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COMBINATORIAL PURE SUBRINGS

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Introduction

Let K be a field and $R = K[t_1, \dots, t_d]$ the polynomial ring in d variables over K. Let A be a homogeneous affine semigroup ring generated by monomials belonging to R. If T is a nonempty subset of $[d] = \{1, \dots, d\}$, then we write R_T for the polynomial ring $K[\{t_j\,;\, j \in T\}]$ with the restricted variables. A subring of A of the form $A \cap R_T$ with $\emptyset \neq T \subset [d]$ is called a combinatorial pure subring of A.

The most reasonable question is which ring-theoretical properties are inherited by combinatorial pure subrings. First of all, in Section 1, this problem will be discussed. One of the most fundamental observation on combinatorial pure subrings is that the elimination technique of Gröbner bases can be always applied to combinatorial pure subrings. Namely, if I_A is the defining ideal of A and if G is any reduced Gröbner basis of I_A, then, for any combinatorial pure subring B of A, $G \cap I_B$ is the reduced Gröbner basis of I_B, where I_B is the defining ideal of B.

Let Σ_A denote the infinite divisor poset (partially ordered set) of A; that is to say, Σ_A is the infinite poset consisting of all monomials belonging to A, ordered by divisibility. It then follows immediately that if B is a combinatorial pure subring of A and if $\alpha \in \Sigma_B$, then any element $\beta \in \Sigma_A$ with $\beta \leq \alpha$ belongs to Σ_B. Hence, the closed interval $[1, \alpha]$ of Σ_B coincides with the closed interval $[1, \alpha]$ of Σ_A. This simple observation enables us to show that all combinatorial pure subrings of a Koszul semigroup ring are again Koszul. Moreover, it will be proved that if a homogeneous semigroup ring is (i) normal, (ii) strongly Koszul, (iii) sequentially Koszul, or (iv) extendable sequentially Koszul, then any of its combinatorial pure subrings inherits each of these properties.

In Section 2, we are interested in a homogeneous semigroup ring coming from a poset, i.e., a homogeneous semigroup ring having an initial ideal which is the Stanley-Reisner ideal of a finite poset. By virtue of the elimination technique together with a combinatorial criterion for a squarefree quadratic monomial ideal to be the Stanley-Reisner ideal of a finite poset, we can prove that if A comes from a poset, then all combinatorial pure subrings of A come from posets. We will apply this basic fact to so-called squarefree Veronese subrings.

Let $2 \leq q < d$. The q-th squarefree Veronese subring of order d is the affine semigroup ring $\mathcal{R}_d^{(q)}$ which is generated by all squarefree monomials of degree q belonging to the polynomial ring $K[t_1, \dots, t_d]$. It is known [13] that each $\mathcal{R}_d^{(q)}$ has an initial ideal generated by squarefree quadratic monomials. However, it seems to be unknown if each $\mathcal{R}_d^{(q)}$ comes from a poset. Observing that $\mathcal{R}_d^{(q)}$ is a combinatorial pure subring of $\mathcal{R}_{d'}^{(q)}$ if $d < d'$, we show that the q-th squarefree Veronese subring of order d comes from a poset if and only if either $q = 2$ and $3 \leq d \leq 4$, or $q \geq 3$ and $d = q + 1$. See Theorem 2.3. In addition, it will be proved that the q-th squarefree Veronese subring of order d is Golod if and only if $d = q + 1$. See Corollary 2.7.

The topic of Section 3 is the Lawrence lifting of homogeneous semigroup rings. Let $A = K[f_1, \dots, f_n]$ be a homogeneous semigroup ring generated by monomials $f_1, \dots, f_n$. Then, the Lawrence lifting of A is the homogeneous semigroup ring $K[f_1 z_1, \dots, f_n z_n, z_1, \dots, z_n]$, where $z_1, \dots, z_n$ are variables over K. A crucial observation is that if B is any subring of A generated by a subset of $\{f_1, \dots, f_n\}$, then the Lawrence lifting of B is a combinatorial pure subring of the Lawrence lifting of A. Thus, the technique of combinatorial pure subrings will be useful for the study of Lawrence liftings of homogeneous semigroup rings. The main result of Section 3 is Theorem 3.4 which guarantees that the Lawrence lifting of a homogeneous semigroup ring A is normal if and only if A is unimodular, i.e., all initial ideals of the defining ideal of A are squarefree. A quite effective criterion for a homogeneous semigroup ring A to be unimodular is known: A homogeneous semigroup ring A is unimodular if and only if every circuit belonging to the defining ideal of A is squarefree. (Here, a circuit is an irreducible binomial with a minimal support and a binomial is called squarefree if each of the monomials of the binomial is squarefree.) See Proposition 3.3. We conclude this paper with some examples of unimodular semigroup rings arising from combinatorial commutative algebra.

