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ON NON-SINGULAR HYPERPLANE SECTIONS OF SOME HERMITIAN SYMMETRIC SPACES

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Let $P^k(\mathbf{C})$ denote a complex projective space of dimension k . The product space $P^m(\mathbf{C}) \times P^n(\mathbf{C})$ has a natural imbedding in $P^{mn+m+n}(\mathbf{C})$, called the Segre imbedding. Let V be a non-singular hyperplane section of $P^m(\mathbf{C}) \times P^n(\mathbf{C})$ in $P^{mn+m+n}(\mathbf{C})$. The identity connected component $\text{Aut}_0(V)$ of the group of all holomorphic automorphisms of V has been determined by J.-I. Hano [3]. For an irreducible Hermitian symmetric space M of compact type we have the canonical equivariant imbedding $j: M \rightarrow P^N(\mathbf{C})$. Now take a non-singular hyperplane section V of M in $P^N(\mathbf{C})$. In this note we shall determine the structure of the Lie algebra of $\text{Aut}(V)$ for the cases when M is a complex Grassmann manifold $G_{m,2}(\mathbf{C})$ of 2-planes in \mathbf{C}^m and when M is $\text{SO}(10)/\text{U}(5)$, by applying Hano's method. In particular, using Lichnerowicz-Matsushima's theorem, we prove the following.

1) For the case M is $G_{m,2}(\mathbf{C})$ ($m \geq 4$), if m is odd a non-singular hyperplane section V does not admit any Kähler metric with constant scalar curvature, and if m is even V is a Kählerian C -space.

2) For the case M is $\text{SO}(10)/\text{U}(5)$, V does not admit any Kähler metric with constant scalar curvature.

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1. Preliminaries

A simply connected compact homogeneous complex manifold is called a C -space. A C -space is said to be Kählerian if it admits a Kähler metric. We recall some known facts on Kählerian C -spaces and holomorphic line bundles on these complex manifolds (cf. [1], [4]).

Fact 1. *Every holomorphic line bundle on a Kählerian C -space M is homogeneous. If we denote by $H^1(M, \theta^*)$ the group of all isomorphism classes of holomorphic line bundles on M and by $c_1(F)$ the Chern class of a holomorphic line bundle F , then the homomorphism $F \rightarrow c_1(F): H^1(M, \theta^*) \rightarrow H^2(M, \mathbf{Z})$ is bijective.*

Fact 2. *Every ample holomorphic line bundle on a Kählerian C -space M is*

very ample. Moreover for each very ample holomorphic line bundle the corresponding holomorphic imbedding of M can be realized as an orbit space of the irreducible representation of all holomorphic automorphism group $\text{Aut}(M)$ of M .

From now on we assume that M is a kählerian C -space with the second Betti number $b_2(M)=1$. In this case there is a unique very ample holomorphic line bundle L on M which is a generator of the group $H^1(M, \theta^*)$. The corresponding holomorphic imbedding for L is called the *canonical* imbedding of M and denoted by $j: M \rightarrow P^N(\mathbf{C})$. Let $h=c_1(L)$. Then h is a generator of $H^2(M, \mathbf{Z})$. For a divisor D on M let $\{D\}$ be the holomorphic line bundle on M associated to D . Then for a positive divisor D on M there is a positive integer $a(D)$ such that $c_1(\{D\})=a(D)h$. The integer $a(D)$ is called the *degree* of D .

Fact 3. *Let $j: M \rightarrow P^N(\mathbf{C})$ be the canonical imbedding of a kählerian C -space M with $b_2(M)=1$. Then for each positive divisor D on M of degree a there exists a homogeneous polynomial F on \mathbf{C}^{N+1} of degree a such that D is the pull back of the divisor on $P^N(\mathbf{C})$ defined by the zero points of F by the canonical imbedding j .*

For a non-singular hypersurface V of M the degree of the positive divisor defined by V is called the *degree* of V . Let $K(V)$ and $K(M)$ denote the canonical line bundles on V and M respectively. It is known that the first Chern class $c_1(M)$ of M is given by $c_1(M)=\kappa h$ for some positive integer κ . Since $K(V)=\iota^*(K(M) \otimes \{V\})$ where $\iota: V \rightarrow M$ is inclusion, the first Chern class $c_1(V)$ of V is given by $c_1(V)=(\kappa-a)\iota^*h$ if the degree of V is a . In particular, if V is a non-singular hypersurface of degree $a < \kappa$, the first Chern class $c_1(V)$ of V is positive. It is also known that irreducible Hermitian symmetric spaces of compact type are kählerian C -spaces with the second Betti number 1 and the positive number $\kappa \geq 2$. Therefore if V is a non-singular hyperplane section of an irreducible Hermitian symmetric space M of compact type for the canonical imbedding $j: M \rightarrow P^N(\mathbf{C})$, the first Chern class $c_1(V)$ of V is positive.

Let $T(M)$ and $T(V)$ be the holomorphic tangent bundles of M and V respectively. Given a holomorphic vector bundle E , we denote by $\Omega^0(E)$ the sheaf of germs of local holomorphic sections of E .

Fact 4 (Kimura [5]). *Let M be an irreducible Hermitian symmetric space of compact type. Assume that M is not a complex projective space $P^n(\mathbf{C})$ or a complex quadric $Q^n(\mathbf{C})$. Then for a non-singular hypersurface V of M the exact sequence of sheaves on M*

$$0 \rightarrow \Omega^0(T(M) \otimes \{V\}^{-1}) \rightarrow \Omega^0(T(M)) \rightarrow \Omega^0(T(M)|V) \rightarrow 0$$

induces the exact sequence of cohomologies

$$0 \rightarrow H^0(M, T(M)) \rightarrow H^0(V, T(M)|V) \rightarrow 0$$

Moreover $H^1(V, T(M)|V) = (0)$.

REMARK. If V is a non-singular hypersurface $Q^n(\mathbf{C}) (n > 3)$ of degree $a \neq 2$, the same result as in Fact 4 holds.

2. The case M is a complex Grassmann manifold $G_{m,2}(\mathbf{C})$

Let ρ be the natural representation of $SL(m, \mathbf{C})$ on \mathbf{C}^m and consider the p -th exterior representation $\Lambda^p \rho: SL(m, \mathbf{C}) \rightarrow GL(\Lambda^p \mathbf{C}^m)$ induced by ρ . Note that $\Lambda^p \rho$ is an irreducible representation of $SL(m, \mathbf{C})$. Fix a highest weight vector $v_0 \in \Lambda^p \mathbf{C}^m$ and consider the subgroup U of $SL(m, \mathbf{C})$ defined by

$$\{h \in SL(m, \mathbf{C}) \mid (\Lambda^p \rho)(h) v_0 = c v_0 \text{ for some } c \in \mathbf{C} - (0)\}.$$

Then the map $j: SL(m, \mathbf{C})/U \rightarrow P(\Lambda^p \mathbf{C}^m)$ defined by

$$j(gU) = [\Lambda^p \rho(g)(v_0)] \quad \text{for } g \in SL(m, \mathbf{C}),$$

where $[w]$ ($w \in \Lambda^p \mathbf{C}^m$) denotes the line determined by w , is the canonical imbedding of the Grassmann manifold $M = G_{m,p}(\mathbf{C})$ and is called the Plücker imbedding of M .

From now on we assume that M is a complex Grassmann manifold of 2-planes in \mathbf{C}^m which is not a complex projective space, so we may assume $m \geq 4$. We may also regard M as a non-singular projective subvariety of $P(\Lambda^2 \mathbf{C}^m)$ by the canonical imbedding.

