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ON NON-SINGULAR HYPERPLANE SECTIONS OF SOME HERMITIAN SYMMETRIC SPACES

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Let $P^k(\boldsymbol{C})$ denote a complex projective space of dimension k. The product space $P^m(\boldsymbol{C})\times P^n(\boldsymbol{C})$ has a natural imbedding in $P^{mn+m+n}(\boldsymbol{C})$, called the Segre imbedding. Let V be a non-singular hyperplane section of $P^m(\boldsymbol{C})\times P^n(\boldsymbol{C})$ in $P^{mn+m+n}(\boldsymbol{C})$. The identity connected component $\mathrm{Aut}_0(V)$ of the group of all holomorphic automorphisms of V has been determined by *J-I.* Hano [3]. For an irreducible Hermitian symmetric space M of compact type we have the canonical equivariant imbedding $j: M\rightarrow P^N(\boldsymbol{C})$. Now take a non-singular hyperplane section V of M in $P^N(\boldsymbol{C})$. In this note we shall determine the structure of the Lie algebra of $\mathrm{Aut}(V)$ fro the cases when M is a complex Grassmann manifold $G_{m,2}(\boldsymbol{C})$ of 2-planes in $\boldsymbol{C}^m$ and when M is SO(10)/U(5), by applying Hano's method. In particular, using Lichnerowicz-Matsushima's theorem, we prove the following.

1) For the case M is $G_{m,2}(\boldsymbol{C})$ $(m\geq 4)$, if m is odd a non-singular hyperplane section V does not admit any Kähler metric with constant scalar curvature, and if m is even V is a kählerian C-space.

2) For the case M is SO(10)/U(5), V does not admit any Kähler metric with constant scalar curvature.

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1. Preliminaries

A simply connected compact homogeneous complex manifold is called a C-space. A C-space is said to be kählerian if it admits a Kähler metric. We recall some known facts on kählerian C-spaces and holomorphic line bundles on these complex manifolds (cf. [1], [4]).

Fact 1. *Every holomorphic line bundle on a kählerian C-space M is homogeneous. If we denote by $H^1(M, \theta^*)$ the group of all isomorphism classes of holomorphic line bundles on M and by $c_1(F)$ the Chern class of a holomorphic line bundle F, then the homomorphism $F\rightarrow c_1(F)$: $H^1(M, \theta^*)\rightarrow H^2(M, \boldsymbol{Z})$ is bijective.*

Fact 2. *Every ample holomorphic line bundle on a kählerian C-space M is* *very ample. Moreover for each very ample holomorphic line bundle the corresponding holomorphic imbedding of M can be realized as an orbit space of the irreducible representation of all holomorphic automorphism group $Aut(M)$ of M.*

From now on we assume that M is a kählerian C-space with the second Betti number $b_2(M)=1$. In this case there is a unique very ample holomorphic line bundle L on M which is a generator of the group $H^1(M, \theta^*)$. The corresponding holomorphic imbedding for L is called the *canonical* imbedding of M and denoted by j: $M \rightarrow P^N(\boldsymbol{C})$. Let $h=c_1(L)$. Then h is a generator of $H^2(M, \boldsymbol{Z})$. For a divisor D on M let $\{D\}$ be the holomorphic line bundle on Massociated to D. Then for a positive divisor D on M there is a positive integer $a(D)$ such that $c_1(\{D\})=a(D)h$. The integer $a(D)$ is called the *degree* of D.

Fact 3. *Let j: $M \rightarrow P^N(\boldsymbol{C})$ be the canonical imbedding of a kählerian C-space M with $b_2(M)=1$. Then for each positive divisor D on M of degree a there exists a homogeneous polynomial F on $\boldsymbol{C}^{N+1}$ of degree a such that D is the pull back of the divisor on $P^N(\boldsymbol{C})$ defined by the zero points of F by the canonical imbedding j.*

For a non-singular hypersurface V of M the degree of the positive divisor defined by V is called the *degree* of V. Let $K(V)$ and $K(M)$ denote the canonical line bundles on V and M respectively. It is known that the first Chern class $c_1(M)$ of M is given by $c_1(M)=\kappa h$ for some positive integer κ. Since $K(V)=\iota^*(K(M)\otimes\{V\})$ where ι: $V \rightarrow M$ is inclusion, the first Chern class $c_1(V)$ of V is given by $c_1(V)=(\kappa-a)\ \iota^*h$ if the degree of V is a. In particular, if V is a non-singular hypersurface of degree $a<\kappa$, the first Chern class $c_1(V)$ of V is positive. It is also known that irreducible Hermitian symmetric spaces of compact type are kählerian C-spaces with the second Betti number 1 and the positive number $\kappa \geq 2$. Therefore if V is a non-singular hyperplane section of an irreducible Hermitian symmetric space M of compact type for the canonical imbedding j: $M \rightarrow P^N(\boldsymbol{C})$, the first Chern class $c_1(V)$ of V is positive.

Let $T(M)$ and $T(V)$ be the holomorphic tangent bundles of M and V respectively. Given a holomorphic vector bundle E, we denote by $\Omega^0(E)$ the sheaf of germs of local holomorphic sections of E.

Fact 4 (Kimura [5]). *Let M be an irreducible Hermitian symmetric space of compact type. Assume that M is not a complex projective space $P^n(\boldsymbol{C})$ or a complex quadric $Q^n(\boldsymbol{C})$. Then for a non-singular hypersurface V of M the exact sequence of sheaves on M*

$$0 \rightarrow \Omega^0(T(M)\otimes\{V\}^{-1}) \rightarrow \Omega^0(T(M)) \rightarrow \Omega^0(T(M)\,|\,V) \rightarrow 0$$

induces the exact sequence of cohomologies

$$0 \rightarrow H^0(M,\ T(M)) \rightarrow H^0(V,\ T(M)\,|\,V) \rightarrow 0$$

Moreover $H^1(V, T(M)|V)=(0)$.

REMARK. If V is a non-singular hypersurface $Q^n(\boldsymbol{C})(n>3)$ of degree $a\neq 2$, the same result as in Fact 4 holds.

2. The case M is a complex Grassmann manifold $G_{m,2}(\boldsymbol{C})$

Let ρ be the natural representation of $SL(m, \boldsymbol{C})$ on $\boldsymbol{C}^m$ and consider the p-th exterior representation $\Lambda^p\rho$: $SL(m, \boldsymbol{C})\to GL(\Lambda^p\boldsymbol{C}^m)$ induced by ρ. Note that $\Lambda^p\rho$ is an irreducible representation of $SL(m, \boldsymbol{C})$. Fix a highest weight vector $v_0\in\Lambda^p\boldsymbol{C}^m$ and consider the subgroup U of $SL(m, \boldsymbol{C})$ defined by

$$\{h\in SL(m, \boldsymbol{C})\,|\,(\Lambda^p\rho)(h)\,v_0 = cv_0 \text{ for some } c\in\boldsymbol{C}-(0)\}\ .$$

Then the map j: $SL(m, \boldsymbol{C})/U\to P(\Lambda^p\boldsymbol{C}^m)$ defined by

$$j(gU) = [\Lambda^p\rho(g)\,(v_0)] \quad \text{for } g\in SL(m, \boldsymbol{C})\,,$$

where $[w]$ ($w\in\Lambda^p\boldsymbol{C}^m$) denotes the line determined by w, is the canonical imbedding of the Grassmann manifold $M=G_{m,p}(\boldsymbol{C})$ and is called the Plücker imbedding of M.

