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Author(s)	Morimoto, Kanji
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ON THE ADDITIVITY OF h -GENUS OF KNOTS

KANJI MORIMOTO

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Introduction

We say that $(V_1, V_2; F)$ is a Heegaard splitting of the 3-sphere S^3 , if both V_1 and V_2 are handlebodies, $S^3 = V_1 \cup V_2$ and $V_1 \cap V_2 = \partial V_1 = \partial V_2 = F$. Then F is called a Heegaard surface of S^3 .

Let K be a knot in S^3 . Then it is well known that there exists a Heegaard surface of S^3 which contains K . Thus we define $h(K)$ as the minimum genus among all Heegaard surfaces of S^3 containing K , and we call it the h -genus of K . We note here that any two Heegaard surfaces of S^3 with the same genus are mutually ambient isotopic ([11]).

By the definition, it follows that $h(K) = 0$ if and only if K is a trivial knot and that $h(K) = 1$ if and only if K is a torus knot. Hence if $h(K) = 1$ then K is prime. In this paper we show:

Theorem. *Let K_1 and K_2 be non-trivial knots in S^3 . If $h(K_1 \# K_2) = 2$, then $h(K_1) = h(K_2) = 1$.*

On the other hand, we show the following two propositions.

Proposition 1. *Let K_1 and K_2 be non-trivial knots in S^3 with $(1, 1)$ -decompositions. Suppose neither K_1 nor K_2 are torus knots. Then $h(K_1) = h(K_2) = 2$ and $h(K_1 \# K_2) = 3$.*

Here, we say that a knot K admits a (g, b) -decomposition, if there is a genus g Heegaard splitting $(V_1, V_2; F)$ of S^3 such that $V_i \cap K$ is a b -string trivial arc system in V_i ($i = 1, 2$) (cf. [2] and [6]).

REMARK. Since every 2-bridge knot admits a $(1, 1)$ -decomposition, there are infinitely many knots satisfying the hypothesis of Proposition 1.

Proposition 2. *Let n be an integer greater than 1 and K_n the knot illustrated in Figure 1. Then $h(K_n) = 3$ and $h(K_n \# K) = 3$ for any 2-bridge knot K .*

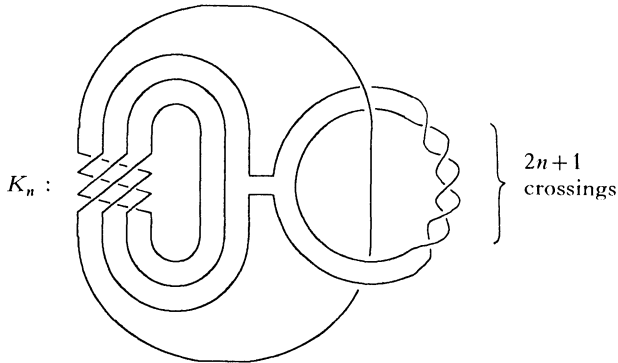


Figure 1

By Propositions 1 and 2, concerning h -genus we have the following “equalities”: $2+2=3$, $3+1=3$ and $3+2=3$. Hence it seems difficult to determine $h(K_1)$ and $h(K_2)$ when $h(K_1\#K_2)=3$.

Next, let $t(K)$ be the tunnel number of a knot K in S^3 . Here the tunnel number of K is the minimum number of mutually disjoint arcs properly embedded in the exterior of K in S^3 whose complementary space is a handlebody. We call the family of such arcs an unknotting tunnel system for K . Concerning the relation between $t(K)$ and $h(K)$, C. Morin and M. Saito pointed out the following fact.

Fact. $t(K) \leq h(K) \leq t(K) + 1$.

By Fact, we have the Venn diagram illustrated in Figure 2. For behavior of tunnel number of knots under connected sum, see [4], [5], [6], [7] and [9].

