 (resp. 2.5), we must choose one of the square roots of $ad - bc$ (resp. $\alpha\delta - \beta\gamma$) to represent $\tilde{b}_j$'s. But this choice has no influence on not only the Weierstrass data of $\tilde{X}$ but also the value of each $\tilde{b}_j\tilde{b}_k$. Therefore, in the case of conformal transformations of $\hat{\mathbf{C}}$, we have

$$\begin{aligned}
\tilde{w}_{jk} &= \tilde{b}_j\tilde{b}_k \frac{p_k - p_j}{\tilde{q}_k - \tilde{q}_j} = \sqrt{\psi'(q_j)}\, b_j \sqrt{\psi'(q_k)}\, b_k \frac{p_k - p_j}{\psi(q_k) - \psi(q_j)} \\
&= b_j b_k \sqrt{\psi'(q_j)\psi'(q_k)} \frac{p_k - p_j}{(aq_k + b)/(cq_k + d) - (aq_j + b)/(cq_j + d)} \\
&= b_j b_k \frac{ad - bc}{(cq_j + d)(cq_k + d)} \frac{p_k - p_j}{\{(ad - bc)(q_k - q_j)\}/\{(cq_j + d)(cq_k + d)\}} = w_{jk}.
\end{aligned}$$

In the case of orthogonal transformations of $\mathbf{R}^3$, we have

$$\begin{aligned}\tilde{w}_{jk} &= \tilde{b}_j\tilde{b}_k\frac{\tilde{p}_k-\tilde{p}_j}{q_k-q_j} = \frac{1}{\sqrt{F'(p_j)}}b_j\frac{1}{\sqrt{F'(p_k)}}b_k\frac{F(p_k)-F(p_j)}{q_k-q_j}\\ &= b_jb_k\frac{1}{\sqrt{F'(p_j)F'(p_k)}}\frac{(\alpha p_k+\beta)/(\gamma p_k+\delta)-(\alpha p_j+\beta)/(\gamma p_j+\delta)}{q_k-q_j}\\ &= b_jb_k\frac{(\gamma p_j+\delta)(\gamma p_k+\delta)}{\alpha\delta-\beta\gamma}\frac{\{(\alpha\delta-\beta\gamma)(p_k-p_j)\}/\{(\gamma p_j+\delta)(\gamma p_k+\delta)\}}{q_k-q_j} = w_{jk}.\square\end{aligned}$$

By using w_{jk}, we can rewrite the condition (2.2) as follows:

$$\text{(2.3)}\qquad \begin{cases}\displaystyle\sum_{k=1;k\neq j}^{n} w_{jk}=a_j\\ \displaystyle\sum_{k=1;k\neq j}^{n} w_{jk}\frac{\overline{p}_jp_k+1}{p_k-p_j}=0\end{cases}\qquad (j=1,\dots,n).$$

We note here that the absolute value of each term in the left-hand side of the second equality of (2.3) is also invariant under the conformal transformations of $\hat{\mathbf{C}}$ and the congruent transformations of $\mathbf{R}^3$.

We also note here that the Hopf differential of X is represented as follows:

$$\begin{aligned}\eta\cdot dg &= \sum_{j<k} w_{jk}\left(\frac{1}{z-q_j}-\frac{1}{z-q_k}\right)^2 dz^2\\ &= \sum_{j=1}^{n}\left\{\frac{a_j}{(z-q_j)^2}+2\left(\sum_{k=1;k\neq j}^{n}\frac{w_{jk}}{q_k-q_j}\right)\frac{1}{z-q_j}\right\}dz^2.\end{aligned}$$

Hence we can regard w_{jk}'s as coefficients of $\eta\cdot dg$ in a sense. But we cannot determine w_{jk}'s only by $\eta\cdot dg$ when $n>5$.

3. Lengths of the images of the circles

In general, it is difficult to calculate the length of the minimal closed geodesic in each homology class. However, if its length is short enough, then it is expected that the minimal closed geodesic is approximated by the image of some circle in the domain M. Therefore, we calculate the lengths of the images of such asymptotic circles, to estimate the lengths of the minimal closed geodesics from above.

Lemma 3.1. *Let $X\colon M=\hat{\mathbf{C}}\setminus\{q_1,\dots,q_n\}\longrightarrow\mathbf{R}^3$ be an n-end catenoid of genus 0 given by* (2.1) *with* (2.2). *Let γ be the circle $z=z_0+Re^{\sqrt{-1}\,\theta}$ $(0\le\theta\le2\pi)$ in $\hat{\mathbf{C}}$. If $q_1,\dots,q_m$ is included in the inside of γ, and if $q_{m+1},\dots,q_n$ is included in the outside* *of γ, then the length l' of the image of the circle γ by X is given by the following formula*:

$$l' = 2\pi R\left\{\sum_{j=1}^{m}\sum_{k=1}^{m}\frac{-(1+p_j\overline{p}_k)b_j\overline{b}_k}{(q_j-z_0)(\overline{q}_k-\overline{z}_0)-R^2} - \sum_{j=m+1}^{n}\sum_{k=m+1}^{n}\frac{-(1+p_j\overline{p}_k)b_j\overline{b}_k}{(q_j-z_0)(\overline{q}_k-\overline{z}_0)-R^2}\right\}.$$

If

$$\operatorname{Re} q_j \begin{cases} > 0 & (j=1,\ldots,m) \\ < 0 & (j=m+1,\ldots,n), \end{cases}$$

then the length l'' of the image of the imaginary axis of $\hat{\mathbf{C}}$ by X is given by the following formula:

$$l'' = 2\pi\left\{\sum_{j=1}^{m}\sum_{k=1}^{m}\frac{(1+p_j\overline{p}_k)b_j\overline{b}_k}{q_j+\overline{q}_k} - \sum_{j=m+1}^{n}\sum_{k=m+1}^{n}\frac{(1+p_j\overline{p}_k)b_j\overline{b}_k}{q_j+\overline{q}_k}\right\}.$$

Proof. Recall here that the line element ds of the minimal surface $X(M)$ is given by

$$ds = (1+|g|^2)|\eta|.$$

By this and (2.2), we have

$$\frac{ds}{|dz|} = \left|\sum_{j=1}^{n}\frac{b_j}{z-q_j}\right|^2 + \left|\sum_{j=1}^{n}\frac{p_jb_j}{z-q_j}\right|^2 \tag{3.1}$$

for any n-end catenoid of genus 0.

For any circle $\gamma : z = z_0 + Re^{\sqrt{-1}\,\theta}$ $(0 \le \theta \le 2\pi)$, it holds that

$$\begin{aligned}
\int_\gamma\left|\sum_{j=1}^{n}\frac{b_j}{z-q_j}\right|^2|dz| &= \int_0^{2\pi}\left|\sum_{j=1}^{n}\frac{b_j}{z_0+Re^{\sqrt{-1}\,\theta}-q_j}\right|^2\left|\sqrt{-1}\,Re^{\sqrt{-1}\,\theta}\,d\theta\right| \\
&= \int_0^{2\pi}\sum_{j=1}^{n}\frac{b_j}{z_0+Re^{\sqrt{-1}\,\theta}-q_j}\sum_{k=1}^{n}\frac{\overline{b}_k}{\overline{z}_0+Re^{-\sqrt{-1}\,\theta}-\overline{q}_k}R\,d\theta \\
&= \int_0^{2\pi}\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\frac{1}{e^{\sqrt{-1}\,\theta}-(q_j-z_0)/R} - \frac{1}{e^{\sqrt{-1}\,\theta}-R/(\overline{q}_k-\overline{z}_0)}\right) \\
&\qquad\times\frac{-b_j\overline{b}_kR}{(q_j-z_0)(\overline{q}_k-\overline{z}_0)-R^2}e^{\sqrt{-1}\,\theta}\,d\theta \\
&= \int_{|\zeta|=1}\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\frac{1}{\zeta-(q_j-z_0)/R} - \frac{1}{\zeta-R/(\overline{q}_k-\overline{z}_0)}\right)
\end{aligned}$$

$$\times \frac{-b_j\overline{b}_k R}{(q_j - z_0)(\overline{q}_k - \overline{z}_0) - R^2} \frac{1}{\sqrt{-1}} d\zeta.$$