From now on we assume that M is a complex Grassmann manifold of 2-planes in $\boldsymbol{C}^m$ which is not a complex projective space, so we may assume $m\geq 4$. We may also regard M as a non-singular projective subvariety of $P(\Lambda^2\boldsymbol{C}^m)$ by the canonical imbedding.

Theorem 1. *For an integer $m\geq 4$ let V be a non-singular hyperplane section of $G_{m,2}(\boldsymbol{C})$ in $P(\Lambda^2\boldsymbol{C}^m)$.*

(1) *If m is even, V is a kählerian C-space $Sp(n, \boldsymbol{C})/P$ with the second Betti number* 1 *where $n=m/2$ and P is a parabolic subgroup of $Sp(n, \boldsymbol{C})$.*

(2) *If m is odd, the group $Aut(V)$ of all holomorphic transformations of V is not reductive and thus V does not admit any Kähler metric with constant scalar curvature. Moreover we have $H^1(V, T(V))=(0)$.*

Proof. By the Lefschetz theorem on hyperplane sections, we have $b_2(V)=1$ since $b_2(G_{m,2}(\boldsymbol{C}))=1$. From the fact 4 we see that every holomorphic vector field on V can be extended uniquely to a holomorphic vector field on M. Let $A=\{g\in \mathrm{Aut}(M)\,|\,g(V)=V\}$. Then the Lie algebra $\mathfrak{a}$ of A can be identified with the Lie algebra of all holomorphic vector fields on V. By means of irreducible representation $\Lambda^2\rho$: $SL(m, \boldsymbol{C})\to GL(\Lambda^2\boldsymbol{C}^m)$ each element of $SL(m, \boldsymbol{C})$ maps a hyperplane of $P(\Lambda^2\boldsymbol{C}^m)$ to another hyperplane. Take a hyperplane H of $P(\Lambda^2\boldsymbol{C}^m)$ such that $V=H\cap M$. Note that such a hyperplane H in $P(\Lambda^2\boldsymbol{C}^m)$ is determined uniquely since the canonical imbedding j: $M\to P(\Lambda^2\boldsymbol{C}^m)$ is full. Thus the Lie algebra $\mathfrak{a}$ of A coincides with the Lie algebra of $A'=\{g\in SL(m, \boldsymbol{C})\,|\,g\cdot H=H\}$. A hyperplane H is the zero locus of non-zero linear form B on $\Lambda^2\boldsymbol{C}^m$. If we let

$$b(z, w) = B(z \wedge w) \quad (z, w \in \boldsymbol{C}^m),$$

b is a skew-symmetric form on $\boldsymbol{C}^m$. Therefore

$$A' = \{g \in SL(m, \boldsymbol{C}) \mid b(g \cdot z, g \cdot w) = \lambda(g)\, b(z, w),\ z, w \in \boldsymbol{C}^m$$

for some non-zero constant $\lambda(g) \in \boldsymbol{C}\}$.

Now we choose coordinates on $\boldsymbol{C}^m$ in such a way as

$$b(z, w) = \sum_{i=1}^{k} (z_i\, w_{k+i} - z_{k+i}\, w_i) \quad \text{where } 1 \leq k \leq [m/2]$$

(that is, if $p_{\alpha\beta}$ denote Plücker coordinates, the hyperplane H is defined by $p_{1k+1} + \cdots + p_{k2k}) = 0$.

We claim that $k = [m/2]$ if V is non-singular. Suppose that $k < [m/2]$. Then $2k \leq m-2$. We can take vectors $z, w \in \boldsymbol{C}^m$ given by

$$z_1 = \cdots = z_{2k} = 0,\ z_{2k+1} = 1,\ z_{2k+2} = \cdots = z_m = 0,$$

$$w_1 = \cdots = w_{2k+1} = 0,\ w_{2k+2} = 1,\ w_{2k+3} = \cdots = w_m = 0,$$

respectively. The $z \wedge w$ determines a point of V which is singular, since

$$db = \sum_{j=1}^{k} (w_{k+i}\, dz_i + z_i\, dw_{k+i} - w_i\, dz_{k+i} - z_{k+i}\, dw_i)$$

vanishes at this point. Hence $k = [m/2]$.

Now we consider the cases where m is even or odd separately.

Case 1 $m = 2n$

In this case the Lie algebra $\mathfrak{a}$ is given by the Lie algebra of

$$\left\{ g \in SL(2n, \boldsymbol{C}) \mid {}^t g \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} g = \lambda(g) \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \right\}$$

where 1_n denotes $n \times n$ identity matrix. We may write an $m \times m$ matrix X in the form

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A, B, C and D are $n \times n$ matrices. Thus we see that $X \in \mathfrak{a}$ if and only if $C = {}^tC$, $B = {}^tB$ and ${}^tA + D = \mu(X)\, 1_n$ for some $\mu(X) \in \boldsymbol{C}$. Since $tr(X) = 0$, we have $\mu(X) = 0$ and hence $X \in \mathfrak{a}$ if and only if

$$X \in \mathfrak{sp}(n, \boldsymbol{C}) = \left\{ X \mid {}^tX \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} X = 0 \right\}.$$

Therefore we may identify the connected component of the identity of A' with $Sp(n, \boldsymbol{C})$. Take two vectors $e_1={}^t(1, 0, \cdots, 0)$ and $e_2={}^t(0, 1, 0, \cdots, 0)$ of $\boldsymbol{C}^m$. The $e_1\wedge e_2$ determines a point x_0 of V (that is, in the Plücker coordinates, x_0 is given by $P_{12}\neq 0$ and $p_{\alpha\beta}=0$ otherwise). Let P be the isotropy subgroup at x_0. Then it is not difficult to see that P is a parabolic subgroup of $Sp(n, \boldsymbol{C})$. Since dim $Sp(n, \boldsymbol{C})/P=2(2n-2)-1$, dim $V=2(2n-2)-1$ and V is compact, we see $V=Sp(n, \boldsymbol{C})/P$.

Case 2 $m=2n+1$

We may write a $(2n+1)\times(2n+1)$ matrix X in the form

$$X=\begin{pmatrix} A & \alpha \\ \beta & \gamma \end{pmatrix}$$

where A is a $2n\times 2n$ matrix. Then $X\in\mathfrak{a}$ if and only if $\alpha=0$ and

$${}^tA\begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}+\begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}A=\mu(X)\begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$$

for some $\mu(X)\in\boldsymbol{C}$. Thus we get

$$\mathfrak{a}=\left\{\begin{pmatrix} A & 0 \\ \beta & \gamma \end{pmatrix}\in\mathfrak{gl}(2n+1, \boldsymbol{C})\;\middle|\; A=\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}\quad {}^tX_2=X_2\right.$$
$$\left. {}^tX_3=X_3,\ X_1+{}^tX_4=-(\gamma/n)1_n,\ {}^t\beta\in\boldsymbol{C}^{2n},\ \gamma\in\boldsymbol{C}\right\}$$

and dim $\mathfrak{a}=2n^2+3n+1$. Let

$\mathfrak{n}=\left\{\begin{pmatrix} 0 & 0 \\ \beta & 0 \end{pmatrix} \middle| {}^t\beta\in\boldsymbol{C}^{2n}\right\}$. Then $\mathfrak{n}$ is an abelian ideal of $\mathfrak{a}$. On the other hand the center $\mathfrak{z}$ of $\mathfrak{a}$ is given by $\{a\,1_{2n+1}\,|\,a\in\boldsymbol{C}\}$. Since $\mathfrak{n}\cap\mathfrak{z}=(0)$, $\mathfrak{a}$ is not reductive. By a theorem of Lichnerowicz-Matsushima [76], we see that V does not admit any Kähler metric with constant scalar curvature.

