INCOMPRESSIBILITY OF CLOSED SURFACES
IN TOROIDALLY ALTERNATING
LINK COMPLEMENTS

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1. Introduction

It has been known that alternating knots and links have many nice properties, and there have been many generalizations of the notion of alternating links. For instance, homogeneous, adequate, augmented alternating and almost alternating links. C.C. Adams introduced the notion of toroidally alternating links in [1]. The class of toroidally alternating links turned out to contain those of alternating links, almost alternating links, pretzel links and Montesinos links (See [1] and [3]). C. Hayashi also studied alternating links on surfaces of arbitrary positive genera in [3].

The purpose of this paper is to determine incompressibility of given closed orientable surfaces in toroidally alternating link complements.

Let M be a lens space or the 3-sphere S^3, and let T be a torus which gives a Heegaard splitting of M. This torus is unique up to isotopy (See [2]). A link L in M is called *toroidally alternating* (*with respect to T*) if it can be isotoped into a neighborhood $T \times I$ of T so that it has an alternating diagram $\pi(L)$ on T such that $T - \pi(L)$ consists of open discs with respect to a projection $\pi : T \times I \to T$.

Let $F \subset M - L$ be an embedded connected closed surface. F is called *incompressible* in $M - L$ if either F is a 2-sphere and F does not bound a 3-ball in $M - L$, or F is not a 2-sphere and for each disc $D \subset M - L$ with $D \cap F = \partial D$, there is a disc $D' \subset F$ with $\partial D = \partial D'$. F is called *pairwise incompressible* if there does not exist a disc $D \subset M$ meeting L transversely in one point with $D \cap F = \partial D$. An embedded disconnected surface F in $M - L$ is called *incompressible* in $M - L$ (resp. *pairwise incompressible*) if every component of F is incompressible in $M - L$ (resp. pairwise incompressible).

A link $L \subset M$ is called *non-split* if every 2-sphere embedded in $M - L$ bounds a 3-ball in $M - L$.

W. Menasco defined the notion of *standard position* for surfaces either without boundary or with meridional boundary in alternating knot and link complements
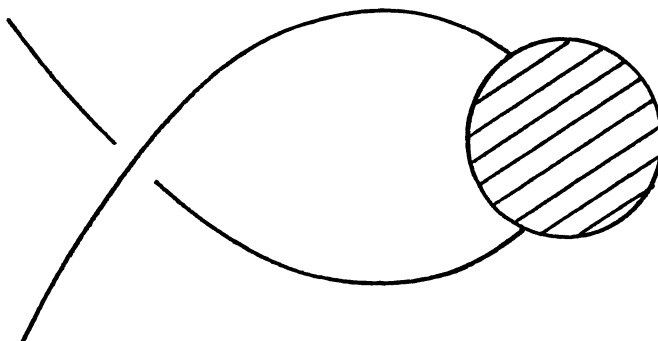

Fig. 2.1.

in [4], and he showed that there is no closed incompressible and pairwise incompressible surface in non-split prime alternating link complements. He also gave a sufficient condition for surfaces with meridional boundary in standard position to be incompressible in [5]. Adams proved that there is no closed incompressible and pairwise incompressible surface in some toroidally alternating knot complements. But it is known that there are toroidally alternating link complements which contain incompressible and pairwise incompressible surfaces. Adams and Hayashi gave examples of such surfaces in [1] and [3] independently. In this paper we define the notion of standard position for closed orientable surfaces in toroidally alternating link complements, and we see that we can easily determine whether given surfaces are incompressible or not using the notion of standard position.

Theorem.   *Let M be S^3 or a lens space (excluding $S^2 \times S^1$). Let L be a non-split toroidally alternating link in M, and let $F \subset M - L$ be a closed orientable surface in standard position. Then F is incompressible in $M - L$ and pairwise incompressible.*

In §2 we consider some properties which incompressible and pairwise incompressible surfaces satisfy, and define for closed orientable surfaces in toroidally alternating link complements being in standard position as satisfying such properties. In §3 we define the notion of standard position for compressing ·discs, and finally show that there exists no compressing disc of closed orientable surfaces in standard position in toroidally alternating link complements in §4, §5 and §6.

2.   Standard position

Let $L \subset M$ be a non-split toroidally alternating link with respect to T. Throughout this paper we assume that L has been isotoped so as to eliminate *nugatory crossings* shown in Fig. 2.1.

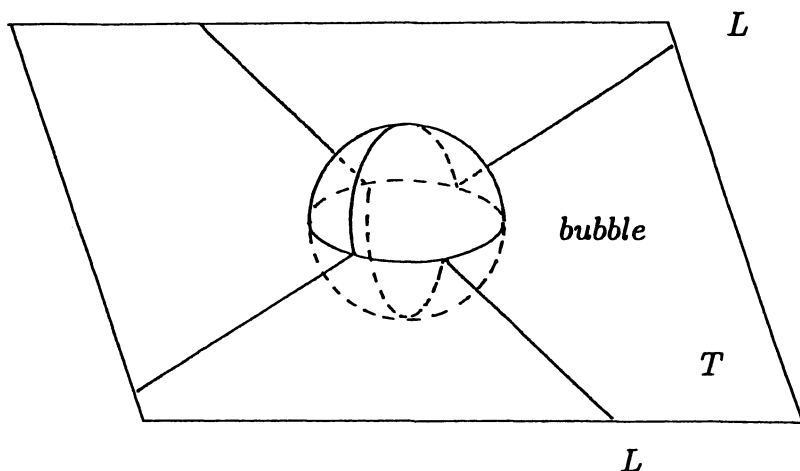As in [4], we place a bubble at each crossing of the diagram $\pi(L)$ and isotope·

Fig. 2.2.

L so that overstrand at the crossing runs on the upper hemisphere and understrand runs on the lower hemisphere as shown in Fig. 2.2.

Let T_+ (respectively T_-) be the torus obtained from T with each equatorial disc inside a bubble replaced by the upper (resp. lower) hemisphere of the bubble. We define V_+ (resp. V_-) to be the solid torus bounded by T_+ (resp. T_-) which does not intersect the interiors of the bubbles. We use the notation $T_\pm$ (resp. $V_\pm$) to mean T_+ or T_- (resp. V_+ or V_-).

Let $F \subset M - L$ be an incompressible, pairwise incompressible closed orientable surface. Following [4], we isotope F to be in good position with respect to the diagram $\pi(L)$.

Proposition 2.1.   *We can isotope F so that*;

(i)   *F meets $T_\pm$ transversely in a pairwise disjoint collection of simple closed curves, and*

(ii)   *F meets each 3-ball bounded by a bubble in a collection of saddle-shaped discs* (Fig. 2.3).

Such a saddle-shaped disc is called a *saddle*.

For a surface F satisfying (i), (ii) of Proposition 2.1, we define the *complexity* $c(F)$ of F to be the lexicographically ordered pair (t, u), where t is the number of the saddles of F and u is the total number of components of $F \cap T_+$ and $F \cap T_-$. We say F has *minimal complexity* if $c(F) \le c(F')$ for any surface F' isotopic to F.

Lemma 2.2.   *Suppose F has minimal complexity. Then each loop of $F \cap T_\pm$*
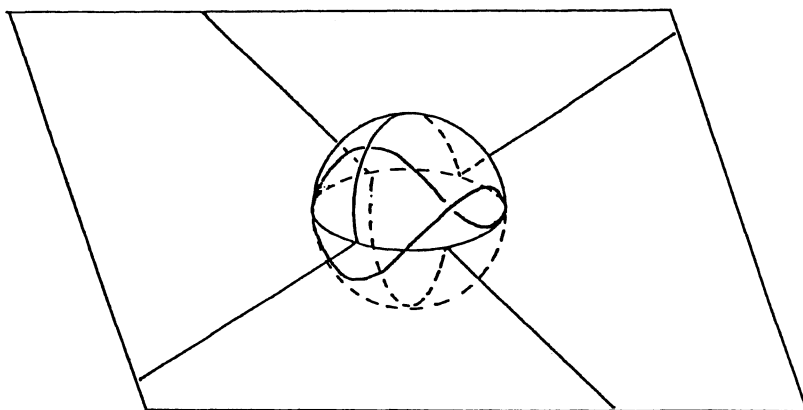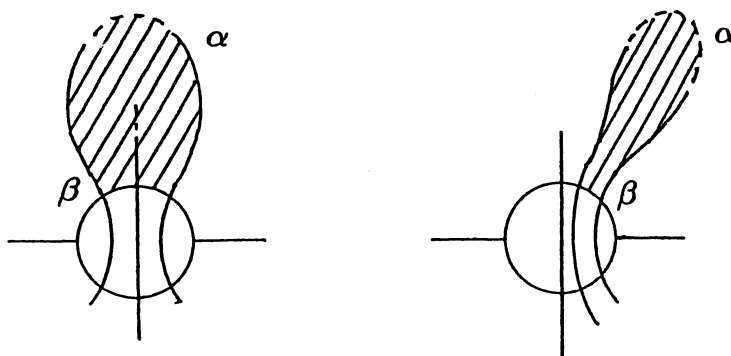
Fig. 2.3.

Fig. 2.4.

is a component of boundary of one of a peripheral disc, a peripheral annulus or a meridian disc.

Proof.   It follows from the incompressibility of F and the non-splittability of L.   □

We call a loop $C \subset F \cap T_{\pm}$ a *trivial loop* if C is boundary of a peripheral disc, and a *non-trivial loop* otherwise.

Let H be a upper(or a lower) hemisphere of a bubble. We say a loop $C \subset F \cap T_{\pm}$ *meets a bubble twice with encircling a disc* if there are arcs $\alpha \subset C$ and $\beta \subset \partial H$ such that $\alpha \cup \beta$ forms a loop bounding a disc on T as shown in Fig. 2.4.

Proposition 2.3.   *Suppose F has minimal complexity. Then no loop of $F \cap T_\pm$ meets a bubble twice with encircling a disc.*

Proof.   See [4, Lemma 1(ii)].   □

Since L is toroidally alternating and $T - \pi(L)$ consists of open discs, $F \cap T_\pm$ has the following *alternating property*:
(∗)   Suppose that a loop $C \subset F \cap T_\pm$ crosses two bubbles B_1 and B_2 (they are possibly the same bubble) in succession. Then two arcs of $L \cap T_\pm$ in B_1 and B_2 lie on opposite sides of C with respect to an arbitrary orientation of C.

Proposition 2.4.   *Suppose F has minimal complexity. Then each loop of $F \cap T_\pm$ is boundary of a meridian disc of $F \cap V_\pm$.*

The proof of Proposition 2.4 is obtained by Lemma 2.2 and the following two Lemmas.

Lemma 2.5.   *Suppose F has minimal complexity. Then there is no trivial loop in $F \cap T_\pm$.*

Proof.   See [4, Lemma 1].   □

Lemma 2.6.   *Suppose F has minimal complexity. Then there is no non-trivial loop which is a component of boundary of a peripheral annulus of $F \cap V_\pm$.*

Proof.   See proof of [1, Theorem 3.1].   □

Proof of Proposition 2.4.   Proposition 2.4 follows immediately from Lemmas 2.2, 2.5 and 2.6.   □

Let $F \subset M - L$ be a closed orientable surface. We say F is in *standard position* if F satisfies the conclusions of Propositions 2.1, 2.3 and 2.4. That is;
(i) F meets $T_\pm$ transversely in a pairwise disjoint collection of simple closed curves, and
(ii) F meets each 3-ball bounded by a bubble in a collection of saddle-shaped discs. (Fig. 2.3), and
(iii)   No loop of $F \cap T_\pm$ meets a bubble twice with encircling a disc, and
(iv)   Each loop of $F \cap T_\pm$ is boundary of a meridian disc of $F \cap V_\pm$.
   In this section we showed that every closed orientable incompressible and pairwise incompressible surface in a toroidally alternating link complement can be isotoped into standard position.

3. Compressing discs in standard position

We prove Theorem by reduction to absurdity. Let $F \subset M - L$ be a closed surface in standard position. From now on we examine compressing discs and pairwise compressing discs for F. Menasco also defined the notion of standard position for compressing discs in [5].

First as in [5], we place a compressing or a pairwise compressing disc in suitable position. We omit the proof of Proposition 3.1.

Proposition 3.1.   *Suppose F has a compressing or a pairwise compressing disc D. Then we can isotope D so that;*

(i)   *D meets $T_{\pm}$ transversely in a pairwise disjoint collection of simple closed curves and properly embedded arcs in D, and*

(ii)   *D meets each 3-ball bounded by a bubble in a collection of saddles.*

For a compressing or a pairwise compressing disc D of F satisfying (i) and (ii) of Proposition 3.1, we define the *complexity* $c(D)$ of D to be the lexicographically ordered pair (t, u), where t is the number of the saddles of D and u is the total number of components of $D \cap T_+$ and $D \cap T_-$. We say D has *minimal complexity* if $c(D) \leq c(D')$ for every compressing or pairwise compressing disc D' of F. From now on we assume that D has minimal complexity.

We call a component of $T - \pi(L)$ a *region*.

Proposition 3.2.   *D satisfies the following;*

(i)   *No loop or arc of $D \cap T_{\pm}$ meets a bubble twice with encircling a disc (see §2 for the definition of this situation), and*

(ii)   *No arc component of $D \cap T_{\pm}$ that is contained in a single region cobounds a disc in the region together with a subarc of a loop of $F \cap T_{\pm}$ (Fig. 3.1), and*

(iii)   *Each component of $D \cap V_{\pm}$ is incompressible in $V_{\pm}$.*

Proof.   For (iii), suppose $D \cap V_+$, say, is compressible in V_+. Since M does not contain a non-separating 2-sphere, the boundary of a compression disc of $D \cap V_+$ bounds a disc also on $D - (L \cap D)$. As in Lemma 2.2, we can isotope D so that $D \cap V_+$ is incompressible in V_+. This isotopy reduces the number of the components of $D \cap T_{\pm}$, which contradicts the minimal complexity of D.

For (i) and (ii), see Lemma 4(b) and (a) in [5].   □

We say a compressing or a pairwise compressing disc D is in *standard position with respect to F* if D satisfies the conclusions of Propositions 3.1 and 3.2.
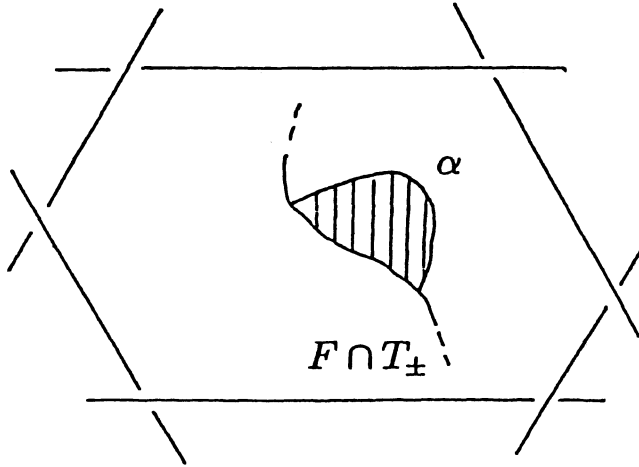
Fig. 3.1.

4. Loops in $D \cap T_{\pm}$ (I)

In §4 and §5, we will prove Proposition 4.2, that is, there is no loop in $D \cap T_{\pm}$. In §4 we make some preparations. First we prove that if there were a loop of $D \cap T_{\pm}$, then it would bound a meridian disc of $V_{\pm}$ in Proposition 4.1. Second we study some parts of $(F \cup D) \cap T_{\pm}$ and prove three Propositions 4.3, 4.6 and 4.9. Proposition 4.2 is proved in the end of §5.

From now on we assume that D is in standard position with respect to F.

Since L is a toroidally alternating link, $D \cap T_{\pm}$ also satisfies the following *alternating property*:

Suppose $\alpha \subset D \cap T_{\pm}$ crosses two bubbles B_1 and B_2 in succession, then,

$(**)$ (i)   If two arcs of $L \cap T_{\pm}$ in B_1 and B_2 lie on opposite sides of α, then α does not cross the puncture $D \cap L$ between B_1 and B_2.

       (ii)   If two arcs of $L \cap T_{\pm}$ in B_1 and B_2 lie on the same side of α, then α crosses the puncture $D \cap L$ between B_1 and B_2.

Note that since a loop of $F \cap T_{\pm}$ satisfies the alternating property $(*)$, a subarc of $(F \cup D) \cap T_{\pm}$ satisfies the alternating property $(**)$.