Theorem 1. *For an integer $m \geq 4$ let V be a non-singular hyperplane section of $G_{m,2}(\mathbf{C})$ in $P(\Lambda^2 \mathbf{C}^m)$.*

(1) *If m is even, V is a kählerian \mathbf{C} -space $Sp(n, \mathbf{C})/P$ with the second Betti number 1 where $n = m/2$ and P is a parabolic subgroup of $Sp(n, \mathbf{C})$.*

(2) *If m is odd, the group $Aut(V)$ of all holomorphic transformations of V is not reductive and thus V does not admit any Kähler metric with constant scalar curvature. Moreover we have $H^1(V, T(V)) = (0)$.*

Proof. By the Lefschetz theorem on hyperplane sections, we have $b_2(V) = 1$ since $b_2(G_{m,2}(\mathbf{C})) = 1$. From the fact 4 we see that every holomorphic vector field on V can be extended uniquely to a holomorphic vector field on M . Let $A = \{g \in Aut(M) \mid g(V) = V\}$. Then the Lie algebra \mathfrak{a} of A can be identified with the Lie algebra of all holomorphic vector fields on V . By means of irreducible representation $\Lambda^2 \rho: SL(m, \mathbf{C}) \rightarrow GL(\Lambda^2 \mathbf{C}^m)$ each element of $SL(m, \mathbf{C})$ maps a hyperplane of $P(\Lambda^2 \mathbf{C}^m)$ to another hyperplane. Take a hyperplane H of $P(\Lambda^2 \mathbf{C}^m)$ such that $V = H \cap M$. Note that such a hyperplane H in $P(\Lambda^2 \mathbf{C}^m)$ is determined uniquely since the canonical imbedding $j: M \rightarrow P(\Lambda^2 \mathbf{C}^m)$ is full. Thus the Lie algebra \mathfrak{a} of A coincides with the Lie algebra of $A' = \{g \in SL(m, \mathbf{C}) \mid g \cdot H = H\}$. A hyperplane H is the zero locus of non-zero linear form B on $\Lambda^2 \mathbf{C}^m$. If we let

$$b(z, w) = B(z \wedge w) \quad (z, w \in \mathbf{C}^m),$$

b is a skew-symmetric form on \mathbf{C}^m . Therefore

$$A' = \{g \in SL(m, \mathbf{C}) \mid b(g \cdot z, g \cdot w) = \lambda(g) b(z, w), z, w \in \mathbf{C}^m$$

for some non-zero constant $\lambda(g) \in \mathbf{C}\}$.

Now we choose coordinates on \mathbf{C}^m in such a way as

$$b(z, w) = \sum_{i=1}^k (z_i w_{k+i} - z_{k+i} w_i) \quad \text{where } 1 \leq k \leq [m/2]$$

(that is, if $p_{\alpha\beta}$ denote Plücker coordinates, the hyperplane H is defined by $p_{1k+1} + \dots + p_{k2k} = 0$).

We claim that $k = [m/2]$ if V is non-singular. Suppose that $k < [m/2]$. Then $2k \leq m-2$. We can take vectors $z, w \in \mathbf{C}^m$ given by

$$z_1 = \dots = z_{2k} = 0, z_{2k+1} = 1, z_{2k+2} = \dots = z_m = 0,$$

$$w_1 = \dots = w_{2k+1} = 0, w_{2k+2} = 1, w_{2k+3} = \dots = w_m = 0,$$

respectively. The $z \wedge w$ determines a point of V which is singular, since

$$db = \sum_{j=1}^k (w_{k+i} dz_i + z_i dw_{k+i} - w_i dz_{k+i} - z_{k+i} dw_i)$$

vanishes at this point. Hence $k = [m/2]$.

Now we consider the cases where m is even or odd separately.

Case 1 $m = 2n$

In this case the Lie algebra \mathfrak{a} is given by the Lie algebra of

$$\left\{ g \in SL(2n, \mathbf{C}) \mid {}^t g \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} g = \lambda(g) \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \right\}$$

where 1_n denotes $n \times n$ identity matrix. We may write an $m \times m$ matrix X in the form

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A, B, C and D are $n \times n$ matrices. Thus we see that $X \in \mathfrak{a}$ if and only if $C = {}^t C$, $B = {}^t B$ and ${}^t A + D = \mu(X) 1_n$ for some $\mu(X) \in \mathbf{C}$. Since $\text{tr}(X) = 0$, we have $\mu(X) = 0$ and hence $X \in \mathfrak{a}$ if and only if

$$X \in \mathfrak{sp}(n, \mathbf{C}) = \left\{ X \mid {}^t X \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} X = 0 \right\}.$$

Therefore we may identify the connected component of the identity of A' with $Sp(n, \mathbf{C})$. Take two vectors $e_1 = {}^t(1, 0, \dots, 0)$ and $e_2 = {}^t(0, 1, 0, \dots, 0)$ of \mathbf{C}^m . The $e_1 \wedge e_2$ determines a point x_0 of V (that is, in the Plücker coordinates, x_0 is given by $P_{12} \neq 0$ and $p_{\alpha\beta} = 0$ otherwise). Let P be the isotropy subgroup at x_0 . Then it is not difficult to see that P is a parabolic subgroup of $Sp(n, \mathbf{C})$. Since $\dim Sp(n, \mathbf{C})/P = 2(2n-2)-1$, $\dim V = 2(2n-2)-1$ and V is compact, we see $V = Sp(n, \mathbf{C})/P$.

Case 2 $m = 2n+1$

We may write a $(2n+1) \times (2n+1)$ matrix X in the form

$$X = \begin{pmatrix} A & \alpha \\ \beta & \gamma \end{pmatrix}$$

where A is a $2n \times 2n$ matrix. Then $X \in \mathfrak{a}$ if and only if $\alpha = 0$ and

$${}^tA \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} A = \mu(X) \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$$

for some $\mu(X) \in \mathbf{C}$. Thus we get

$$\mathfrak{a} = \left\{ \begin{pmatrix} A & 0 \\ \beta & \gamma \end{pmatrix} \in \mathfrak{sl}(2n+1, \mathbf{C}) \mid A = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}, {}^tX_2 = X_2, \right. \\ \left. {}^tX_3 = X_3, X_1 + {}^tX_4 = -(\gamma/n)1_n, {}^t\beta \in \mathbf{C}^{2n}, \gamma \in \mathbf{C} \right\}$$

and $\dim \mathfrak{a} = 2n^2 + 3n + 1$. Let

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & 0 \\ \beta & 0 \end{pmatrix} \mid {}^t\beta \in \mathbf{C}^{2n} \right\}. \text{ Then } \mathfrak{n} \text{ is an abelian ideal of } \mathfrak{a}. \text{ On the other hand}$$

the center \mathfrak{z} of \mathfrak{a} is given by $\{a \cdot 1_{2n+1} \mid a \in \mathbf{C}\}$. Since $\mathfrak{n} \cap \mathfrak{z} = (0)$, \mathfrak{a} is not reductive. By a theorem of Lichnerowicz-Matsushima [76], we see that V does not admit any Kähler metric with constant scalar curvature.