Now the exact sequence of sheaves

$$0\to\Omega^0(T(V))\to\Omega^0(T(M)\,|\,V)\to\Omega^0(\{V\}\,|\,V)\to 0$$

induces the exact sequence of cohomologies

$$\begin{aligned}0&\to H^0(V,T(V))\to H^0(V, T(M)\,|\,V)\to H^0(V,\{V\}\,|\,V)\\ &\to H^1(V, T(V))\to H^1(V, T(M)\,|\,V)\to\cdots.\end{aligned}$$

Since $H^1(V, T(M)\,|\,V)=(0)$, $H^0(V, T(M)\,|\,V)\cong H^0(M, T(M))$ by the fact 4 and $h^0(V,\{V\}\,|\,V)=h^0(M,\{V\})-1$, we get

$$\begin{aligned} h^1(V, T(V)) &= h^0(V, T(V)) - h^0(M, T(M)) + h^0(V, \{V\} \mid V) \\ &= 2n^2+3n+1-((2n+1)^2-1)+\binom{2n+1}{2}-1 = 0 \end{aligned}$$

q.e.d.

3. The case M is $SO(10)/U(5)$

Let M be an irreducible Hermitian symmetric space of compact type of type DIII. It is known that M is diffeomorphic to $SO(2n)/U(n)$ $(n \geq 4)$. Note that M is a complex quadric $Q^6(\boldsymbol{C})$ if $n=4$.

Consider a semi-spin representation of the complex simple Lie algebra $\mathfrak{g}$ of type D_n and the corresponding representation ρ of the simply connected complex Lie Group G with the Lie algebra $\mathfrak{g}$. Fix a highest weight vector v_0 and let U be the subgroup of G defined by $\{g \in G \mid \rho(g)\, v_0 = cv_0$ form some $c \in \boldsymbol{C}-(0)\}$. Then a map

$$j\colon G/U \to P(\boldsymbol{C}^{2^{n-1}})$$

defined by $j(gU) = [\rho(g)\, v_0]$ for $g \in G$, is the canonical imbedding of $M=G/U$.

We recall semi-spin representations of type D_n (cf. [2], chap. VIII, §13), so that we can fix our notations. Let W be a $2n$-dimensional complex vector space and Φ a non-degenerate symmetric bilinear form on W. Then W is a direct sum of maximal totally isotropic subspaces F and F' of W; $W=F \oplus F'$. Let $\{e_1, \cdots, e_n, e_{-n}, \cdots, e_{-1}\}$ be a Witt basis of W, that is, $\{e_1, \cdots, e_n\}$ and $\{e_{-n}, \cdots, e_{-1}\}$ are bases of F and F' respectively which satisfy the relation $\Phi(e_i, e_{-j}) = \delta_{ij}$ for $i, j = 1, \cdots, n$. The corresponding matrix of Φ with respect to a Witt basis is given as

$$\begin{pmatrix} 0 & s \\ s & 0 \end{pmatrix} \text{ where } s = \begin{pmatrix} 0 & & 1 \\ & 1 & \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}$$

and the Lie algebra $\mathfrak{g}$ can be given by

$$\left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \middle| B = -s\,{}^tBs,\ C = -s\,{}^tCs,\ D = -s\,{}^tAs \right\}.$$

Let $E_{p,q}$ be a matrix unit, that is, the (k, l)-component of $E_{p,q}$ is given by $\delta_{kp}\,\delta_{lq}$. Put $\mathfrak{h} = \{X \in \mathfrak{g} \mid X$ is a diagonal matrix$\}$ and $H_i = E_{i,i} - E_{-i,-i}$ for $i=1, \cdots, n$. Then $\{H_1, \cdots, H_n\}$ is a basis of $\mathfrak{h}$. Let $\{\varepsilon_1, \cdots, \varepsilon_n\}$ be the dual basis of the dual space $\mathfrak{h}^*$.

Put

$$X_{\varepsilon_i-\varepsilon_j}=E_{i,j}-E_{-j,-i}$$
$$X_{-\varepsilon_i+\varepsilon_j}=-E_{j,i}+E_{-i,-j}$$
$$X_{\varepsilon_i+\varepsilon_j}=E_{i,-j}-E_{j,-i}$$
$$X_{-\varepsilon_i-\varepsilon_j}=-E_{-j,i}+E_{-i,j}$$
$$\text{for } 1\leq i<j\leq n\,.$$

Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and the root system Σ of $\mathfrak{g}$ relative to $\mathfrak{h}$ is given by $\Sigma=\{\pm\varepsilon_i\pm\varepsilon_j\,|\,1\leq i<j\leq n\}$. Let $\alpha_1=\varepsilon_1-\varepsilon_2$, $\alpha_2=\varepsilon_2-\varepsilon_3$, $\cdots$, $\alpha_{n-1}=\varepsilon_{n-1}-\varepsilon_n$, $\alpha_n=\varepsilon_{n-1}+\varepsilon_n$. Then $\{\alpha_1,\cdots,\alpha_n\}$ is a fundamental root system Π of Σ and the fundamental weights corresponding Π to are

$$\Lambda_{\alpha_i}=\varepsilon_1+\cdots+\varepsilon_i\quad(1\leq i\leq n-2)$$
$$\Lambda_{\alpha_{n-1}}=\frac{1}{2}(\varepsilon_1+\cdots+\varepsilon_{n-2}+\varepsilon_{n-1}-\varepsilon_n)$$
$$\Lambda_{\alpha_n}=\frac{1}{2}(\varepsilon_1+\cdots+\varepsilon_{n-2}+\varepsilon_{n-1}+\varepsilon_n)$$

Now semi-spin representations are irreducible representations of $\mathfrak{g}$ with the highest weight $\Lambda_{\alpha_{n-1}}$ and Λ_{α_n} respectively.

Let Q be the quadric form defined by $x\rightarrow\Phi(x,x)/2$ and let $C(Q)$ denote the Clifford alegbra of W relative to Q. Let N be the exterior algebra of the maximal totally isotropic subspace F'. We shall identify F and the dual of F' via Φ. For $x\in F'$ and $y\in F$ let $\lambda(x)$ and $\lambda(y)$ denote the left exterior product by x and left interior product by y in N respectively; so that for $x\in F'$ and $y\in F$

$$\lambda(x)\,a_1\wedge\cdots\wedge a_k=x\wedge a_1\wedge\cdots\wedge a_k$$
$$\lambda(y)\,a_1\wedge\cdots\wedge a_k=\sum_{i=1}^{k}(-1)^{i-1}\,\Phi(a_i,y)\,a_1\wedge\cdots\wedge\hat{a}_i\wedge\cdots\wedge a_k$$

where $a_1,\cdots,a_k\in F'$.