Let ξ be a component of $F \cap T_+$ (resp. $F \cap T_-$) or $D \cap T_+$ (resp. $D \cap T_-$) which meets some bubble B. We define the *mate to ξ at B* to be the component of $F \cap T_+$ (resp. $F \cap T_-$) or $D \cap T_+$ (resp. $D \cap T_-$) which meets the other side of B and contain a subarc of the boundary of the saddle which is incident to ξ at B.

From (i) of Proposition 3.1, $D \cap T_{\pm}$ consists of loops and properly embedded arcs. First we deal with loops. From (iii) of Proposition 3.2, each loop of $D \cap T_{\pm}$ is a component of boundary of one of a peripheral disc, an incompressible peripheral

annulus or a meridian disc.

Proposition 4.1.   *No loop of $D \cap T_\pm$ is a component of boundary of a peripheral disc or a peripheral annulus.*

Proof.   Suppose there exists a trivial loop of $D \cap T_+$ which is boundary of a peripheral disc. Let C be an innermost one of such loops on T_+ and d the disc bounded by C on T_+. If C intersects no bubbles, then C bounds peripheral discs in both V_+ and V_-, which is impossible. So C intersects bubbles. If C does not intersect L or intersects more than two bubbles, then by the alternating property $(**)$ and Proposition 3.2(i) we can find a mate to C in d. Let C' be the mate. Suppose C' is a loop of $D \cap T_+$. Then C' cannot exit from d, C' is a trivial loop in d, which contradicts the choice of d. Hence C' is an arc of $D \cap T_+$. Then C' connects with a loop of $F \cap T_+$ in d, and the loop of $F \cap T_+$ cannot exit from d, which contradicts that F is in standard position (See Proposition 2.4). Suppose C intersects L and intersects exactly one bubble. Then there is a loop or an arc of $D \cap T_-$ which violates Proposition 3.2(i). Therefore trivial loop cannot exist.

Suppose there exists a non-trivial loop which is a component of boundary of a peripheral annulus. Note that an incompressible annulus has a non-meridional boundary slope. Since every loop of $F \cap T_\pm$ bounds a meridian disc of $V_\pm$, the annuli of $D \cap V_\pm$ intersect $F \cap V_\pm$. Then $(\mathrm{int} D) \cap F$ is not empty. But it contradicts the choice of D.   □

Hence every loop of $D \cap T_\pm$ would bound a meridian disc of $V_\pm$.

Proposition 4.2.   *There is no loop component in $D \cap T_\pm$.*

To prove Proposition 4.2, we need some preparations. From now on we study some parts of $(F \cup D) \cap T_\pm$.

First we consider a part of $(F \cup D) \cap T_\pm$ shown in Fig. 4.1. That is, there are a bubble B, an arc α of $D \cap T_+$ (resp. $D \cap T_-$) and a loop β of $F \cap T_+$ (resp. $F \cap T_-$) such that α meets B, both ends of α connect with β, and α and a subarc of β cobound a disc d on T_+ (resp. T_-). Moreover $(L \cap B) \cap d = \phi$ and there is no saddle incident to $\beta \cap \partial d$.

Proposition 4.3.   *If there exists a part of $(F \cup D) \cap T_\pm$ shown in* Fig. 4.1, *then α meets the puncture $D \cap L$ and no bubble other than B.*

To prove Proposition 4.3, we assume for a contradiction that $\alpha \cap (D \cap L) = \phi$. We need following two Lemmas 4.4 and 4.5.
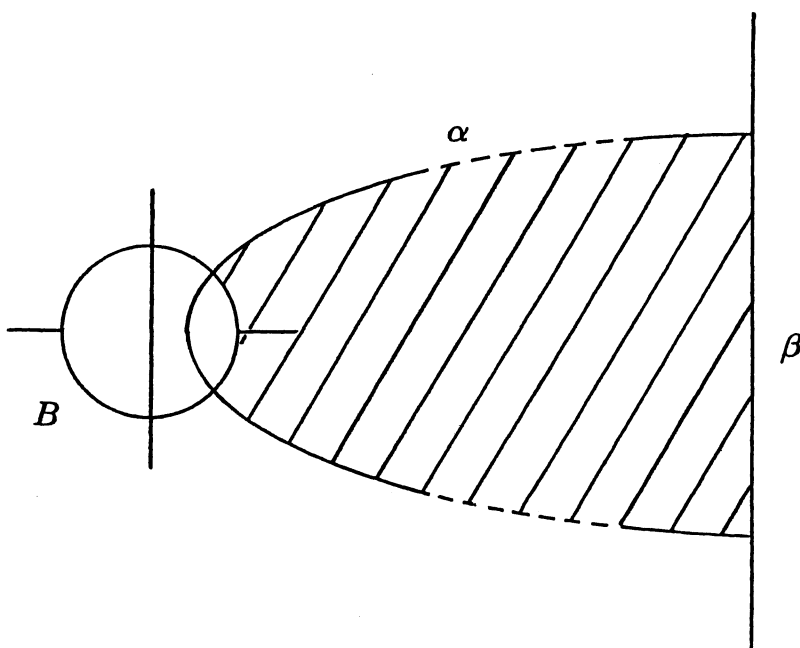
Fig. 4.1.

Lemma 4.4. *We can find in d a smaller copy of the pattern as shown in* Fig. 4.1. *That is, there is another arc ξ of $D \cap T_+$ which cobounds a disc d_1 in d together with a subarc of $\beta \cap \partial d$ such that ξ meets a bubble B' and $(L \cap B') \cap d_1 = \phi$.*

The same argument below will do for the case of $(F \cup D) \cap T_-$.

Proof. Suppose there is a part of $(F \cup D) \cap T_+$ shown in Fig. 4.1. There is a part of $L \cap T$ which enters d at the bubble B. Since the part of $L \cap T$ exits from d, α meets another bubble than B. Let B' be the bubble. Then by $(**)$ and Proposition 3.2(i), there is the mate to α at B' in d, which is not α. Let ξ be the mate. If ξ were a loop, it could not exit from d, which violates Proposition 4.1. Hence ξ is another arc and its both ends connect with $\beta \cap \partial d$ and it cobounds a disc $d_1 \subset d$ together with a subarc of $\beta \cap \partial d$. $\square$

Note that the arc ξ may intersect L. We can find a collection of discs in d which are cobounded by arcs of $D \cap T_+$ and subarcs of $\beta \cap \partial d$ as shown in Fig. 4.1.

We can define a partial ordering on the set of such discs by declaring that $d' < d''$ if and only if $d' \subset d''$, where d' and d'' are such discs. We take one of discs which are minimal with respect to this partial ordering. Let d_n be the disc and ζ

the arc of $D \cap T_+$ which cobounds d_n together with a subarc of $\beta \cap \partial d$. Let B_n be the bubble which ζ meets.

Lemma 4.5.   *The arc ζ meets the puncture $D \cap L$ and no bubble other than B_n.*

Proof.   From the definition of ζ, there is a part of $L \cap T$ entering d_n at B_n. Suppose first that ζ does not contain the puncture. Then we can apply Lemma 4.4 to $\zeta \cup (\beta \cap \partial d_n)$, and we have a contradiction to the fact that we take d_n to be innermost on d. Therefore ζ contains the puncture. Suppose ζ meets another bubble than B_n. If ζ meets at least two bubbles other than B_n, we can find a mate to ζ in d. Then as we did above we can find a mate of ζ which violates that d_n is innermost. Hence ζ meets at most one bubble other than B_n. If ζ does meet a bubble other than B_n, then let B'_n be the bubble. Then there are exactly one or three parts of $L \cap T$ entering d_n at B'_n. Therefore there are exactly three or five parts of $L \cap T$ entering d_n at B_n, at B'_n and at the puncture, which is a contradiction. Hence ζ meets the puncture and no bubble other than B_n. This completes the proof of Lemma 4.5.   □

Proof of Proposition 4.3.   We take the disc d_{n-1} such that $d_n < d_{n-1}$ and there does not exist a disc d_i other than d_n or d_{n-1} satisfying $d_n < d_i < d_{n-1}$. Note that we can take d_{n-1} uniquely. Suppose there is another such disc d_j than d_n satisfying $d_j < d_{n-1}$. We take the disc in d_j which is minimal with respect to
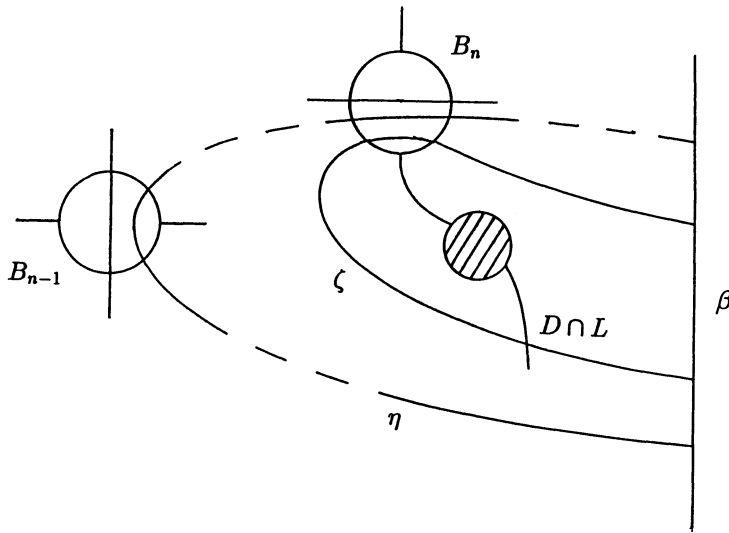

Fig. 4.2.

the partial ordering. Let d_k be the disc. Since ζ contains the puncture, ∂d_k does not contain the puncture. Then we apply Lemma 4.4 to d_k and have a contradiction. Hence there is no such disc other than d_n in d_{n-1}. Note that this means there is no other component of $D \cap T_+$ in $\text{int}(d_{n-1} - d_n)$ which meets a bubble. If there is a component of $D \cap T_+$ in $\text{int}(d_{n-1} - d_n)$ which meets no bubbles, then it violates Proposition 4.1 or (ii) of Proposition 3.2.

Let η be the arc of $D \cap T_+$ which cobounds d_{n-1} together with a subarc of $\beta \cap \partial d$. From the way of taking η, η meets B_n, otherwise we can find the mate to ξ at B_n in $\text{int}(d_{n-1} - d_n)$. Suppose η and ζ meet the same side of B_n. See Fig. 4.2. From the way of taking d_{n-1}, there is a part of $L \cap T$ entering d_{n-1} at some bubble B_{n-1} such that $(L \cap B_{n-1}) \cap d_{n-1} = \phi$. Suppose B_{n-1} coincides with B_n. Since there is a part of $L \cap T$ existing from d_n at the puncture, we need at least one more saddle incident to η other than B_n. Then η violates (i) of Proposition 3.2 or we can find a mate to η in $\text{int}(d_{n-1} - d_n)$. In both cases we have a contradiction. Hence B_{n-1} is different from B_n. Since η satisfies the alternating property (**) and since η does not contain the puncture, η meets another bubble between meeting B_{n-1} and B_n. Then we can find another component of $D \cap T_+$ in $\text{int}(d_{n-1} - d_n)$, which is a contradiction.

Therefore η and ζ meet distinct sides of B_n. Let η_a (resp. η_b) the subarc of η which connect B_n with β and does not meet B_{n-1} (resp. meets B_{n-1}). Let d_a (resp. d_b) be a disc in d_{n-1} which is cobounded by η_a (resp. η_b), and subarcs of ζ and ∂H_n, where H_n is the upper hemisphere of B_n. See Fig. 4.3. Suppose $\zeta \cap \partial d_b$
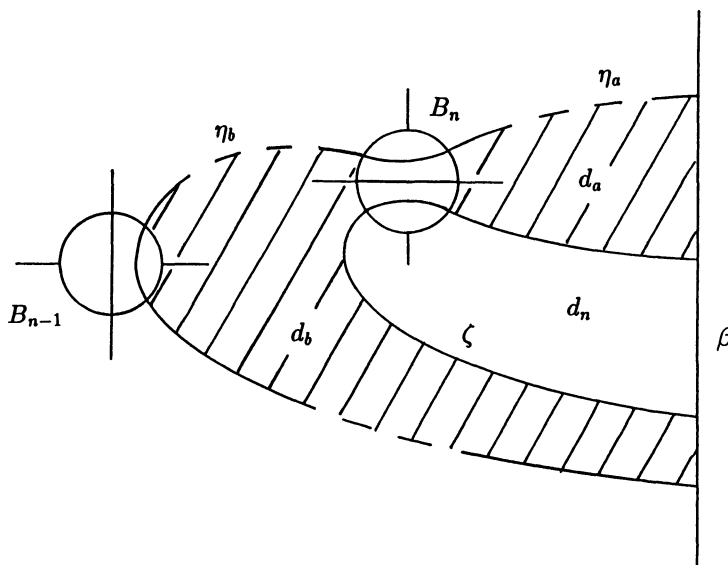

Fig. 4.3.

contains the puncture. Since there are three parts of $L \cap T$ entering d_b at B_{n-1}, at B_n and at the puncture, we need at least one more saddle incident to η_b. Then it follows either η_b violates (i) of Proposition 3.2 or we can find a mate to η_b in $\mathrm{int} d_b$ $\subset \mathrm{int}(d_{n-1} - d_n)$, which is a contradiction. Hence $\zeta \cap \partial d_a$ contains the puncture. Suppose η_a meets another bubble B_i than B_n. Then from the alternating property $(**)$, there is a part of $L \cap T$ entering d_a. It follows that there are three parts of $L \cap T$ entering d_a at B_n, at B_i and at the puncture. Then we need at least one more saddle incident to η_a. Then either η_a violates (i) of Proposition 3.2 or we can find a mate to η_a in $\mathrm{int} d_a \subset \mathrm{int}(d_{n-1} - d_n)$, which is a contradiction. Therefore η_a meets no other bubble. Then the crossing at B_n is a nugatory crossing, which contradicts our assumption. See Fig. 4.4.

Hence it follows that α meets the puncture. If there is another component of $D \cap T_+$ in d, by Propositions 3.2(ii), 4.1 or Lemma 4.4 we have a contradiction. Hence d is minimal with respect to the partial ordering. Then as in Lemma 4.5, we can show the proposition. This completes the proof. $\square$

Next we study another part of $(F \cup D) \cap T_\pm$ shown in Fig. 4.5. That is, there are two bubbles B_1 and B_2, an arc α of $D \cap T_+$ (resp. $D \cap T_-$) and a loop β of $F \cap T_+$ (resp. $F \cap T_-$) such that α meets B_1, β meets B_2 and both ends of α connect with β with cobounding a disc d on T_+ (resp. T_-) together with a subarc of β. Furthermore ∂d meets B_1 and B_2, $\beta \cap \partial d$ meets no bubble other than B_2, $(L \cap B_1) \cap d = \phi$ and $(L \cap B_2) \cap d = \phi$.

Proposition 4.6. *There does not exist a part of* $(F \cup D) \cap T_\pm$ *shown in* Fig. 4.5.

To prove Proposition 4.6, we need two Lemmas.

Lemma 4.7. *Suppose there is a part of* $(F \cup D) \cap T_+$ *shown in* Fig. 4.5. *Then there is an arc* ξ *of* $D \cap T_+$ *in* d *which meets a bubble and cobounds a disc* d_1 *in* d *together with a subarc of* $\beta \cap \partial d$.

Proof.   Since ∂d satisfies the alternating property $(**)$ and $\beta \cap \partial d$ meets only B_2 and $(L \cap B_1) \cap d = \phi$ and $(L \cap B_2) \cap d = \phi$, it follows that α meets another bubble, say B', than B_1. Then there is the mate to α at B' in d. Let ξ be the mate. Then, by Proposition 4.1, ξ is another arc and its both ends connect with $\beta \cap \partial d$. $\square$

Note that in Lemma 4.7 the arc $\beta \cap \partial d_1$ may not meet the bubble B_2.