1. Basic results on combinatorial pure subrings

Let K be a field and $K[\mathbf{t}] = K[t_1, \dots, t_d]$ the polynomial ring in d variables over K. Let $\mathcal{A} = \{f_1, \dots, f_n\}$ be a set of monomials belonging to $K[\mathbf{t}]$ and suppose that the affine semigroup ring $K[\mathcal{A}] = K[f_1, \dots, f_n]$ is a homogeneous K-algebra, i.e., $K[\mathcal{A}]$ is a graded algebra $K[\mathcal{A}] = (K[\mathcal{A}])_0 \bigoplus (K[\mathcal{A}])_1 \bigoplus \cdots$ with $(K[\mathcal{A}])_0 = K$ and with each $f_i \in (K[\mathcal{A}])_1$. Such a semigroup ring $K[\mathcal{A}]$ is called a *homogeneous* semigroup ring. Let $K[\mathbf{x}] = K[x_1, \dots, x_n]$ be the polynomial ring in n variables over K with each $\deg x_i = 1$ and let $I_{\mathcal{A}}$ denote the kernel of the surjective homomorphism $\pi : K[\mathbf{x}] \to K[\mathcal{A}]$ defined by $\pi(x_i) = f_i$ for all $1 \leq i \leq n$. We call $I_{\mathcal{A}}$ the *defining ideal* of $K[\mathcal{A}]$.

Let $[d] = \{1, \dots, d\}$. If T is a nonempty subset of $[d]$, then we write $\mathcal{A}_T$ for the subset $\mathcal{A} \cap K[\{t_j ; j \in T\}]$ of $\mathcal{A}$. A subring of $K[\mathcal{A}]$ of the from $K[\mathcal{A}_T]$ with $\emptyset \neq T \subset [d]$ is called a *combinatorial pure subring* of $K[\mathcal{A}]$. If $\mathcal{A}_T = \{f_{i_1}, f_{i_2}, \dots, f_{i_r}\}$, then we set $K[\mathbf{x}_T] = K[x_{i_1}, x_{i_2}, \dots, x_{i_r}]$. Thus $I_{\mathcal{A}_T} = I_{\mathcal{A}} \cap K[\mathbf{x}_T]$.

Let $<$ be an arbitrary term order on $K[\mathbf{x}]$ and $g \in I_{\mathcal{A}}$ a binomial of $K[\mathbf{x}]$. If the initial monomial $in_<(g)$ of g belongs to $K[\mathbf{x}_T]$, then g must belong to $K[\mathbf{x}_T]$. In fact, if $g = u - v$ where u and v are monomials of $K[\mathbf{x}]$, then $\pi(u) = \pi(v)$ since $g \in I_{\mathcal{A}}$.

Thus $\pi(u) \in K[\{t_j\,;\, j \in T\}]$ if and only if $\pi(v) \in K[\{t_j\,;\, j \in T\}]$. Since $\pi(x_i) \in K[\{t_j\,;\, j \in T\}]$ if and only if $i \in \{i_1, i_2, \ldots, i_r\}$, it follows that $\pi(u) \in K[\{t_j\,;\, j \in T\}]$ if and only if $u \in K[\mathbf{x}_T]$.

This simple observation yields the fundamental result on elimination of Gröbner bases for combinatorial pure subrings.

Proposition 1.1. *If G is the reduced Gröbner basis of $I_{\mathcal{A}}$ with respect to a term order $<$ on $K[\mathbf{x}]$, then $G \cap K[\mathbf{x}_T]$ is the reduced Gröbner basis of $I_{\mathcal{A}_T}$ (with respect to the term order on $K[\mathbf{x}_T]$ induced by $<$).*

Proof. Let $h \in I_{\mathcal{A}_T} = I_{\mathcal{A}} \cap K[\mathbf{x}_T]$. Since $h \in I_{\mathcal{A}}$, we can find $g \in G$ such that $in_<(g)$ divides $in_<(h)$. Thus, in particular, $in_<(g) \in K[\mathbf{x}_T]$. Since g is a binomial with $g \in I_{\mathcal{A}}$, g must belong to $K[\mathbf{x}_T]$. Thus $g \in G \cap K[\mathbf{x}_T]$. Hence, $G \cap K[\mathbf{x}_T]$ is the reduced Gröbner basis of $I_{\mathcal{A}_T}$ as required. $\square$

Proposition 1.2. *If $K[\mathcal{A}]$ is normal, then any combinatorial pure subring of $K[\mathcal{A}]$ is normal.*