If we set $\zeta_j := (q_j - z_0)/R$, then we have

$$\begin{aligned}\int_\gamma \left|\sum_{j=1}^n \frac{b_j}{z-q_j}\right|^2 |dz| &= \frac{1}{\sqrt{-1}} \int_{|\zeta|=1} \sum_{j=1}^n \sum_{k=1}^n \left(\frac{1}{\zeta-\zeta_j} - \frac{1}{\zeta - \overline{\zeta}_k^{\,-1}}\right) \frac{-b_j\overline{b}_k R^{-1}}{\zeta_j\overline{\zeta}_k - 1} d\zeta \\ &= \frac{1}{\sqrt{-1}} \int_{|\zeta|=1} \left(\sum_{j=1}^n \frac{1}{\zeta-\zeta_j} \sum_{k=1}^n \frac{-b_j\overline{b}_k R^{-1}}{\zeta_j\overline{\zeta}_k - 1} - \sum_{k=1}^n \frac{1}{\zeta - \overline{\zeta}_k^{\,-1}} \sum_{j=1}^n \frac{-b_j\overline{b}_k R^{-1}}{\zeta_j\overline{\zeta}_k - 1}\right) d\zeta.\end{aligned}$$

Now, since $q_1, \dots, q_m$ is included in the inside of γ and $q_{m+1}, \dots, q_n$ is included in the outside of γ, it holds that $|\zeta_j| < 1$ $(j = 1, \dots, m)$ and $|\zeta_k| > 1$ $(k = m+1, \dots, n)$. Hence, by the residue theorem, we get

$$\begin{aligned}\int_\gamma \left|\sum_{j=1}^n \frac{b_j}{z-q_j}\right|^2 |dz| &= \frac{2\pi\sqrt{-1}}{\sqrt{-1}\,R} \left(\sum_{j=1}^m \sum_{k=1}^n \frac{-b_j\overline{b}_k}{\zeta_j\overline{\zeta}_k - 1} - \sum_{k=m+1}^n \sum_{j=1}^n \frac{-b_j\overline{b}_k}{\zeta_j\overline{\zeta}_k - 1}\right) \\ &= \frac{2\pi}{R} \left(\sum_{j=1}^m \sum_{k=1}^m \frac{-b_j\overline{b}_k}{\zeta_j\overline{\zeta}_k - 1} - \sum_{k=m+1}^n \sum_{j=m+1}^n \frac{-b_j\overline{b}_k}{\zeta_j\overline{\zeta}_k - 1}\right).\end{aligned}$$

Note here that the imaginary axis $z = -\sqrt{-1}\,t$ $(-\infty \le t \le +\infty)$ is the limit of the family of circles $z = R + Re^{\sqrt{-1}\,\theta}$ $(0 \le \theta \le 2\pi)$. For this family, we have

$$R(\zeta_j\overline{\zeta}_k - 1) = \frac{q_j\overline{q}_k - (q_j + \overline{q}_k)R}{R} \to -(q_j + \overline{q}_k) \quad \text{as} \quad R \to +\infty.$$

Therefore, if $\operatorname{Re} q_j > 0$ $(j = 1, \dots, m)$ and if $\operatorname{Re} q_k < 0$ $(k = m+1, \dots, n)$, then we get

$$\int_{\operatorname{Re} z=0} \left|\sum_{j=1}^n \frac{b_j}{z-q_j}\right|^2 |dz| = 2\pi \left(\sum_{j=1}^m \sum_{k=1}^m \frac{b_j\overline{b}_k}{q_j + \overline{q}_k} - \sum_{k=m+1}^n \sum_{j=m+1}^n \frac{b_j\overline{b}_k}{q_j + \overline{q}_k}\right).$$

We can show the similar equalities for the line integrals of the second term of the right-hand side of (3.1), and we get our assertion. □

When $n = 4$, we may choose

$$q_1 = q,\ q_2 = -q,\ q_3 = \frac{1}{q},\ q_4 = -\frac{1}{q}$$

for some $q \in \mathbf{C} \setminus \{0, \pm 1, \pm\sqrt{-1}\,\}$ without loss of generality. For the length l' and l'', we can show the following formula by Lemma 3.1 and direct computation.

Corollary 3.2. *Under the assumption above, if* $|q| < 1$, *then the length of the image of the unit circle centered at the origin* 0 *is given by*

$$l' = \frac{2\pi}{1-|q|^4}\{|b_1+b_2|^2 + |p_1b_1+p_2b_2|^2 + |q|^2(|b_1-b_2|^2 + |p_1b_1-p_2b_2|^2) + |q|^2(|b_3-b_4|^2 + |p_3b_3-p_4b_4|^2) + |q|^4(|b_3+b_4|^2 + |p_3b_3+p_4b_4|^2)\}.$$

If $\operatorname{Re} q > 0$, *then the length of the image of the imaginary axis is given by*

$$l'' = \frac{2\pi}{(q+\overline{q})(1+|q|^2)}\{|b_1+q^2b_3|^2 + |p_1b_1+q^2p_3b_3|^2 + |b_2+q^2b_4|^2 + |p_2b_2+q^2p_4b_4|^2 + |q|^2(|b_1+b_3|^2 + |p_1b_1+p_3b_3|^2 + |b_2+b_4|^2 + |p_2b_2+p_4b_4|^2)\}.$$

Now, if the ends $q_1,\dots,q_m$ (resp. the ends $q_{m+1},\dots,q_n$) approach to a point 1 (resp. -1) in M, then we can take the image of the imaginary axis as a loop that separates these two groups of the ends, and if the ends are concentrated to the two points ± 1, then the length l'' is estimated as follows:

Lemma 3.3. *Under the assumption of* Lemma 3.1, *assume*

$$\begin{cases} |q_j - 1| < \epsilon & (j = 1,\dots,m) \\ |q_k + 1| < \epsilon & (k = m+1,\dots,n) \end{cases}$$

for some $0 < \epsilon < 1$. *Then it holds that*

$$\left| l'' - \pi\left(\left|\sum_{j=1}^m b_j\right|^2 + \left|\sum_{k=m+1}^n b_k\right|^2 + \left|\sum_{j=1}^m p_jb_j\right|^2 + \left|\sum_{k=m+1}^n p_kb_k\right|^2\right)\right| \leq \frac{\pi\epsilon}{1-\epsilon}\left\{\left(\sum_{j=1}^m |b_j|\right)^2 + \left(\sum_{k=m+1}^n |b_k|\right)^2 + \left(\sum_{j=1}^m |p_jb_j|\right)^2 + \left(\sum_{k=m+1}^n |p_kb_k|\right)^2\right\}.$$

Proof. Under the assumption, we have

$$\begin{aligned}
\left|\sum_{j=1}^m\sum_{k=1}^m \frac{b_j\overline{b}_k}{q_j+\overline{q}_k} - \frac{1}{2}\left|\sum_{j=1}^m b_j\right|^2\right| &= \left|\sum_{j=1}^m\sum_{k=1}^m \frac{b_j\overline{b}_k}{q_j+\overline{q}_k} - \sum_{j=1}^m\sum_{k=1}^m \frac{b_j\overline{b}_k}{1+1}\right| \\
&= \left|\sum_{j=1}^m\sum_{k=1}^m b_j\overline{b}_k\left(\frac{1}{q_j+\overline{q}_k} - \frac{1}{2}\right)\right| = \left|\sum_{j=1}^m\sum_{k=1}^m b_j\overline{b}_k\frac{2-q_j-\overline{q}_k}{2(q_j+\overline{q}_k)}\right| \\
&\leq \sum_{j=1}^m\sum_{k=1}^m |b_j\overline{b}_k|\frac{|1-q_j|+|1-\overline{q}_k|}{2(2-|1-q_j|-|1-\overline{q}_k|)} \\
&\leq \sum_{j=1}^m\sum_{k=1}^m |b_j\overline{b}_k|\frac{2\epsilon}{2(2-2\epsilon)} = \frac{\epsilon}{2(1-\epsilon)}\left(\sum_{j=1}^m |b_j|\right)^2.
\end{aligned}$$