Then in the same way as we did in the paragraph before Lemma 4.5, for the set of discs in d which are cobounded by subarcs of $\beta \cap \partial d_1$ and arcs of $D \cap T_+$ which meets a bubble, we define a partial ordering. We take one of the discs which
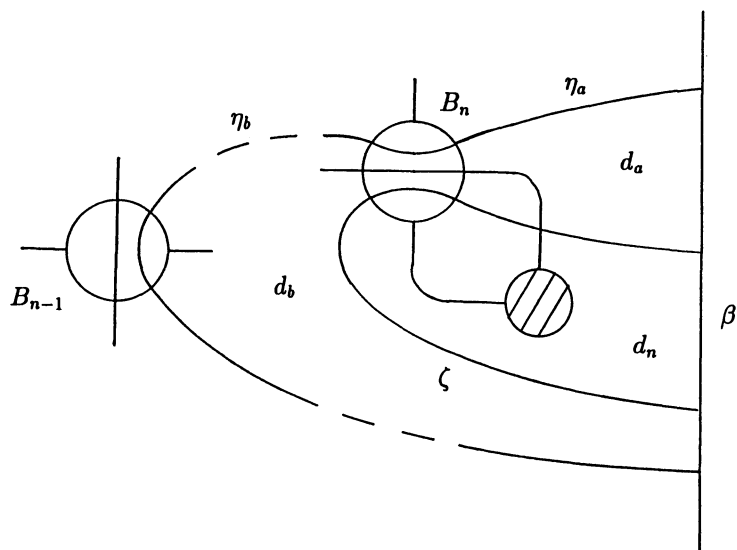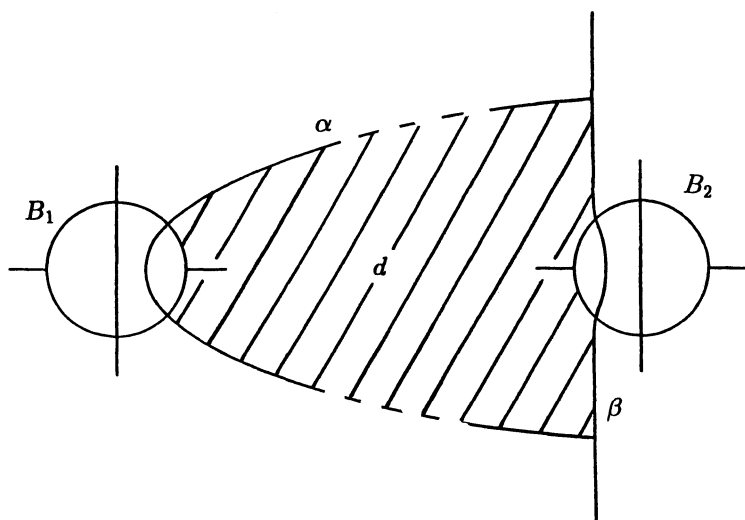
Fig. 4.4.

Fig. 4.5.

are minimal with respect to this partial ordering. Let d_n be the disc and ζ the arc of $D \cap T_+$ which cobounds d_n together with a subarc of $\beta \cap \partial d$.

Lemma 4.8.   *The subarc $\beta \cap \partial d_n$ does not meet B_2.*

Proof.   Follows from Lemma 4.7   □

Proof of Proposition 4.6.   From Lemma 4.8, $\beta \cap \partial d_n$ does not meet B_2. By Proposition 4.3, ζ contains the puncture. From the definition of ζ, it meets exactly one bubble. Let B_n be the bubble. As in proof of Proposition 4.3, we take the disc d_{n-1}. Then we can prove Proposition 4.6 similarly as in the proof of Proposition 4.3 by hand.   □

Next we study a particular part of $(F \cup D) \cap T_+$ (resp. $(F \cup D) \cap T_-$) shown in Fig. 4.6. To be more precise, there are a bubble B, a loop ξ of $F \cap T_+$ (resp. $F \cap T_-$) and an arc η of $D \cap T_+$ (resp. $D \cap T_-$) such that η connects with ξ, both ξ and η meet B and subarcs of ξ, η and ∂H cobound a disc d_0, where H is the upper (resp. lower) hemisphere of B. Let s and s' be the saddles inside B which η and ξ are incident to respectively.

We say a part of $(F \cup D) \cap T_+$ shown in Fig. 4.6 satisfies the *condition X* if the arc $\xi \cap H$ is nearer to the arc $L \cap H$ than the arc $\eta \cap H$, that is, the saddle s' is upper than the saddle s.

We also study parts of $(F \cup D) \cap T_\pm$ shown in Fig. 4.7 which are special cases of Fig. 4.6. In (i), (ii) and (iii) of Fig. 4.7, we assume that ξ and η meet the same side of B.

Moreover in (i) of Fig. 4.7, $\mathrm{int}(\xi \cap \partial d_0)$ meets no bubble, $\mathrm{int}(\eta \cap \partial d_0)$ meets the puncture $D \cap L$ and only one bubble, and the bubble is nearer to B than the puncture.

In (ii) of Fig. 4.7, $\mathrm{int}(\xi \cap \partial d_0)$ meets only one bubble and $\mathrm{int}(\eta \cap \partial d_0)$ meets the puncture and no bubble.

In (iii) of Fig. 4.7, $\mathrm{int}(\xi \cap \partial d_0)$ meets no bubble, $\mathrm{int}(\eta \cap \partial d_0)$ meets the puncture and only one bubble, and the puncture is nearer to B than the bubble.

Proposition 4.9.   *Suppose there is a part of $(F \cup D) \cap T_\pm$ shown in* Fig. 4.6. *Then there is a part of $(F \cup D) \cap T_\pm$ shown in* Fig. 4.7.
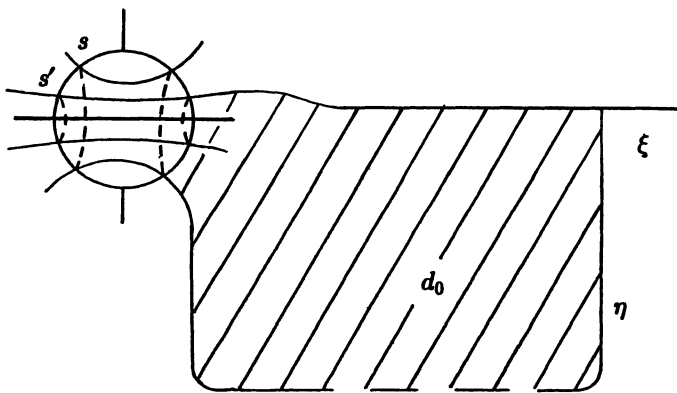
To prove Proposition 4.9, we need following seven Lemmas.

Lemma 4.10.   *Suppose there is a part of $(F \cup D) \cap T_\pm$ shown in* Fig. 4.6. *Suppose ξ and η meet distinct sides of B (*Fig. 4.6(ii)*). Then we can take another part of $(F \cup D) \cap T_\pm$ shown in* Fig. 4.6 *in d_0.*

Proof.   Without loss of generality, suppose $(F \cup D) \cap T_+$ contains the part shown in Fig. 4.6(ii). We can find the mate to ξ or to η at B entering d_0. First
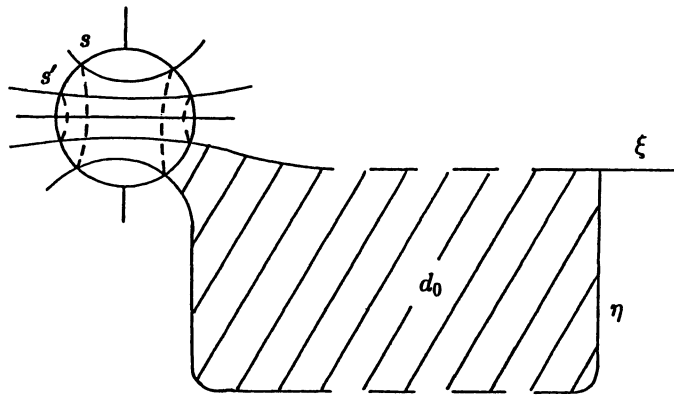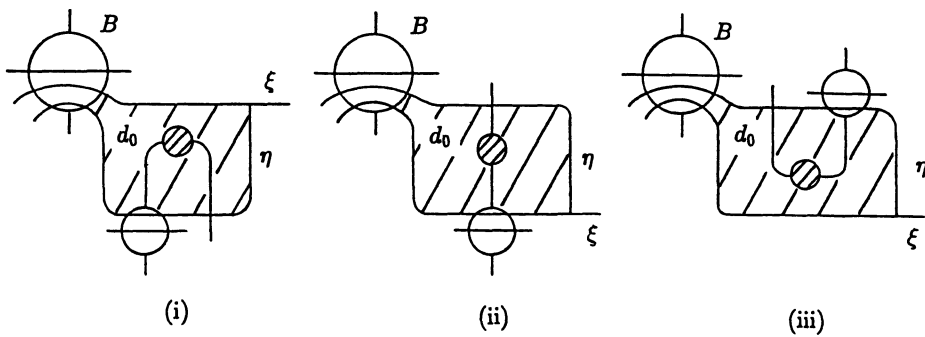
Fig. 4.6.

Fig. 4.7.

suppose that it is the mate to ξ. Let ξ' be the mate. Since ξ' is a loop of $F \cap T_+$, ξ' exits from d_0 at B. It violates Proposition 2.3. Suppose secondly that there is the mate to η entering d_0. Let η' be the mate. Then it follows (1) η' connects with another loop of $F \cap T_+$ than ξ in $\mathrm{int}d_0$, (2) η' exits from d_0 at B or (3) η' connects with $\xi \cap \partial d_0$. In the case (1), the loop exits from d_0 at B, which violates Proposition 2.3. In the case (2), η' violates Proposition 3.2(i). In the case (3), we can find another part of $(F \cup D) \cap T_\pm$ shown in Fig. 4.6 in d_0. Note that ξ and η' meets the same side of B.   □

From now on, in case there is another part of $(F \cup D) \cap T_\pm$ shown in Fig. 4.6 in d_0, we take one of the innermost such parts with respect to the discs they bound. Let d be the disc the part bounds. Hence we can assume ξ and η meet the same side of B as in Fig. 4.6(i). First we consider the case where a part of $(F \cup D) \cap T_\pm$ satisfies the condition X in Lemmas 4.11, 4.12 and 4.13. Later the other case in Lemmas 4.15, 4.16 and 4.17. Finally we give a proof of Proposition 4.9 in the end of this section.

Lemma 4.11.   *Suppose there is a part of* $(F \cup D) \cap T_\pm$ *shown in* Fig. 4.6 *which satisfies the condition* X. *Then there is no saddle incident to* $\mathrm{int}(\xi \cap \partial d)$.

Proof.   Suppose for a contradiction that there is a saddle $\tilde{s}$ incident to $\mathrm{int}(\xi \cap \partial d)$. Let $\tilde{B}$ be the bubble which contains $\tilde{s}$. From the alternating property (∗), (1) we can find the mate to ξ at $\tilde{B}$ in d or (2) both ξ and η meet $\tilde{B}$. In the case (1), since every loop of $F \cap T_+$ is non-trivial, the loop exits from d at B, which violates Proposition 2.3. In the case (2), it contradicts that d is innermost. Hence there is no saddle incident to $\mathrm{int}(\xi \cap \partial d)$.   □

Lemma 4.12.   *Suppose there is a part of* $(F \cup D) \cap T_+$ *shown in* Fig. 4.6 *which satisfies the condition* X *and there is no saddle incident to* $\mathrm{int}(\xi \cap \partial d)$. *Suppose* $\mathrm{int}(\eta \cap \partial d)$ *contains the puncture* $D \cap L$. *Then we can find a part of* $(F \cup D) \cap T_+$ *shown in* (i) *of* Fig. 4.7.

Proof.   First suppose that $\mathrm{int}(\eta \cap \partial d)$ meets more than one bubble. Since there is a part of $L \cap T$ entering d at the puncture, $\mathrm{int}(\eta \cap \partial d)$ meets odd numbers of bubbles. Then from the alternating property (∗∗), there is a mate to $\eta \cap \partial d$ in d. Let η' be the mate. Then (1) η' connects with another loop of $F \cap T_+$ than ξ in d, (2) both ends of η' exits from d at B, (3) one end of η' connects with ξ and the other exits from d, or (4) both ends of η' connect with ξ. In the case (1), the loop exits from d at B, which violates Proposition 2.3. In the case (2), η' violates (i) of Proposition 3.2. In the case (3), we can find a part of $(F \cup D) \cap T_+$ shown in Fig. 4.6 in d, which contradicts that d is innermost. In the case (4), η' and a subarc of ξ

cobounds a disc in d as shown in Fig. 4.1. Since η contains the puncture, η' does not contain the puncture. This violates Proposition 4.3.

Suppose secondly that $\text{int}(\eta \cap \partial d)$ meets only one bubble. Let $\tilde{B}$ be the bubble. Suppose the subarc of $\eta \cap \partial d$ between B and $\tilde{B}$ contains the puncture. It follows from (ii) of the alternating property $(**)$ that $(L \cap \tilde{B})$ is contained in d. Hence we can find a mate to $\eta \cap \partial d$ in d. Then as we did above we have a contradiction. Thus subarc of $\eta \cap \partial d$ between B and $\tilde{B}$ does not contain the puncture. There are exactly two parts of $L \cap T$ entering d at $\tilde{B}$ and at the puncture. Hence we can find a part of $(F \cup D) \cap T_+$ shown in (i) of Fig. 4.7. $\square$

Lemma 4.13. *Suppose there is a part of $(F \cup D) \cap T_+$ shown in* Fig. 4.6 *which satisfies the condition X and there is no saddle incident to $\text{int}(\xi \cap \partial d)$. Suppose $\text{int}(\eta \cap \partial d)$ does not contain the puncture. Then there is no saddle incident to $\text{int}(\eta \cap \partial d)$.*

To prove Lemma 4.13 we need following Sublemma 4.14.

Sublemma 4.14. *Suppose that there is a part of $(F \cup D) \cap T_+$ shown in* Fig. 4.6 *which satisfies the condition X, that there is no saddle incident to $\text{int}(\xi \cap \partial d)$, and that $\text{int}(\eta \cap \partial d)$ does not contain the puncture. Suppose there is a saddle incident to $\text{int}(\eta \cap \partial d)$. Then there is an arc of $D \cap T_+$ in $\text{int} d$ which is a mate of $\eta \cap \partial d$, and both ends of the arc connect with $\xi \cap \partial d$.*

Proof.   If there is a saddle incident to $\text{int}(\eta \cap \partial d)$, then we can find a mate to η at, say $\tilde{B}$, in d. Apply similar argument in the first paragraph of Proof of Lemma 4.12. $\square$

Let ζ be the mate at $\tilde{B}$ in Sublemma 4.14. We assume that there is no such bubble between B and $\tilde{B}$.

Proof of Lemma 4.13.   Suppose for a contradiction that there exists a saddle incident to $\text{int}(\eta \cap \partial d)$.

Then ζ and a subarc of $\xi \cap \partial d$ cobound a disc $d_1 \subset d$. See Fig. 4.8. There is a part of $L \cap T$ which enters d_1 at the bubble $\tilde{B}$. By using Proposition 4.3, we can show that ζ meets the puncture and no bubble other than $\tilde{B}$. There is a part of $L \cap T$ exiting from d_1 at the puncture. Note that $d - (d_1 \cup \tilde{B})$ consists of two discs which are cobounded by subarcs of ξ, η and ζ. Let d_2 be the disc which a part of $L \cap T$ enters at the puncture.

Suppose ∂d_2 meets B. Then from the definition of $\tilde{B}$ there is exactly three parts of $L \cap T$ entering d_2 at $\tilde{B}$, at the puncture and at the bubble which is between B and $\tilde{B}$. This is a contradiction.