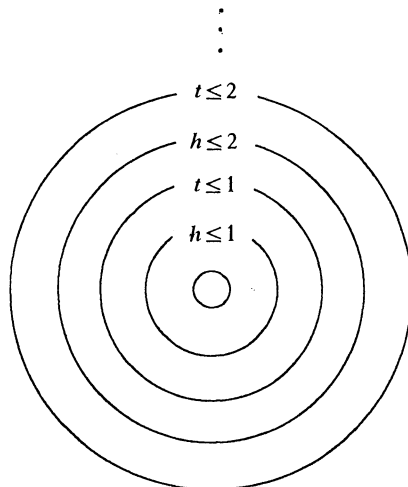


Figure 2

1. Proof of Fact and Propositions 1 and 2

Proof of Fact. Let $\{\gamma_1, \gamma_2, \dots, \gamma_{t(K)}\}$ be an unknotting tunnel system for K . Put $V_1=N(K) \cup N(\gamma_2 \cup \gamma_2 \cup \dots \cup \gamma_{t(K)})$ and $V_2=cl(S^3 - V_1)$, where $N(K)$ is a regular neighborhood of K in S^3 and $N(\gamma_1 \cup \gamma_2 \cup \dots \cup \gamma_{t(K)})$ a regular neighborhood of $\gamma_1 \cup \gamma_2 \cup \dots \cup \gamma_{t(K)}$ in $E(K)=cl(S^3 - N(K))$. Then by the definition of the tunnel number $t(K)$, (V_1, V_2) is a genus $t(K)+1$ Heegaard splitting of S^3 . Since K is a core of a handle of V_1 , K is ambient isotopic to a loop in ∂V_1 . Hence we have $h(K) \leq t(K)+1$.

Conversely, let $(V_1, V_2; F)$ be a genus $h(K)$ Heegaard splitting of S^3 such that K is contained in F . Let Γ be a core graph of V_1 , i.e. $cl(V_1 - N(\Gamma))$ is homeomorphic to $F \times I$, where I is a unit interval. Let α be a "trivial" arc connecting a point in K and a point in Γ . Then, since $cl(V_1 - N(\Gamma))$ is homeomorphic to $F \times I$, $cl(S^3 - N(\Gamma \cup \alpha \cup K))$ is a genus $h(K)+1$ handlebody. This shows that K has an unknotting tunnel system consisting of $h(K)$ arcs. Hence we have $t(K) \leq h(K)$. This completes the proof of the fact. ■

To prove Propostion 1, we prepare a lemma.

Lemma 1. *A knot K admits a (1, 1)-decomposition if and only if there is a genus two Heegaard splitting $(V_1, V_2; F)$ of S^3 satisfying the following conditions: K is contained in F , and there is a cancelling disk pair (D_1, D_2) of (V_1, V_2) such that $D_1 \cap K$ is a single point.*

Here, we say that (D_1, D_2) is a cancelling disk pair of (V_1, V_2) if D_i is a non-separating disk properly embedded in V_i ($i=1, 2$) and $D_1 \cap D_2 = \partial D_1 \cap \partial D_2$ is a single point.

Proof of Lemma 1. Suppose K admits a (1, 1)-decomposition. Then there is a genus one Heegaard splitting (W_1, W_2) of S^3 such that $W_i \cap K$ is a trivial arc properly embedded in W_i , say α_i , ($i=1, 2$). Let $N(\alpha_1)$ be a regular neighborhood of α_1 in W_1 . Let C_1 and C'_1 be the components of $N(\alpha_1) \cap \partial W_1$. Then $C_1 \cup C'_1$ is two disks which is a regular neighborhood of $\partial \alpha_1$ in ∂W_1 . Since α_1 is a trivail arc in W_1 , there is a disk in W_1 , say E , such that ∂E is a union of α_1 and an arc in ∂W_1 , say γ_1 . We may assume that $\gamma_1 \cap C_1$ ($\gamma_1 \cap C'_1$ resp.) is an arc, say β_1 (β'_1 resp.). Put $\Delta_1 = E \cap N(\alpha_1)$ and $D_1 = cl(E - \Delta_1)$.