Now the exact sequence of sheaves

$$0 \rightarrow \Omega^0(T(V)) \rightarrow \Omega^0(T(M)|V) \rightarrow \Omega^0(\{V\}|V) \rightarrow 0$$

induces the exact sequence of cohomologies

$$0 \rightarrow H^0(V, T(V)) \rightarrow H^0(V, T(M)|V) \rightarrow H^0(V, \{V\}|V) \\ \rightarrow H^1(V, T(V)) \rightarrow H^1(V, T(M)|V) \rightarrow \dots$$

Since $H^1(V, T(M)|V) = (0)$, $H^0(V, T(M)|V) \cong H^0(M, T(M))$ by the fact 4 and $h^0(V, \{V\}|V) = h^0(M, \{V\}) - 1$, we get

$$\begin{aligned} h^1(V, T(V)) &= h^0(V, T(V)) - h^0(M, T(M)) + h^0(V, \{V\} | V) \\ &= 2n^2 + 3n + 1 - ((2n+1)^2 - 1) + \binom{2n+1}{2} - 1 = 0 \end{aligned}$$

q.e.d.

3. The case M is $SO(10)/U(5)$

Let M be an irreducible Hermitian symmetric space of compact type of type DIII. It is known that M is diffeomorphic to $SO(2n)/U(n)$ ($n \geq 4$). Note that M is a complex quadric $Q^6(\mathbb{C})$ if $n=4$.

Consider a semi-spin representation of the complex simple Lie algebra \mathfrak{g} of type D_n and the corresponding representation ρ of the simply connected complex Lie Group G with the Lie algebra \mathfrak{g} . Fix a highest weight vector v_0 and let U be the subgroup of G defined by $\{g \in G \mid \rho(g)v_0 = cv_0 \text{ for some } c \in \mathbb{C} - \{0\}\}$. Then a map

$$j: G/U \rightarrow P(\mathbb{C}^{2^{n-1}})$$

defined by $j(gU) = [\rho(g)v_0]$ for $g \in G$, is the canonical imbedding of $M = G/U$.

We recall semi-spin representations of type D_n (cf. [2], chap. VIII, §13), so that we can fix our notations. Let W be a $2n$ -dimensional complex vector space and Φ a non-degenerate symmetric bilinear form on W . Then W is a direct sum of maximal totally isotropic subspaces F and F' of W ; $W = F \oplus F'$. Let $\{e_1, \dots, e_n, e_{-n}, \dots, e_{-1}\}$ be a Witt basis of W , that is, $\{e_1, \dots, e_n\}$ and $\{e_{-n}, \dots, e_{-1}\}$ are bases of F and F' respectively which satisfy the relation $\Phi(e_i, e_{-j}) = \delta_{ij}$ for $i, j = 1, \dots, n$. The corresponding matrix of Φ with respect to a Witt basis is given as

$$\begin{pmatrix} 0 & s \\ s & 0 \end{pmatrix} \text{ where } s = \begin{pmatrix} 0 & & 1 \\ & 1 & \\ & \ddots & \\ 1 & & 0 \end{pmatrix}$$

and the Lie algebra \mathfrak{g} can be given by

$$\left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid B = -s {}^t B s, C = -s {}^t C s, D = -s {}^t A s \right\}.$$

Let $E_{p,q}$ be a matrix unit, that is, the (k, l) -component of $E_{p,q}$ is given by $\delta_{kp} \delta_{lq}$. Put $\mathfrak{h} = \{X \in \mathfrak{g} \mid X \text{ is a diagonal matrix}\}$ and $H_i = E_{i,i} - E_{-i,-i}$ for $i = 1, \dots, n$. Then $\{H_1, \dots, H_n\}$ is a basis of \mathfrak{h} . Let $\{\varepsilon_1, \dots, \varepsilon_n\}$ be the dual basis of the dual space \mathfrak{h}^* .

Put

$$\begin{aligned}
X_{\varepsilon_i - \varepsilon_j} &= E_{i,j} - E_{-j,-i} \\
X_{-\varepsilon_i + \varepsilon_j} &= -E_{j,i} + E_{-i,-j} \\
X_{\varepsilon_i + \varepsilon_j} &= E_{i,-j} - E_{j,-i} \\
X_{-\varepsilon_i - \varepsilon_j} &= -E_{-j,i} + E_{-i,j} \\
&\text{for } 1 \leq i < j \leq n.
\end{aligned}$$

Then \mathfrak{h} is a Cartan subalgebra of \mathfrak{g} and the root system Σ of \mathfrak{g} relative to \mathfrak{h} is given by $\Sigma = \{\pm \varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n\}$. Let $\alpha_1 = \varepsilon_1 - \varepsilon_2$, $\alpha_2 = \varepsilon_2 - \varepsilon_3$, ..., $\alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n$, $\alpha_n = \varepsilon_{n-1} + \varepsilon_n$. Then $\{\alpha_1, \dots, \alpha_n\}$ is a fundamental root system Π of Σ and the fundamental weights corresponding Π to are

$$\begin{aligned}
\Lambda_{\alpha_i} &= \varepsilon_1 + \dots + \varepsilon_i \quad (1 \leq i \leq n-2) \\
\Lambda_{\alpha_{n-1}} &= \frac{1}{2} (\varepsilon_1 + \dots + \varepsilon_{n-2} + \varepsilon_{n-1} - \varepsilon_n) \\
\Lambda_{\alpha_n} &= \frac{1}{2} (\varepsilon_1 + \dots + \varepsilon_{n-2} + \varepsilon_{n-1} + \varepsilon_n)
\end{aligned}$$

Now semi-spin representations are irreducible representations of \mathfrak{g} with the highest weight $\Lambda_{\alpha_{n-1}}$ and Λ_{α_n} respectively.

Let Q be the quadric form defined by $x \rightarrow \Phi(x, x)/2$ and let $C(Q)$ denote the Clifford algebra of W relative to Q . Let N be the exterior algebra of the maximal totally isotropic subspace F' . We shall identify F and the dual of F' via Φ . For $x \in F'$ and $y \in F$ let $\lambda(x)$ and $\lambda(y)$ denote the left exterior product by x and left interior product by y in N respectively; so that for $x \in F'$ and $y \in F$

$$\begin{aligned}
\lambda(x) a_1 \wedge \dots \wedge a_k &= x \wedge a_1 \wedge \dots \wedge a_k \\
\lambda(y) a_1 \wedge \dots \wedge a_k &= \sum_{i=1}^k (-1)^{i-1} \Phi(a_i, y) a_1 \wedge \dots \wedge \hat{a}_i \wedge \dots \wedge a_k
\end{aligned}$$

where $a_1, \dots, a_k \in F'$.