Suppose $\partial d_2 \cap B = \phi$. There is a part of $L \cap T$ entering d_2 at $\tilde{B}$. Suppose $\eta \cap \partial d_2$
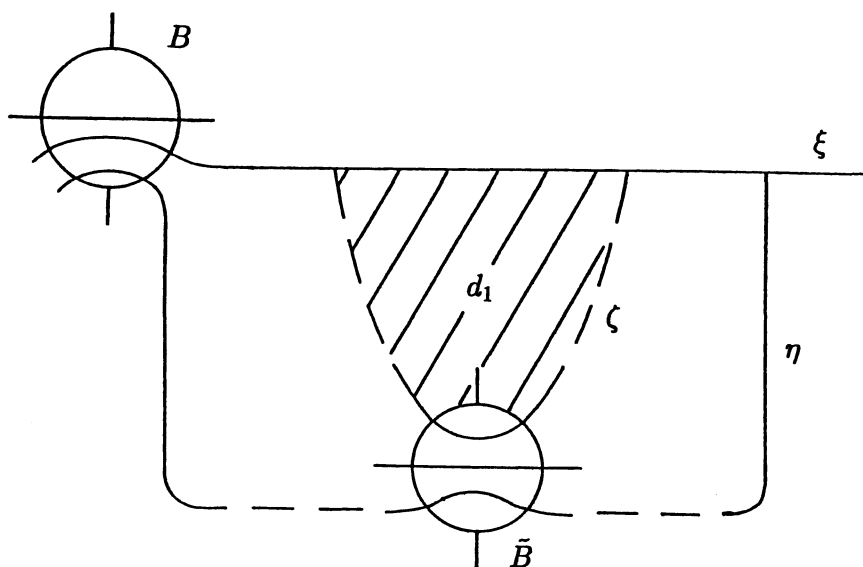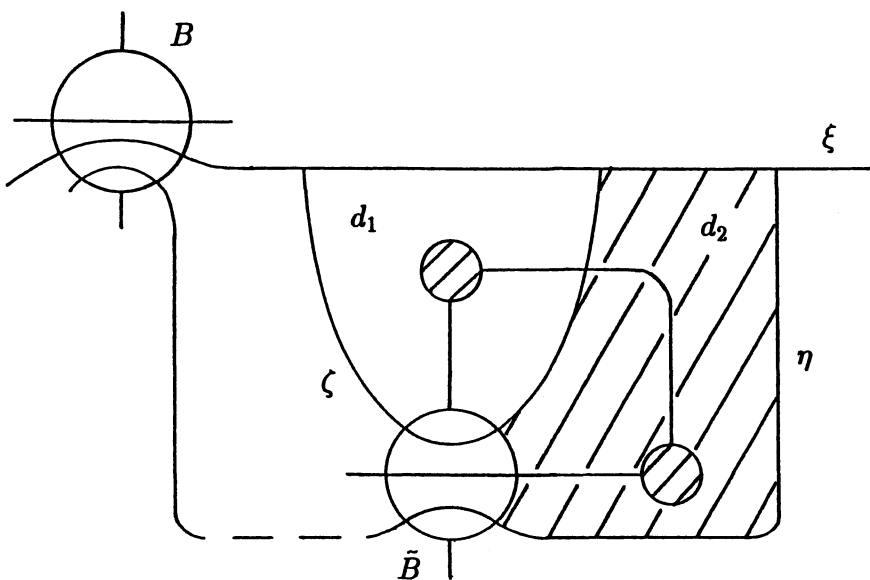
Fig. 4.8.

Fig. 4.9.

meets no bubble. Then the crossing of $\tilde{B}$ is a nugatory crossing, which contradicts our assumption. See Fig. 4.9. Hence $\eta \cap \partial d_2$ meets a bubble. Then we can find a
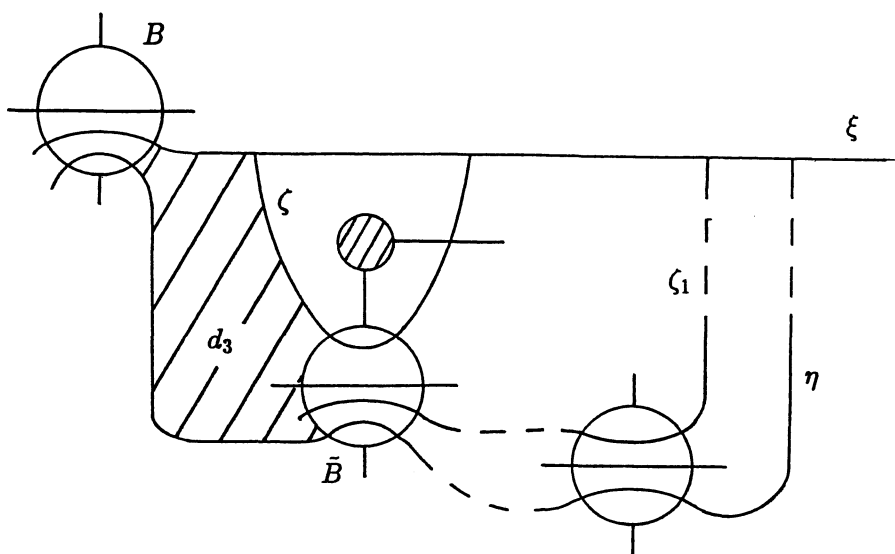
Fig. 4.10.

mate ζ_1 to η in d_2.

By using Propositions 2.3, 3.2(i) and 4.3, it follows that one end connects with $\xi \cap \partial d_2$ and the other exits from d_2 at $\tilde{B}$. Let d_3 be one of the component of $d - (d_1 \cup \tilde{B})$ which is not d_2. See Fig. 4.10. Then it follows (1) ζ_1 exits from d_3 at B, (2) ζ_1 exits from d_3 at $\tilde{B}$, (3) ζ_1 connects with $\xi \cap \partial d_3$, (4) ζ_1 connects with another loop of $F \cap T_+$ than ξ, say C, in d_3 and C exits from d_3 only at B, or (5) C exits from d_3 at B and at $\tilde{B}$. In the case (1), we can find a part of $(F \cup D) \cap T_\pm$ shown in Fig. 4.6 in d which consists of subarcs of ξ and ζ_1 and B, which contradicts that d is innermost. In the case (2), ζ_1 meets $\tilde{B}$ twice with encircling a disc, which violates (i) of Proposition 3.2. In the case (3), ζ_1 and a subarc of $\xi \cap \partial d$ bounds a disc, say d_4, as shown in Fig. 4.1. Since there is no saddle incident to $\xi \cap \partial d$ and ζ_1 does not contain the puncture, we have a contradiction to Proposition 4.3. In the case (4), C violates Proposition 2.3. In the case (5), the subarc of C loop which exits from d_3 at $\tilde{B}$ enters d_2. Then it exits from d_2 at $\tilde{B}$. Hence it meets $\tilde{B}$ twice with encircling a disc, which violates Proposition 2.3. Thus we proved Lemma 4.13.   □

Hence in case a part of $(F \cup D) \cap T_\pm$ satisfies condition X, either there is a part of $(F \cup D) \cap T_\pm$ shown in (i) of Fig. 4.7 or we can show that there is no saddle incident to $\text{int}(\xi \cap \partial d)$ nor $\text{int}(\eta \cap \partial d)$ and $\eta \cap \partial d$ does not meet the puncture $D \cap L$.

Next we consider the case where a part of $(F \cup D) \cap T_\pm$ does not satisfy the condition X.

Lemma 4.15. *Suppose there is a part of* $(F \cup D) \cap T_+$ *shown in* Fig. 4.6 *which does not satisfy the condition* X. *Suppose that there is no saddle incident to* int($\xi \cap \partial d$) *and that* $\eta \cap \partial d$ *contains the puncture. Then we can find a part of* $(F \cup D) \cap T_+$ *shown in* (iii) *of* Fig. 4.7.

Proof.   If int($\eta \cap \partial d$) meets more than one bubble, then as we did in the proof of Lemma 4.12, we have a contradiction. Hence int($\eta \cap \partial d$) meets only one bubble. Let $\tilde{B}$ be the bubble. Suppose first that the subarc of $\eta \cap \partial d$ between B and $\tilde{B}$ does not contain the puncture. Then by the alternating property (**), it follows that we can find a mate to $\eta \cap \partial d$ in d. Then as we did in the proof of Lemma 4.12, we have a contradiction. Suppose secondly that the subarc of $\eta \cap \partial d$ between B and $\tilde{B}$ contains the puncture. There are exactly two parts of $L \cap T$ entering d at $\tilde{B}$ and at the puncture. Hence we can find a part of $(F \cup D) \cap T_+$ shown in (iii) of Fig. 4.7.   □

Lemma 4.16. *Suppose there is a part of* $(F \cup D) \cap T_+$ *shown in* Fig. 4.6 *which does not satisfy the condition* X. *Suppose there is no saddle incident to* int($\xi \cap \partial d$) *and* $\eta \cap \partial d$ *does not contain the puncture. Then there is no saddle incident to* int($\eta \cap \partial d$).

Proof.   Suppose for a contradiction that there is a saddle incident to int($\eta \cap \partial d$). We take a saddle which is nearest to B. Let $\tilde{B}$ be the bubble which contains the saddle. Then it follows that we can find a mate to $\eta \cap \partial d$ in int$d$. Let ζ be the mate. Then as in Sublemma 4.14 we can show that both ends of ζ connects with $\xi \cap \partial d$ with cobounding a disc d_1 together with a subarc of $\xi \cap \partial d$. As we did in Proof of Lemma 4.13, we can show that ζ meets the puncture and no bubble other than $\tilde{B}$. Let d_2 be the component of $d - (d_1 \cup \tilde{B})$ which a part of $L \cap T$ enters at the puncture. Hence ∂d_2 meets B. Then from the definition of $\tilde{B}$ there are exactly two parts of $L \cap T$ entering d_2 at $\tilde{B}$ and at the puncture. Then it follows that the crossing of $\pi(L)$ at $\tilde{B}$ is a nugatory crossing, which contradicts our assumption. Suppose $\partial d_2 \cap B = \phi$. Then from the definition of $\tilde{B}$ there is exactly one part of $L \cap T$ entering another component of $d - (d_1 \cup \tilde{B})$ at $\tilde{B}$, which is a contradiction.   □

Lemma 4.17. *Suppose there is a part of* $(F \cup D) \cap T_+$ *shown in* Fig. 4.6 *which does not satisfy the condition* X. *Suppose there is a saddle incident to* int($\xi \cap \partial d$). *Then we can find a part of* $(F \cup D) \cap T_+$ *shown in* (ii) *of* Fig. 4.7.

Proof.   First suppose int($\xi \cap \partial d$) meets more than one bubble. Then (1) we can find a mate to $\xi \cap \partial d$ in d or (2) both ξ and η meet the same bubble other than B. In the case (1), the mate is a loop of $F \cap T_+$ and it exits from d at B, which violates Proposition 2.3. In the case (2), it contradicts that d is innermost. Hence int($\xi \cap \partial d$) meets only one bubble, say $\tilde{B}$.

Suppose for a contradiciton that int($\eta \cap \partial d$) does not contain the puncture. Since there is a part of $L \cap T$ entering d at $\tilde{B}$, int($\eta \cap \partial d$) meets a bubble. Let $\tilde{B}'$ be the bubble such that η meets no other bubble between B and $\tilde{B}'$. It follows that we can find a mate, say η', to $\eta \cap \partial d$ at $\tilde{B}'$ in d. Then as in Proof of Lemma 4.12, we can show that both ends of η' connects with $\xi \cap \partial d$. Let d_1 be the disc in d cobounded by η' and a subarc of $\xi \cap \partial d$.

Suppose $\xi \cap \partial d_1$ does not meet $\tilde{B}$. Then we can .find a part of $(F \cup D) \cap T_+$ shown in Fig. 4.1. Hence by applying Proposition 4.3, η' contains the puncture. As we did in Proof of Lemma 4.5, we can show that η' meets no bubble other than $\tilde{B}'$. Note that $d - (d_1 \cup \tilde{B}')$ consists of two discs which are cobounded by subarcs of ξ, η and η'. Let d_2 be the disc which a part of $L \cap T$ enters at the puncture and let d_3 be the other. See Fig. 4.11. Then similar argument in Proof of Lemma 4.13 will do. However we need a slight change. That is, we must use Proposition 4.6 rather than Proposition 4.3.

Therefore $\xi \cap \partial d_1$ meets $\tilde{B}$. By applying Proposition 4.6, we have a contradiction.

Hence $\eta \cap \partial d$ contains the puncture. Suppose int($\eta \cap \partial d$) meets a bubble. Since there are two parts of $L \cap T$ entering d at $\tilde{B}$ and at the puncture, int($\eta \cap \partial d$) meets at least two bubbles. It follows we can find a mate to $\eta \cap \partial d$ in int$d$. Let η' be the mate. Then as we did in the proof of Lemma 4.12 we can show that η' is an arc of $D \cap T_+$ and both ends of η' connects with $\xi \cap \partial d$. Then by Propositions 4.3 and 4.6 we have a contradiction. Hence int($\eta \cap \partial d$) meets no bubble. Since there is exactly two parts of $L \cap T$ entering d, they connect. Hence we can find a part of $(F \cup D) \cap T_+$ shown in (ii) of Fig. 4.7. $\square$
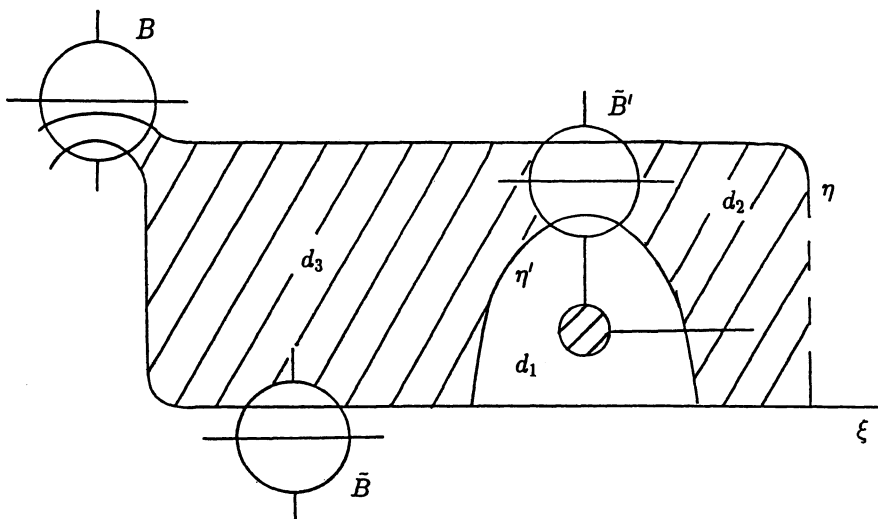

Fig. 4.11.

Proof of Proposition 4.9.   Suppose there is a part of $(F \cup D) \cap T_+$ shown in Fig. 4.6. We assume d is innermost. Suppose the part satisfies the condition X. By applying Lemmas 4.12 and 4.13, we can find a part of $(F \cup D) \cap T_+$ shown in (i) of Fig. 4.7 or we can show that $\eta \cap \partial d$ does not contain the puncture and there is no saddle incident to $\mathrm{int}(\xi \cap \partial d)$ nor $\mathrm{int}(\eta \cap \partial d)$. Suppose the part does not satisfy the condition X. By applying Lemmas 4.15, 4.16 and 4.17, we can find a part of $(F \cup D) \cap T_+$ shown in (ii) or (iii) of Fig. 4.7 or we can show that $\eta \cap \partial d$ does not contain the puncture and there is no saddle incident to $\mathrm{int}(\xi \cap \partial d)$ nor $\mathrm{int}(\eta \cap \partial d)$.

Now we show that we can isotope D so as to reduce the complexity in case where $\eta \cap \partial d$ does not contain the puncture and there is no saddle incident to $\mathrm{int}(\xi \cap \partial d)$ nor $\mathrm{int}(\eta \cap \partial d)$.

We can assume that s and s' are adjacent saddles inside B. That is, there is no component of $F \cap T_+$ or $D \cap T_+$ between ξ and η at B. Otherwise it violates Propositions 2.3, 3.2(i) or contradicts the way of taking the parts. Suppose there is a part of $L \cap T$ in int$d$. Since $\eta \cap \partial d$ does not contain the puncture and there is no saddle incident to $\mathrm{int}(\xi \cap \partial d)$ nor $\mathrm{int}(\eta \cap \partial d)$, ∂d becomes a simple closed curve on T which does not cross L. Hence the projection $\pi(L)$ is split by the simple closed curve on T. Then there is a region (see §2 for the definition) which is not a disc, which contradicts the definition of a toroidally alternating link. Hence there is no part of $L \cap T$ in int$d$. If there were a component of $D \cap T_\pm$ in int$d$, it would violate Proposition 4.1 or (ii) of Proposition 3.2.