Put $V_1 = cl(W_1 - N(\alpha_1))$. Then V_1 is a genus two handlebody and D_1 is a non-separating disk properly embedded in V_1 . Put $V_2 = W_2 \cup N(\alpha_1)$. Then (V_1, V_2) is a genus two Heegaard splitting of S^3 . Let $C_2 \cup C'_2$ be the image of $C_1 \cup C'_1$ in ∂W_2 . Since α_2 is a trivial arc in W_2 , there is disk in W_2 , say Δ_2 , such that $\partial \Delta_2$ is a union of α_2 and an arc in ∂W_2 , say γ_2 . We may assume that $\gamma_2 \cap C_2$ ($\gamma_2 \cap C'_2$ resp.) is an arc, say β_2 (β'_2 resp.). Moreover we may assume that β_1 (β'_1 resp.) is identified with β_2 (β'_2 resp.).

Put $A = \Delta_1 \cup \Delta_2$ in V_2 . Then by the above observation, A is an annulus in V_2 such that ∂A is a union of K and a loop in ∂V_2 , say K' . Then we can regard K' as K . Let D_2 be a disk properly embedded in $N(\alpha_1)$ parallel to C_1 . Then D_2 is a non-separating disk properly embedded in V_2 intersecting K' in a single point. Moreover by the definition of D_1 and D_2 , we see that (D_1, D_2) is a cancelling disk pair of the Heegaard splitting (V_1, V_2) . This completes the proof of "if" part of the lemma.

Conversely by tracing back the above argument, we complete the proof of the lemma. ■

Proof of Propostion 1. By Lemma 1, for $i=1, 2$, we have a genus two Heegaard splitting $(V_1^i, V_2^i; F^i)$ of S^3 satisfying the following conditions: K_i is contained in F_i , $V_1^i \cap V_2^i = \emptyset$ and there is a cancelling disk pair (D_1^i, D_2^i) of (V_1^i, V_2^i) such that $D_1^i \cap K_i$ is a single point. Hence $h(K_1) \leq 2$ and $h(K_2) \leq 2$. Let $N(D_1^i)$ be a regular neighborhood of D_1^i in V_1^i ($i=1, 2$), and put $U_1^i = cl(V_1^i - N(D_1^i))$. Let W_1 be a genus three handlebody in S^3 obtained from U_1^1 and U_1^2 by identifying $cl(\partial U_1^1 - \partial V_1^1)$ with $cl(\partial U_1^2 - \partial V_1^2)$, and put $W_2 = cl(S^3 - W_1)$. Then since (D_1^i, D_2^i) is a cancelling disk pair of (V_1^i, V_2^i) , (W_1, W_2) is a genus three Heegaard splitting of S^3 . Moreover since $D_1^i \cap K_i$ is a single point, $K_1 \# K_2$ is contained in ∂W_1 (see Figure 3). Hence we have $h(K_1 \# K_2) \leq 3$. On the other hand, since K_i is not a torus knot ($i=1, 2$), we have $h(K_1) \geq 2$ and $h(K_2) \geq 2$. And by Theorem we have $h(K_1 \# K_2) \geq 3$. This completes the proof of the proposition. ■