Proof. Let $K[\mathcal{A}_T]$ be a combinatorial pure subring of $K[\mathcal{A}]$ and choose a monomial u belonging to the quotient field of $K[\mathcal{A}_T]$ such that u is integral over $K[\mathcal{A}_T]$. Since $K[\mathcal{A}_T]$ is a subring of $K[\mathcal{A}]$, the monomial u belongs to the quotient field of $K[\mathcal{A}]$ and is integral over $K[\mathcal{A}]$. Thus, u belongs to $K[\mathcal{A}]$ since $K[\mathcal{A}]$ is normal. Since u belongs to the quotient field of $K[\mathcal{A}_T]$, it follows that no variable t_j with $j \notin T$ appears in u. Hence, u must belongs to $K[\mathcal{A}_T]$ since $K[\mathcal{A}_T]$ is a combinatorial pure subring of $K[\mathcal{A}]$. Thus, $K[\mathcal{A}_T]$ is normal as desired. $\square$

Proposition 1.3. *If $K[\mathcal{A}]$ is Koszul, then any combinatorial pure subrings of $K[\mathcal{A}]$ is Koszul.*

Proof. Let $\Sigma_{K[\mathcal{A}]}$ denote the infinite divisor poset of $K[\mathcal{A}]$; that is to say, $\Sigma_{K[\mathcal{A}]}$ is the infinite poset consisting of all monomials belonging to $K[\mathcal{A}]$, ordered by divisibility. It is known, e.g., [12] that $K[\mathcal{A}]$ is Koszul if and only if, for all $\alpha \in \Sigma_{K[\mathcal{A}]}$, the closed interval $[1, \alpha]$ of $\Sigma_{K[\mathcal{A}]}$ is Cohen-Macaulay. If $K[\mathcal{A}_T]$ is a combinatorial pure subring of $K[\mathcal{A}]$ and if $\alpha \in \Sigma_{K[\mathcal{A}_T]}$, then any element $\beta \in \Sigma_{K[\mathcal{A}]}$ with $\beta \leq \alpha$ belongs to $\Sigma_{K[\mathcal{A}_T]}$. Hence, the closed interval $[1, \alpha]$ of $\Sigma_{K[\mathcal{A}_T]}$ coincides with the closed interval $[1, \alpha]$ of $\Sigma_{K[\mathcal{A}]}$. Thus, if $K[\mathcal{A}]$ is Koszul, then $K[\mathcal{A}_T]$ is Koszul, as desired. $\square$

Let S be a graded K-algebra and $R \subset S$ a graded K-subalgebra. Then, R is called an *algebra retract of S* if there exists a surjective homomorphism of graded K-algebras $\varepsilon : S \to R$ such that $\varepsilon_{|R} = \mathrm{id}_R$. Note that a combinatorial pure subring $K[\mathcal{A}_T]$

of a homogeneous semigroup ring $K[\mathcal{A}]$ is an algebra retract. In fact, the image of $K[\mathcal{A}]$ under the natural epimorphism $p : K[\mathbf{t}] \to K[\{t_j\,;\, j \in T\}]$ is just $K[\mathcal{A}_T]$, and the restriction of p to $K[\mathcal{A}_T]$ is the identity. Hence, $\varepsilon = p_{|K[\mathcal{A}]}$ is a retraction map for $K[\mathcal{A}_T] \subset K[\mathcal{A}]$.

Let R be a finitely generated homogeneous K-algebra and M a finitely generated graded R-module. Set

$$P_R^M(s,t) = \sum_{i,j\geq 0} \dim_K \operatorname{Tor}_i^R(M,K)_j s^i t^j.$$

This formal power series is called the *graded Poincaré series of M*. Since for each i there exist only finitely many j with $\operatorname{Tor}_i^R(M,K)_j \neq 0$, we can write

$$P_R^M(s,t) = \sum_{i\geq 0} p_i^M(t) s^i,$$

where each $p_i^M(t)$ is a polynomial in t.

Following Backelin [3] we define

$$\operatorname{rate}(R) = \sup\left\{ \frac{\deg p_i^K(t) - 1}{i-1}\,;\, i \geq 2 \right\}.$$

It is clear that $\operatorname{rate}(R) = 1$ if and only if R is Koszul.

The following result generalizes Proposition 1.3.