Then we get that $\lambda(x)^2 = \lambda(y)^2$ and $\lambda(x) \lambda(y) + \lambda(y) \lambda(x) = \Phi(x, y) 1$, and there exist a unique homomorphism of $C(Q)$ into $\text{End}(N)$, denoted also by λ , which is a prolongation of the map $\lambda: F \cup F' \rightarrow \text{End}(N)$. Let $C^+(Q)$ denote the subalgebra of $C(Q)$ spanned by even elements and put

$$N_+ = \sum_{p: \text{even}} \Lambda^p F', \quad N_- = \sum_{p: \text{odd}} \Lambda^p F'.$$

Now N_+ and N_- are stable for the restriction of λ to $C^+(Q)$, and the representations λ_+ and λ_- of $C^+(Q)$ in N_+ and N_- respectively are called semi-spin representations of $C^+(Q)$. These are simple $C^+(Q)$ -modules. There also exists a canonical linear map $f: \mathfrak{g} \rightarrow C^+(Q)$ which satisfies $[f(X), f(Y)] = f([X, Y])$ for X

and Y in \mathfrak{g} and $f(\mathfrak{g})$ generates the associative algebra $C^+(Q)$. Furthermore if N is a left $C^+(Q)$ -module and ρ is the corresponding homomorphism of $C^+(Q)$ into $\text{End}(N)$, then $\rho \circ f$ is a representation of \mathfrak{g} in N (cf. [2], p. 195, Lemma 1). Thus $\rho_+ = \lambda_+ \circ f$ and $\rho_- = \lambda_- \circ f$ are irreducible representations of \mathfrak{g} . In particular, the action of \mathfrak{g} on N is given as follows:

$$\begin{aligned} X_{\varepsilon_i - \varepsilon_j}(e_{-i_1} \wedge \cdots \wedge e_{-i_k}) &= \lambda(e_i)(e_{-j} \wedge e_{-i_1} \wedge \cdots \wedge e_{-i_k}) \\ X_{-\varepsilon_i + \varepsilon_j}(e_{-i_1} \wedge \cdots \wedge e_{-i_k}) &= -\lambda(e_j)(e_{-i} \wedge e_{-i_1} \wedge \cdots \wedge e_{-i_k}) \\ X_{-\varepsilon_i - \varepsilon_j}(e_{-i_1} \wedge \cdots \wedge e_{-i_k}) &= e_{-i} \wedge e_{-j} \wedge e_{-i_1} \wedge \cdots \wedge e_{-i_k} \\ X_{\varepsilon_i + \varepsilon_j}(e_{-i_1} \wedge \cdots \wedge e_{-i_k}) &= \lambda(e_i)\lambda(e_j)(e_{-i_1} \wedge \cdots \wedge e_{-i_k}) \end{aligned}$$

where $1 \leq i < j \leq n$ and

$$\begin{aligned} &H(e_{-i_1} \wedge \cdots \wedge e_{-i_k}) \\ &= \left(\frac{1}{2} (\varepsilon_1 + \cdots + \varepsilon_n) - (\varepsilon_{i_1} + \cdots + \varepsilon_{i_k}) \right) (H)(e_{-i_1} \wedge \cdots \wedge e_{-i_k}) \end{aligned}$$

for $H \in \mathfrak{h}$. Particularly we see that the highest weights of ρ_+ and ρ_- are Λ_{α_n} and $\Lambda_{\alpha_{n-1}}$ respectively. The representation ρ_- is the contragradient representation of ρ_+ .

From now on we consider the case $n=5$ exclusively.

Theorem 2. *Let V be a non-singular hyperplane section of $M^{10} = SO(10)/U(5)$ in $P^{15}(\mathbb{C})$ via the canonical imbedding. Then the group $\text{Aut}_0(V)$ is not reductive and thus V does not admit any Kähler metric with constant scalar curvature. Moreover $H^1(V, T(V)) = (0)$.*

In order to prove Theorem 2 we shall first classify the hyperplanes of N_+ by means of the action of the Lie group G . For a linear form $B: N_+ \rightarrow \mathbb{C}$ and $g \in G$ let g^*A denote the linear form defined by $(g^*A)(n) = A(g \cdot n)$ for $n \in N_+$. Now linear forms B and B_1 are called G -equivalent if there is an element $g \in G$ such that $B_1 = g^*B$.

Lemma. *Let $B: N_+ = \mathbb{C} \cdot 1 + \Lambda^2 F' + \Lambda^4 F' \rightarrow \mathbb{C}$ be a linear form. Then B is G -equivalent to either a linear form on $\mathbb{C} \cdot 1$ or a linear form on $\Lambda^2 F'$.*

Proof. We may assume $B \neq 0$. Take a basis $\{e_{-1}, \dots, e_{-5}\}$ of F' and fix it. A basis of N_+ is now given by $\{1, e_{-i} \wedge e_{-j}, e_{-1} \wedge \cdots \wedge \hat{e}_{-k} \wedge \cdots \wedge e_{-5} \mid 1 \leq i < j \leq 5, k=1, \dots, 5\}$ and the corresponding dual basis of $(N_+)^*$ will be denoted by

$$\{1, (e_{-i} \wedge e_{-j})^*, (e_{-1} \wedge \cdots \wedge \hat{e}_{-k} \wedge \cdots \wedge e_{-5})^* \mid 1 \leq i < j \leq 5, k=1, \dots, 5\}$$

Step 1. We claim the linear form B is G -equivalent to

$$\alpha \cdot 1 + \sum_{i < j} \beta_{ij}(e_{-i} \wedge e_{-j})^* + \sum_k \gamma_k(e_{-1} \wedge \cdots \wedge \hat{e}_{-k} \wedge \cdots \wedge e_{-5})^*$$

with $\alpha \neq 0$.

The linear form B can be written as

$$B = \beta \cdot 1 + \sum_{i < j} \tilde{\beta}_{ij}(e_{-i} \wedge e_{-j})^* + \sum_k \tilde{\gamma}_k(e_{-1} \wedge \cdots \wedge \hat{e}_{-k} \wedge \cdots \wedge e_{-5})^*$$

We may assume that $\beta = 0$. Let $X = \sum_{k < l} p_{kl} X_{-e_k - e_l}$ be an element of \mathfrak{g} . Then we have

$$\exp X(1) = 1 + \sum_{k < l} p_{kl} e_{-k} \wedge e_{-l} + \frac{1}{2} \sum_{k < l} \sum_{i < j} p_{kl} p_{ij} e_{-i} \wedge e_{-j} \wedge e_{-k} \wedge e_{-l}.$$

(a) The case when $\tilde{\beta}_{ij} \neq 0$ for some (i, j) .

Let $p_{kl} = 0$ for $(k, l) \neq (i, j)$ and $p_{ij} = 1$.

Then $B(\exp X(1)) = \tilde{\beta}_{ij} \neq 0$ and the linear form

$(\exp X)^* B$ has the required property.

(b) The case when $\tilde{\beta}_{kl} = 0$ for all (k, l) .

Take $\gamma_k \neq 0$ and choose $\{i, j, s, t\}$ such a way as $i < j < s < t$ and $i, j, s, t \neq k$.

Let $X = X_{-e_i - e_j} + X_{-e_s - e_t}$. Then $B(\exp X(1)) = \gamma_k \neq 0$ and the linear form $(\exp X)^* B$ has the required property.

Step. 2. We claim the linear form B is G -equivalent to

$$\alpha \cdot 1 + (e_{-1} \wedge e_{-2})^* + (e_{-3} \wedge e_{-4})^* + \sum_k \gamma'_k(e_{-1} \wedge \cdots \wedge \hat{e}_{-k} \wedge \cdots \wedge e_{-5})^*$$

with $\alpha \neq 0$ and for some $\gamma'_k \in \mathbb{C}$.

By Step 1 we may assume that B is given by

$$\alpha \cdot 1 + \sum_{i < j} \beta_{ij}(e_{-i} \wedge e_{-j})^* + \sum_k \gamma_k(e_{-1} \wedge \cdots \wedge \hat{e}_{-k} \wedge \cdots \wedge e_{-5})^*$$

with $\alpha \neq 0$. Let $Y = \sum_{k < l} q_{kl} X_{e_k + e_l}$ be an element of \mathfrak{g} . Then we have

$B(\exp Y(e_{-i} \wedge e_{-j})) = B(e_{-i} \wedge e_{-j} + Y(e_{-i} \wedge e_{-j})) = \beta_{ij} - q_{ij} \alpha$ and $B(\exp Y(1)) = B(1) = \alpha$. Hence we can choose Y in such a way as $(\exp Y)^* B = \alpha \cdot 1 + (e_{-1} \wedge e_{-2})^* + (e_{-3} \wedge e_{-4})^* + \sum_k \gamma'_k(e_{-1} \wedge \cdots \wedge \hat{e}_{-k} \wedge \cdots \wedge e_{-5})^*$.