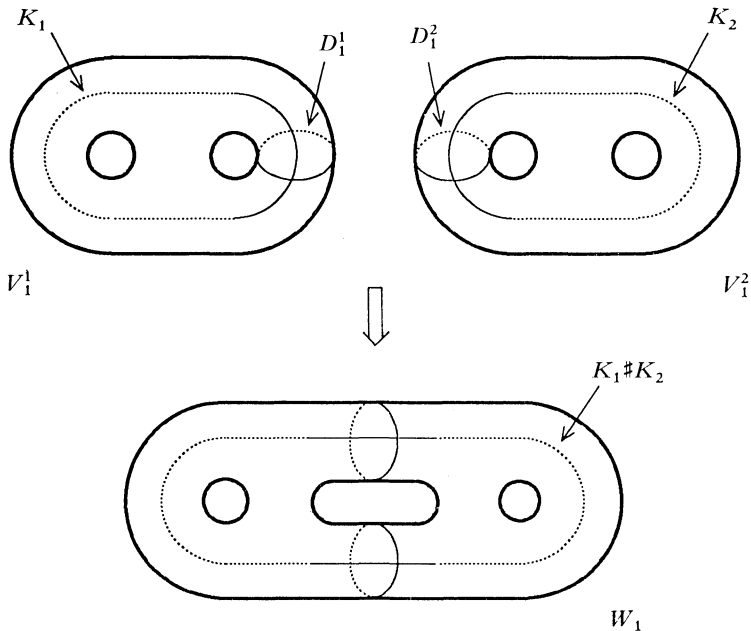


Figure 3

Proof of Propostion 2. By Theorem 3 of [5], we have $t(K_n)=2$ and $t(K_n\#K)=2$ for any 2-bridge knot K . Hence by Fact, $2\leq h(K_n)\leq 3$ and $2\leq h(K_n\#K)\leq 3$. If $h(K_n\#K)=2$, then by Theorem, we have $h(K_n)=1$, a contradiction. Hence we have $h(K_n\#K)=3$.

Suppose $h(K_n)=2$. Then there is a genus two Heegaard surface F of S^3 containing K_n . Then we have the following two cases.

Case 1 : K_n is a separating loop in F .

In this case, K_n bounds a punctured torus in S^3 . Then by Ch.8 of [8], the degree of the Alexander polynomial of K_n is at most two. However, the degree of the Alexander polynomial of K_n is $2n+10$. This is a contradiction, and hence Case 1 does not occur. Since the calculation of the Alexander polynomial is a routine matter, we leave it to the readers.

Case 2 : K_n is a non-separating loop in F .

Since, the orientation preserving mapping class group of F is generated by Dehn twists along the loops a_1, b_1, a_2, b_2 and a_3 indicated in Figure 4 ([3]), K_n is an image of the loop a_1 after a sequence of the Dehn twists. This shows that the orientation preserving involution h of S^3 indicated in Figure 4 fixes K_n setwise, and reverses the orientation of K_n (cf. [1] and [10]). Then by the proof of Theorem 3 of [5], we have a contradiction. This completes the proof of the proposition. ■

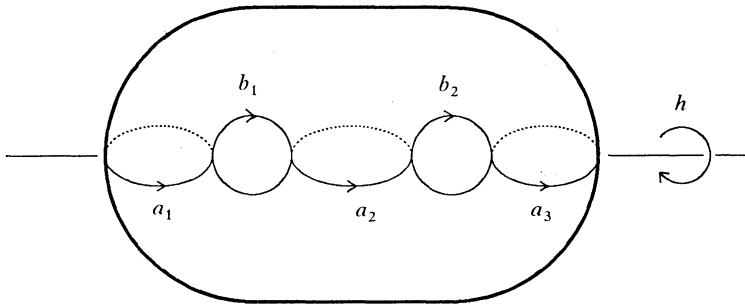


Figure 4

2. Proof of Theorem

Lemma 2. *Let V be a solid torus in S^3 and K a non-trivial knot in S^3 contained in ∂V . If K intersects a meridian of V more than once algebraically, then K is prime.*

Proof of Lemma 2. Put $\partial V=F$. Let S be a 2-sphere in S^3 intersecting K in two points. Then we may assume that each component of $S \cap F$ is a loop and that $\#(S \cap F)$ is minimum among all 2-spheres ambient isotopic rel. K to S , where $\#(\cdot)$ denotes the number of the components. Since S

intersects K in two points, we have the following two cases (see Figure 5).

Case I : $\mathcal{S} \cap F = C_0^* \cup C_1 \cup \dots \cup C_n$

Case II : $\mathcal{S} \cap F = C_1^* \cup C_2^* \cup C_1 \cup \dots \cup C_n$,

where C_i^* ($i=0, 1, 2$) is a loop intersecting K and C_i ($i=1, 2, \dots, n$) is a loop not intersecting K .