Then we get that $\lambda(x)^2=\lambda(y)^2$ and $\lambda(x)\,\lambda(y)+\lambda(y)\,\lambda(x)=\Phi(x,y)\,1$, and there exist a unique homomorphism of $C(Q)$ into $\mathrm{End}(N)$, denoted also by λ, which is a prolongation of the map $\lambda\colon F\cup F'\rightarrow\mathrm{End}\,(N)$. Let $C^+(Q)$ denote the subalgebra of $C(Q)$ spanned by even elements and put

$$N_+=\sum_{p\,:\,\mathrm{even}}\Lambda^pF',\ N_-=\sum_{p\,:\,\mathrm{odd}}\Lambda^pF'\,.$$

Now N_+ and N_- are stable for the restriction of λ to $C^+(Q)$, and the representations λ_+ and λ_- of $C^+(Q)$ in N_+ and N_- respectively are called semi-spin representations of $C^+(Q)$. These are simple $C^+(Q)$-modules. There also exists a canonical linear map $f\colon\mathfrak{g}\rightarrow C^+(Q)$ which satisfies $[f(X),f(Y)]=f([X,Y])$ for X

and Y in $\mathfrak{g}$ and $f(\mathfrak{g})$ generates the associative algebra $C^+(Q)$. Furthermore if N is a left $C^+(Q)$-module and ρ is the corresponding homomorphism of $C^+(Q)$ into $\mathrm{End}(N)$, then $\rho\circ f$ is a representation of $\mathfrak{g}$ in N (cf. [2], p. 195, Lemma 1). Thus $\rho_+=\lambda_+\circ f$ and $\rho_-=\lambda_-\circ f$ are irreducible representations of $\mathfrak{g}$. In particular, the action of $\mathfrak{g}$ on N is given as follows:

$$X_{\varepsilon_i-\varepsilon_j}(e_{-i_1}\wedge\cdots\wedge e_{-i_k})=\lambda(e_i)\,(e_{-j}\wedge e_{-i_1}\wedge\cdots\wedge e_{-i_k})$$

$$X_{-\varepsilon_i+\varepsilon_j}(e_{-i_1}\wedge\cdots\wedge e_{-i_k})=-\lambda(e_j)\,(e_{-i}\wedge e_{-i_1}\wedge\cdots\wedge e_{-i_k})$$

$$X_{-\varepsilon_i-\varepsilon_j}(e_{-i_1}\wedge\cdots\wedge e_{-i_k})=e_{-i}\wedge e_{-j}\wedge e_{-i_1}\wedge\cdots\wedge e_{-i_k}$$

$$X_{\varepsilon_i+\varepsilon_j}(e_{-i_1}\wedge\cdots\wedge e_{-i_k})=\lambda(e_i)\,\lambda(e_j)\,(e_{-i_1}\wedge\cdots\wedge e_{-i_k})$$

where $1\leq i<j\leq n$ and

$$H(e_{-i_1}\wedge\cdots\wedge e_{-i_k})$$
$$=\left(\frac{1}{2}\,(\varepsilon_1+\cdots+\varepsilon_n)-(\varepsilon_{i_1}+\cdots+\varepsilon_{i_k})\right)(H)\,(e_{-i_1}\wedge\cdots\wedge e_{-i_k})$$

for $H\in\mathfrak{h}$. Particularly we see that the highest weights of ρ_+ and ρ_- are Λ_{α_n} and $\Lambda_{\alpha_{n-1}}$ respectively. The representation ρ_- is the contragradient representation of ρ_+.

From now on we consider the case $n=5$ exclusively.

Theorem 2. *Let V be a non-singular hyperplane section of $M^{10}=SO(10)/U(5)$ in $P^{15}(\boldsymbol{C})$ via the canonical imbedding. Then the group $\mathrm{Aut}_0(V)$ is not reductive and thus V does not admit any Kähler metric with constant scalar curvature. Moreover $H^1(V,T(V))=(0)$.*

In order to prove Theorem 2 we shall first classify the hyperplanes of N_+ by means of the action of the Lie group G. For a linear form B: $N_+\to\boldsymbol{C}$ and $g\in G$ let g^*A denote the linear form defined by $(g^*A)\,(n)=A(g\cdot n)$ for $n\in N_+$. Now linear forms B and B_1 are called G-equivalent if there is an element $g\in G$ such that $B_1=g^*B$.

Lemma. *Let $B: N_+=\boldsymbol{C}\cdot 1+\Lambda^2F'+\Lambda^4F'\to\boldsymbol{C}$ be a linear form. Then B is G-equivalent to either a linear form on $\boldsymbol{C}\cdot 1$ or a linear form on Λ^2F'.*

Proof. We may assume $B\neq 0$. Take a basis $\{e_{-1},\cdots,e_{-5}\}$ of F' and fix it. A basis of N_+ is now given by $\{1,\, e_{-i}\wedge e_{-j},\, e_{-1}\wedge\cdots\wedge\hat{e}_{-k}\wedge\cdots\wedge e_{-5}\,|\,1\leq i<j\leq 5,\, k=1,\cdots,5\}$ and the corresponding dual basis of $(N_+)^*$ will be denoted by

$$\{1,\,(e_{-i}\wedge e_{-j})^*,\,(e_{-1}\wedge\cdots\wedge\hat{e}_{-k}\wedge\cdots\wedge e_{-5})^*\,|\,1\leq i<j\leq 5,\, k=1,\,\cdots,\,5\}$$

Step 1. We claim the linear form B is G-equivalent to

$$\alpha\cdot 1+\sum_{i<j}\beta_{ij}(e_{-i}\wedge e_{-j})^*+\sum_{k}\gamma_k(e_{-1}\wedge\cdots\wedge\hat{e}_{-k}\wedge\cdots\wedge e_{-5})^*$$

with $\alpha\neq 0$.

The linear form B can be written as

$$B=\beta\cdot 1+\sum_{i<j}\tilde{\beta}_{ij}(e_{-i}\wedge e_{-1})^*+\sum_{k}\tilde{\gamma}_k(e_{-1}\wedge\cdots\wedge\hat{e}_{-k}\wedge\cdots\wedge e_{-5})^*$$

We may assume that $\beta=0$. Let $X=\sum_{k<l}p_{kl}X_{-\varepsilon_k-\varepsilon_l}$ be an element of $\mathfrak{g}$. Then we have

$$\exp X(1)=1+\sum_{k<l}p_{kl}\,e_{-k}\wedge e_{-l}+\frac{1}{2}\sum_{k<l}\sum_{i<j}p_{kl}\,p_{ij}\,e_{-i}\wedge e_{-j}\wedge e_{-k}\wedge e_{-l}\,.$$

(a) The case when $\tilde{\beta}_{ij}\neq 0$ for some (i,j).

Let $p_{kl}=0$ for $(k,l)\neq(i,j)$ and $p_{ij}=1$.

Then $B(\exp X(1))=\tilde{\beta}_{ij}\neq 0$ and the linear form $(\exp X)^*B$ has the required property.

(b) The case when $\tilde{\beta}_{kl}=0$ for all (k,l).

Take $\gamma_k\neq 0$ and choose $\{i,j,s,t\}$ such a way as $i<j<s<t$ and $i,j,s,t\neq k$. Let $X=X_{-\varepsilon_i-\varepsilon_j}+X_{-\varepsilon_s-\varepsilon_t}$. Then $B(\exp X(1))=\gamma_k\neq 0$ and the linear form $(\exp X)^*B$ has the required property.

Step. 2. We claim the linear form B is G-equivalent to

$$\alpha\cdot 1+(e_{-1}\wedge e_{-2})^*+(e_{-3}\wedge e_{-4})^*+\sum_{k}\gamma'_k(e_{-1}\wedge\cdots\wedge\hat{e}_{-k}\wedge\cdots\wedge e_{-5})^*$$

with $\alpha\neq 0$ and for some $\gamma'_k\in\boldsymbol{C}$.