Step 3. We claim the linear form B is G -equivalent to

$$\alpha \cdot 1 + \sum_{i < j} \beta'_{ij}(e_{-i} \wedge e_{-j})^* \text{ for some } \beta'_{ij} \in \mathbb{C}.$$

We may assume B is given by

$$\alpha \cdot 1 + (e_{-1} \wedge e_{-2})^* + (e_{-3} \wedge e_{-4})^* + \sum_k \gamma'_k(e_{-1} \wedge \cdots \wedge \hat{e}_{-k} \wedge \cdots \wedge e_{-5})^*.$$

Let $Y_1 = q'_{12} X_{e_1 + e_2}$ be an element of \mathfrak{g} . Then

$$(\exp Y_1)^* B = \alpha \cdot 1 + (1 - q'_{12} \alpha) (e_{-1} \wedge e_{-2})^* + (e_{-3} \wedge e_{-4})^* \\ + (\gamma_5 - q'_{12}) (e_{-1} \wedge e_{-2} \wedge e_{-3} \wedge e_{-4})^* + \sum_{k \leq 4} \gamma'_k (e_{-1} \wedge \cdots \wedge \hat{e}_{-k} \wedge \cdots \wedge e_{-5})^*.$$

Let $q'_{12} = \gamma_5$. Then we have

$$(\exp Y_1)^* B = \alpha \cdot 1 + \mu_{12} (e_{-1} \wedge e_{-2})^* + (e_{-3} \wedge e_{-4})^* \\ + \sum_{k \leq 4} \gamma'_k (e_{-1} \wedge \cdots \wedge \hat{e}_{-k} \wedge \cdots \wedge e_{-5})^*$$

where $\mu_{12} = 1 - \gamma_5 \alpha$.

Let $Y_2 = q'_{25} X_{e_2+e_5} + q'_{15} X_{e_1+e_5}$. Then we have

$$\begin{aligned} (\exp Y_2)^* (\exp Y_1)^* B (e_{-2} \wedge e_{-3} \wedge e_{-4} \wedge e_{-5}) &= \gamma'_1 - q'_{25} \\ (\exp Y_2)^* (\exp Y_1)^* B (e_{-1} \wedge e_{-3} \wedge e_{-4} \wedge e_{-5}) &= \gamma'_2 - q'_{15} \\ (\exp Y_2)^* (\exp Y_1)^* B (e_{-1} \wedge e_{-2} \wedge e_{-4} \wedge e_{-5}) &= \gamma'_3 \\ (\exp Y_2)^* (\exp Y_1)^* B (e_{-1} \wedge e_{-2} \wedge e_{-3} \wedge e_{-5}) &= \gamma'_4 \\ (\exp Y_2)^* (\exp Y_1)^* B (e_{-1} \wedge e_{-2} \wedge e_{-3} \wedge e_{-4}) &= 0 \\ (\exp Y_2)^* (\exp Y_1)^* B (e_{-i} \wedge e_{-j}) &= (\exp Y_1)^* B (e_{-i} \wedge e_{-j}) \\ &\text{if } (i, j) \neq (1, 5), (2, 5) \\ (\exp Y_2)^* (\exp Y_1)^* B (e_{-2} \wedge e_{-5}) &= -q'_{25} \alpha \\ (\exp Y_2)^* (\exp Y_1)^* B (e_{-1} \wedge e_{-5}) &= -q'_{15} \alpha \end{aligned}$$

Thus setting $q'_{15} = \gamma'_2$ and $q'_{25} = \gamma'_1$, we get

$$(\exp Y_2)^* (\exp Y_1)^* B = \alpha \cdot 1 + \mu_{12} (e_{-1} \wedge e_{-2})^* + (e_{-3} \wedge e_{-4})^* - \gamma'_2 \alpha (e_{-1} \wedge e_{-5})^* \\ - \gamma'_1 \alpha (e_{-2} \wedge e_{-5})^* + \gamma'_3 (e_{-1} \wedge e_{-2} \wedge e_{-4} \wedge e_{-5})^* + \gamma'_4 (e_{-1} \wedge e_{-2} \wedge e_{-3} \wedge e_{-5})^*.$$

(a) Now we consider the case $\mu_{12} \neq 0$, $\gamma'_2 \neq 0$ or $\gamma'_1 \neq 0$.

Let $Y_3 = q'_{45} X_{e_4+e_5} + q'_{35} X_{e_3+e_5}$. Then we have

$(\exp Y_3)^* (\exp Y_2)^* (\exp Y_1)^* B = \alpha \cdot 1 + \sum_{i < j} \beta'_{ij} (e_{-i} \wedge e_{-j})^* + (\gamma'_4 - q'_{35} \mu_{12}) (e_{-1} \wedge e_{-2} \wedge e_{-3} \wedge e_{-5})^* + (\gamma'_3 - q'_{45} \mu_{12}) (e_{-1} \wedge e_{-2} \wedge e_{-4} \wedge e_{-5})^*$ for some $\beta'_{ij} \in \mathbf{C}$. If $\mu_{12} \neq 0$, let $q'_{35} = \gamma'_4 / \mu_{12}$ and $q'_{45} = \gamma'_3 / \mu_{12}$, then $(\exp Y_3)^* (\exp Y_2)^* (\exp Y_1)^* B$ has the required property. Similarly if $\gamma'_2 \neq 0$, let $Y_3 = q'_{24} X_{e_2+e_4} + q'_{23} X_{e_2+e_3}$ where $q'_{24} = -\gamma'_3 / \gamma'_2 \alpha$ and $q'_{23} = -\gamma'_4 / \gamma'_2 \alpha$, then $(\exp Y_3)^* (\exp Y_2)^* (\exp Y_1)^* B$ has the required property. And if $\gamma'_1 \neq 0$, let $Y_3 = q'_{14} X_{e_1+e_4} + q'_{13} X_{e_1+e_3}$ where $q'_{14} = \gamma'_3 / \gamma'_1 \alpha$ and $q'_{13} = \gamma'_4 / \gamma'_1 \alpha$, then $(\exp Y_3)^* (\exp Y_2)^* (\exp Y_1)^* B$ has the required property.

(b) Now we consider the case $\mu_{12} = \gamma'_2 = \gamma'_1 = 0$.

Let $Y_3 = \tilde{q}_{12} X_{e_1+e_2} + \tilde{q}_{35} X_{e_3+e_5} + \tilde{q}_{45} X_{e_4+e_5}$.

Then $(\exp Y_3)^* (\exp Y_2)^* (\exp Y_1)^* B = \alpha \cdot 1 - \tilde{q}_{12} \alpha (e_{-1} \wedge e_{-2})^* + (e_{-3} \wedge e_{-4})^* - \tilde{q}_{35} \alpha (e_{-3} \wedge e_{-5})^* - \tilde{q}_{45} \alpha (e_{-4} \wedge e_{-5})^* - \tilde{q}_{12} (e_{-1} \wedge e_{-2} \wedge e_{-3} \wedge e_{-4})^* + (\gamma'_4 + \tilde{q}_{12} \tilde{q}_{35}) (e_{-1} \wedge e_{-2} \wedge e_{-3} \wedge e_{-5})^* + (\gamma'_3 + \tilde{q}_{12} \tilde{q}_{45}) (e_{-1} \wedge e_{-2} \wedge e_{-4} \wedge e_{-5})^*$.