Now we look at the other sides of the saddles s and s'. Then there is a subarc of a loop ξ' of $F \cap T_+$ and a subarc of a component η' of $D \cap T_+$ such that they are mates to ξ and to η respectively.
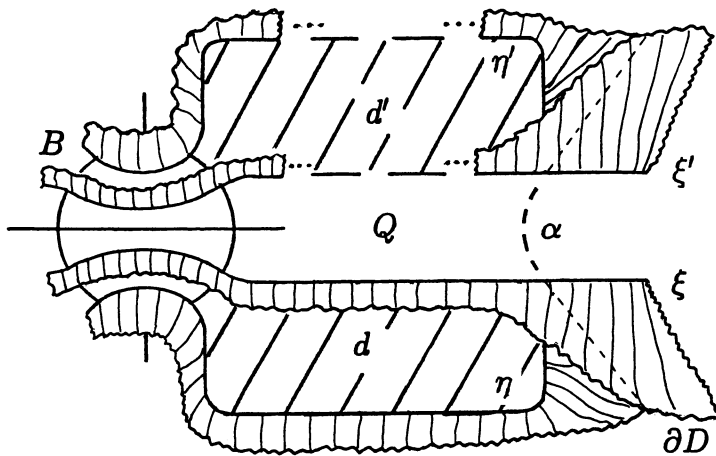

Fig. 4.12.

First we consider a special case, where ξ', η' and ∂D are placed as shown Fig. 4.12. That is, η' connects with ξ' with cobounding a disc together with subarcs of ξ' and ∂H, $\eta' \cap \partial d'$ does not contain the puncture and there is no saddle incident to $\text{int}(\xi' \cap \partial d')$ nor $\text{int}(\eta' \cap \partial d')$. Moreover $\xi \cap d$ and $\xi' \cap d'$ are subarcs of boundary of a disc Q of $F \cap V_-$ and the two points $\xi \cap \eta \cap d$ and $\xi' \cap \eta' \cap d'$ are connected by an arc α of $\partial D \cap Q$. Doing the same argument above, we can show that there is no component of $F \cap T_+$ nor $D \cap T_+$ in d'. We take an arc λ on $F \cap V_-$ near B which is parallel to a subarc of ∂H and is not contained in Q as shown in Fig. 4.13. There is a disc $\tilde{Q}$ on F which contains Q and is cobounded by λ, α and subarcs of ξ and ξ'. We isotope ∂D along $\tilde{Q}$ so as to eliminate the saddle s as shown in Fig. 4.14. This contradicts minimality of complexity of D.

For general case, we can apply the above argument as in the following mannar. Take a triangle xyz on T_- as shown in Fig. 4.15, where x is at the point η connects with ξ, $y \in \eta' \cap \partial H$ and $z \in \xi' \cap \partial H$. We isotope this triangle into V_- so that $x \in \partial D \cap V_-$, $y \in D \cap T$, $z \in F \cap T$ and $xy \subset D \cap V_-$, $yz \subset T$, $zx \subset F \cap V_-$.
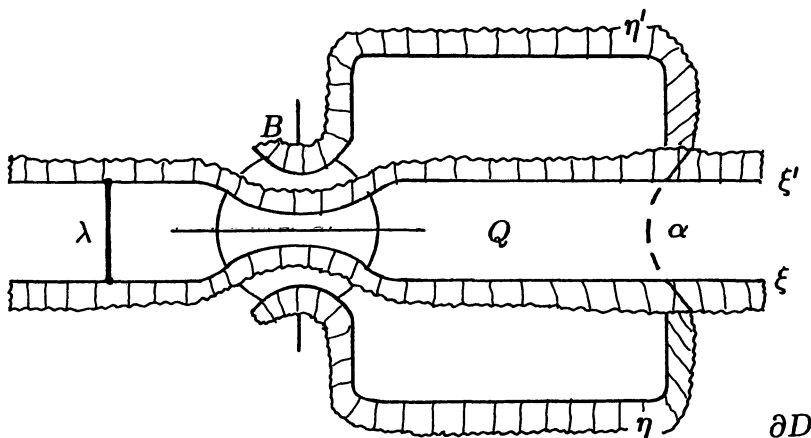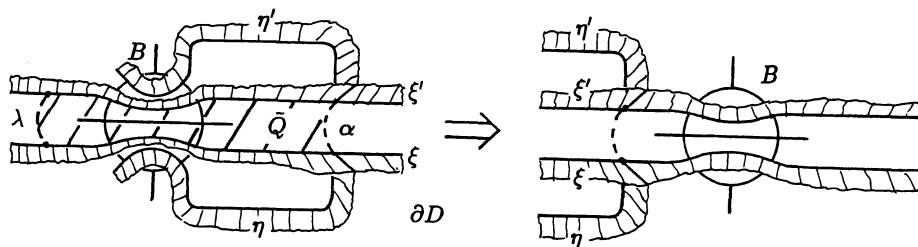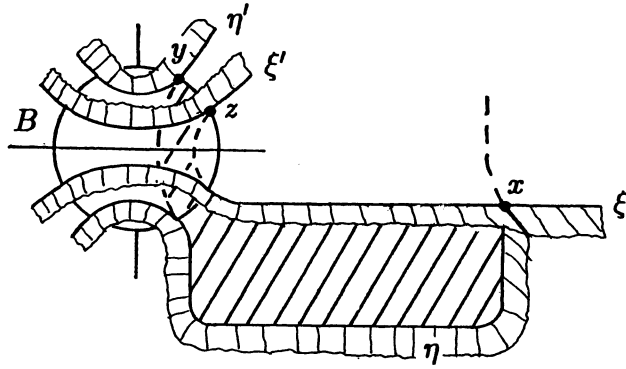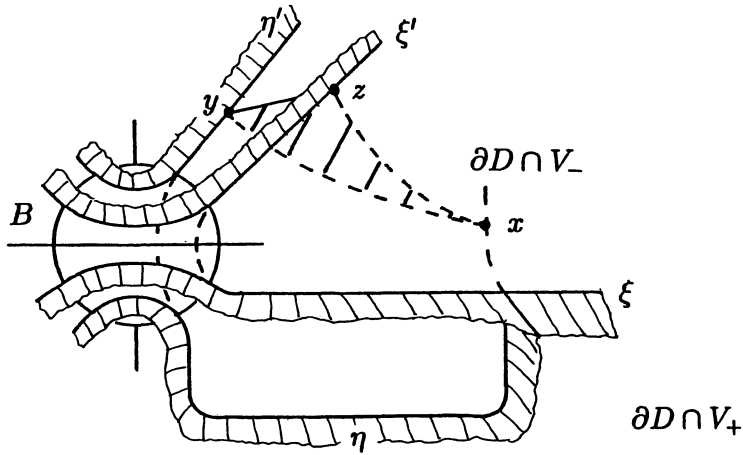

Fig. 4.13.

Fig. 4.14.

Fig. 4.15.

Fig. 4.16.

See Fig. 4.16. We isotope D along this triangle so that $F \cap T_\pm$, $D \cap T_\pm$ and ∂D near B is as shown in Fig. 4.17. Now as we did above, we isotope D so as to eliminate the saddle s. Then this contradicts minimality of complexity of D. Hence we have proved Proposition 4.9.   □

In the proof of Proposition 4.9, we proved the following Lemma.

Lemma 4.18. *Suppose there is a part of* $(F \cup D) \cap T_+$ *shown in* Fig. 4.6. *Suppose there is no saddle incident to* int$(\xi \cap \partial d)$ *and* int$(\eta \cap \partial d)$. *Then we can isotope* D *so as to eliminate the saddle* s.
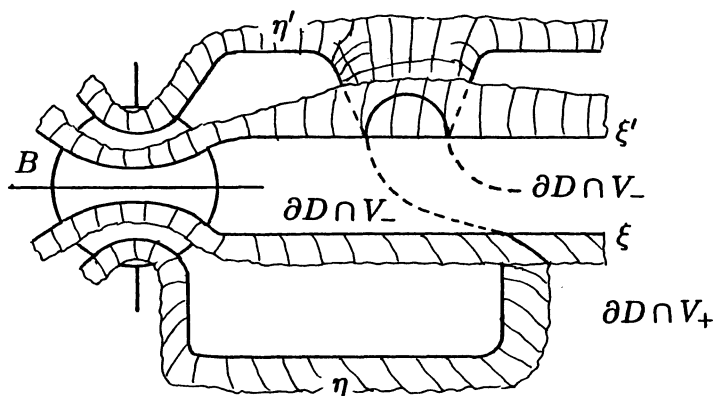
Fig. 4.17.

5. Loops in $D \cap T_\pm$ (II)

In §5 we give a proof of Proposition 4.2.

We prove Proposition 4.2 as follows. We show that if there were a loop of $D \cap T_\pm$, then we could find two of "bigons" or "trigons" as in Fig. 5.2 satisfying some property. In Proposition 5.1, we prove that if there were such a bigon or a trigon then we could find a part of $(F \cup D) \cap T_\pm$ shown in Fig. 4.7 in it. Then it follows that there would be two punctures, which is a contradiction.

First we define bigons and trigons. A *bigon* consists of two bubbles and two subarcs of loops or arcs of $F \cap T_\pm$ or $D \cap T_\pm$. See for example (i) or (iii) of Fig. 5.2. A *trigon* consists of two bubbles and three subarcs of loops or arcs of $F \cap T_\pm$ or $D \cap T_\pm$. See (ii) or (iv) of Fig. 5.2.

Next we add some restriction to the shape of bigons or trigons. Before starting that we introduce some notation.

Let ξ be a component of $F \cap T_+$ or $D \cap T_+$ and η a component of $F \cap T_-$ or $D \cap T_-$ such that ξ and η meet the same bubble, say B. We say ξ *crosses η at B* and η *crosses ξ at B* if ξ and η connect with distinct saddles s and s' at B respectively and s is upper than s'. See Fig. 5.1.

Now we consider bigons or trigons described as blow. Typical examples are shown in Fig. 5.2.

We mean, for (i) of Fig. 5.2, there are two bubbles B_1 and B_2, a component α of $F \cap T_+$ or $D \cap T_+$ and a component β of $F \cap T_-$ or $D \cap T_-$ such that α crosses β at B_1 and B_2 and subarcs of α, β, ∂H_1 and ∂H_2 cobound a disc d_0, where H_1 and H_2 are upper hemispheres of B_1 and B_2 respectively.

For (ii), there are two bubbles B_1 and B_2, a component α of $F \cap T_+$ or $D \cap T_+$ (resp. $F \cap T_-$ or $D \cap T_-$), an arc β of $D \cap T_-$ (resp. $D \cap T_+$) and a loop γ of $F \cap T_-$ (resp. $F \cap T_+$) such that β connects with γ, β crosses α at B_1, γ crosses α at B_2 and
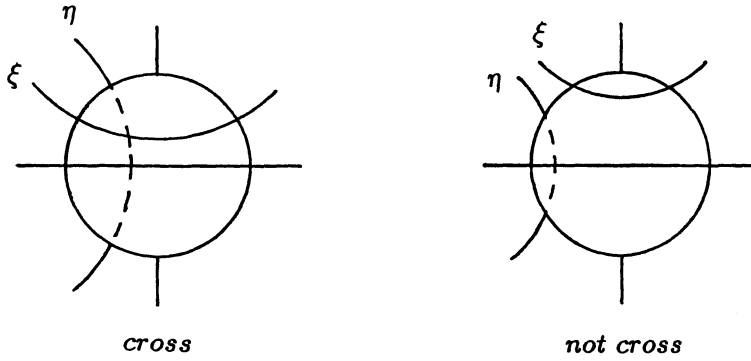
Fig. 5.1.
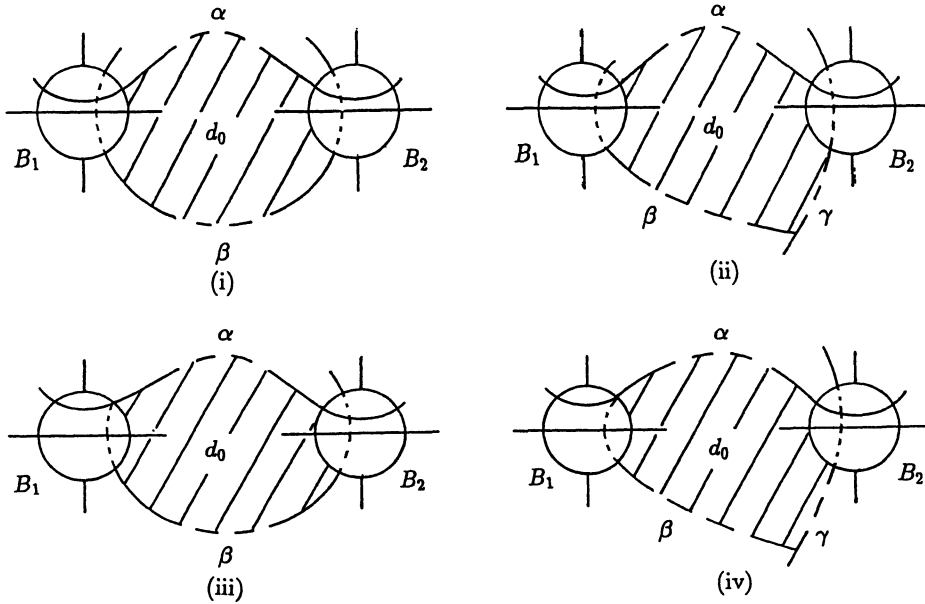
Fig. 5.2.

subarcs of α, β, γ, ∂H_1 and ∂H_2 cobound a disc d_0.

For (iii), there are two bubbles B_1 and B_2, a component α of $F \cap T_+$ (resp. $D \cap T_+$) and a component β of $F \cap T_-$ (resp. $D \cap T_-$) such that α and β connect with the same saddle, say s, at B_1, α crosses β at B_2 and subarcs of α, β, ∂H_1 and ∂H_2 cobound a disc d_0.

For (iv), there are two bubbles B_1 and B_2, a component α of $F \cap T_+$ (resp. $D \cap T_+$), a component β of $F \cap T_-$ (resp. $D \cap T_-$) and a component γ of $D \cap T_-$ (resp. $F \cap T_-$) such that γ connects with β (resp. β connects with γ), α and β connects

with the same saddle, say s, at B_1, γ crosses α at B_2 and subarcs of α, β, γ, ∂H_1 and ∂H_2 cobound a disc d_0. We also consider the case where α is a component of $F \cap T_-$ (resp. $D \cap T_-$), β is a component of $F \cap T_+$ (resp. $D \cap T_+$) and γ is a component of $D \cap T_+$ (resp. $F \cap T_-$).

From now on we study these bigons and trigons.

Proposition 5.1.   *Suppose there is a part of* $(F \cup D) \cap T_\pm$ *shown in* Fig. 5.2. *Then we can find a part of* $(F \cup D) \cap T_\pm$ *shown in* Fig. 4.7 *in* d_0.

To prove Proposition 5.1, we need the following seven Lemmas.

First we consider a part of $F \cap T_\pm$ or $D \cap T_\pm$ shown in Fig. 5.3, which is a kind of bigons. That is, there are two bubbles B_1 and B_2, a component α of $F \cap T_+$ (resp. $D \cap T_+$) and a component β of $F \cap T_-$ (resp. $D \cap T_-$) such that α and β connect with the same saddles at B_1 and at B_2, and subarcs of α, β, ∂H_1 and ∂H_2 cobound a disc d_0, where H_1 and H_2 are upper hemispheres of B_1 and B_2 respectively.

Lemma 5.2.   *Suppose there is a part of* $F \cap T_\pm$ *or* $D \cap T_\pm$ *shown in* Fig. 5.3. *Then there is a part of* $(F \cup D) \cap T_\pm$ *shown in* Fig. 5.2 *in* d_0.

Proof.   Suppose there is a part of $F \cap T_\pm$ or $D \cap T_\pm$ shown in Fig. 5.3. If there is another part of $F \cap T_\pm$ or $D \cap T_\pm$ shown in Fig. 5.3 in d_0, we take one of the innermost such discs in d_0 with respect to the discs they bound. Let d be the disc.