We can show the similar estimates for the other terms, and we get our assertion. $\square$

4. Other lemmas

We also use the following two lemmas:

Lemma 4.1. *For any v and $v' \in \mathbf{S}^2$, if $\angle(v, v') \geq C_3$, then $|\sigma(v) - \sigma(v')| \geq 2C_4$, where $C_4 := \tan(C_3/4)$.*

Proof. Choose an orthogonal transformation P such that $\sigma(P(v)) = -\sigma(P(v'))$ and $|\sigma(P(v))| = |\sigma(P(v'))| \leq 1$. Set $\tilde{p} := \sigma(P(v))$. Now, it holds that

$$\sigma \circ P^{-1} \circ \sigma^{-1}(\zeta) = \beta \frac{\overline{\alpha}\zeta + 1}{\zeta - \alpha} \qquad \text{or} \qquad \beta\zeta \qquad (\zeta \in \hat{\mathbf{C}})$$

for some $\alpha \in \mathbf{C}$ and $|\beta| = 1$. In the former case, we have

$$\begin{aligned} |\sigma(v) - \sigma(v')| &= \left| \beta \frac{\overline{\alpha}\tilde{p} + 1}{\tilde{p} - \alpha} - \beta \frac{\overline{\alpha}(-\tilde{p}) + 1}{-\tilde{p} - \alpha} \right| = \left| \frac{\overline{\alpha}\tilde{p} + 1}{\tilde{p} - \alpha} - \frac{\overline{\alpha}(-\tilde{p}) + 1}{-\tilde{p} - \alpha} \right| \\ &= \left| \frac{-2(|\alpha|^2 + 1)\tilde{p}}{\alpha^2 - \tilde{p}^2} \right| \geq \frac{2(|\alpha|^2 + 1)|\tilde{p}|}{|\alpha|^2 + |\tilde{p}|^2} \geq 2|\tilde{p}|, \end{aligned}$$

and in the latter case, we have

$$|\sigma(v) - \sigma(v')| = |\beta\tilde{p} - \beta(-\tilde{p})| = 2|\tilde{p}|.$$

Set $\theta := \angle(v, v')$. Then $|\tilde{p}| = \tan(\theta/4)$ and we get our assertion. $\square$

Lemma 4.2. *For any $v_1, \ldots, v_n \in \mathbf{S}^2$, there exists an orthogonal transformation P such that $|\sigma(P(v_j))| \leq \sqrt{n-1}$ $(j = 1, \ldots, n)$.*

Proof. For any $v_j \in \mathbf{S}^2$ and $0 \leq \theta_0 \leq \pi$, the area of the closed domain $\{v \in \mathbf{S}^2 \mid \angle(v, v_j) \leq \theta_0\}$ is $2\pi(1 - \cos\theta_0)$. Hence if $\theta_0 = \mathrm{Cos}^{-1}(1 - 2/n)$, then $2\pi(1 - \cos\theta_0) \times n = 4\pi$ and

$$\bigcup_{j=1}^{n} \{v \in \mathbf{S}^2 \mid \angle(v, v_j) < \theta_0\} \neq \mathbf{S}^2.$$

Therefore there exists v_0 such that $\angle(v_j, v_0) \geq \theta_0$ for any $j = 1, \ldots, n$. Hence, we have that if we choose an orthogonal transformation P such that $P(v_0)$ is the north pole, then

$$|\sigma(P(v_j))| \leq \tan\frac{\pi - \theta_0}{2} = \frac{\sin\theta_0}{1 - \cos\theta_0} = \frac{(2/n)\sqrt{n-1}}{2/n} = \sqrt{n-1} \qquad (j = 1, \ldots, n). \quad \square$$

5. Upper estimate

Now, let us prove our first main theorem.

Proof of Theorem 1.1. Set $p_j := \sigma(v_j)$ $(j = 1, \dots, n)$. By the assumption (1.4) and Lemma 4.1, we have $|p_j - p_k| \geq 2C_4$ $(j, k = 1, \dots, n\ ;\ j \neq k)$. Set $C_5 := \max |p_j|$. By Lemma 4.2, we may assume $C_5 \leq \sqrt{n-1}$ without loss of generality. Assume $\epsilon_2 < C_1C_4/4\sqrt{3}\,C_5$.

Arrangement of q_j's. Now, we may assume $q_1 = 1$ and $q_{m+1} = -1$. We may also assume

$$
(5.1)\quad \begin{cases} \angle\bigl(\sigma^{-1}(q_1), \sigma^{-1}(q_j)\bigr) \leq \angle\bigl(\sigma^{-1}(q_1), \sigma^{-1}(q_2)\bigr) & (j = 3, \dots, m) \\ \angle\bigl(\sigma^{-1}(q_{m+1}), \sigma^{-1}(q_k)\bigr) \leq \angle\bigl(\sigma^{-1}(q_{m+1}), \sigma^{-1}(q_{m+2})\bigr) & (k = m+3, \dots, n) \end{cases}
$$

and

$$
|q_1 - q_2| = |q_{m+1} - q_{m+2}| =: t.
$$

By the assumption (1.3), we have

$$
\frac{{\epsilon_1}^2}{{C_2}^2} \leq \left| \frac{w_{1\,m+1} w_{2\,m+2}}{w_{12} w_{m+1\,m+2}} \right| = \frac{|p_1 - p_{m+1}||p_2 - p_{m+2}||q_1 - q_2||q_{m+1} - q_{m+2}|}{|p_1 - p_2||p_{m+1} - p_{m+2}||q_1 - q_{m+1}||q_2 - q_{m+2}|} \leq \frac{{\epsilon_2}^2}{{C_1}^2},
$$

and hence

$$
\frac{{\epsilon_1}^2}{{C_2}^2} \leq \frac{2C_5 \cdot 2C_5 \cdot t \cdot t}{2C_4 \cdot 2C_4 \cdot 2 \cdot (2 - 2t)} = \frac{{C_5}^2}{4{C_4}^2} \frac{t^2}{1 - t}
$$

and

$$
\frac{{C_4}^2}{4{C_5}^2} \frac{t^2}{1 + t} = \frac{2C_4 \cdot 2C_4 \cdot t \cdot t}{2C_5 \cdot 2C_5 \cdot 2 \cdot (2 + 2t)} \leq \frac{{\epsilon_2}^2}{{C_1}^2}.
$$

Set $\epsilon_3 := (2C_4/C_2C_5)\epsilon_1$ and $\epsilon_4 := (2C_5/C_1C_4)\epsilon_2$. Then we have ${\epsilon_3}^2(1 - t) \leq t^2 \leq {\epsilon_4}^2(1 + t)$, from which it follows that

$$
\frac{\epsilon_3}{1 + \epsilon_3} \leq \frac{2}{\epsilon_3 + \sqrt{{\epsilon_3}^2 + 4}}\epsilon_3 \leq t \leq \frac{\epsilon_4 + \sqrt{{\epsilon_4}^2 + 4}}{2}\epsilon_4 \leq (1 + \epsilon_4)\epsilon_4.
$$