Proposition 1.4. *Let $R \subset S$ be an algebra retract of graded K-algebras with retraction map ε. Then, we have*

(a) $\operatorname{rate}(R) \leq \operatorname{rate}(S)$;

(b) *Consider R as an S-module via ε. Then, the following conditions are equivalent*:
   - (i) *R is Koszul*;
   - (ii) *S is Koszul and R has a linear S-resolution.*

Proof. For the proof we use a graded version of the following result from [7]:

$$P_S^K(s,t) = P_S^R(s,t) P_R^K(s,t).$$

Write $P_S^K(s,t) = \sum_{i\geq 0} p_i(t)s^i$, $P_S^R(s,t) = \sum_{i\geq 0} q_i(t)s^i$ and $P_R^K(s,t) = \sum_{i\geq 0} r_i(t)s^i$. Then

$$p_i(t) = \sum_{j=0}^{i} q_j(t) p_{i-j}(t) \quad \text{for all} \quad i.$$

Since the coefficients of the polynomials q_i and r_i are all non-negative integers, it follows that

$$\deg p_i(t) = \max\{\deg q_j(t) + \deg r_{i-j}(t)\,;\ j = 0, \ldots, i\}.$$

From this equation both assertion of the proposition follow at once. $\square$

We refer the reader to [1], [2] and [8] for the fundamental information about strongly Koszul, sequentially Koszul and extendable sequentially Koszul algebras.

Proposition 1.5. *Let $R \subset S$ be an algebra retract of graded K-algebras with retraction map ε. If S is strongly Koszul, sequentially Koszul or extendable sequentially Koszul with respect to the sequence $\mathbf{x} = x_1, \ldots, x_n$ (forming a K-basis of S_1), such that there exists a subset $\mathbf{x}' = x_{i_1}, \ldots, x_{i_k}$ of $\mathbf{x}$ with $\varepsilon(x_j) = x_j$ for $x_j \in \mathbf{x}'$ and $\varepsilon(x_j) = 0$ for $x_j \notin \mathbf{x}'$. Then, R is strongly Koszul, sequentially Koszul or extendable sequentially Koszul with respect to the sequence $\mathbf{x}'$, respectively.*

Corollary 1.6. *Let $K[\mathcal{A}_T]$ be a combinatorial pure subring of $K[\mathcal{A}]$. If $K[\mathcal{A}]$ is strongly Koszul, sequentially Koszul or extendable sequentially Koszul, then $K[\mathcal{A}_T]$ has this property, too.*

Before proving Proposition 1.5 we note the following

Lemma 1.7. *Let $R \subset S$ be an algebra retract of graded K-algebras with retraction map ε, let $I \subset S$ be an ideal and $x \in R$. If $\varepsilon(I) \subset I$, then*

$$\varepsilon(I :_S x) = \varepsilon(I) :_R x.$$

Proof. Suppose $a \in I :_S x$; then $ax \in I$, and so $\varepsilon(a)\varepsilon(x) \in \varepsilon(I)$. Since $x \in I$, we have $x = \varepsilon(x)$, and so $\varepsilon(a) \in \varepsilon(I) :_R x$. Conversely, let $a \in \varepsilon(I) :_R x$. Then $ax \in \varepsilon(I) \subset I$, and hence $a \in I :_S x$, so that $a = \varepsilon(a) \in \varepsilon(I :_S x)$. $\square$

Proof. [Proof of Proposition 1.5] Since $\mathbf{x}$ is K-basis of S_1, it follows at once that $\mathbf{x}'$ is a K-basis of R_1, and hence a minimal set of generators of the K-algebra R.

Suppose S is strongly Koszul with respect to $\mathbf{x}$, and let $x_i, x_j \in \mathbf{x}'$. Then $(x_i) :_S x_j$ is generated by a subset of $\mathbf{x}$. By Lemma 1.7 we have $\varepsilon((x_i) :_S x_j) = (x_j) :_R x_i$, and it follows that $(x_j) :_R x_i$ is generated by a subset of $\mathbf{x}'$, as desired.

Next suppose that S is sequentially Koszul with respect to $\mathbf{x}$. In order to prove that R is sequentially Koszul with respect to $\mathbf{x}'$, we have to show that all derived sequences of $\mathbf{x}'$ have linear quotients. This will be a consequence of the following assertion and Lemma 1.7: Let $\mathbf{y}'$ be an i-th derived sequence of $\mathbf{x}'$ in R. Then, there exists an i-th derived sequence of $\mathbf{x}$ in S with $\varepsilon(\mathbf{y}) = \mathbf{y}'$. In fact, there exists an $(i-1)$-th derived sequence $\mathbf{z}' = z_1, \ldots, z_l$ of $\mathbf{x}'$ (with $z_i \in \{x_1, \ldots, x_n\}$) such that