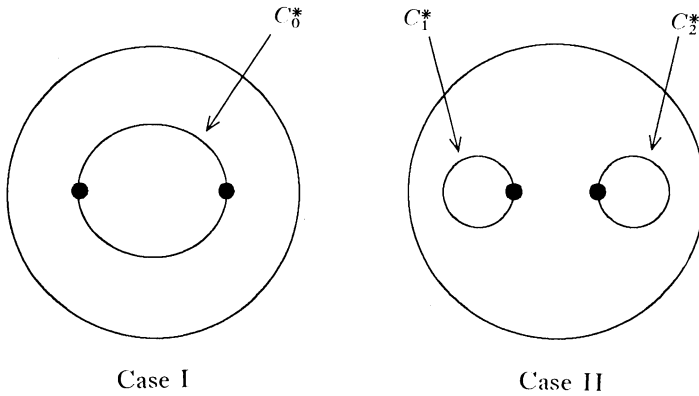


Figure 5

Claim 1 : There is no component of $\{C_i\}_{i=1}^n$ which is innermost in \mathcal{S} .

Proof. Suppose there is an innermost component of $\{C_i\}_{i=1}^n$, say C_k , and let D be the disk in \mathcal{S} bounded by C_k such that $D \cap (\mathcal{S} \cap F - C_k) = \emptyset$. Then D is a disk properly embedded in V or in $cl(\mathcal{S}^3 - V)$. By the minimality of $\#(\mathcal{S} \cap F)$, D is essential in V or in $cl(\mathcal{S}^3 - V)$. If D is in V , then D is a meridian disk of V . Then by the hypothesis of the lemma, D intersects K , a contradiction. If D is in $cl(\mathcal{S}^3 - V)$, then ∂D is a preferred longitude of V . Then by the hypothesis of the lemma, D intersects K , a contradiction. This completes the proof of the claim.

Claim 2 : Case II does not occur.

Proof. Suppose we are in Case II. Then by Claim 1, C_1^* bounds a disk in \mathcal{S} , say D , such that $D \cap (\mathcal{S} \cap F - C_1^*) = \emptyset$. Since $\partial D (= C_1^*)$ intersects K in a single point, D is a non-separating disk properly embedded in V or in $cl(\mathcal{S}^3 - V)$. If D is in V , then D is a meridian disk of V . This contradicts the hypothesis of the lemma. If D is in $cl(\mathcal{S}^3 - V)$, then V is an unknotted solid torus. Then K is a $(n, 1)$ -torus knot for some integer n . Hence K is a trivial knot, a contradiction. This completes the proof of the claim.

Suppose we are in Case I. By Claim 1, we have $\mathcal{S} \cap F = C_0^*$. Let D_1 and D_2 be the two disks in \mathcal{S} bounded by C_0^* . We may assume that D_1 is in V . If D_1 is a meridian disk of V , then since a core of V intersects D_1 in a single point, \mathcal{S} is a non-separating 2-sphere in \mathcal{S}^3 , a contradiction. Hence D_1 is a separating disk in V . Then D_1 is isotopic rel. ∂D_1 to a disk in ∂V , say

D. Let *B* be the 3-ball in S^3 bounded by *S* containing *D*. Then $(B, B \cap K)$ is a trivial ball pair because $B \cap K$ is an arc properly embedded in $D \subset B$. This completes the proof of the lemma. ■

Lemma 3. *Let $(V_1, V_2; F)$ be a genus two Heegaard splitting of S^3 and *K* a non-trivial knot in S^3 contained in *F*. Suppose there is a non-separating disk properly embeded in V_1 , say *D*, such that $D \cap K$ consists of at most one point. Then *K* is prime.*

Proof of Lemma 3. Let $N(D)$ be a regular neighborhood of *D* in V_1 such that $N(D) \cap K = \emptyset$ or an arc according as $D \cap K = \emptyset$ or a point.

Case I : $N(D) \cap K = \emptyset$.

Put $V = cl(V_1 - N(D))$. Then *V* is a solid torus and *K* is a knot in ∂V . Since *K* is a non-trivial knot, *K* intersects a meridian of *V* algebraically. If *K* intersects a meridian of *V* more than once algebraically, then by Lemma 2, *K* is prime.

Suppose *K* intersects a meridian of *V* in a single point. Then *K* is ambient isotopic to a core of *V*, say *K'*. Since V_1 is obtained by attaching a 1-handle $N(D)$ to *V*, and $S^3 - V_1 = V_2$ is a handlebody, we see that *K'* is a tunnel number one knot. Then, since tunnel number one knots are prime ([7]), *K'* is prime. Hence *K* is prime. This completes the proof of Case I.