Since we assume $\epsilon_2 < C_1C_4/4\sqrt{3}\,C_5$, we have $\epsilon_4 < 1/2\sqrt{3}$ and $t < (2\sqrt{3}/3)\epsilon_4 < 1/3$. Now, by the assumption (5.1), $q_2, \dots, q_m$ (resp. $q_{m+2}, \dots, q_n$) are included in the closed ball centered on the real axis whose boundary circle passes $1 - t$ and $1/(1 - t)$ (resp. $-(1 - t)$ and $-1/(1 - t)$). Hence it holds that

$$
\operatorname{Re} q_j \begin{cases} > 0 & (j = 1, \dots, m) \\ < 0 & (j = m+1, \dots, n) \end{cases}
$$

and

$$|q_1 - q_j|,\ |q_{m+1} - q_k| \leq \frac{t}{1-t} \qquad (j = 1, \dots, m\ ;\ k = m+1, \dots, n).$$

Therefore we get

$$|q_j - q_k| \leq \begin{cases} \dfrac{2t}{1-t} & (j, k = 1, \dots, m \text{ or } j, k = m+1, \dots, n) \\ \dfrac{2}{1-t} & (j = 1, \dots, m\ ;\ k = m+1, \dots, n). \end{cases}$$

Estimates for b_j's. By the estimate above, we have

$$\begin{aligned} |b_j b_k| &= \left| w_{jk} \frac{q_k - q_j}{p_k - p_j} \right| \leq C_2 \frac{2t/(1-t)}{2C_4} \\ &= \frac{C_2}{C_4} \frac{t}{1-t} \qquad (j, k = 1, \dots, m \text{ or } j, k = m+1, \dots, n\ ;\ j \neq k). \end{aligned}$$

Let j_0 and k_0 be indices that satisfy

$$|b_{j_0}| = \max\{|b_j| \mid j = 1, \dots, m\}, \qquad |b_{k_0}| = \max\{|b_k| \mid k = m+1, \dots, n\}.$$

Then it holds that

$$|b_j|,\ |b_k| \leq \sqrt{\frac{C_2}{C_4} \frac{t}{1-t}} =: t_1 \qquad (j = 1, \dots, m\ ;\ j \neq j_0\ ;\ k = m+1, \dots, n\ ;\ k \neq k_0).$$

On the other hand, we also have

$$|b_{j_0} b_{k_0}| = \left| w_{j_0 k_0} \frac{q_{k_0} - q_{j_0}}{p_{k_0} - p_{j_0}} \right| \leq \epsilon_2 \frac{2/(1-t)}{2C_4} = \frac{1}{C_4} \frac{\epsilon_2}{1-t}.$$

In the case when $|b_{j_0}| \geq |b_{k_0}|$, since

$$\begin{aligned} C_1 &\leq |w_{k_0 k}| = \left| b_{k_0} b_k \frac{p_k - p_{k_0}}{q_k - q_{k_0}} \right| \\ &\leq \frac{1}{|b_{j_0}|} \frac{1}{C_4} \frac{\epsilon_2}{1-t} \sqrt{\frac{C_2}{C_4} \frac{t}{1-t}}\ \frac{2C_5}{|q_{k_0} - q_k|} \qquad (k = m+1, \dots, n\ ;\ k \neq k_0), \end{aligned}$$

we have

$$t = |q_{m+1} - q_{m+2}| \leq |q_{m+1} - q_{k_0}| + |q_{k_0} - q_{m+2}| \leq 2 \frac{1}{|b_{j_0}|} \frac{2C_5}{C_1} \sqrt{\frac{C_2}{{C_4}^3} \frac{t}{(1-t)^3}}\, \epsilon_2.$$

Therefore we get

$$|b_{j_0}| \leq \frac{4C_5}{C_1} \sqrt{\frac{C_2}{{C_4}^3}}\ \frac{\epsilon_2}{\sqrt{(1-t)^3 t}} =: t_2.$$

Also in the case when $|b_{j_0}| \leq |b_{k_0}|$, we get the same estimate for $|b_{k_0}|$.

Estimate for the length l''. Now, by using Lemma 3.3, we get

$$\begin{aligned} l'' &\leq \left(\pi + \frac{\pi\{t/(1-t)\}}{1-t/(1-t)}\right) \\ &\quad \times \left\{ \left(\sum_{j=1}^{m} |b_j|\right)^2 + \left(\sum_{k=m+1}^{n} |b_k|\right)^2 + \left(\sum_{j=1}^{m} |p_j b_j|\right)^2 + \left(\sum_{k=m+1}^{n} |p_k b_k|\right)^2 \right\} \\ &\leq \pi \left(1 + \frac{t}{1-2t}\right) \left[(1+{C_5}^2)\{(m-1)t_1 + t_2\}^2 + (1+{C_5}^2)\{(n-m-1)t_1 + t_2\}^2\right] \\ &= \pi \frac{1-t}{1-2t}(1+{C_5}^2)[\{(m-1)^2 + (n-m-1)^2\}{t_1}^2 + 2(n-2)t_1 t_2 + 2{t_2}^2], \end{aligned}$$

where

$$\begin{aligned} {t_1}^2 &= \frac{C_2}{C_4}\frac{t}{1-t} \leq \frac{C_2}{C_4}\left(1+\frac{3}{2}t\right)t \leq \frac{C_2}{C_4}\left\{1+\frac{3}{2}(1+\epsilon_4)\epsilon_4\right\}(1+\epsilon_4)\epsilon_4, \\ t_1 t_2 &= \frac{4C_2C_5}{C_1{C_4}^2}\frac{\epsilon_2}{(1-t)^2} = \frac{2C_2}{C_4}\frac{\epsilon_4}{(1-t)^2} \\ &\leq \frac{2C_2}{C_4}\left(1+\frac{15}{4}t\right)\epsilon_4 \leq \frac{2C_2}{C_4}\left\{1+\frac{15}{4}(1+\epsilon_4)\epsilon_4\right\}\epsilon_4, \\ {t_2}^2 &= \frac{16C_2{C_5}^2}{{C_1}^2{C_4}^3}\frac{{\epsilon_2}^2}{(1-t)^3 t} \leq \frac{4C_2}{C_4}\frac{{\epsilon_4}^2}{(1-t)^3}\frac{1+\epsilon_3}{\epsilon_3} \\ &\leq \frac{4C_2}{C_4}\left(1+\frac{57}{8}t\right)\frac{(1+\epsilon_3){\epsilon_4}^2}{\epsilon_3} \leq \frac{4C_2}{C_4}\left\{1+\frac{57}{8}(1+\epsilon_4)\epsilon_4\right\}(1+\epsilon_3)\frac{\epsilon_4}{\epsilon_3}\epsilon_4. \end{aligned}$$

Combining these estimates, we get

$$l \leq l'' \leq C\epsilon_2$$

for a positive constant $C = C(C_1, C_2, C_3, \epsilon_2/\epsilon_1, n)$. □

In the statement of Theorem 1.1, we assume (1.4) since w_{jk} vanishes automatically when $v_j = v_k$ (i.e. $p_j = p_k$). But this assumption excludes Karcher's example and some others. To treat these case at the same time, we have only to replace $|w_{jk}|$ by

$$m_{jk} := \max\left\{|w_{jk}|, \left|w_{jk}\frac{\overline{p}_j p_k + 1}{p_k - p_j}\right|\right\}.$$

Theorem 5.1. *Let $X\colon M = \hat{\mathbf{C}} \setminus \{q_1, \ldots, q_n\} \longrightarrow \mathbf{R}^3$ be an n-end catenoid of genus* 0 *satisfying* (1.2) ($n \geq 4$), *and m_{jk} as above. Assume that there exist positive* *numbers* C_1, C_2, *and* ϵ_1, ϵ_2 *small enough satisfying*

$$\begin{cases} C_1 \leq m_{jk} \leq C_2 & (j,k=1,\dots,m \text{ or } j,k=m+1,\dots,n\ ;\ j\neq k) \\ \epsilon_1 \leq m_{jk} \leq \epsilon_2 & (j=1,\dots,m\ ;\ k=m+1,\dots,n) \end{cases}$$