$(\mathbf{y}') = (z_1, \ldots, z_{j-1}) :_R z_j$ for some $j \leq l$. Inducting on i, we may assume there exists an $(i-1)$-th derived sequence $\mathbf{z}$ of $\mathbf{x}$ with $\varepsilon(\mathbf{z}) = \mathbf{z}'$. In particular, $\mathbf{z}'$ is a subsequence of $\mathbf{z}$. Let $\tilde{\mathbf{z}}$ be the largest initial sequence of $\mathbf{z}$ not containing z_j, and let $\mathbf{y}$ be the sequence generating the colon ideal $(\tilde{\mathbf{z}}) : z_j$. Since $\varepsilon(\tilde{\mathbf{z}}) = z_1, \ldots, z_{j-1}$, it follows from Lemma 1.7 that $\varepsilon(\mathbf{y}) = \mathbf{y}'$.

In a similar way one shows that R is extendable sequentially Koszul with respect to $\mathbf{x}'$ if S has this property with respect to $\mathbf{x}$. We only note that if a sequence $\mathbf{y}$ is extended to a sequence $\mathbf{y}_1$ with linear quotients, then $\varepsilon(\mathbf{y})$ can extended to $\varepsilon(\mathbf{y}_1)$, having again linear quotients. □

2. Squarefree Veronese subrings

Let $K[\mathbf{x}] = K[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field K and suppose that I is an ideal of $K[\mathbf{x}]$ which is generated by squarefree quadratic monomials. We say that I is the Stanley-Reisner ideal of the order complex of a finite poset if there exists a partial order on $[n]$ such that I is generated by those squarefree quadratic monomials $x_i x_j$ such that i and j are incomparable in the partial order. Let $\Gamma(I)$ denote the graph on the vertex set $[n] = \{1, \ldots, n\}$ with the edge set consisting of all $\{i, j\}$ such that $x_i x_j \notin I$.

Let G be a finite graph and suppose that G has no loop and no multiple edge. A *quasi-cycle* of G of *length* k is a finite sequence of vertices $(a_1, a_2, \ldots, a_k)$ of G such that (i) all of the edges $\{a_i, a_{i+1}\}$ with $1 \leq i \leq k-1$ and the edge $\{a_k, a_1\}$ belong to G and (ii) if $a_i = a_j$ with $i, j < k$ and $i \neq j$, then $a_{i+1} \neq a_{j+1}$, and if $a_i = a_k$ with $i < k$, then $a_{i+1} \neq a_1$. A quasi-cycle is called *odd* if its length is odd. Note that a vertex may appear more than once in a quasi-cycle. A *triangular chord* of a quasi-cycle $(a_1, a_2, \ldots, a_k)$ is an edge of G of the form either $\{a_i, a_{i+2}\}$ with $1 \leq i \leq k-2$ or $\{a_{k-1}, a_1\}$ or $\{a_k, a_2\}$.

Now, the criterion, e.g., [5] guarantees that

Lemma 2.1. *Let I be an ideal of $K[\mathbf{x}]$ which is generated by squarefree quadratic monomials. Then, I is the Stanley-Reisner ideal of the order complex of a finite poset if and only if the following condition $(*)$ is satisfied: $(*)$ Every odd quasi-cycle of $\Gamma(I)$ of length ≥ 5 has at least one triangular chord.*

We say that a homogeneous semigroup ring $K[\mathcal{A}]$ *comes from a poset* if $I_{\mathcal{A}}$ possesses an initial ideal which is the Stanley-Reisner ideal of the order complex of a finite poset. For example, every monomial ASL (algebra with straightening laws) discussed in, e.g., [2] comes from a poset. It is shown in [12] that if $K[\mathcal{A}]$ comes from a poset, then the infinite divisor poset of $K[\mathcal{A}]$ is shellable. Here, the infinite divisor poset of $K[\mathcal{A}]$ is the infinite poset consisting of all monomials of $K[\mathcal{A}]$, ordered by divisibility.

Proposition 2.2. *If a homogeneous semigroup ring $K[\mathcal{A}]$ comes from a poset, then any combinatorial pure subring of $K[\mathcal{A}]$ comes from a poset.*

Proof. Suppose that a homogeneous semigroup ring $K[\mathcal{A}]$ comes from a poset and choose a term order $<$ on $K[\mathbf{x}]$ such that the initial ideal $in_<(I_{\mathcal{A}})$ is the Stanley-Reisner ideal of the order complex of a finite poset. Let $K[\mathcal{A}_T]$ be a combinatorial pure subring of $K[\mathcal{A}]$. It then follows from Proposition 1.1 that $in_<(I_{\mathcal{A}_T}) = in_<(I_{\mathcal{A}}) \cap K[\mathbf{x}_T]$. Hence, $\Gamma(I_{\mathcal{A}_T})$ is an induced subgraph of $\Gamma(I_{\mathcal{A}})$. Since $\Gamma(I_{\mathcal{A}})$ satisfies the condition $(*)$ of Lemma 2.1, its induced subgraph $\Gamma(I_{\mathcal{A}_T})$ also satisfies the condition $(*)$ as desired. □