If there were no part of $(F \cup D) \cap T_\pm$ shown in Fig. 5.2 in d, then we can show that it is as in Fig. 5.4 by hand, which violates Proposition 3.2(i). $\square$
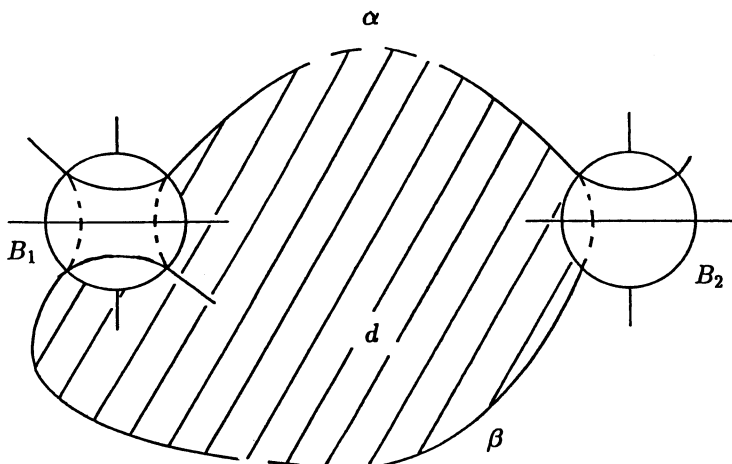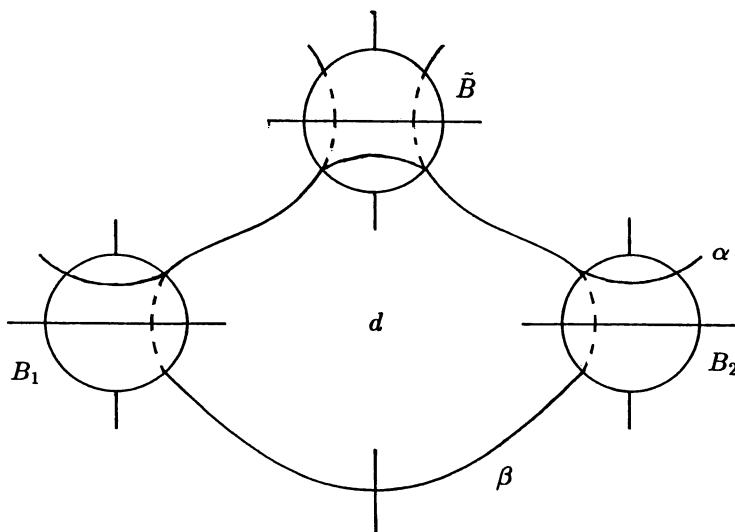

Fig. 5.3.

Fig. 5.4.

We assume without loss of generality $\alpha \subset F \cap T_+$ or $D \cap T_+$, $\beta \subset F \cap T_-$ or $D \cap T_-$, and $\gamma \subset F \cap T_-$ or $D \cap T_-$ in Fig. 5.2 rather than $\alpha \subset F \cap T_-$ or $D \cap T_-$ and so forth.

Lemma 5.3. *Suppose there is a part of $(F \cup D) \cap T_\pm$ shown in Fig. 5.2. We assume the part is innermost one with respect to the disc which it bounds. Suppose there is a mate to α, β or γ at B_1 or B_2 in* $\mathrm{int} d_0$. *Then we can find a part of $(F \cup D) \cap T_\pm$ shown in* Fig. 4.7.

Proof.   Suppose there is a mate of α, β and γ at B_1 or B_2 in $\mathrm{int} d_0$. Let ξ be the mate. Then it follows either (a) ξ enters d_0 crossing one of α, β and γ at B_1 or B_2 (See Fig. 5.5), (b) both ends of a subarc of ξ near the bubble are contained in d_0 (See Fig. 5.6), or (c) α and β connect with the same saddle at B_1 and there are mates to α and to β at B_1 which enter d_0 (See Fig. 5.7).

In each cases we can show that we can find a part of $(F \cup D) \cap T_\pm$ as in Fig. 4.7 or another part as in Fig. 5.2 by examining the mate. For example, suppose the condition (a) holds. Suppose furthermore that ξ connects with γ. See Fig. 5.7. By applying Proposition 4.9, we can find a part of $(F \cup D) \cap T_\pm$ shown in Fig. 4.7 in d_0.   □

From now on we assume there is no mate to α, β nor γ at B_1 and B_2 in $\mathrm{int} d_0$.

Lemma 5.4. *Suppose there is a part of $(F \cup D) \cap T_\pm$ shown in* Fig. 5.2. *If we*
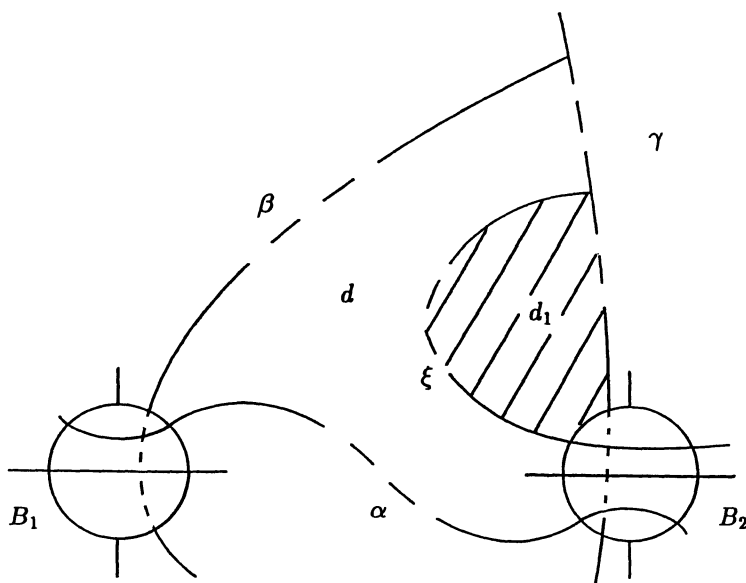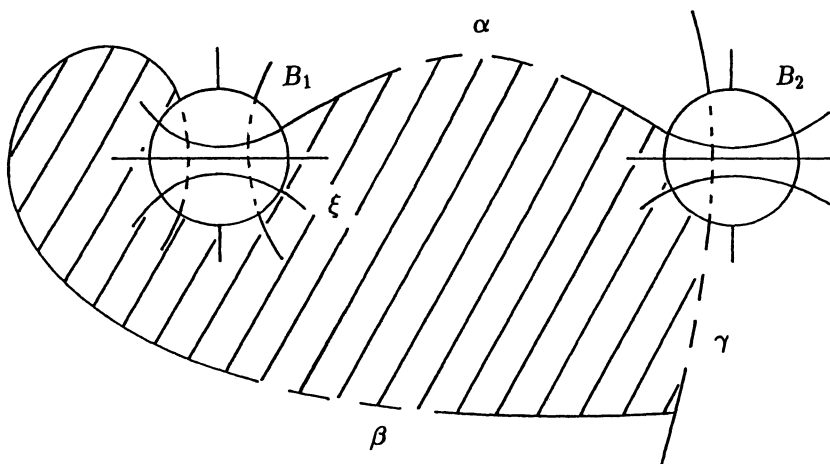
Fig. 5.5.

Fig. 5.6.

take the part to be innermost one, then it is not one as follows;
  α and β connects with the same saddle s at B_1 as in (iii) or (iv) of Fig. 5.2,
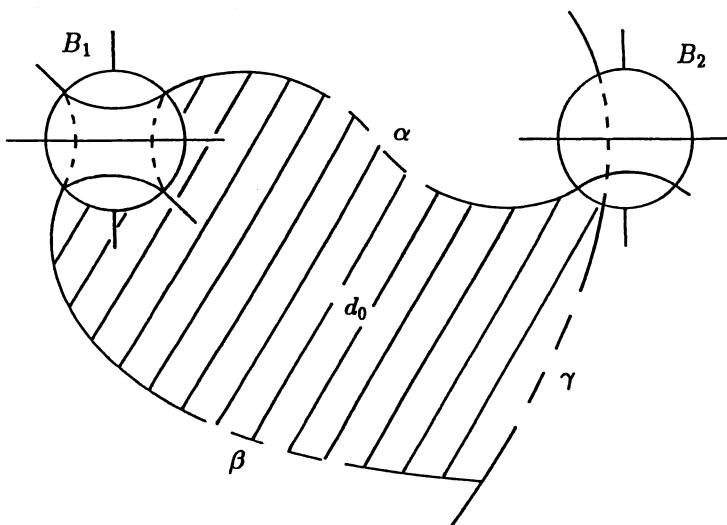 and a subarc of $\alpha \cap \beta \subset D \cap T$ or $F \cap T$ enters $\text{int} d_0$ at B_1.
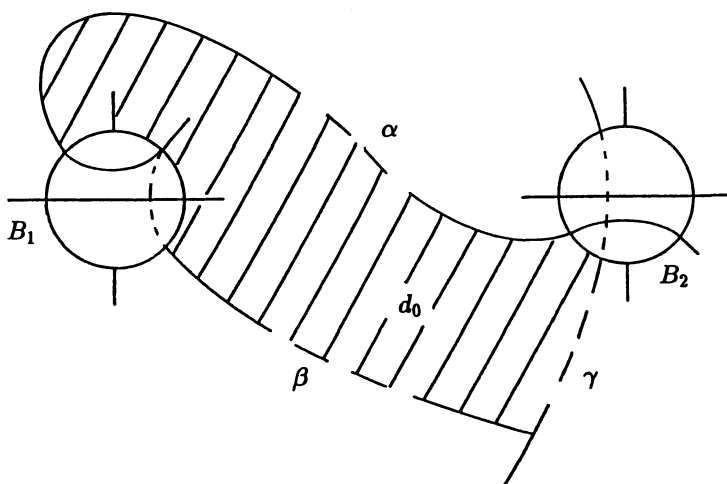
Fig. 5.7.

Fig. 5.8.

Proof.   Suppose there is a subarc of $\alpha \cap \beta$ entering $\mathrm{int} d_0$ at B_1 which connects with the saddle s. For example, see Fig. 5.8. Then one end of β enters $\mathrm{int} d_0$ at B_1. It follows (1) the end exits from d_0 crossing α, (2) the end connects with a saddle incident to α, (3) the end of β connects with another loop of $F \cap T_-$ than γ in d_0, or (4) the end connects with γ. In the case (1), it contradicts the choice of d_0. In the case (2), from Lemma 5.2 we can find another part of $(F \cup D) \cap T_\pm$ shown in
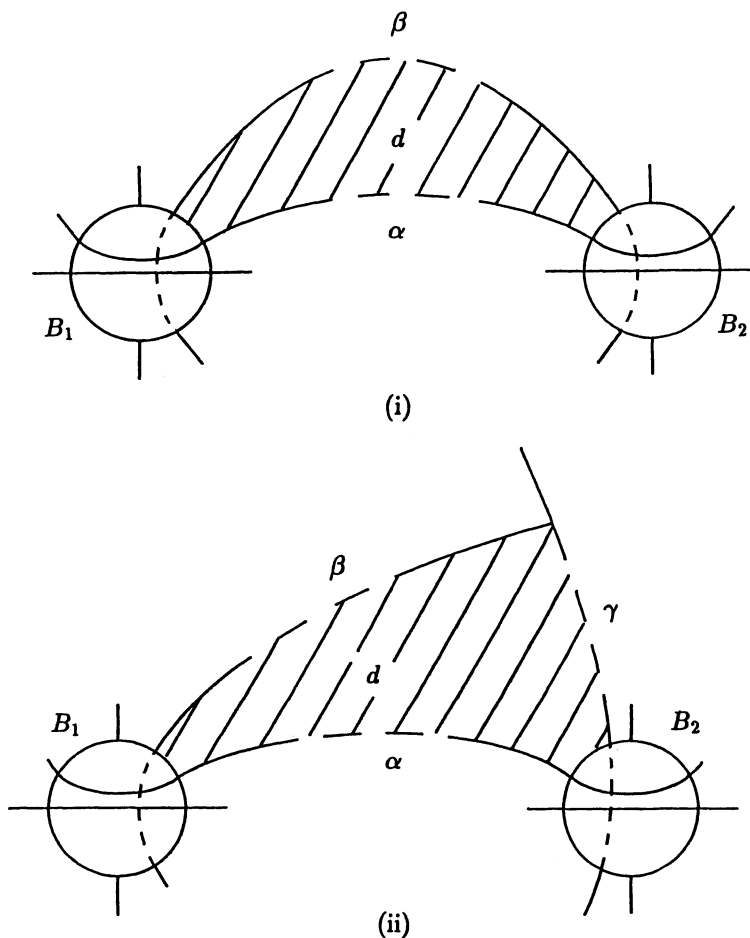
Fig. 5.9.

Fig. 5.2 in d_0, which contradicts the choice of d_0. In the case (3), it violates Proposition 2.3 or contradicts the choice of d_0. In the case (4), this contradicts the choice of d_0. □

By taking one of the innermost such discs in d_0 and by applying Lemmas 5.3 and 5.4, either we can find a part of $(F \cup D) \cap T_\pm$ shown in Fig. 4.7, or we can assume that the part is shown in Fig. 5.9.

For (i) and (ii), we mean there is no part of $L \cap T$ which enters d at B_1 and B_2, where d is the disc the part bounds.

For (iii) and (iv), we mean there is one part of $L \cap T$ which enters d at B_1 and there is no part of $L \cap T$ which enters d at B_2.
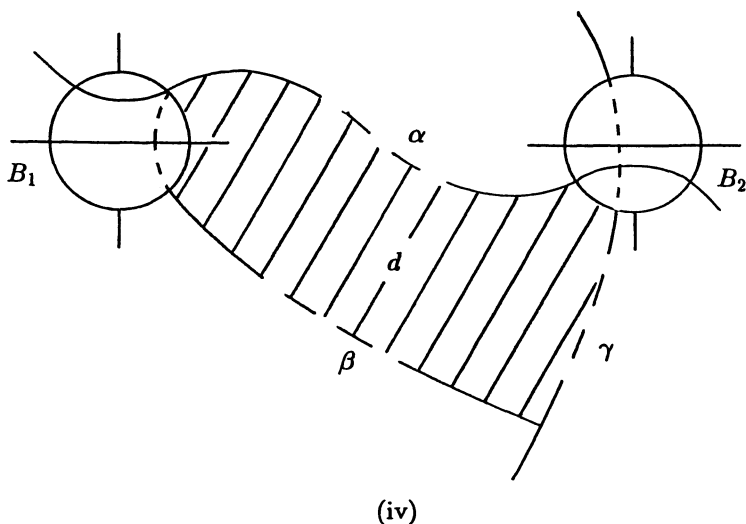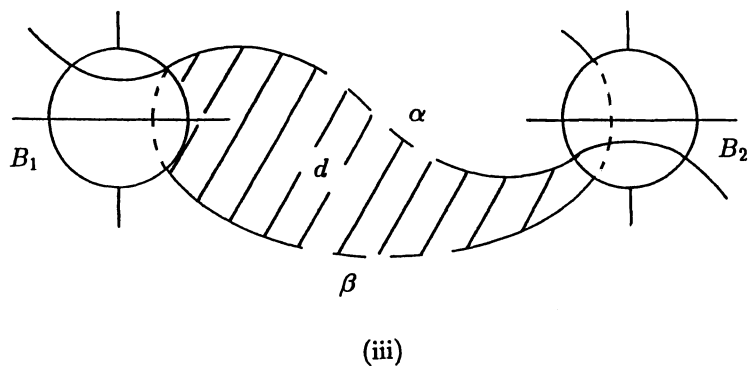
(iii)

(iv)

Fig. 5.9.

Lemma 5.5.   *Suppose there is a part of $(F \cup D) \cap T_{\pm}$ shown in* Fig. 5.2 *such that it is innermost and as shown in* (i) *of* Fig. 5.2. *Then we can find a part of* $(F \cup D) \cap T_{\pm}$ *shown in* Fig. 4.7.

Proof.   We can assume that the part is as shown in (i) of Fig. 5.9. Let d be the disc the part bounds.

From the alternating property $(*)$ or $(**)$, there is a saddle incident to $\text{int}(\beta \cap \partial d)$ or $\text{int}(\beta \cap \partial d)$ meets the puncture. Suppose there is a saddle incident to $\text{int}(\beta \cap \partial d)$. Let $\tilde{B}$ be the bubble which contains the saddle. Then from the alternating property $(*)$ or $(**)$, we can find subarcs of components of $F \cap T_+$ or $D \cap T_+$ entering d at $\tilde{B}$. Let ξ be one of the subarcs. It follows (1) ξ connects with another saddle incident

to β, (2) ξ exits from d crossing β, (3) ξ connects with another loop of $F \cap T_+$ than α in int$d$, (4) ξ connects with α. In the case (1), we can find a part of $F \cap T_\pm$ or $D \cap T_\pm$ shown in Fig. 5.3 in d. Then from Lemma 5.2, we can find another part of $(F \cup D) \cap T_\pm$ shown in Fig. 5.2 in d, which violates the choice of d. In the case (2), we can find another part of $(F \cup D) \cap T_\pm$ shown in Fig. 5.2 in d. In the case (3), the loop exits from d crossing β. Then the loop violates Proposition 2.3 or there is another part of $(F \cup D) \cap T_\pm$ shown in Fig. 5.2 in d, which contradicts that d is innermost. Thus there is no saddle incident to int$(\beta \cap \partial d)$. It follows int$(\beta \cap \partial d)$ contains the puncture. This means $\beta \subset D \cap T_-$ rather than $\beta \subset F \cap T_-$.