$(2 \leq m \leq n-2)$. *Then there exists a positive number* $C = C(C_1, C_2, \epsilon_2/\epsilon_1, n)$ *such that the length* l *of the minimal closed geodesic that separates the surface* $X(M)$ *to the side of the ends* $q_1, \dots, q_m$ *and the side of the ends* $q_{m+1}, \dots, q_n$ *satisfies*

$$l \leq C\epsilon_2.$$

Outline of proof. Set $p_{jk} := \max\{|p_k - p_j|, |\overline{p}_j p_k + 1|\}$. Then

$$m_{jk} = \left| b_j b_k \frac{p_{jk}}{q_k - q_j} \right|.$$

By the definition, we have

$$2(\sqrt{2}-1) = 2\tan\frac{\pi}{8} \leq p_{jk} \leq C_5{}^2 + 1 \ (\leq n).$$

Replacing the estimate $2C_4 \leq |p_j - p_k| \leq 2C_5$ in the proof of Theorem 1.1 by the estimate above, we can show the asseretion of Theorem 5.1. □

6. Lower estimate

In this section, we give a proof for a more general version of Theorem 1.2 stated as follows:

Theorem 6.1. *Let* $X\colon M = \hat{\mathbf{C}} \setminus \{q_1, \dots, q_n\} \longrightarrow \mathbf{R}^3$ *be an* n*-end catenoid of genus* 0 *satisfying* (1.2) $(n \geq 4)$, *and* w_{jk} *the relative weight. Assume that there exist complex numbers* w_k^j $(j,k = 1,\dots,n)$, *a positive number* ϵ *small enough, and a positive number* C_3 *satisfying*

$$\text{(6.1)} \qquad \begin{cases} w_k^j w_l^k = w_l^j & (j,k,l = 1,\dots,n) \\ \left| \dfrac{w_{jl}}{w_{kl} w_k^j} - 1 \right| \leq \epsilon & (j,k,l=1,\dots,n\ ;\ l \neq j,k) \end{cases}$$

and (1.4). *Moreover, assume that* $\beta_j := (1/n)\sum_{k=1}^n w_j^k \neq 0$ *and* $|w_1^j| \geq 1$ $(j = 1,\dots,n)$. *Set* $W := \max|w_k^j|$ *and* $w := w_{12}w_2^1$. *Then there exists a positive number* $C' = C'(C_3, n, W, W/|\beta_1|)$ *such that the length* l *of the minimal closed geodesic of the surface* $X(M)$ *satisfies*

$$l \geq \min\{n^2 C_3 |\beta_1|^2 |w| (1 - C'\epsilon), 4\pi \min|a_j|\}.$$

Proof. Set $p_j := \sigma(v_j)$ $(j = 1, \dots, n)$. By the assumption (1.4) and Lemma 4.1, we have $|p_j - p_k| \geq 2C_4$ $(j, k = 1, \dots, n\ ;\ j \neq k)$ as before. Set $C_5 := \max |p_j|$, and assume $C_5 \leq \sqrt{n-1}$ as before. In particular in the case when $\dim\langle v_1, \dots, v_n\rangle \leq 2$, we may assume $C_5 = 1$. Set $C_6 := 2C_5/C_4$. Since $C_5 \geq C_4$ holds in general, we have $C_6 \geq 2$. Assume $\epsilon < 1/(1 + C_6 + 3{C_6}^2)$.

Note here that $w_j^k = 1/w_k^j$ $(j, k = 1, \dots, n)$ and $w_j^j = 1$ $(j = 1, \dots, n)$ hold automatically by the definition.

Arrangement of q_j's. Set

$$\epsilon_{jk,l} := \frac{w_{jl}}{w_{kl} w_k^j} - 1 \qquad (l \neq j, k).$$

Then $|\epsilon_{jk,l}| \leq \epsilon < 1$. Set

$$\epsilon_{jklm} := \frac{w_{jk} w_{lm}}{w_{jl} w_{km}} - 1 = \frac{1 + \epsilon_{kl,j}}{1 + \epsilon_{kl,m}} - 1 = \frac{\epsilon_{kl,j} - \epsilon_{kl,m}}{1 + \epsilon_{kl,m}} \qquad (j \neq k, l\ ;\ m \neq k, l).$$

Then it holds that

$$|\epsilon_{jklm}| \leq \frac{|\epsilon_{kl,j}| + |\epsilon_{kl,m}|}{1 - |\epsilon_{kl,m}|} \leq \frac{2\epsilon}{1 - \epsilon} \qquad (j \neq k, l\ ;\ m \neq k, l).$$

By the assumption (1.4), p_j's are different from each other. Hence we may assume $q_j = p_j$ $(j = 1, 2, 3)$. Now, for any $j = 4, \dots, n$,

$$\begin{aligned} 1 + \epsilon_{123j} &= \frac{w_{12} w_{3j}}{w_{13} w_{2j}} \\ &= \frac{(p_1 - p_2)(p_3 - p_j)(q_1 - q_3)(q_2 - q_j)}{(p_1 - p_3)(p_2 - p_j)(q_1 - q_2)(q_3 - q_j)} = \frac{(p_3 - p_j)(p_2 - q_j)}{(p_2 - p_j)(p_3 - q_j)}, \end{aligned}$$

and hence

$$p_j - q_j = \frac{(p_2 - p_j)(p_3 - p_j)}{(p_3 - p_2) - (p_2 - p_j)\epsilon_{123j}} \epsilon_{123j}.$$

Therefore we get

$$\begin{aligned} |p_j - q_j| &\leq \frac{(|p_2| + |p_j|)(|p_3| + |p_j|)}{|p_3 - p_2| - (|p_2| + |p_j|)|\epsilon_{123j}|} |\epsilon_{123j}| \\ &\leq \frac{(2C_5)^2}{2C_4 - 2C_5\{2\epsilon/(1-\epsilon)\}} \frac{2\epsilon}{1 - \epsilon} \leq \frac{C_4 {C_6}^2 \epsilon}{1 - (1 + C_6)\epsilon} \end{aligned}$$

and

$$\begin{aligned} |q_j - q_k| &\geq -|q_j - p_j| + |p_j - p_k| - |p_k - q_k| \\ &\geq 2C_4 - \frac{2C_4 {C_6}^2}{1 - (1 + C_6)\epsilon} \epsilon \qquad (j, k = 1, \dots, n\ ;\ j \neq k). \end{aligned}$$

Estimates for b_j's. Set

$$\xi_{jk} := \frac{b_j b_k}{w_{jk}} - 1 = \frac{q_k - q_j}{p_k - p_j} - 1 = \frac{(p_j - q_j) - (p_k - q_k)}{p_k - p_j} \qquad (k \neq j).$$

Then it holds that

$$|\xi_{jk}| \leq \frac{|p_j - q_j| + |p_k - q_k|}{|p_k - p_j|} \leq \frac{2}{2C_4} \frac{C_4 {C_6}^2 \epsilon}{1 - (1 + C_6)\epsilon} = \frac{{C_6}^2 \epsilon}{1 - (1 + C_6)\epsilon} =: \xi \qquad (k \neq j).$$

Since we assume $\epsilon < 1/(1 + C_6 + 3{C_6}^2)$, we have $\xi < \epsilon/3 < 1$. The above estimates for the arrangement of q_j's are rewritten as follows:

$$\text{(6.2)} \qquad |p_j - q_j| \leq C_4 \xi \qquad (j = 1, \dots, n),$$

$$\text{(6.3)} \qquad |q_j - q_k| \geq 2(C_4 - C_4 \xi) \qquad (j, k = 1, \dots, n\ ;\ j \neq k).$$