Let $K[t_1, \ldots, t_d]$ be the polynomial ring in d variables over a field K with each $\deg t_j = 1$. Let $2 \leq q < d$. The q-th *squarefree Veronese subring* of order d is the affine semigroup ring $\mathcal{R}_d^{(q)}$ which is generated by all squarefree monomials of degree q belonging to $K[t_1, \ldots, t_d]$. It is known [13] that each $\mathcal{R}_d^{(q)}$ has an initial ideal generated by squarefree quadratic monomials. However, it seems to be unknown if each $\mathcal{R}_d^{(q)}$ comes from a poset.

Theorem 2.3. *Let $2 \leq q < d$. The q-th squarefree Veronese subring of order d comes from a poset if and only if either* (i) *$q = 2$ and $3 \leq d \leq 4$, or* (ii) *$q \geq 3$ and $d = q + 1$.*

Proof. First of all, note that the squarefree Veronese subring $\mathcal{R}_{q+1}^{(q)}$ is the polynomial ring in $q + 1$ variables over K. Thus, $\mathcal{R}_{q+1}^{(q)}$ comes from a poset in the obvious way. Moreover, $\mathcal{R}_4^{(2)}$ comes from a poset since the defining ideal of $\mathcal{R}_4^{(2)}$ has an initial ideal (x_1x_2, x_3x_4).

To show the "only if" part, we first show that $\mathcal{R}_5^{(2)}$ does not come from a poset. It is discussed in [4] and [2, Example 4.3 (b)] that there exist only two quadratic initial ideals (up to symmetry) of $\mathcal{R}_5^{(2)}$; they are

$$I_1 = (x_1x_8, x_1x_9, x_1x_{10}, x_3x_5, x_3x_9, x_4x_5, x_4x_6, x_4x_8, x_6x_9, x_7x_8);$$
$$I_2 = (x_1x_{10}, x_2x_6, x_2x_7, x_2x_{10}, x_3x_5, x_3x_7, x_3x_9, x_4x_5, x_5x_{10}, x_7x_8).$$

The graph $\Gamma(I_1)$ has the odd cycle $(3, 4, 9, 5, 6)$ of length 5 with no chord, and $\Gamma(I_2)$ has the odd cycle $(2, 3, 10, 7, 5)$ of length 5 with no chord. Hence, neither $\Gamma(I_1)$ nor $\Gamma(I_1)$ satisfies the condition $(*)$ of Lemma 2.1. Hence, $\mathcal{R}_5^{(2)}$ does not come from a poset, as required.

Now, if $d \geq 6$, then $\mathcal{R}_5^{(2)}$ is a combinatorial pure subring of $\mathcal{R}_d^{(2)}$. Hence, if $d \geq 6$, then $\mathcal{R}_d^{(2)}$ does not come from a poset by Proposition 2.2.

Since $\mathcal{R}_{q+2}^{(q)} \cong \mathcal{R}_{q+2}^{(2)}$, we know that $\mathcal{R}_{q+2}^{(q)}$ does not come from a poset if $q \geq 3$. Since $\mathcal{R}_{q+2}^{(q)}$ is a combinatorial pure subring of $\mathcal{R}_d^{(q)}$ if $d \geq q + 2$, it follows that $\mathcal{R}_d^{(q)}$

does not come from a poset if $q \geq 3$ and $d \geq q + 2$. □

The initial ideal I_1 of $\mathcal{R}_5^{(2)}$ in the above proof of Theorem 2.3 can be obtained by a reverse lexicographic term order as well as by a lexicographic term order. The initial ideal I_2 of $\mathcal{R}_5^{(2)}$ can be obtained by a lexicographic term order, but cannot be obtained by a reverse lexicographic term order since all variables x_i appear in the system of generators of I_2.

In [2], it is proved that all second squarefree Veronese subrings $\mathcal{R}_d^{(2)}$ are extendable sequentially Koszul. Hence, the infinite divisor posets of the second squarefree Veronese subrings are shellable ([2, Theorem 4.1]). However, the shellability of the infinite divisor posets of $\mathcal{R}_d^{(2)}$ cannot follow from [12] if $d \geq 5$. It remains open if the infinite divisor posets of *all* squarefree Veronese subrings $\mathcal{R}_d^{(q)}$ with $q \geq 3$ are shellable.