From the alternating property $(**)$, there is a saddle incident to int$(\alpha \cap \partial d)$ or int$(\alpha \cap \partial d)$ contains the puncture. Suppose there is a saddle incident to int$(\alpha \cap \partial d)$. Then as we did above we have a contradiction. Hence int$(\alpha \cap \partial d)$ contains the puncture. Since there is exactly one puncture, and since $\beta \cap \partial d$ also has the puncture, $\alpha \cap \partial d$ coincide with $\beta \cap \partial d$. But it is impossible.   □

Lemma 5.6.   *Suppose there is a part of $(F \cup D) \cap T_\pm$ shown in* Fig. 5.2 *such that it is innermost and as shown in* (ii), (iii) *or* (iv) *of* Fig. 5.2. *Then we can find a part of $(F \cup D) \cap T_\pm$ shown in* Fig. 4.7.

Proof.   Similarly by using the arguments in §4 and §5, we can prove this lemma by hand.   □

Proof of Proposition 5.1.   Lemmas 5.5, and 5.6 form a proof of Proposition 5.1.   □

Let S_1 be the set of all the saddles incident to the loops of $D \cap T_\pm$, and S_2 the set of all the saddles incident to the arcs of $D \cap T_\pm$.

Lemma 5.7.   *Suppose S_1 is not empty. Then each of S_2 and $S_1 \cap S_2$ is not empty.*

Proof.   Suppose for a contradiction that $S_2 = \phi$. Then $(D \cap T_\pm) \cup (saddles)$ consists of four-valent graphs which are disjoint from ∂D and possibly arcs properly embedded in D. See Fig. 5.10(i). Then we can find a loop of $D \cap T_\pm$ which does not bound a meridian disc of $V_\pm$ in $D \cap V_\pm$, which contradicts Proposition 4.1.

Suppose $S_1 \cap S_2 = \phi$. We can find a component of $(D \cap T_\pm) \cup (saddles)$ which is a four-valent graph and disjoint from ∂D. See Fig. 5.10(ii). Then we can again find a loop of $D \cap T_\pm$ which does not bound a meridian disc. This is a contradiction.   □

Proof of Proposition 4.2.   Suppose there is a loop component in $D \cap T_\pm$. From
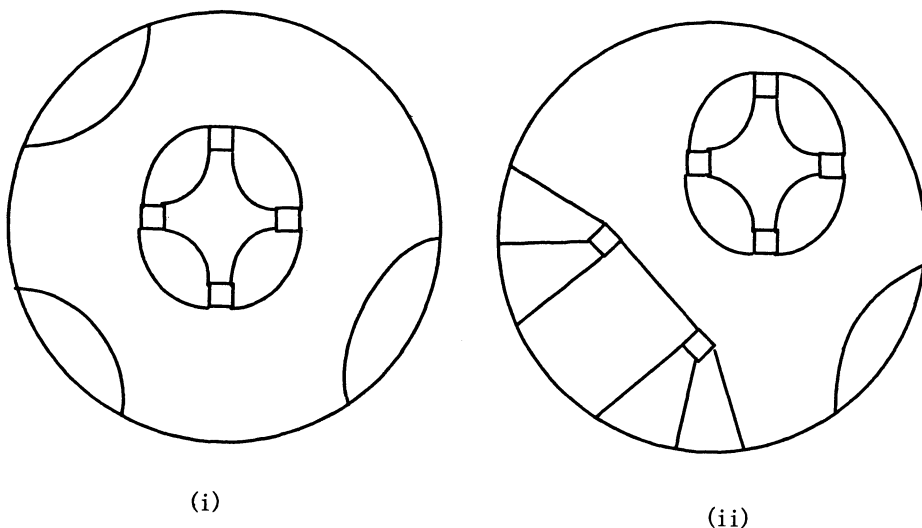
(i)

(ii)

Fig. 5.10.

Proposition 4.1, a loop bounds a meridian disc of $V_\pm$.

From Lemma 5.7, each of S_2 and $S_1 \cap S_2$ is not empty. We choose a saddle s of $S_1 \cap S_2$, a loop C of $D \cap T_\pm$ and a subarc η of an arc of $D \cap T_\pm$ such that η connects s with ∂D, that s is incident to C, that if $C \subset D \cap T_+$ (resp. $D \cap T_-$) then $\eta \subset D \cap T_-$ (resp. $D \cap T_+$) and that intη does not meet a saddle which is incident to a loop of $D \cap T_+$ (resp. $D \cap T_-$). Note that intη may meet a bubble incident to a loop of $D \cap T_-$ (resp. $D \cap T_+$). We call η a *nice arc of s*. Without loss of generality, we can assume $C \subset D \cap T_+$. The same argument below will do in case s is incident to a loop of $D \cap T_-$. It follows that η is a subarc of an arc of $D \cap T_-$. Let C' be a loop of $F \cap T_-$ which η connects with. The behavior of η is classified into the following four cases. See Fig. 5.11.

(i) η connects with C' before crossing C, and there are subarcs $\alpha \subset C$ and $\beta \subset C'$ such that η connects with β, (intβ)$\cap C = \phi$ and $\alpha \cup \beta$ forms a loop bounding a disc d_1 on T.

(ii) η connects with C' before crossing C, and C' crosses C without cobounding a disc together with a subarc of C.

(iii) There are subarcs $\gamma \subset \eta$ and $\alpha \subset C$ such that (intγ)$\cap C = \phi$ and $\alpha \cup \gamma$ forms a loop bounding a disc d_3 on T.

(iv) η crosses C without cobounding a disc together with a subarc of C before it connects with C'.

For (i), C' crosses C. Then we can find a part of $(F \cup D) \cap T_\pm$ shown in Fig. 5.2. Since C' bounds a meridian disc in $D \cap V_-$, C' crosses C once more so that a subarc of C' cobounds a disc together with a subarc of C. Then we can find another part
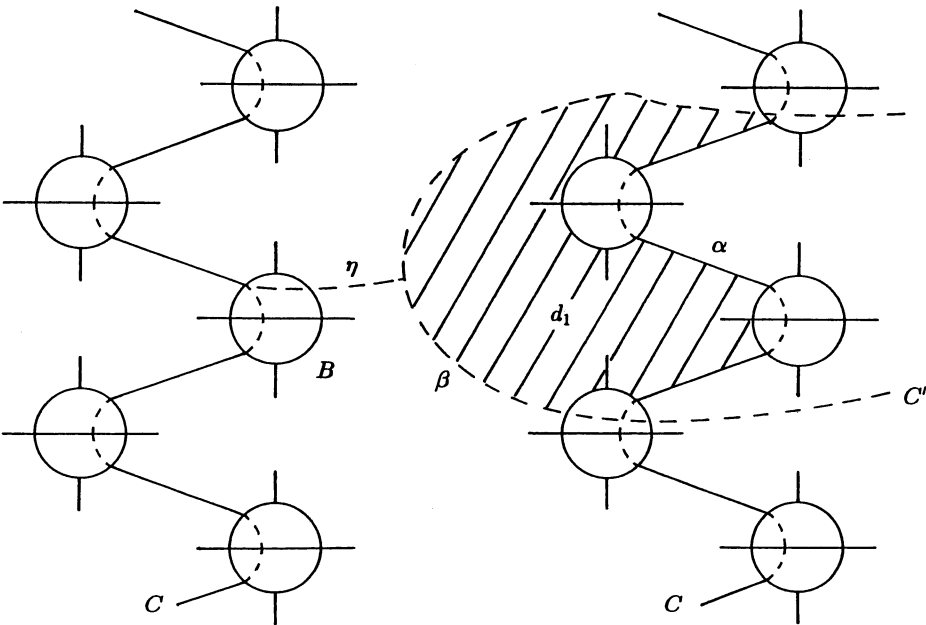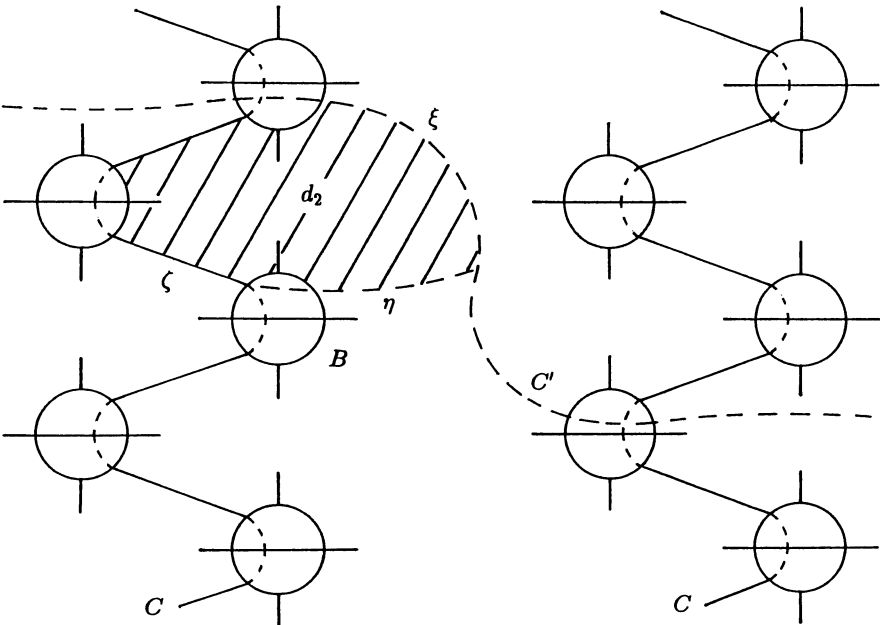
Fig. 5.11.   (i)

Fig. 5.11.   (ii)

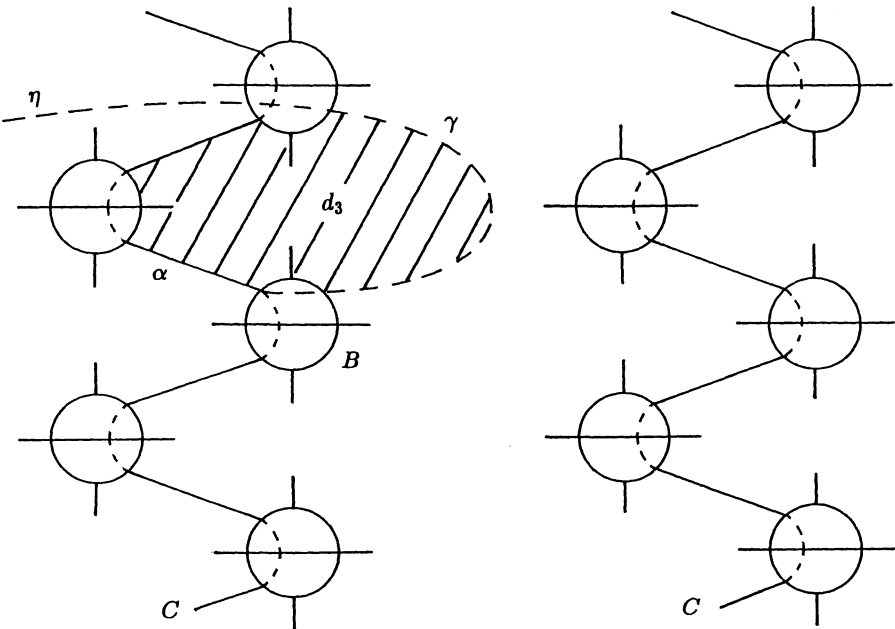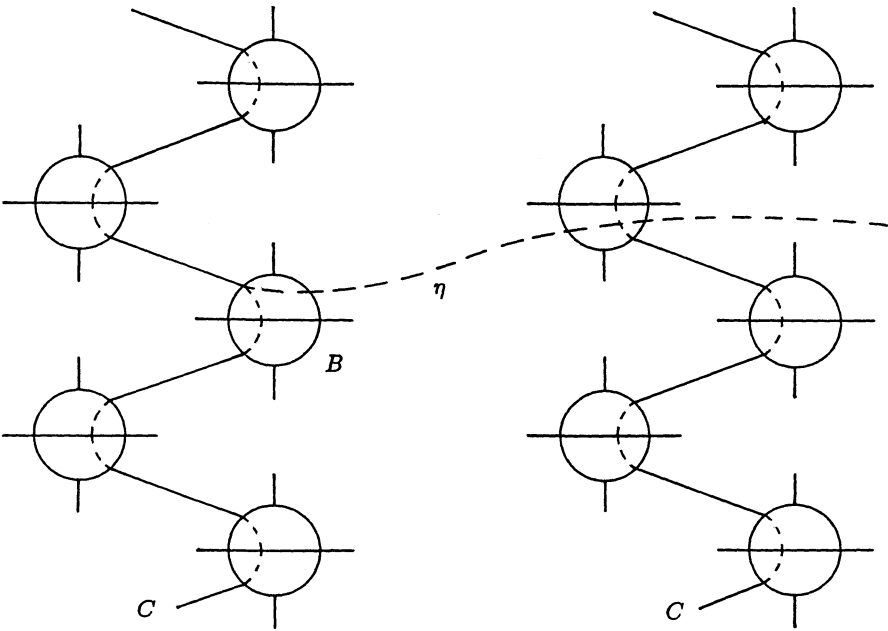Fig. 5.11.   (iii)

Fig. 5.11.   (iv)

of $(F \cup D) \cap T_\pm$ shown in Fig. 5.2. Since we can take two disjoint discs as shown in Fig. 5.2 by taking another discs if necessary, by applying Proposition 5.1 we can find two punctures, which is a contradiction. Therefore the case (i) never happens.

For (ii), there are subarcs $\xi \subset C'$ and $\zeta \subset C$ such that η, ξ and ζ form a loop bounding a disc d_2. We will find a disc as shown in Fig. 4.7 in d_2 as follows. This means the existence of the puncture in d_2. We can find a part of $(F \cup D) \cap T_\pm$ shown in (iv) of Fig. 5.2, or ξ crosses C at B. In the former case, by applying Proposition 5.1, we can find a part of $(F \cup D) \cap T_\pm$ shown in Fig. 4.7 in d_2. In the latter case, by applying Proposition 4.9 we can find a part of $(F \cup D) \cap T_\pm$ shown in Fig. 4.7 in d_2. Thus we can find a disc as shown in Fig. 4.7 in d_2 in the case (ii). It follows there is the puncture in d_2.

For (iii), by applying Proposition 5.1 to d_3 we can find the puncture in d_3. We consider the behavior of a subarc of C' which contains the point at which η connects with C'. It follows (1)there are subarcs $\alpha \subset C$ and $\beta \subset C'$ such that η connects with β, $(\text{int}\beta) \cap C = \phi$ and $\alpha \cup \beta$ forms a loop bounding a disc d_5 on T, or (2)C' meets C without cobounding a disc together with a subarc of C. In the case (1), as we did in the case (i), we have a contradiction. In the case (2), there are subarcs $\xi \subset C$, $\eta' \subset \eta$ and $\zeta \subset C'$ such that ξ, η' and ζ form a loop bounding a disc d_6 and $\text{int}\eta'$ and $\text{int}\zeta$ do not cross C. Then as we did in the case (ii), we can find the puncture in d_6. Hence we found the puncture in d_3 and in d_6.