Since

$$\frac{b_j}{b_k w_k^j} = \frac{b_j b_l}{b_k b_l w_k^j} = \frac{w_{jl}(1 + \xi_{jl})}{w_{kl}(1 + \xi_{kl}) w_k^j} = \frac{(1 + \xi_{jl})(1 + \epsilon_{jk,l})}{1 + \xi_{kl}} \qquad (k \neq j\ ;\ l \neq j, k),$$

we have

$$\text{(6.4)} \qquad \frac{(1 - \xi)(1 - \epsilon)}{1 + \xi} \leq \left| \frac{b_j}{b_k w_k^j} \right| \leq \frac{(1 + \xi)(1 + \epsilon)}{1 - \xi} \qquad (k \neq j)$$

and

$$\text{(6.5)} \qquad \left| \frac{b_j}{b_k w_k^j} - 1 \right| \leq \frac{2\xi + (1 + \xi)\epsilon}{1 - \xi} \qquad (k \neq j).$$

On the other hand, since

$$\frac{{b_j}^2}{w_{jk} w_k^j} = \frac{b_j b_k b_j b_l}{b_k b_l w_{jk} w_k^j} = \frac{w_{jk}(1 + \xi_{jk}) w_{jl}(1 + \xi_{jl})}{w_{kl}(1 + \xi_{kl}) w_{jk} w_k^j} = \frac{(1 + \xi_{jk})(1 + \xi_{jl})}{1 + \xi_{kl}} \frac{w_{jl}}{w_{kl} w_k^j}$$
$$= \frac{(1 + \xi_{jk})(1 + \xi_{jl})(1 + \epsilon_{jk,l})}{1 + \xi_{kl}} \qquad (k \neq j\ ;\ l \neq j, k),$$

it holds that

$$\frac{(1 - \xi)^2(1 - \epsilon)}{1 + \xi} \leq \left| \frac{{b_j}^2}{w_{jk} w_k^j} \right| \leq \frac{(1 + \xi)^2(1 + \epsilon)}{1 - \xi} \qquad (k \neq j)$$

and

$$\left| \frac{{b_j}^2}{w_{jk} w_k^j} - 1 \right| \leq \frac{(3 + \xi)\xi + (1 + \xi)^2 \epsilon}{1 - \xi} \qquad (k \neq j).$$

Moreover, since

$$\frac{(b_j w_j^1)^2}{w} = \frac{{b_j}^2}{w_{j1} w_1^j} \frac{w_{j1}}{w_{12} w_2^1 w_1^j} = \frac{{b_j}^2}{w_{j1} w_1^j} \frac{w_{j1}}{w_{21} w_2^j} = \frac{{b_j}^2}{w_{j1} w_1^j}(1 + \epsilon_{j2,1}) \qquad (j = 2, \dots, n),$$

we have

$$\text{(6.6)} \qquad \frac{(1-\xi)^2(1-\epsilon)^2}{1+\xi} \leq \left| \frac{(b_j w_j^1)^2}{w} \right| \leq \frac{(1+\xi)^2(1+\epsilon)^2}{1-\xi} \qquad (j = 1, \dots, n)$$

and

$$\text{(6.7)} \qquad \left| \frac{(b_j w_j^1)^2}{w} - 1 \right| \leq \frac{(3+\xi)\xi + (1+\xi)^2(2+\epsilon)\epsilon}{1-\xi} \qquad (j = 1, \dots, n).$$

By (6.5) and (6.7), we see that b_1 and all the $b_j w_j^1$'s are close to a common square root $\sqrt{w}$ of $w = w_{12} w_2^1$ if we choose an ϵ small enough. Set

$$\delta_j := \frac{b_j w_j^1}{\sqrt{w}} - 1 \qquad (j = 1, \dots, n).$$

Then we have

$$\text{(6.8)} \qquad |\delta_j| \leq \frac{2\xi + (1+\xi)\epsilon}{1-\xi} \qquad (j = 1, \dots, n).$$

Estimate for the length l. Now, recall (3.1). Note here that

$$\sum_{j=1}^{n} \frac{p_j b_j}{z - q_j} = z \sum_{j=1}^{n} \frac{b_j}{z - q_j} - \sum_{j=1}^{n} b_j - \sum_{j=1}^{n} \frac{(q_j - p_j) b_j}{z - q_j}.$$

Since it holds that

$$|t|^2 + |at - b|^2 \geq \frac{|b|^2}{|a|^2 + 1} \qquad (a, b, t \in \mathbf{C}),$$

we have

$$\frac{ds}{|dz|} \geq \frac{1}{|z|^2 + 1} \left| \sum_{j=1}^{n} b_j + \sum_{j=1}^{n} \frac{(q_j - p_j) b_j}{z - q_j} \right|^2.$$

Denote the line element of the standard sphere by $ds_{\mathbf{S}^2}$. Then we get

$$\frac{ds}{ds_{\mathbf{S}^2}} = \frac{ds}{\{2/(1 + |z|^2)\}|dz|}$$

$$= \frac{1+|z|^2}{2}\frac{ds}{|dz|} \geq \frac{1}{2}\max\left\{\left|\sum_{j=1}^{n}\frac{b_j}{z-q_j}\right|^2, \left|\sum_{j=1}^{n} b_j + \sum_{j=1}^{n}\frac{(q_j-p_j)b_j}{z-q_j}\right|^2\right\}.$$

Since $\xi < 1/3$, we have $C_4 - C_4\xi > 2C_4/3 > 0$. Hence, by (6.3), the closed domains $B_j := \{z \in \hat{\mathbf{C}} \mid |z-q_j| \leq R\}$ $(j = 1, \ldots, n)$ are disjoint for any positive number $R < C_4 - C_4\xi$. Now, for any $z \in B_j$, since

$$\begin{aligned} |z-q_k| &\geq -|z-q_j| - |q_j - p_j| + |p_j - p_k| - |p_k - q_k| \\ &\geq 2(C_4 - C_4\xi) - R = 2C_4(1-\xi) - R \qquad (k \neq j), \end{aligned}$$

it holds that

$$\left|\frac{z-q_j}{z-q_k}\right| \leq \frac{R}{2C_4(1-\xi)-R} = \frac{R}{2C_4-R}\frac{1}{1-\{2C_4/(2C_4-R)\}\xi} \qquad (k \neq j).$$

Hence we have, by (6.4) and (6.6),

$$\begin{aligned} \left|\sum_{k=1}^{n}\frac{b_k}{z-q_k}\right| &\geq \frac{|b_j|}{|z-q_j|}\left(1 - \sum_{k=1;k\neq j}^{n}\left|\frac{b_k}{b_j}\right|\left|\frac{z-q_j}{z-q_k}\right|\right) \\ &\geq \frac{\sqrt{|w|}\,|w_1^j|(1-\xi)(1-\epsilon)}{\sqrt{1+\xi}\,R} \\ &\qquad \times\left\{1 - \sum_{k=1;k\neq j}^{n}|w_j^k|\frac{(1+\xi)(1+\epsilon)}{1-\xi}\frac{R}{2C_4-R}\frac{1}{1-\{2C_4/(2C_4-R)\}\xi}\right\} \\ &\geq \frac{\sqrt{|w|}\,(1-\xi)(1-\epsilon)}{\sqrt{1+\xi}\,R} \\ &\qquad \times\left\{1 - (n-1)W\frac{(1+\xi)(1+\epsilon)}{1-\xi}\frac{R}{2C_4-R}\frac{1}{1-\{2C_4/(2C_4-R)\}\xi}\right\}. \end{aligned}$$