Case II : $N(D) \cap K$ is an arc.

Put $\alpha = cl(K - N(D))$ and $cl(\partial N(D) - \partial V_1) = D_1 \cup D_2$. Then α is an arc in ∂V_1 connecting the disks D_1 and D_2 . Let $N(D_1 \cup D_2 \cup \alpha)$ be a regular neighborhood of $D_1 \cup D_2 \cup \alpha$ in V_1 and put $cl(\partial N(D_1 \cup D_2 \cup \alpha) - \partial V_1) = D_1^* \cup D_2^* \cup E$, where D_i^* is a disk parallel to D_i ($i=1, 2$). Then *E* is a disk properly embedded in V_1 which splits V_1 into two solid tori $N(D \cup K)$ and *W*, where $N(D \cup K)$ is a regular neighborhood of $D \cup K$ in V_1 and $W = cl(V_1 - N(D \cup K))$. Then since *K* is isotopic to a core of *W*, *K* is a tunnel number one knot. Hence *K* is prime, and this completes the proof of the lemma. ■

Proof of Theorem. Put $K = K_1 \# K_2$. Let $(V_1, V_2; F)$ be a genus two Heegaard splitting of S^3 whose Heegarrd surface contains *K*, and let *S* be a 2-sphere which gives the non-trivial connected sum of *K*. We may assume that each component of $S \cap F$ is a loop and that $\#(S \cap F)$ is minimum among all 2-spheres ambient isotopic rel. *K* to *S*. Then similarly to the proof of Lemma 2, we have the following two cases (see Figure 5).

Case I : $S \cap F = C_0^* \cup C_1 \cup \dots \cup C_n$

Case II : $S \cap F = C_1^* \cup C_2^* \cup C_1 \cup \dots \cup C_n$,

where C_i^* ($i=0, 1, 2$) is a loop interescting *K* and C_i ($i=1, 2, \dots, n$) is a loop not intersecting *K*.

Claim 1: There is no component of $\{C_i\}_{i=1}^n$ which is innermost in *S*.

Proof. Suppose there is an innermost component of $\{C_i\}_{i=1}^n$, say C_k , and let D be the disk in S bounded by C_k such that $D \cap (S \cap F - C_k) = \emptyset$. Then we may assume that D is a disk properly embedded in V_1 . By the minimality of $\#(S \cap F)$, D is an essential disk in V_1 . If D is a non-separating disk of V_1 , then by Lemma 3 K is prime, a contradiction. If D splits V_1 into two solid tori, say W_1 and W_2 . Then we may assume that K is contained in W_1 . Let D' be a meridian disk of W_2 with $D' \cap D = \emptyset$. Then D' is a non-separating disk of V_1 such that $D' \cap K = \emptyset$. Then by Lemma 3, K is prime. This contradiction completes the proof of the claim.

Claim 2 : Case II does not occur.

Proof. Suppose we are in Case II. Then by Claim 1, C_1^* bounds a disk in S , say D , such that $D \cap (S \cap F - C_1^*) = \emptyset$. Then we may assume that D is properly embedded in V_1 . Since $\partial D (= C_1^*)$ intersects K in a single point, D is a non-separating disk of V_1 , and satisfies the hypothesis of Lemma 3. Hence K is prime. This contradiction completes the proof of the claim.

Now suppose we are in Case I. By Claim 1, we have $S \cap F = C_0^*$. Let D_1 and D_2 be the two disks in S bounded by C_0^* . We may assume that D_i is contained in V_i ($i=1, 2$). For $i=1$ or 2 , if D_i is a non-separating disk in V_i , then since a core of a handle of V_i intersects D_i in a single point, S is a non-separating 2-sphere in S^3 , a contradiction. Hence both D_1 and D_2 are separating disks in V_1 and in V_2 respectively. This shows that both K_1 and K_2 are contained in genus one Heegaard surfaces and completes the proof of Theorem. ■

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Department of Mathematics
Takushoku University
Tatemachi, Hachioji,
Tokyo 193, Japan