In the rest of this section, we will discuss the problem of finding all squarefree Veronese subrings which are Golod. If T is a homogeneous K-algebra, then we write $H(T) = H(\mathbf{x}; T)$ for the Koszul homology of T with respect to a K-basis of generators $\mathbf{x}$ of T_1. Recall that $H(T)$ is a skew symmetric graded K-algebra. This algebra is unique up to isomorphisms, i.e., it does not depend on the particular chosen basis of T_1. In the category of skew-symmetric algebras we can define an algebra retract just as in the commutative case.

Proposition 2.4. *Let $R \subset S$ be an algebra retract of homogeneous K-algebras. Then the inclusion $R \subset S$ induces an algebra retract $H(R) \subset H(S)$.*

Proof. Let $\varepsilon : S \to R$ be the retraction map. We may choose a K-basis $\mathbf{x}' = x_1, \ldots, x_n$ of S_1 such that for some $m \leq n$ the sequence $\mathbf{x} = x_1, \ldots, x_m$ is a K-basis of R_1, and such that $\varepsilon(x_i) = x_i$ for $i = 1, \ldots, m$, and $\varepsilon(x_i) = 0$ for $i = m+1, \ldots, n$. The natural inclusion $R \subset S$ induces an algebra homomorphism $\iota : H(\mathbf{x}; R) \to H(\mathbf{x}'; S)$ and the retraction map ε induces an algebra homomorphism $\eta : H(\mathbf{x}'; S) \to H(\varepsilon(\mathbf{x}'); R)$. By the choice of the basis $\mathbf{x}'$ we have $\varepsilon(\mathbf{x}') = x_1, \ldots, x_m, 0, \ldots, 0$. From this it follows easily that $H(\varepsilon(\mathbf{x}'); R)$ is isomorphic to the graded tensor product $H(\mathbf{x}; R) \otimes \bigwedge V$, where V is a K-vector space of dimension $n - m + 1$. In particular, $H(\mathbf{x}; R)$ is a subalgebra of $H(\varepsilon(\mathbf{x}'); R)$, and in fact is precisely the image of $\eta \circ \iota$, as desired. □

Corollary 2.5. *Let $R \subset S$ be an algebra retract of homogeneous K-algebras. Write $R = A/I$ and $S = B/J$, where A and B are polynomial rings over K, and I and J are graded ideals containing no forms of degree 1. Then for the graded Betti numbers of I and J we have*

$$\beta_{ij}^A(I) \leq \beta_{ij}^B(J) \quad \textit{for all} \quad i \quad \textit{and} \quad j.$$

Proof. The assertion follows from 2.4 and the fact that $\beta_{ij}^A(I) = \dim_K H_{i+1}(R)_j$, and $\beta_{ij}^B(J) = \dim_K H_{i+1}(S)_j$ for all i and j. □

Corollary 2.6. *Let $R \subset S$ be an algebra retract of homogenous K-algebras. If S is Golod, then R is Golod.*

Proof. By definition, S is Golod, if all Massey operations in the Koszul complex $K(S)$ vanish. The Massey operations $\mu(z_1, \dots, z_r)$ of order r (which are cycles in $K(S)$) are defined on all r-tuples of cycles z_i of $K(S)$, provided all Massey operations of order $r - 1$ are defined and are even boundaries (in which case one says that the Massey operations of order $< r$ vanish). Note that $\mu(z_1, z_2)$ is just $z_1 z_2$, so that $H(S)$ has trivial multiplication if S is Golod. We refer the reader to [6] for a full definition of Massey operations.

Now we wish to show that Massey operations in $K(R)$ of order r vanish for all r. We prove this by induction on r. Since $H(R)$ is a subalgebra of $H(S)$, it has trivial multiplication, too. Hence the Massey operations of order 2 vanish. Now suppose that $r > 2$, and that the Massey operations of order $r - 1$ vanish. Then $\mu(z_1, \dots, z_r)$ is defined and is a cycle. Since $\mu(z_1, \dots, z_r)$ is also a Massey operation in $K(S)$ and is a boundary in $K(S)$, and since the natural map $H(R) \to H(S)$ is injective, it follows that $\mu(z_1, \dots, z_r)$ is a boundary in $K(R)$, too, as we wanted to show. □

Corollary 2.7. *Let $2 \le q < d$. The q-th squarefree Veronese subring of order d is Golod if and only if $d = q + 1$.*