Now choose $\tilde{q}_{12} \neq 0$, \tilde{q}_{35} and \tilde{q}_{45} such that $\gamma'_4 + \tilde{q}_{12} \tilde{q}_{35} = 0$ and $\gamma'_3 + \tilde{q}_{12} \tilde{q}_{45} = 0$, so that $(\exp Y_3)^* (\exp Y_2)^* (\exp Y_1)^* B = \alpha \cdot 1 - \tilde{q}_{12} \alpha (e_{-1} \wedge e_{-2})^* + (e_{-3} \wedge e_{-4})^*$

$$-\tilde{q}_{35}\alpha(e_{-3}\wedge e_{-5})^*-\tilde{q}_{45}\alpha(e_{-4}\wedge e_{-5})^*-\tilde{q}_{12}(e_{-1}\wedge e_{-2}\wedge e_{-3}\wedge e_{-4})^*.$$

Let $Y_4=-\tilde{q}_{12}X_{e_1+e_2}$. Then

$$(\exp Y_4)^*(\exp Y_3)^*(\exp Y_2)^*(\exp Y_1)^*B(e_{-1}\wedge e_{-2}\wedge e_{-3}\wedge e_{-4})=-\tilde{q}_{12}+\tilde{q}_{12}\times 1=0$$

and hence

$(\exp Y_4)^*(\exp Y_3)^*(\exp Y_2)^*(\exp Y_1)^*B$ has the required property.

Step 4. Now we may assume B is given by $\alpha\cdot 1+\sum\beta'_{ij}(e_{-i}\wedge e_{-j})^*$. If $\beta'_{ij}=0$ for all (i,j) , B is a linear form on $\mathbf{C}\cdot 1$. We may assume there is (i,j) such that $\beta'_{ij}\neq 0$. Let

$X_1=p'_{ij}X_{-e_i-e_j}$. Then

$$\begin{aligned} (\exp X_1)^*B(1) &= \alpha-p'_{ij}\beta'_{ij} \\ (\exp X_1)^*B(e_{-k}\wedge e_{-l}) &= B(e_{-k}\wedge e_{-l}) \text{ for each } (k,l) \\ (\exp X_1)^*B(e_{-1}\wedge\cdots\wedge\hat{e}_{-k}\wedge\cdots\wedge e_{-5}) &= 0 \text{ for each } k. \end{aligned}$$

Letting $p'_{ij}=\alpha/\beta'_{ij}$, $(\exp X_1)^*B$ can be regarded as a linear form on Λ^2F' .

q.e.d.

Proof of Theorem 2. From the fact 4 we see that every holomorphic vector field on a non-singular hyperplane section V can be extended uniquely to a holomorphic vector field on M . Let $A=\{g\in\text{Aut}(M)\mid g(V)=V\}$. Then the Lie algebra of A can be identified with the Lie algebra of all holomorphic vector fields on V . Take the hyperplane H of $P(N_+)$ such that $V=M\cap H$ and let $A'=\{g\in G\mid gH=H\}$. A hyperplane H is the zero locus of non-zero linear form B on N_+ and thus the Lie algebra \mathfrak{a} of A' is given by $\mathfrak{a}(B)=\{X\in\mathfrak{so}(10,\mathbf{C})\mid B(X\cdot n)=c(X)B(n), n\in N_+ \text{ for some } c(X)\in\mathbf{C}\}$. Note also that if linear forms B and B' on N_+ are G -equivalent the Lie algebras $\mathfrak{a}(B)$ and $\mathfrak{a}(B')$ are isomorphic. Therefore by Lemma we may assume that B is a linear form on $\mathbf{C}\cdot 1$ or a linear form on Λ^2F' . If $B=\alpha\cdot 1$ ($\alpha\neq 0$) we can see the variety $M\cap H$ has a singular point (see Appendix). Thus we may assume B is a linear form on Λ^2F' . Now we can take a basis $\{e_{-1}, e_{-2}, e_{-3}, e_{-4}, e_{-5}\}$ of F' such that $B=(e_{-1}\wedge e_{-2})^*+(e_{-3}\wedge e_{-4})^*$ or $B=(e_{-1}\wedge e_{-2})^*$. We claim if $M\cap H$ is non-singular $B=(e_{-1}\wedge e_{-2})^*+(e_{-3}\wedge e_{-4})^*$. Since a generic hyperplane section of M is non-singular, it is sufficient to see that if $B=(e_{-1}\wedge e_{-2})^*$, $M\cap H$ has a singular point. Let $X=X_{-e_1-e_2}$ and $Y=X_{e_1+e_2}$. Then $(\exp Y)^*(\exp X)^*B=1$, and thus B is G -equivalent to a linear form on $\mathbf{C}\cdot 1$. Hence, $M\cap H$ has a singularity.

Now we shall compute the Lie algebra $\mathfrak{a}(B)$ for $B=(e_{-1}\wedge e_{-2})^*+(e_{-3}\wedge e_{-4})^*$. We may write an element X of $\mathfrak{g}=\mathfrak{so}(10,\mathbf{C})$ as

$$\begin{aligned} X &= \sum_{i<j} a_{ij} X_{e_i-e_j} + \sum_{i<j} b_{ij} X_{-e_i+e_j} + \sum_{i<j} c_{ij} X_{e_i+e_j} + \sum_{i<j} d_{ij} X_{-e_i-e_j} \\ &\quad + \sum_i l_i H_i. \end{aligned}$$

Since $B(1)=0, B(X\cdot 1)=B(\sum_{i<j} d_{ij} e_{-i}\wedge e_{-j})=d_{12}+d_{34}=0$. Since $B(e_{-1}\wedge\cdots\wedge\hat{e}_{-k}\wedge\cdots\wedge e_{-5})=0$, we see that

$$\begin{aligned}
B(X \cdot e_{-1} \wedge e_{-2} \wedge e_{-3} \wedge e_{-4}) &= -c_{12} - c_{34} = 0 \\
B(X \cdot e_{-1} \wedge e_{-2} \wedge e_{-3} \wedge e_{-5}) &= -c_{35} = 0 \\
B(X \cdot e_{-1} \wedge e_{-2} \wedge e_{-4} \wedge e_{-5}) &= -c_{45} = 0 \\
B(X \cdot e_{-1} \wedge e_{-3} \wedge e_{-4} \wedge e_{-5}) &= -c_{15} = 0 \\
B(X \cdot e_{-2} \wedge e_{-3} \wedge e_{-4} \wedge e_{-5}) &= -c_{25} = 0.
\end{aligned}$$

Moreover

$$\begin{aligned}
B(X \cdot e_{-1} \wedge e_{-2}) &= \left(\frac{1}{2} (l_1 + l_2 + l_3 + l_4 + l_5) - (l_1 + l_2) \right) = c(X) \\
B(X \cdot e_{-3} \wedge e_{-4}) &= \left(\frac{1}{2} (l_1 + l_2 + l_3 + l_4 + l_5) - (l_3 + l_4) \right) = c(X) \\
B(X \cdot e_{-1} \wedge e_{-3}) &= a_{14} + a_{23} = 0, \quad B(X \cdot e_{-1} \wedge e_{-4}) = -a_{13} + b_{24} = 0, \\
B(X \cdot e_{-1} \wedge e_{-5}) &= b_{25} = 0, \quad B(X \cdot e_{-2} \wedge e_{-3}) = a_{24} - b_{13} = 0, \\
B(X \cdot e_{-2} \wedge e_{-4}) &= -a_{23} - b_{14} = 0, \quad B(X \cdot e_{-2} \wedge e_{-5}) = -b_{15} = 0, \\
B(X \cdot e_{-3} \wedge e_{-5}) &= b_{45} = 0, \quad B(X \cdot e_{-4} \wedge e_{-5}) = -b_{35} = 0.
\end{aligned}$$