By Step 1 we may assume that B is given by

$$\alpha\cdot 1+\sum_{i<j}\beta_{ij}(e_{-i}\wedge e_{-j})^*+\sum_{k}\gamma_k(e_{-1}\wedge\cdots\wedge\hat{e}_{-k}\wedge\cdots\wedge e_{-5})^*$$

with $\alpha\neq 0$. Let $Y=\sum_{k<l}q_{kl}X_{\varepsilon_k+\varepsilon_l}$ be an element of $\mathfrak{g}$. Then we have $B(\exp Y(e_{-i}\wedge e_{-j}))=B(e_{-i}\wedge e_{-j}+Y(e_{-i}\wedge e_{-j}))=\beta_{ij}-q_{ij}\,\alpha$ and $B(\exp Y(1))=B(1)=\alpha$. Hence we can choose Y in such a way as $(\exp Y)^*B=\alpha\cdot 1+(e_{-1}\wedge e_{-2})^*+(e_{-3}\wedge e_{-4})^*+\sum_{k}\gamma'_k(e_{-1}\wedge\cdots\wedge\hat{e}_{-k}\wedge\cdots\wedge e_{-5})^*$.

Step 3. We claim the linear form B is G-equivalent to

$$\alpha\cdot 1+\sum_{i<j}\beta'_{ij}(e_{-i}\wedge e_{-j})^*\text{ for some }\beta'_{ij}\in\boldsymbol{C}\,.$$

We may assume B is given by

$$\alpha\cdot 1+(e_{-1}\wedge e_{-2})^*+(e_{-3}\wedge e_{-4})^*+\sum_{k}\gamma'_k(e_{-1}\wedge\cdots\wedge\hat{e}_{-k}\wedge\cdots\wedge e_{-5})^*\,.$$

Let $Y_1=q'_{12}X_{\varepsilon_1+\varepsilon_2}$ be an element of $\mathfrak{g}$. Then

$$(\exp Y_1)^*B=\alpha\cdot 1+(1-q'_{12}\alpha)(e_{-1}\wedge e_{-2})_*+(e_{-3}\wedge e_{-4})^*$$
$$+(\gamma_5-q'_{12})(e_{-1}\wedge e_{-2}\wedge e_{-3}\wedge e_{-4})^*+\sum_{k\leq 4}\gamma'_k(e_{-1}\wedge\cdots\wedge\hat{e}_{-k}\wedge\cdots\wedge e_{-5})^*\,.$$

Let $q'_{12}=\gamma_5$. Then we have

$$(\exp Y_1)^*B=\alpha\cdot 1+\mu_{12}(e_{-1}\wedge e_{-2})^*+(e_{-3}\wedge e_{-4})^*$$
$$+\sum_{k\leq 4}\gamma'_k(e_{-1}\wedge\cdots\wedge\hat{e}_{-k}\wedge\cdots\wedge e_{-5})^*$$

where $\mu_{12}=1-\gamma_5\alpha$.

Let $Y_2=q'_{25}\,X_{\varepsilon_2+\varepsilon_5}+q'_{15}\,X_{\varepsilon_1+\varepsilon_5}$. Then we have

$$(\exp Y_2)^*\,(\exp Y_1)^*B(e_{-2}\wedge e_{-3}\wedge e_{-4}\wedge e_{-5})=\gamma'_1-q'_{25}$$
$$(\exp Y_2)^*\,(\exp Y_1)^*B(e_{-1}\wedge e_{-3}\wedge e_{-4}\wedge e_{-5})=\gamma'_2-q'_{15}$$
$$(\exp Y_2)^*\,(\exp Y_1)^*B(e_{-1}\wedge e_{-2}\wedge e_{-4}\wedge e_{-5})=\gamma'_3$$
$$(\exp Y_2)^*\,(\exp Y_1)^*B(e_{-1}\wedge e_{-2}\wedge e_{-3}\wedge e_{-5})=\gamma'_4$$
$$(\exp Y_2)^*\,(\exp Y_1)^*B(e_{-1}\wedge e_{-2}\wedge e_{-3}\wedge e_{-4})=0$$
$$(\exp Y_2)^*\,(\exp Y_1)^*B(e_{-i}\wedge e_{-j})=(\exp Y_1)^*\,B(e_{-i}\wedge e_{-j})$$
$$\text{if } (i,j)\neq(1,5)\ (2,5)$$
$$(\exp Y_2)^*\,(\exp Y_1)^*B(e_{-2}\wedge e_{-5})=-q'_{25}\alpha$$
$$(\exp Y_2)^*\,(\exp Y_1)^*B(e_{-1}\wedge e_{-5})=-q'_{15}\alpha$$

Thus setting $q'_{15}=\gamma'_2$ and $q'_{25}=\gamma'_1$, we get

$(\exp Y_2)^*\,(\exp Y_1)^*\,B=\alpha\cdot 1+\mu_{12}(e_{-1}\wedge e_{-2})^*+(e_{-3}\wedge e_{-4})^*-\gamma'_2\alpha(e_{-1}\wedge e_{-5})^*-\gamma'_1\alpha(e_{-2}\wedge e_{-5})^*+\gamma'_3(e_{-1}\wedge e_{-2}\wedge e_{-4}\wedge e_{-5})^*+\gamma'_4(e_{-1}\wedge e_{-2}\wedge e_{-3}\wedge e_{-5})^*$.

(a) Now we consider the case $\mu_{12}\neq 0$, $\gamma'_2\neq 0$ or $\gamma'_1\neq 0$.

Let $Y_3=q'_{45}\,X_{\varepsilon_4+\varepsilon_5}+q'_{35}\,X_{\varepsilon_3+\varepsilon_5}$. Then we have $(\exp Y_3)^*\,(\exp Y_2)^*\,(\exp Y_1)^*B=\alpha\cdot 1+\sum_{i<j}\beta'_{ij}(e_{-i}\wedge e_{-j})^*+(\gamma'_4-q'_{35}\mu_{12})\,(e_{-1}\wedge e_{-2}\wedge e_{-3}\wedge e_{-5})^*+(\gamma'_3-q'_{45}\mu_{12})\,(e_{-1}\wedge e_{-2}\wedge e_{-4}\wedge e_{-5})^*$ for some $\beta'_{ij}\in\boldsymbol{C}$. If $\mu_{12}\neq 0$, let $q'_{35}=\gamma'_4/\mu_{12}$ and $q'_{45}=\gamma'_3/\mu_{12}$, then $(\exp Y_3)^*\,(\exp Y_2)^*\,(\exp Y_1)^*\,B$ has the required property. Similarly if $\gamma'_2\neq 0$, let $Y_3=q'_{24}\,X_{\varepsilon_2+\varepsilon_4}+q'_{23}\,X_{\varepsilon_2+\varepsilon_3}$ where $q'_{24}=-\gamma'_3/\gamma'_2\alpha$ and $q'_{23}=-\gamma'_4/\gamma'_2\alpha$, then $(\exp Y_3)^*\,(\exp Y_2)^*\,(\exp Y_1)^*\,B$ has the required property. And if $\gamma'_1\neq 0$, let $Y_3=q'_{14}\,X_{\varepsilon_1+\varepsilon_4}+q'_{13}\,X_{\varepsilon_1+\varepsilon_3}$ where $q'_{14}=\gamma'_3/\gamma'_2\alpha$ and $q'_{13}=\gamma'_4/\gamma'_2\alpha$, then $(\exp Y_3)^*\,(\exp Y_2)^*\,(\exp Y_1)^*\,B$ has the required property.

(b) Now we consider the case $\mu_{12}=\gamma'_2=\gamma'_1=0$.