Suppose d_3 and d_6 are disjoint. Then there are two punctures, which is a contradiction. Hence d_3 and d_6 are not disjoint. Suppose $(\text{int}d_3) \cap (\text{int}d_6) = \phi$ and d_3 and d_6 have an edge in common, where an edge means a subarc of the boundary of d_3 or d_6. Let e be the edge. Suppose the puncture is not contained in e. Then it follows that there are two punctures, which is a contradiction. Hence the puncture is contained in e. It follows that the part of $(F \cup D) \cap T_\pm$ shown in Fig. 4.7 in d_3 and d_6 have e in common. Since d_3 and d_6 cannot have a subarc of $\gamma \subset \eta$ in common, e is contained in C. That is, the puncture is contained in the loop C. Suppose at least one of the parts of $(F \cup D) \cap T_\pm$ contained in d_3 and d_6 is as in (i) or (ii) of Fig. 4.7. Then since η does not meet a bubble between the point $\eta \cap \xi$ and the puncture, the puncture is contained in arcs of $D \cap T_+$ and $D \cap T_-$ rather than in a loop of $D \cap T_\pm$, which is a contradiction. Hence both parts are as in (iii) of Fig. 4.7. Then we can also show that the components of $D \cap T_+$ and $D \cap T_-$ which contain the puncture are arcs as shown in Fig. 5.12, which is a contradiction. Then one disc is contained in another. Suppose d_6 is contained in d_3. Since C' exits from d_3 crossing α, a subarc of C' cobounds a disc in d_3 together with a subarc of $\alpha \subset C$. See Fig. 5.13. Then as we did in the case (i), we have a contradiction. Suppose d_3 is contained in d_6. Then a subarc of η cobounds a disc together with a subarc of C such that interiors of the disc and d_3 are disjoint. Then as we did above we have a contradiction. Hence the case (iii) never happens.

For (iv), we consider the behavior of a subarc of C' which contains the point

Fig. 5.12.

Fig. 5.13.

at which η connects with C'. It follows (1) there are subarcs $\alpha \subset C$ and $\beta \subset C'$ such that η connects β, $(\text{int}\beta) \cap C = \phi$ and $\alpha \cup \beta$ forms a loop bounding a disc d_7 on T, or (2) C' meets C without cobounding a disc together with a subarc of C. In the case (1), as we did in the case (i), we have a contradiction. In the case (2), there are subarcs $\xi \subset C$, $\eta' \subset \eta$ and $\zeta \subset C'$ such that ξ, η' and ζ form a loop bounding

a disc d_4. Then as we did in the case (ii), we can find a disc as shown in Fig. 4.7. Hence in the case (iv), we can find the puncture in d_4.

In the cases (ii) and (iv), by using similar arguments above we can show that there are two disjoint part of $(F \cup D) \cap T_\pm$ shown in Fig. 4.7, which is a contradiction.

$\square$

Therefore there exists no loop in $D \cap T_\pm$.

6. Arcs in $D \cap T_\pm$

Hence $D \cap T_\pm$ is composed of properly embedded arcs. In §6 we deal with arcs in $D \cap T_\pm$ and we give the proof of Theorem. Let $G = (D \cap (T_+ \cup T_-)) \cup (saddles)$.

Lemma 6.1.   *Suppose D is in standard position. Then G satisfies at least one of the following properties;*
(i)   *There is a component of $D - G$ whose boundary is an arc in $D \cap T_\pm$ and an arc in ∂D, or*
(ii)   *There are two adjacent components of $D - G$, E_+ and E_-, such that $\partial E_\pm = \gamma_\pm \cup \zeta_\pm$, where $\gamma_\pm \subset D \cap T_\pm$ and $\zeta_\pm \subset \partial D \cap V_\pm$ are arcs. Furthermore each of γ_+ and γ_- meets one single bubble, $\gamma_+ \cap \gamma_-$ consists of an arc and $\gamma_\pm$ does not contain the puncture, or*
(iii)   *G is a single arc which intersects L as the puncture.*

Proof.   We consider G as a kind of four-valent graph on D as follows. We regard a saddle as a vertex. Each segment of $D \cap T_+ \cap T_-$ which is between two adjacent saddles, between a saddle and a point of $\partial D \cap T_+ \cap T_-$ or between two points of $\partial D \cap T_+ \cap T_-$ is considered as an edge of G.

If there is no vertex in G, G satisfies (i) or (iii). In case there is a vertex, a standard outermost fork argument will do and the lemma follows. $\square$

Proposition 6.2.   *$D \cap T_\pm$ is empty.*

The proof of Proposition 6.2 is divided into three Lemmas according to (i), (ii) and (iii) of Lemma 6.1.

Lemma 6.3.   *G is not as shown in (iii) of Fig. 6.1.*

Proof.   Suppose G is as shown in (iii) of Fig. 6.1. Let γ_+ (resp. γ_-) be the arc of $D \cap T_+$ (resp. $D \cap T_-$) and let δ_+ (resp. δ_-) be the arc $\partial D \cap V_+$ (resp. $\partial D \cap V_-$). See Fig. 6.2.

First we claim there are two loops $\alpha \subset F \cap T_+$ and $\beta \subset F \cap T_-$ with subarcs $a, b \subset \alpha \cap \beta$ such that (1) a and b contained in adjacent regions, and (2) there is
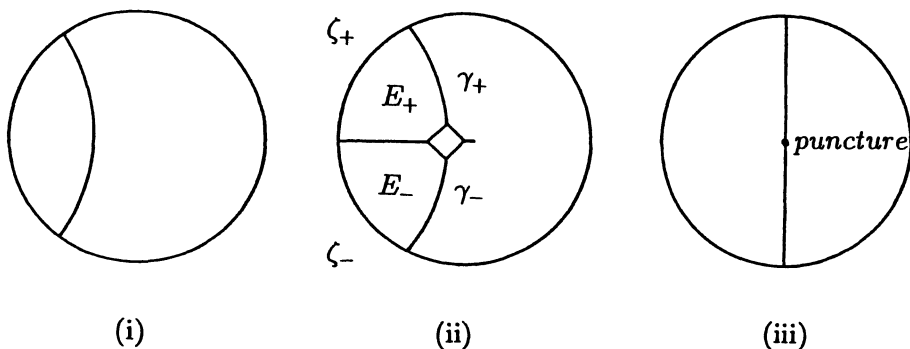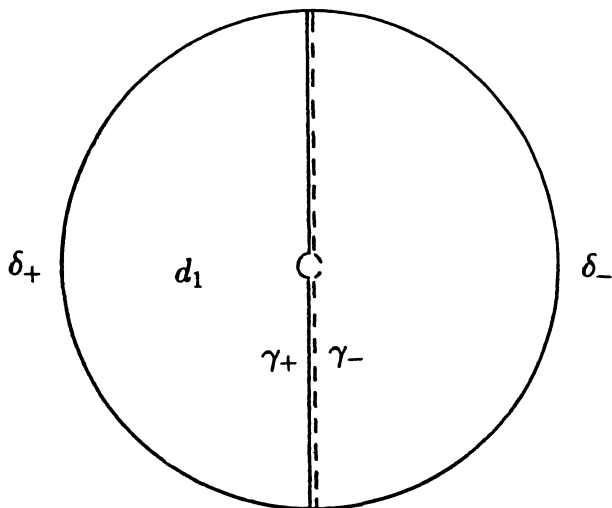
Fig. 6.1.

Fig. 6.2.

a subarc of α (resp. β) which cobounds a disc d (resp. d') together with γ_+ (resp. γ_-). See Fig. 6.3. Since every component of $F \cap T_\pm$ bounds a meridian disc of $V_\pm$, and $\gamma_\pm$ has only one puncture and no saddle, we can take α, β, a and b satisfying the former condition (1). We will show that α and β satisfy the latter condition (2). Since the meridian disc which α bounds in $F \cap V_+$ is boundary incompressible in V_+, and since there is a disc $d_1 = D \cap V_+$ cobounded by δ_+ and γ_+ as shown in Fig. 6.2, there is a subarc $\eta_+ \subset \alpha$ such that $\delta_+ \cup \eta_+$ forms a loop bounding a disc d'_1 in $F \cap V_+$. Then $d_1 \cup d'_1$ is a peripheral disc or a meridian disc of V_+. In case of a peripheral disc, $\gamma_+ \cup \eta_+$ forms a loop bounding a disc on T_+. In case of a meridian disc, let η'_+ be the complement of η_+ in α. Since $\alpha = \eta_+ \cup \eta'_+$ bounds a meridian disc, $\gamma_+ \cup \eta'_+$ forms a loop bounding a disc on T_+. Hence in both cases α
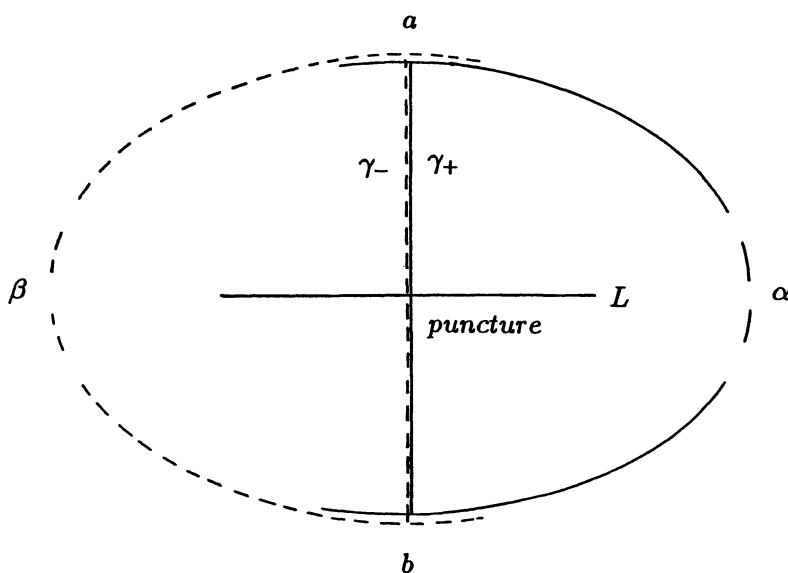
Fig. 6.3.

satisfies the condition (2). In exactly the same way, we can show β also satisfies the condition (2).

Let η_+ be the subarc of α which cobounds d together with γ_+. There is a part of $L \cap T$ entering d at the puncture. Since the part of $L \cap T$ goes out of d, η_+ meets a bubble. If η_+ meets more than one bubbles, then either η_+ violates Proposition 2.3 or we can find a mate to η_+ in d. The mate is a loop of $F \cap T_+$ and it cannot exit from d. Since it is a trivial loop, which contradicts F is in standard position. Hence η_+ meets only one bubble.

Suppose subarcs of α and β cobound discs together with γ_+ and γ_- respectively on the same side of $\gamma_\pm$. In this case β meets the bubble twice with encircling a disc in d', which violates Proposition 2.3.

Therefore subarcs of α and β cobounds discs d, d' together with γ_+ and γ_- respectively on distinct sides of $\gamma_\pm$. Then β also meets the bubble twice with encircling a disc in $d \cup d'$. Hence (iii) cannot exist. $\square$

Lemma 6.4. *G does not have a part shown in* (i) *of* Fig. 6.1.

Proof.   Suppose (i) exists. We call the arc α. Since there is no saddle incident to α, α is contained in a single region, say R. Suppose without loss of generality that the component of $D - (D \cap T_\pm)$ which is cobounded by α and a subarc of ∂D is contained in V_+. Since every loop of $F \cap T_\pm$ bounds a meridian disc of $V_\pm$, both

ends of α connect with the same component, say C, of $F \cap T_+$. If C connects the two points $\partial \alpha$ without exiting from R, it violates Proposition 3.2(ii). Hence C goes out of R. We consider two loops γ_1 and γ_2 on T_+ which are composed of α and halves of C. Let β_1 and β_2 be halves of C such that $\gamma_1 = \alpha \cup \beta_1$ and $\gamma_2 = \alpha \cup \beta_2$. Suppose γ_1 and γ_2 do not bound a disc on T_+. Since C bounds a meridian disc of V_+, each loop has a different slope from the meridian of V_+. Hence if there is another loop of $F \cap T_+$ than C, then it intersects γ_1 or γ_2. Since every closed surface in S^3 or a lens space (excluding $S^2 \times S^1$) is separating, and F cannot be contained in $V_\pm$ because of the incompressibility of F, there are at least two component of $F \cap T_+$ (Each core of $V_\pm$ must intersect F at least twice). Hence we have a contradiction. Therefore at least one of γ_1 and γ_2, say γ_1, bounds a disc d on T_+. Since β_1 exits from R and enters R, there are at least two saddles incident to β_1. From the alternating property $(*)$, there is a mate to β_1 in d. The mate is a loop of $F \cap T_+$ and it cannot exit from d, which violates Lemma 2.5. Therefore (i) cannot exist.   □

Lemma 6.5.   *G does not have a part shown in* (ii) *of* Fig. 6.1.

Proof.   Suppose (ii) exists. Let γ_+ (resp. γ_-) be the arc of $D \cap T_+$ (resp. $D \cap T_-$) and s be the saddle which is incident to $\gamma_\pm$. Let B be the bubble which contains s. If we see the circumstance around B in $T_\pm$, there is a loop $\alpha \subset F \cap T_+$ which connects two points $\partial \gamma_+$ and a loop $\beta \subset F \cap T_-$ which connects two points $\partial \gamma_-$. We take subarcs $a, c \subset \alpha$ and $b, c \subset \beta$ which contain $\partial \gamma_\pm$. Let L_1 and L_2 be parts of $L \cap T$ which are incident to B. See Fig. 6.4.

We separate α into two subarcs α_1 and α_2 so that $\alpha_1 \cup \gamma_+$ and $\alpha_2 \cup \gamma_+$ form loops on T_+ in the same way as we did in the proof of Lemma 6.4. Then we can
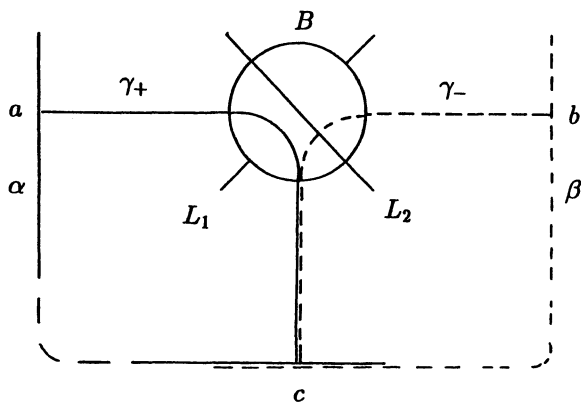

Fig. 6.4.

show either $\alpha_1 \cup \gamma_+$ or $\alpha_2 \cup \gamma_+$ bounds a disc d_α on T_+. Note that $\alpha_1 \cup \gamma_+$ (resp. $\alpha_2 \cup \gamma_+$) bounds d_α so that d_α does not contain α_2 (resp. α_1). Otherwise the loop α is contained in d_α, which is a contradiction. We also do the same things for β.

Then we can show that, say $\gamma_+ \cup \alpha_1$ (resp. $\gamma_- \cup \beta_1$) forms a loop bounding a disc d_α (resp. d_β) which does not contain L_1 (resp. L_2) and that there is no saddle incident to α_1 and β_1 by hand. It follows that the crossing at B is a nugatory crossing as in Fig. 6.5, which is a contradiction. $\square$

Proof of Proposition 6.2.   From Lemma 6.1 G has at least one of the parts shown in Fig. 6.1. But Lemmas 6.3, 6.4 and 6.5 prohibit that. $\square$

Proof of the Theorem.   We take a compressing or a pairwise compressing disc D of F which has minimal complexity and is in standard position. By Propositions 6.2, $D \cap T_\pm$ is empty. Hence D is contained in $V_\pm$. Since $D \cap L = \phi$, if D is a pairwise compressing disc, we have a contradiction. Hence D is a compressing disc. Then $F \cap V_\pm$ is compressible in $V_\pm$. However since $F \cap V_\pm$ consists of meridian discs, $F \cap V_\pm$ is incompressible in $V_\pm$. Hence we have a contradiction. Thus D does not exist. Therefore F is incompressible and pairwise incompressible. $\square$
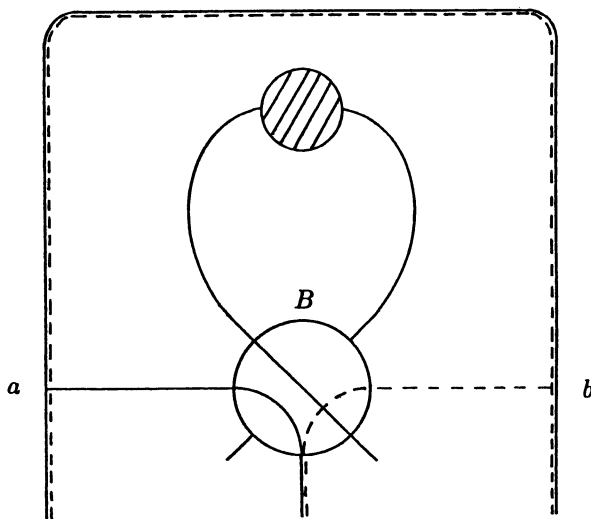

Fig. 6.5.

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