Set $R := C_4/3nW$. Then $R < 2C_4/3 < C_4 - C_4\xi$,

$$\frac{R}{2C_4-R} = \frac{1}{6nW-1}, \qquad \frac{2C_4}{2C_4-R} = \frac{6nW}{6nW-1}$$

and

$$\begin{aligned} \left|\sum_{k=1}^{n}\frac{b_k}{z-q_k}\right| &\geq \frac{\sqrt{|w|}\,(1-\xi)(1-\epsilon)}{\sqrt{1+\xi}}\,\frac{3nW}{C_4} \\ &\qquad \times\left\{1 - \frac{(1+\xi)(1+\epsilon)}{1-\xi}\frac{(n-1)W}{6nW-1}\frac{1}{1-\{6nW/(6nW-1)\}\xi}\right\}. \end{aligned}$$

Now, since

$$\frac{6nW-1}{6(n-1)W} = \frac{6(n-1)W + 6W - 1}{6(n-1)W} > 1,$$

$$\frac{6nW}{6nW-1} = \frac{1}{1-1/6nW} \le \frac{1}{1-1/12} = \frac{12}{11},$$
$$\xi = \frac{{C_6}^2\epsilon}{1-(1+C_6)\epsilon} \ge {C_6}^2\epsilon \ge 4\epsilon,$$

it holds that

$$\begin{aligned}
&4\frac{6nW-1}{6(n-1)W}\left(1-\frac{6nW}{6nW-1}\xi\right)(1-\xi)-(1+\xi)(1+\epsilon)\\
&\quad\ge 4\cdot 1\cdot\left(1-\frac{12}{11}\xi\right)(1-\xi)-(1+\xi)\left(1+\frac{1}{4}\xi\right)\\
&\quad=\frac{1}{44}(181\xi^2-423\xi+132)>0 \qquad \left(0\le\xi\le\frac{1}{3}\right),
\end{aligned}$$

which implies that

$$1-\frac{(1+\xi)(1+\epsilon)}{1-\xi}\frac{(n-1)W}{6nW-1}\frac{1}{1-\{6nW/(6nW-1)\}\xi} > 1-\frac{4}{6}=\frac{1}{3}.$$

Hence we get

$$\left|\sum_{k=1}^{n}\frac{b_k}{z-q_k}\right|^2 > \frac{|w|(1-\xi)^2(1-\epsilon)^2}{1+\xi}\left(\frac{3nW}{C_4}\right)^2\frac{1}{9} = \frac{|w|(1-\xi)^2(1-\epsilon)^2}{1+\xi}\left(\frac{nW}{C_4}\right)^2.$$

On the other hand, by (6.8), we have

$$\begin{aligned}
\left|\sum_{j=1}^{n} b_j\right| &= \left|\sum_{j=1}^{n}\sqrt{w}\,w_1^j(1+\delta_j)\right| \ge |\sqrt{w}|\left(\left|\sum_{j=1}^{n} w_1^j\right|-\left|\sum_{j=1}^{n} w_1^j\delta_j\right|\right)\\
&\ge \sqrt{|w|}\left(n|\beta_1|-W\sum_{j=1}^{n}|\delta_j|\right) \ge \sqrt{|w|}\,n|\beta_1|\left\{1-\frac{W}{|\beta_1|}\frac{2\xi+(1+\xi)\epsilon}{1-\xi}\right\}.
\end{aligned}$$

For any $z\in\hat{\mathbf{C}}\setminus\bigcup_{j=1}^{n}B_j$, by (6.2) and (6.6), it holds that

$$\begin{aligned}
\left|\sum_{j=1}^{n}\frac{(q_j-p_j)b_j}{z-q_j}\right| &\le \sum_{j=4}^{n}\frac{|q_j-p_j||b_j|}{|z-q_j|} \le \sum_{j=4}^{n}\frac{C_4\xi|\sqrt{w}\,w_1^j||1+\delta_j|}{R}\\
&\le \frac{(n-3)C_4\sqrt{|w|}\,W(1+\xi)(1+\epsilon)}{R\sqrt{1-\xi}}\xi.
\end{aligned}$$

Hence we have

$$\left|\sum_{j=1}^{n} b_j+\sum_{j=1}^{n}\frac{(q_j-p_j)b_j}{z-q_j}\right| \ge \left|\sum_{j=1}^{n} b_j\right|-\left|\sum_{j=1}^{n}\frac{(q_j-p_j)b_j}{z-q_j}\right|$$

$$\geq \sqrt{|w|}\,n|\beta_1|\left[1-\frac{W}{|\beta_1|}\left\{\frac{2\xi+(1+\xi)\epsilon}{1-\xi}+\frac{(n-3)C_4(1+\xi)(1+\epsilon)}{nR\sqrt{1-\xi}}\xi\right\}\right]$$
$$\geq \sqrt{|w|}\,n|\beta_1|\left[1-\frac{W}{|\beta_1|}\left\{\frac{2\xi+(1+\xi)\epsilon}{1-\xi}+\frac{(n-3)\cdot 3W(1+\xi)(1+\epsilon)}{1-\xi}\xi\right\}\right].$$

Set $C_7 := (1+C_6+3{C_6}^2)/3$. Then $\epsilon \leq 1/3C_7$, $\xi \leq C_7\epsilon \leq 1/3$, and

$$\begin{aligned}
&\frac{2\xi+(1+\xi)\epsilon}{1-\xi}+\frac{(n-3)\cdot 3W(1+\xi)(1+\epsilon)}{1-\xi}\xi\\
&\quad=\frac{1}{1-\xi}\left\{2\xi+\epsilon+\xi\epsilon+3(n-3)W(1+\xi)(1+\epsilon)\xi\right\}\\
&\quad\leq\frac{1}{1-1/3}\left\{2C_7\epsilon+\epsilon+\frac{1}{3}\epsilon+3(n-3)W\left(1+\frac{1}{3}\right)\left(1+\frac{1}{3C_7}\right)C_7\epsilon\right\}\\
&\quad=\left\{3C_7+2+6(n-3)W\left(C_7+\frac{1}{3}\right)\right\}\epsilon.
\end{aligned}$$

Set

$$\begin{aligned}
C_8 &:= \frac{2W}{|\beta_1|}\left\{3C_7+2+6(n-3)W\left(C_7+\frac{1}{3}\right)\right\}\\
&= \frac{2W}{|\beta_1|}\left\{3+C_6+3{C_6}^2+2(n-3)W(2+C_6+3{C_6}^2)\right\},
\end{aligned}$$

and assume $\epsilon < 1/C_8$ additionally. Then we get

$$\left|\sum_{j=1}^{n}b_j+\sum_{j=1}^{n}\frac{(q_j-p_j)b_j}{z-q_j}\right|^2 \geq \left\{\sqrt{|w|}\,n|\beta_1|\left(1-\frac{1}{2}C_8\epsilon\right)\right\}^2 \geq n^2|\beta_1|^2|w|(1-C_8\epsilon)$$

for any $z \in \hat{\mathbf{C}} \setminus \bigcup_{j=1}^{n} B_j$.

Now, since

$$\begin{aligned}
&(1-\xi)^3(1-\epsilon)^2-(1-\xi^2)+2\{2\xi+(1+\xi)\epsilon\}(1+\xi)\\
&\quad=\xi(1-\xi^2)+8\xi^2+2\epsilon\{2\xi(1-\xi)+3\xi+\xi^3\}+\epsilon^2(1-\xi)^3>0,
\end{aligned}$$

it holds that

$$\frac{(1-\xi)^2(1-\epsilon)^2}{1+\xi} > 1-2\frac{2\xi+(1+\xi)\epsilon}{1-\xi} \geq 1-\frac{2W}{|\beta_1|}\frac{2\xi+(1+\xi)\epsilon}{1-\xi}$$

from which it follows that

$$\left|\sum_{k=1}^{n}\frac{b_k}{z-q_k}\right|^2 \;\geq\; n^2\frac{W^2}{{C_4}^2}|w|\frac{(1-\xi)^2(1-\epsilon)^2}{1+\xi}$$

$$\geq n^2W^2|w|\left[1-\frac{2W}{|\beta_1|}\left\{\frac{2\xi+(1+\xi)\epsilon}{1-\xi}+\frac{(n-3)\cdot 3W(1+\xi)(1+\epsilon)}{1-\xi}\xi\right\}\right]$$
$$\geq n^2W^2|w|(1-C_8\epsilon)$$

for any $z \in B_j$, where we use the assumption $C_4 = \tan(C_3/4) \leq 1$.