Let $Y_3=\tilde{q}_{12}\,X_{\varepsilon_1+\varepsilon_2}+\tilde{q}_{35}\,X_{\varepsilon_3+\varepsilon_5}+\tilde{q}_{45}\,X_{\varepsilon_4+\varepsilon_5}$.

Then $(\exp Y_3)^*\,(\exp Y_2)^*\,(\exp Y_1)^*B=\alpha\cdot 1-\tilde{q}_{12}\alpha(e_{-1}\wedge e_{-2})^*+(e_{-3}\wedge e_{-4})^*-\tilde{q}_{35}\alpha(e_{-3}\wedge e_{-5})^*-\tilde{q}_{45}\alpha(e_{-4}\wedge e_{-5})^*-\tilde{q}_{12}(e_{-1}\wedge e_{-2}\wedge e_{-3}\wedge e_{-4})^*+(\gamma'_4+\tilde{q}_{12}\tilde{q}_{35})(e_{-1}\wedge e_{-2}\wedge e_{-3}\wedge e_{-5})^*+(\gamma'_3+\tilde{q}_{12}\tilde{q}_{45})\,(e_{-1}\wedge e_{-2}\wedge e_{-4}\wedge e_{-5})^*$.

Now choose $\tilde{q}_{12}\neq 0$, $\tilde{q}_{35}$ and $\tilde{q}_{45}$ such that $\gamma'_4+\tilde{q}_{12}\tilde{q}_{35}=0$ and $\gamma'_3+\tilde{q}_{12}\tilde{q}_{45}=0$, so that $(\exp Y_3)^*\,(\exp Y_2)^*\,(\exp Y_1)^*B=\alpha\cdot 1-\tilde{q}_{12}\alpha(e_{-1}\wedge e_{-2})^*+(e_{-3}\wedge e_{-4})^*$

$-\tilde{q}_{35}\alpha(e_{-3}\wedge e_{-5})^*-\tilde{q}_{45}\alpha(e_{-4}\wedge e_{-5})^*-\tilde{q}_{12}(e_{-1}\wedge e_{-2}\wedge e_{-3}\wedge e_{-4})^*$.
Let $Y_4=-\tilde{q}_{12}\, X_{\varepsilon_1+\varepsilon_2}$. Then
$(\exp Y_4)^*\,(\exp Y_3)^*\,(\exp Y_2)^*\,(\exp Y_1)^* B\,(e_{-1}\wedge e_{-2}\wedge e_{-3}\wedge e_{-4})=-\tilde{q}_{12}+\tilde{q}_{12}\times 1=0$
and hence
$(\exp Y_4)^*\,(\exp Y_3)^*\,(\exp Y_2)^*\,(\exp Y_1)^*\, B$ has the required property.

Step 4. Now we may assume B is given by $\alpha\cdot 1+\sum \beta'_{ij}(e_{-i}\wedge e_{-j})^*$. If $\beta'_{ij}=0$ for all (i,j), B is a linear form on $\boldsymbol{C}\cdot 1$. We may assume there is (i,j) such that $\beta'_{ij}\neq 0$. Let
$X_1=p'_{ij}\, X_{-\varepsilon_i-\varepsilon_j}$. Then

$$(\exp X_1)^*B(1) = \alpha-p'_{ij}\,\beta'_{ij}$$
$$(\exp X_1)^*B(e_{-k}\wedge e_{-l}) = B(e_{-k}\wedge e_{-l}) \text{ for each } (k, l)$$
$$(\exp X_1)^*B(e_{-1}\wedge\cdots\wedge\hat{e}_{-k}\wedge\cdots\wedge e_{-5}) = 0 \text{ for each } k.$$

Letting $p'_{ij}=\alpha/\beta'_{ij}$, $(\exp X_1)^*B$ can be regarded as a linear form on Λ^2F'.

q.e.d.

Proof of Theorem 2. From the fact 4 we see that every holomorphic vector field on a non-singular hyperplane section V can be extended uniquely to a holomorphic vector field on M. Let $A=\{g\in \mathrm{Aut}(M)\,|\,g(V)=V\}$. Then the Lie algebra of A can be identified with the Lie algebra of all holomorphic vector fields on V. Take the hyperplane H of $P(N_+)$ such that $V=M\cap H$ and let $A'=\{g\in G\,|\,gH=H\}$. A hyperplane H is the zero locus of non-zero linear form B on N_+ and thus the Lie algebra $\mathfrak{a}$ of A' is given by $\mathfrak{a}(B)=\{X\in\mathfrak{so}(10, \boldsymbol{C})\,|\,B(X\cdot n)=c(X)\,B(n),\ n\in N_+$ for some $c(X)\in\boldsymbol{C}\}$. Note also that if linear forms B and B' on N_+ are G-equivalent the Lie algebras $\mathfrak{a}(B)$ and $\mathfrak{a}(B')$ are isomorphic. Therefore by Lemma we may assume that B is a linear form on $\boldsymbol{C}\cdot\cdot 1$ or a linear form on Λ^2F'. If $B=\alpha\cdot 1(\alpha\neq 0)$ we can see the variety $M\cap H$ has a singular point (see Appendix). Thus we may assume B is a linear form on Λ^2F'. Now we can take a basis $\{e_{-1}, e_{-2}, e_{-3}, e_{-4}, e_{-5}\}$ of F' such that $B=(e_{-1}\wedge e_{-2})^*+(e_{-3}\wedge e_{-4})^*$ or $B=(e_{-1}\wedge e_{-2})^*$. We claim if $M\cap H$ is non-snigular $B=(e_{-1}\wedge e_{-2})^*+(e_{-3}\wedge e_{-4})^*$. Since a generic hyperplane section of M is non-isngular, it is sufficient to see that if $B=(e_{-1}\wedge e_{-2})^*$, $M\cap H$ has a singular point. Let $X=X_{-\varepsilon_1-\varepsilon_2}$ and $Y=X_{\varepsilon_1+\varepsilon_2}$. Then $(\exp Y)^*\,(\exp X)^*\,B=1$, and thus B is G-equivalent to a linear form on $\boldsymbol{C}\cdot 1$. Hence, $M\cap H$ hsa a singularity.

Now we shall compute the Lie algebra $\mathfrak{a}(B)$ for $B=(e_{-1}\wedge e_{-2})^*+(e_{-3}\wedge e_{-4})^*$. We may write an element X of $\mathfrak{g}=\mathfrak{so}(10, \boldsymbol{C})$ as

$$X=\sum_{i<j} a_{ij}\, X_{\varepsilon_i-\varepsilon_j}+\sum_{i<j} b_{ij}\, X_{-\varepsilon_i+\varepsilon_j}+\sum_{i<j} c_{ij}\, X_{\varepsilon_i+\varepsilon_j}+\sum_{i<i} d_{ij}\, X_{-\varepsilon_i-\varepsilon_j}$$
$$+\sum_i l_i H_i\,.$$

Since $B(1)=0, B(X\cdot 1)=B(\sum_{i<j} d_{ij}\, e_{-i}\wedge e_{-j})=d_{12}+d_{34}=0$. Since $B(e_{-1}\wedge\cdots\wedge\hat{e}_{-k}\wedge\cdots\wedge e_{-5})=0$, we see that

$$\begin{aligned}
B(X\cdot e_{-1}\wedge e_{-2}\wedge e_{-3}\wedge e_{-4}) &= -c_{12}-c_{34}=0\\
B(X\cdot e_{-1}\wedge e_{-2}\wedge e_{-3}\wedge e_{-5}) &= -c_{35}=0\\
B(X\cdot e_{-1}\wedge e_{-2}\wedge e_{-4}\wedge e_{-5}) &= -c_{45}=0\\
B(X\cdot e_{-1}\wedge e_{-3}\wedge e_{-4}\wedge e_{-5}) &= -c_{15}=0\\
B(X\cdot e_{-2}\wedge e_{-3}\wedge e_{-4}\wedge e_{-5}) &= -c_{25}=0\,.
\end{aligned}$$