Thus the Lie algebra $\mathfrak{a}(B)$ is given by

$$\left\{ \begin{array}{c|c} \begin{array}{cccccc} l_1 & a_{12} & a_{13} & a_{14} & a_{15} & \vdots \\ -b_{12} & l_2 & a_{23} & a_{24} & a_{25} & \vdots \\ -b_{13} - b_{23} & l_3 & a_{34} & a_{35} & & \vdots \\ -b_{14} - b_{24} - b_{34} & l_4 & a_{45} & & & \vdots \\ 0 & 0 & 0 & 0 & l_5 & \vdots \end{array} & \begin{array}{cccccc} 0 & c_{14} & c_{13} & c_{12} & 0 & \vdots \\ 0 & c_{24} & c_{23} & 0 & -c_{12} & \vdots \\ 0 & c_{34} & 0 & -c_{23} - c_{13} & & \vdots \\ 0 & 0 & -c_{34} - c_{24} - c_{14} & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & \vdots \end{array} \\ \hline \begin{array}{cccccc} -d_{15} - d_{25} - d_{35} - d_{45} & 0 & & & & \vdots \\ -d_{14} - d_{24} - d_{34} & 0 & d_{45} & & & \vdots \\ -d_{13} - d_{23} & 0 & d_{34} & d_{35} & & \vdots \\ -d_{12} & 0 & d_{23} & d_{24} & d_{25} & \vdots \\ 0 & d_{12} & d_{13} & d_{14} & d_{15} & \vdots \end{array} & \begin{array}{cccccc} -l_5 - a_{45} - a_{35} - a_{25} - a_{15} & & & & & \vdots \\ 0 & -l_4 & -a_{34} - a_{24} - a_{14} & & & \vdots \\ 0 & -b_{34} & l_3 & -a_{23} - a_{13} & & \vdots \\ 0 & b_{24} & b_{23} - l_2 & -a_{12} & & \vdots \\ 0 & b_{14} & b_{13} & b_{12} - l_1 & & \vdots \end{array} \end{array} \right\} \begin{array}{l} l_1 + l_2 = l_3 + l_4 \\ a_{14} + b_{23} = 0 \\ -a_{13} + b_{24} = 0 \\ a_{24} - b_{13} = 0 \\ a_{23} + b_{14} = 0 \\ d_{12} + d_{34} = 0 \\ c_{12} + c_{34} = 0 \end{array}$$

and, in particular, $\dim \mathfrak{a}(B) = 30$. Let

$$\mathfrak{n} = \left\{ X \in \mathfrak{a}(B) \mid X = \begin{pmatrix} & & \alpha_1 & & & \\ & 0 & \vdots & & 0 & \\ & & \alpha_5 & & & \\ \hline -\beta_1 \cdots -\beta_4 & 0 & & & -\alpha_5 \cdots -\alpha_1 & \\ & & \beta_4 & & & \\ & 0 & \vdots & & 0 & \\ & & \beta_1 & & & \end{pmatrix} \right\}$$

Then \mathfrak{n} is a solvable ideal of $\mathfrak{a}(B)$ such that $[\mathfrak{n}, \mathfrak{n}] \neq (0)$ and $[[\mathfrak{n}, \mathfrak{n}], [\mathfrak{n}, \mathfrak{n}]] = (0)$.

Therefore $\mathfrak{a}(B)$ is not a reductive Lie algebra. By a theorem of Lichnerowicz-Matsushima [6], we see that the hyperplane section V does not admit any Kähler metric with constant scalar curvature.

Now by the same argument as in the proof of Theorem 1, we get

$$\begin{aligned} \dim H^1(V, T(V)) &= h^1(V, T(V)) \\ &= h^0(V, T(V)) - h^0(M, T(M)) + h^0(V, \{V\} | V) \\ &= \dim \mathfrak{a}(B) - \dim \mathfrak{so}(10, \mathbb{C}) + (16 - 1) \\ &= 30 - 45 + 15 = 0. \end{aligned}$$

q.e.d.

Appendix

Let M be an Hermitian symmetric space of compact type and L a very ample holomorphic line bundle on M . Let $j_L: M \rightarrow P^N(\mathbb{C})$ be the imbedding associated to L . Then it is known that the homogeneous ideal of M is generated by quadrics [7]. We shall determine these quadrics in the case when $M = SO(10)/U(5)$ and the imbedding is canonical. Denote by o the point in $P(N_+)$ corresponding to $U(5)$ of M . Let $\mathfrak{m}_- = \sum_{i < j} \mathfrak{g}_{-\varepsilon_i - \varepsilon_j}$ be an abelian subalgebra of $\mathfrak{g} = \mathfrak{so}(10, \mathbb{C})$ and M_- the Lie subgroup corresponding to \mathfrak{m}_- . Fix a basis $\{e_{-1}, e_{-2}, e_{-3}, e_{-4}, e_{-5}\}$ of F' . Then

$$\{1, e_{-i} \wedge e_{-j}, e_{-i_1} \wedge e_{-i_2} \wedge e_{-i_3} \wedge e_{-i_4} \mid i < j, i_1 < i_2 < i_3 < i_4\}$$

is a basis of N_+ . We also denote by $\{x_\lambda\}$ the dual basis of N_+^* . Now consider the orbit $M_- \cdot o = j(\exp \mathfrak{m}_- \cdot U) = [\rho(\exp \mathfrak{m}_-) v_0]$. We may write an element Y of \mathfrak{m}_- as

$$Y = \sum_{i < j} \xi_{-\varepsilon_i - \varepsilon_j} X_{-\varepsilon_i - \varepsilon_j}.$$

Note that the highest vector v_0 is given by $1 \in N_+$ in our case. Then

$$\begin{aligned} \rho(\exp Y) \cdot 1 \\ = 1 + \sum \xi_{-\varepsilon_i - \varepsilon_j} X_{-\varepsilon_i - \varepsilon_j} \cdot 1 + \frac{1}{2} \sum \xi_{-\varepsilon_i - \varepsilon_j} \xi_{-\varepsilon_k - \varepsilon_l} X_{-\varepsilon_i - \varepsilon_j} X_{-\varepsilon_k - \varepsilon_l} \cdot 1. \end{aligned}$$

For simplicity we denote the highest weight $\Lambda_{\mathfrak{m}_-}$ by Λ . Now we get

$$\begin{aligned} x_\Lambda(\rho(\exp Y) \cdot c \cdot 1) &= c \\ x_{\Lambda - \varepsilon_i - \varepsilon_j}(\rho(\exp Y) \cdot c \cdot 1) &= c \xi_{-\varepsilon_i - \varepsilon_j} \\ x_{\Lambda - \varepsilon_i - \varepsilon_j - \varepsilon_k - \varepsilon_l}(\rho(\exp Y) \cdot c \cdot 1) \\ &= c(\xi_{-\varepsilon_i - \varepsilon_j} \xi_{-\varepsilon_k - \varepsilon_l} - \xi_{-\varepsilon_i - \varepsilon_k} \xi_{-\varepsilon_j - \varepsilon_l} + \xi_{-\varepsilon_i - \varepsilon_l} \xi_{-\varepsilon_j - \varepsilon_k}) \end{aligned}$$

where $i < j < k < l$. Thus we see on $M_- \cdot o$

$$\begin{aligned} & \mathcal{X}_\Lambda \mathcal{X}_{\Lambda-(\varepsilon_i+\varepsilon_j+\varepsilon_k+\varepsilon_l)} - \mathcal{X}_{\Lambda-(\varepsilon_i+\varepsilon_j)} \mathcal{X}_{\Lambda-(\varepsilon_k+\varepsilon_l)} \\ & + \mathcal{X}_{\Lambda-(\varepsilon_i+\varepsilon_k)} \mathcal{X}_{\Lambda-(\varepsilon_j+\varepsilon_l)} - \mathcal{X}_{\Lambda-(\varepsilon_i+\varepsilon_l)} \mathcal{X}_{\Lambda-(\varepsilon_j+\varepsilon_k)} = 0 \end{aligned}$$

for $i < j < k < l$.