Now, we get

$$\frac{ds}{ds_{\mathbf{S}^2}} \geq \frac{n^2}{2}|\beta_1|^2|w|(1-C_8\epsilon)$$

in $M = \hat{\mathbf{C}} \setminus \{q_1, \ldots, q_n\}$.

By (6.2) and Lemma 4.1, we have

$$\angle(\sigma^{-1}(p_j), \sigma^{-1}(q_j)) \leq 4\,\mathrm{Tan}^{-1}\frac{C_4\xi}{2} \leq 4\cdot\frac{C_4\xi}{2} = 2C_4\xi \qquad (j = 1, \ldots, n).$$

Hence

$$\begin{aligned}\angle(\sigma^{-1}(q_j), \sigma^{-1}(q_k)) &\geq \angle(\sigma^{-1}(p_j), \sigma^{-1}(p_k)) - \angle(\sigma^{-1}(p_j), \sigma^{-1}(q_j)) \\ &\quad -\angle(\sigma^{-1}(p_k), \sigma^{-1}(q_k)) \\ &\geq C_3 - 2C_4\xi - 2C_4\xi = C_3 - 4C_4\xi \geq C_3 - 4C_4C_7\epsilon \qquad (k \neq j).\end{aligned}$$

Therefore the length of any loop in $X(M)$ surrounding at least two ends satisfies

$$l \geq \frac{n^2}{2}|\beta_1|^2|w|(1-C_8\epsilon)\times 2(C_3 - 4C_4C_7\epsilon) \geq n^2C_3|\beta_1|^2|w|(1-C_9\epsilon),$$

where we set $C_9 := C_8 + 4C_4C_7/C_3$.

On the other hand, the length of any loop in $X(M)$ surrounding only one end satisfies

$$l \geq 4\pi \min|a_j|.$$

Combining these estimates, we get our assertion. □

Proof of Theorem 1.2. Theorem 1.2 is the special case of Theorem 6.1 when all the w_k^j's are equal to 1. In particular, $W = 1$ and $\beta_1 = 1$, and hence we get our assertion. □

7. Examples

As we mentioned in §1, examples of n-end catenoids with short minimal closed geodesics, which satisfy the assumption (1.3) in Theorem 1.1, are found in [3], [5], [7], [10] etc. On the other hand, examples which satisfy the assumption (6.1) in Theorem 6.1 (or (1.5) in Theorem 1.2) are found in [4], [11]. Here we present an example of a family of 4-end catenoids which includes [3, Example 2.3.8], [5, Example 3.7] and more surfaces satisfying (1.3).

By Corollary 3.2 and direct computation, we get the following

EXAMPLE 7.1. For the data

$$\begin{cases} p_1 = p,\ p_2 = -p,\ p_3 = p^{-1}\zeta,\ p_4 = -p^{-1}\zeta, \\ a_1 = a_2 = a_3 = a_4 = 1, \\ \text{where } p \in \mathbf{R},\ 0 < p < +\infty,\ |\zeta| = 1, \end{cases}$$

the equation (2.2) possesses a solution

$$\begin{cases} q_1 = q,\ q_2 = -q,\ q_3 = q^{-1},\ q_4 = -q^{-1}, \\ b_1 = b_2 = qt,\ b_3 = b_4 = \zeta^{-1/2}pt, \\ \text{where } \dfrac{q(q^2+\zeta)}{\zeta^{1/2}(q^4-1)} = \dfrac{p^2-1}{4p},\ t = \sqrt{\dfrac{q^4-1}{q\{p(q^4-1)+2\zeta^{-1/2}q(p^2q^2-\zeta)\}}}. \end{cases}$$

Example 2.3.8 (Figure 2.3.8) in [3] is in the case when $\zeta = 1$, and Example 3.7 (Figure 3.2(a), (b)) in [5] is in the case when $\zeta = \sqrt{-1}$.

In our general case, the relative weight of each end-pair is given by

$$\begin{aligned} w_{12} = w_{34} &= \frac{p(q^4-1)}{w_0}, \\ w_{13} = w_{24} &= \frac{\zeta^{-1/2}q(p^2-\zeta)(q^2+1)}{w_0}, \\ w_{14} = w_{23} &= \frac{\zeta^{-1/2}q(p^2+\zeta)(q^2-1)}{w_0}, \\ \text{where } w_0 &= p(q^4-1) + 2\zeta^{-1/2}q(p^2q^2-\zeta). \end{aligned}$$

When $p \to 1$ and $q \to 0$, the relative weights behave as follows:

$$\begin{aligned} w_{12} = w_{34} &\to 1 && \neq 0, \\ w_{13} = w_{24} &\sim \zeta^{-1/2}(\zeta-1)q && \to 0, \\ w_{14} = w_{23} &\sim \zeta^{-1/2}(\zeta+1)q && \to 0. \end{aligned}$$

In this case, the length l' of the image of the unit circle centered at the origin 0, which separates the ends q_1, q_2 and the ends q_3, q_4, is given by

$$l' = \frac{16\pi|q|^2(1+p^2|q|^2)}{1-|q|^4}|t|^2 \sim 16\pi|q| \to 0.$$

When $p \to 1$ and $q \to \sqrt{-\zeta}$, the relative weights behave as follows:

$$w_{12} = w_{34} \to \frac{(\zeta-1)(\zeta+1)}{\zeta^2 - 1 - 4\zeta\sqrt{-1}} \quad (\neq 0 \text{ if } \zeta \neq \pm 1),$$

$$w_{13} = w_{24} \to \frac{(\zeta-1)^2\sqrt{-1}}{\zeta^2 - 1 - 4\zeta\sqrt{-1}} \quad (\neq 0 \text{ if } \zeta \neq 1),$$

$$w_{14} = w_{23} \to -\frac{(\zeta+1)^2\sqrt{-1}}{\zeta^2 - 1 - 4\zeta\sqrt{-1}} \quad (\neq 0 \text{ if } \zeta \neq -1).$$

When $p \to 0$ and $q \to 1$, the relative weights behave as follows:

$$\begin{aligned} w_{12} = w_{34} &\sim -2\zeta^{-1/2}p(q-1) && \to 0, \\ w_{13} = w_{24} &\to 1 && \neq 0, \\ w_{14} = w_{23} &\sim -(q-1) && \to 0. \end{aligned}$$

In this case, the length l'' of the image of the imaginary axis, which separates the ends q_1, q_3 and the ends q_2, q_4, is given by

$$l'' = \frac{8\pi|q|^2(|pq + \zeta^{1/2}|^2 + |p + \zeta^{1/2}q|^2)}{(q+\overline{q})(1+|q|^2)}|t|^2 \sim 8\pi|q-1| \to 0.$$

To our regret, our estimates are not sharp in general. On the other hand, when we observe other examples of deformations of n-end catenoids, we often find that some of the relative weights tend to 0 or ∞ when a surface X goes near to the boundary of the moduli space of n-end catenoids. Therefore it is expected that there are better estimates under weaker assumptions. If we get such an estimate, then we can understand the relationship between the relative weights and the collapse of n-end catenoids more deeply.

It is also an open problem to introduce the relative weights in the case of higher genus.

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