Moreover

$$\begin{aligned}
&B(X\cdot e_{-1}\wedge e_{-2}) = \left(\frac{1}{2}(l_1+l_2+l_3+l_4+l_5)-(l_1+l_2)\right) = c(X)\\
&B(X\cdot e_{-3}\wedge e_{-4}) = \left(\frac{1}{2}(l_1+l_2+l_3+l_4+l_5)-(l_3+l_4)\right) = c(X)\\
&B(X\cdot e_{-1}\wedge e_{-3}) = a_{14}+a_{23}=0\,, \quad B(X\cdot e_{-1}\wedge e_{-4}) = -a_{13}+b_{24}=0\,,\\
&B(X\cdot e_{-1}\wedge e_{-5}) = b_{25}=0\,, \quad B(X\cdot e_{-2}\wedge e_{-3}) = a_{24}-b_{13}=0\,,\\
&B(X\cdot e_{-2}\wedge e_{-4}) = -a_{23}-b_{14}=0\,, \quad B(X\cdot e_{-2}\wedge e_{-5}) = -b_{15}=0\,,\\
&B(X\cdot e_{-3}\wedge e_{-5}) = b_{45}=0\,, \quad B(X\cdot e_{-4}\wedge e_{-5}) = -b_{35}=0\,.
\end{aligned}$$

Thus the Lie algebra $\mathfrak{a}(B)$ is given by

$$\left\{\left(\begin{array}{ccccc:ccccc}
l_1 & a_{12} & a_{13} & a_{14} & a_{15} & 0 & c_{14} & c_{13} & c_{12} & 0\\
-b_{12} & l_2 & a_{23} & a_{24} & a_{25} & 0 & c_{24} & c_{23} & 0 & -c_{12}\\
-b_{13} & -b_{23} & l_3 & a_{34} & a_{35} & 0 & c_{34} & 0 & -c_{23} & -c_{13}\\
-b_{14} & -b_{24} & -b_{34} & l_4 & a_{45} & 0 & 0 & -c_{34} & -c_{24} & -c_{14}\\
0 & 0 & 0 & 0 & l_5 & 0 & 0 & 0 & 0 & 0\\
\hdashline
-d_{15} & -d_{25} & -d_{35} & -d_{45} & 0 & -l_5 & -a_{45} & -a_{35} & -a_{25} & -a_{15}\\
-d_{14} & -d_{24} & -d_{34} & 0 & d_{45} & 0 & -l_4 & -a_{34} & -a_{24} & -a_{14}\\
-d_{13} & -d_{23} & 0 & d_{34} & d_{35} & 0 & -b_{34} & l_3 & -a_{23} & -a_{13}\\
-d_{12} & 0 & d_{23} & d_{24} & d_{25} & 0 & b_{24} & b_{23} & -l_2 & -a_{12}\\
0 & d_{12} & d_{13} & d_{14} & d_{15} & 0 & b_{14} & b_{13} & b_{12} & -l_1
\end{array}\right) \;\middle|\; \begin{array}{l}
l_1+l_2=l_3+l_4\\
a_{14}+b_{23}=0\\
-a_{13}+b_{24}=0\\
a_{24}-b_{13}=0\\
a_{23}+b_{14}=0\\
d_{12}+d_{34}=0\\
c_{12}+c_{34}=0
\end{array}\right\}$$

and, in particular, $\dim \mathfrak{a}(B)=30$. Let

$$\mathfrak{n} = \left\{ X\in\mathfrak{a}(B) \;\middle|\; X = \left(\begin{array}{cc:c}
0 & \begin{matrix}\alpha_1\\ \vdots\\ \alpha_5\end{matrix} & 0\\
\hdashline
-\beta_1\cdots-\beta_4 & 0 & -\alpha_5\cdots-\alpha_1\\
0 & \begin{matrix}\beta_4\\ \vdots\\ \beta_1\end{matrix} & 0
\end{array}\right)\right\}$$

Then $\mathfrak{n}$ is a solvable ideal of $\mathfrak{a}(B)$ such that $[\mathfrak{n},\mathfrak{n}]\neq(0)$ and $[[\mathfrak{n},\mathfrak{n}],[\mathfrak{n},\mathfrak{n}]]=(0)$.

Therefore $\mathfrak{a}(B)$ is not a reductive Lie algebra. By a theorem of Lichnerowicz-Matsushima [6], we see that the hyperplane section V does not admit any Kähler metric with constant scalar curvature.

Now by the same argument as in the proof of Theorem 1, we get

$$
\begin{aligned}
&\dim H^1(V, T(V)) = h^1(V, T(V)) \\
&= h^0(V, T(V)) - h^0(M, T(M)) + h^0(V, \{V\} \mid V) \\
&= \dim \mathfrak{a}(B) - \dim \mathfrak{so}(10, \boldsymbol{C}) + (16-1) \\
&= 30 - 45 + 15 = 0\,.
\end{aligned}
$$

q.e.d.

Appendix

Let M be an Hermitian symmetric space of compact type and L a very ample holomorphic line bundle on M. Let $j_L\colon M \to P^N(\boldsymbol{C})$ be the imbedding associated to L. Then it is known that the homogeneous ideal of M is generated by quadrics [7]. We shall determine these quadrics in the case when $M = SO(10)/U(5)$ and the imbedding is canonical. Denote by o the point in $P(N_+)$ corresponding to $U(5)$ of M. Let $\mathfrak{m}_- = \sum_{i<j} \mathfrak{g}_{-\varepsilon_i - \varepsilon_j}$ be an abelian subalgebra of $\mathfrak{g} = \mathfrak{so}(10, \boldsymbol{C})$ and M_- the Lie subgroup corresponding to $\mathfrak{m}_-$. Fix a basis $\{e_{-1}, e_{-2}, e_{-3}, e_{-4}, e_{-5}\}$ of F'. Then

$$
\{1,\ e_{-i} \wedge e_{-j},\ e_{-i_1} \wedge e_{-i_2} \wedge e_{-i_3} \wedge e_{-i_4} \mid i<j,\ i_1<i_2<i_3<i_4\}
$$

is a basis of N_+. We also denote by $\{x_\lambda\}$ the dual basis of N_+^*. Now consider the orbit $M_- \cdot o = j(\exp \mathfrak{m}_- \cdot U) = [\rho(\exp \mathfrak{m}_-)\, v_0]$. We may write an element Y of $\mathfrak{m}_-$ as

$$
Y = \sum_{i<j} \xi_{-\varepsilon_i - \varepsilon_j} X_{-\varepsilon_i - \varepsilon_j}\,.
$$

Note that the highest vector v_0 is given by $1 \in N_+$ in our case. Then

$$
\begin{aligned}
&\rho(\exp Y) \cdot 1 \\
&= 1 + \sum \xi_{-\varepsilon_i - \varepsilon_j} X_{-\varepsilon_i - \varepsilon_j} \cdot 1 + \frac{1}{2} \sum \xi_{-\varepsilon_i - \varepsilon_j} \xi_{-\varepsilon_k - \varepsilon_l} X_{-\varepsilon_i - \varepsilon_j} X_{-\varepsilon_k - \varepsilon_l} \cdot 1\,.
\end{aligned}
$$