Since the Zariski closure $\overline{M_- \cdot o}$ of $M_- \cdot o$ in $P(N_+)$ is M , we see that these quadrics vanish on M .

Let $I(M)$ be the homogeneous ideal of M , $S^2(N_+^*)$ the vector space of homogeneous polynomials of degree 2 on N_+ and I_2 the subspace of degree 2 of the ideal $I(M)$. Then $I(M)$, $S^2(N_+^*)$ and I_2 are $\mathfrak{so}(10, \mathbf{C})$ -modules. Now the decomposition of $S^2(N_+^*)$ as $\mathfrak{so}(10, \mathbf{C})$ -modules is given by

$$S^2(N_+^*) = V_{2\Lambda_{\alpha_4}} + V_{\Lambda_{\alpha_1}}$$

where $V_{2\Lambda_{\alpha_4}}$ and $V_{\Lambda_{\alpha_1}}$ denotes $\mathfrak{so}(10, \mathbf{C})$ -modules with the highest weights $2\Lambda_{\alpha_4}$ and Λ_{α_1} respectively, and we see $I_2 = V_{\Lambda_{\alpha_1}}$ as $\mathfrak{so}(10, \mathbf{C})$ -module. (Note that $\Lambda_{\alpha_1} = \varepsilon_1$.) In particular, we have $\dim I_2 = 10$. Applying elements of Weyl group of $\mathfrak{so}(10, \mathbf{C})$, it is not difficult to see that the following 10 quadrics constitute a basis of I_2 :

For $1 \leq i < j < k < l \leq 5$,

$$\begin{aligned} & \mathcal{X}_\Lambda \mathcal{X}_{\Lambda-(\varepsilon_i+\varepsilon_j+\varepsilon_k+\varepsilon_l)} - \mathcal{X}_{\Lambda-(\varepsilon_i+\varepsilon_j)} \mathcal{X}_{\Lambda-(\varepsilon_k+\varepsilon_l)} \\ & + \mathcal{X}_{\Lambda-(\varepsilon_i+\varepsilon_k)} \mathcal{X}_{\Lambda-(\varepsilon_j+\varepsilon_l)} - \mathcal{X}_{\Lambda-(\varepsilon_i+\varepsilon_l)} \mathcal{X}_{\Lambda-(\varepsilon_j+\varepsilon_k)}, \\ & \mathcal{X}_{\Lambda-(\varepsilon_4+\varepsilon_5)} \mathcal{X}_{\Lambda-(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)} - \mathcal{X}_{\Lambda-(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5)} \mathcal{X}_{\Lambda-(\varepsilon_3+\varepsilon_4)} \\ & + \mathcal{X}_{\Lambda-(\varepsilon_1+\varepsilon_3+\varepsilon_4+\varepsilon_5)} \mathcal{X}_{\Lambda-(\varepsilon_2+\varepsilon_4)} - \mathcal{X}_{\Lambda-(\varepsilon_1+\varepsilon_4)} \mathcal{X}_{\Lambda-(\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5)}, \\ & \mathcal{X}_{\Lambda-(\varepsilon_4+\varepsilon_5)} \mathcal{X}_{\Lambda-(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_5)} - \mathcal{X}_{\Lambda-(\varepsilon_1+\varepsilon_2+\varepsilon_4+\varepsilon_5)} \mathcal{X}_{\Lambda-(\varepsilon_3+\varepsilon_5)} \\ & + \mathcal{X}_{\Lambda-(\varepsilon_1+\varepsilon_3+\varepsilon_4+\varepsilon_5)} \mathcal{X}_{\Lambda-(\varepsilon_2+\varepsilon_5)} - \mathcal{X}_{\Lambda-(\varepsilon_1+\varepsilon_5)} \mathcal{X}_{\Lambda-(\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5)}, \\ & \mathcal{X}_{\Lambda-(\varepsilon_3+\varepsilon_5)} \mathcal{X}_{\Lambda-(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)} - \mathcal{X}_{\Lambda-(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_5)} \mathcal{X}_{\Lambda-(\varepsilon_3+\varepsilon_4)} \\ & + \mathcal{X}_{\Lambda-(\varepsilon_1+\varepsilon_3)} \mathcal{X}_{\Lambda-(\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5)} - \mathcal{X}_{\Lambda-(\varepsilon_1+\varepsilon_3+\varepsilon_4+\varepsilon_5)} \mathcal{X}_{\Lambda-(\varepsilon_2+\varepsilon_3)}, \\ & \mathcal{X}_{\Lambda-(\varepsilon_2+\varepsilon_5)} \mathcal{X}_{\Lambda-(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)} - \mathcal{X}_{\Lambda-(\varepsilon_1+\varepsilon_2)} \mathcal{X}_{\Lambda-(\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5)} \\ & + \mathcal{X}_{\Lambda-(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_5)} \mathcal{X}_{\Lambda-(\varepsilon_2+\varepsilon_4)} - \mathcal{X}_{\Lambda-(\varepsilon_1+\varepsilon_2+\varepsilon_4+\varepsilon_5)} \mathcal{X}_{\Lambda-(\varepsilon_2+\varepsilon_3)}, \\ & \mathcal{X}_{\Lambda-(\varepsilon_1+\varepsilon_5)} \mathcal{X}_{\Lambda-(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)} - \mathcal{X}_{\Lambda-(\varepsilon_1+\varepsilon_2)} \mathcal{X}_{\Lambda-(\varepsilon_1+\varepsilon_3+\varepsilon_4+\varepsilon_5)} \\ & + \mathcal{X}_{\Lambda-(\varepsilon_1+\varepsilon_3)} \mathcal{X}_{\Lambda-(\varepsilon_1+\varepsilon_2+\varepsilon_4+\varepsilon_5)} - \mathcal{X}_{\Lambda-(\varepsilon_1+\varepsilon_4)} \mathcal{X}_{\Lambda-(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_5)}. \end{aligned}$$

Now if a hyperplane H is given by $B = \alpha \cdot 1$, that is, $\alpha \cdot x_\Lambda = 0$, then the variety $M \cap H$ has a singular point. In fact, if we take a point $p \in P(N_+)$ defined by

$$x_{\Lambda-(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)}(p) \neq 0 \text{ and } x_\lambda(p) = 0 \text{ otherwise,}$$

then $p \in M \cap H$ is a singular point of $M \cap H$, using the fact M is the zero locus of 10 quadrics above.

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