For simplicity we denote the highest weight Λ_{α_5} by Λ. Now we get

$$
\begin{aligned}
&x_\Lambda(\rho(\exp Y) \cdot c \cdot 1) = c \\
&x_{\Lambda - \varepsilon_i - \varepsilon_j}(\rho(\exp Y) \cdot c \cdot 1) = c\xi_{-\varepsilon_i - \varepsilon_j} \\
&x_{\Lambda - \varepsilon_i - \varepsilon_j - \varepsilon_k - \varepsilon_l}(\rho(\exp Y) \cdot c \cdot 1) \\
&\quad = c(\xi_{-\varepsilon_i - \varepsilon_j} \xi_{-\varepsilon_k - \varepsilon_l} - \xi_{-\varepsilon_i - \varepsilon_k} \xi_{-\varepsilon_j - \varepsilon_l} + \xi_{-\varepsilon_i - \varepsilon_l} \xi_{-\varepsilon_j - \varepsilon_k})
\end{aligned}
$$

where $i<j<k<l$. Thus we see on $M_- \cdot o$

$$\begin{aligned}&x_\Lambda\, x_{\Lambda-(\varepsilon_i+\varepsilon_j+\varepsilon_k+\varepsilon_l)}-x_{\Lambda-(\varepsilon_i+\varepsilon_j)}\, x_{\Lambda-(\varepsilon_k+\varepsilon_l)}\\&\quad+x_{\Lambda-(\varepsilon_i+\varepsilon_k)}\, x_{\Lambda-(\varepsilon_j+\varepsilon_l)}-x_{\Lambda-(\varepsilon_i+\varepsilon_l)}\, x_{\Lambda-(\varepsilon_j+\varepsilon_k)}=0\end{aligned}$$

for $i<j<k<l$.

Since the Zariski closure $\overline{M_-\cdot o}$ of $M_-\cdot o$ in $P(N_+)$ is M, we see that these quadrics vanish on M.

Let $I(M)$ be the homogeneous ideal of M, $S^2(N_+^*)$ the vector space of homogeneous polynomials of degree 2 on N_+ and I_2 the subspace of degree 2 of the ideal $I(M)$. Then $I(M)$, $S^2(N_+^*)$ and I_2 are $\mathfrak{so}(10, \boldsymbol{C})$-modules. Now the decomposition of $S^2(N_+^*)$ as $\mathfrak{so}(10, \boldsymbol{C})$-modules is given by

$$S_2(N_+^*)=V_{2\Lambda_{\alpha_4}}+V_{\Lambda_{\alpha_1}}$$

where $V_{2\Lambda_{\alpha_4}}$ and $V_{\Lambda_{\alpha_1}}$ denotes $\mathfrak{so}(10, \boldsymbol{C})$-modules with the highest weights $2\Lambda_{\alpha_4}$ and Λ_{α_1} respectively, and we see $I_2=V_{\Lambda_{\alpha_1}}$ as $\mathfrak{so}(10, \boldsymbol{C})$-module. (Note that $\Lambda_{\alpha_1}=\varepsilon_1$.) In particular, we have dim $I_2=10$. Applynig elements of Weyl group of $\mathfrak{so}(10, \boldsymbol{C})$, it is not difficult to see that the following 10 quadrics constitute a basis of I_2:
For $1\leq i<j<k<l\leq 5$,

$$\begin{aligned}&x_\Lambda\, x_{\Lambda-(\varepsilon_i+\varepsilon_j+\varepsilon_k+\varepsilon_l)}-x_{\Lambda-(\varepsilon_i+\varepsilon_j)}\, x_{\Lambda-(\varepsilon_k+\varepsilon_l)}\\&\quad+x_{\Lambda-(\varepsilon_i+\varepsilon_k)}\, x_{\Lambda-(\varepsilon_j+\varepsilon_l)}-x_{\Lambda-(\varepsilon_i+\varepsilon_l)}\, x_{\Lambda-(\varepsilon_j+\varepsilon_k)},\\
&x_{\Lambda-(\varepsilon_4+\varepsilon_5)}\, x_{\Lambda-(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)}-x_{\Lambda-(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5)}\, x_{\Lambda-(\varepsilon_3+\varepsilon_4)}\\&\quad+x_{\Lambda-(\varepsilon_1+\varepsilon_3+\varepsilon_4+\varepsilon_5)}\, x_{\Lambda-(\varepsilon_2+\varepsilon_4)}-x_{\Lambda-(\varepsilon_1+\varepsilon_4)}\, x_{\Lambda-(\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5)},\\
&x_{\Lambda-(\varepsilon_4+\varepsilon_5)}\, x_{\Lambda-(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_5)}-x_{\Lambda-(\varepsilon_1+\varepsilon_2+\varepsilon_4+\varepsilon_5)}\, x_{\Lambda-(\varepsilon_3+\varepsilon_5)}\\&\quad+x_{\Lambda-(\varepsilon_1+\varepsilon_3+\varepsilon_4+\varepsilon_5)}\, x_{\Lambda-(\varepsilon_2+\varepsilon_5)}-x_{\Lambda-(\varepsilon_1+\varepsilon_5)}\, x_{\Lambda-(\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5)},\\
&x_{\Lambda-(\varepsilon_3+\varepsilon_5)}\, x_{\Lambda-(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)}-x_{\Lambda-(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_5)}\, x_{\Lambda-(\varepsilon_3+\varepsilon_4)}\\&\quad+x_{\Lambda-(\varepsilon_1+\varepsilon_3)}\, x_{\Lambda-(\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5)}-x_{\Lambda-(\varepsilon_1+\varepsilon_3+\varepsilon_4+\varepsilon_5)}\, x_{\Lambda-(\varepsilon_2+\varepsilon_3)},\\
&x_{\Lambda-(\varepsilon_2+\varepsilon_5)}\, x_{\Lambda-(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)}-x_{\Lambda-(\varepsilon_1+\varepsilon_2)}\, x_{\Lambda-(\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5)}\\&\quad+x_{\Lambda-(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_5)}\, x_{\Lambda-({}_2\varepsilon+\varepsilon_4)}-x_{\Lambda-(\varepsilon_1+\varepsilon_2+\varepsilon_4+\varepsilon_5)}\, x_{\Lambda-(\varepsilon_2+\varepsilon_3)},\\
&x_{\Lambda-(\varepsilon_1+\varepsilon_5)}\, x_{\Lambda-(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)}-x_{\Lambda-(\varepsilon_1+\varepsilon_2)}\, x_{\Lambda-(\varepsilon_1+\varepsilon_3+\varepsilon_4+\varepsilon_5)}\\&\quad+x_{\Lambda-(\varepsilon_1+\varepsilon_3)}\, x_{\Lambda-(\varepsilon_1+\varepsilon_2+\varepsilon_4+\varepsilon_5)}-x_{\Lambda-(\varepsilon_1+\varepsilon_4)}\, x_{\Lambda-(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_5)}.\end{aligned}$$

Now if a hyperplane H is given by $B=\alpha\cdot 1$, that is, $\alpha\cdot x_\Lambda=0$, then the variety $M\cap H$ has a singular point. In fact, if we take a point $p\in P(N_+)$ defined by

$$x_{\Lambda-(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)}(p)\neq 0 \text{ and } x_\lambda(p)=0 \quad \text{otherwise,}$$

then $p\in M\cap H$ is a singular point of $M\cap H$, using the fact M is the zero locus of 10 quadrics above.

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