Proof. If $d = q + 1$, then $\mathcal{R}_{q+1}^{(q)}$ is a polynomial ring and the assertion is trivial. Note that $\mathcal{R}_4^{(2)}$ is a complete intersection defined by two quadratic equations. The Koszul homology of a complete intersection is the exterior algebra of the first Koszul homology, and hence, unless it is a hypersurface ring, has not trivial multiplication. It follows that $\mathcal{R}_4^{(2)}$ is not Golod. Now using Corollary 2.6 we argue as in the proof of Theorem 2.3 to get the desired result. □

3. Lawrence liftings of semigroup rings

Let, as before, $\mathcal{A} = \{f_1, \dots, f_n\}$ be a set of monomials of $K[\mathbf{t}] = K[t_1, \dots, t_d]$ and suppose that the affine semigroup ring $K[\mathcal{A}] = K[f_1, \dots, f_n]$ is a homogeneous semigroup ring. Let $I_{\mathcal{A}} \subset K[\mathbf{x}] = K[x_1, \dots, x_n]$ denote the defining ideal of $K[\mathcal{A}]$.

If $u \in K[\mathbf{x}]$ is a monomial, then we write $\operatorname{supp}(u)$ for the support of u, i.e., $\operatorname{supp}(u)$ is the set of variables x_i which divide u. If $g = u - v$ is a binomial of $K[\mathbf{x}]$, where u and v are monomials of $K[\mathbf{x}]$, then the support of g is $\operatorname{supp}(g) = \operatorname{supp}(u) \cup \operatorname{supp}(v)$.

A binomial $g = u - v \in I_{\mathcal{A}}$ is called *primitive* if there exists no binomial $g' = u' - v' \in I_{\mathcal{A}}$ with $g' \neq g$ such that u' divides u and v' divides v. The set of all primitive binomials of $I_{\mathcal{A}}$ is called the *Graver basis* of $I_{\mathcal{A}}$.

A binomial $g = u - v \in I_{\mathcal{A}}$ is called a *circuit* if g is irreducible and if there exists no binomial $g' = u' - v' \in I_{\mathcal{A}}$ with $\mathrm{supp}(g') \subset \mathrm{supp}(g)$ and with $\mathrm{supp}(g') \neq \mathrm{supp}(g)$.

The *universal Gröbner basis* of $I_{\mathcal{A}}$ is the union of all reduced Gröbner bases of $I_{\mathcal{A}}$. Every circuit of $I_{\mathcal{A}}$ belongs to the universal Gröbner basis of $I_{\mathcal{A}}$, and the universal Gröbner basis of $I_{\mathcal{A}}$ is a subset of the Graver basis of $I_{\mathcal{A}}$. See [13, Proposition 4.11].

Let $\Lambda(\mathcal{A}) = \{f_1 z_1, \ldots, f_n z_n, z_1, \ldots, z_n\}$, where $z_1, \ldots, z_n$ are variables over K. The homogeneous semigroup ring

$$K[\Lambda(\mathcal{A})] = K[f_1 z_1, \ldots, f_n z_n, z_1, \ldots, z_n]$$

is called the *Lawrence lifting* of $K[\mathcal{A}]$.

Let $K[\mathbf{x}, \mathbf{y}] = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ denote the polynomial ring in $2n$ variables over K. If $u = x_{i_1} x_{i_2} \cdots x_{i_k}$ is a monomial of $K[\mathbf{x}]$, then we write $\bar{u}$ for the monomial $y_{i_1} y_{i_2} \cdots y_{i_k}$ of $K[\mathbf{y}]$. Moreover, if $g = u - v$ is a binomial of $K[\mathbf{x}]$, then we define the binomial $\bar{g}$ of $K[\mathbf{x}, \mathbf{y}]$ by $\bar{g} = u\bar{v} - v\bar{u}$. It then follows that the defining ideal $I_{\Lambda(\mathcal{A})}$ of the Lawrence lifting $K[\Lambda(\mathcal{A})]$ of $K[\mathcal{A}]$ is generated by all binomials $\bar{g}$ with $g \in I_{\mathcal{A}}$. Moreover, the Graver basis of $I_{\Lambda(\mathcal{A})}$ coincides with the set of those binomials $\bar{g}$ such that g belongs to the Graver basis of $I_{\mathcal{A}}$, and the set of circuits of $I_{\Lambda(\mathcal{A})}$ coincides with the set of those binomials $\bar{g}$ such that g is a circuit of $I_{\mathcal{A}}$.

In the present section, we are interested in the question when the Lawrence lifting $K[\Lambda(\mathcal{A})]$ of $K[\mathcal{A}]$ is normal.

Lemma 3.1. *If $g = u - v$ is a binomial of $K[\mathbf{x}]$ such that neither u nor v is squarefree and if $I_{\mathcal{A}} = (g)$, then $K[\mathcal{A}]$ is not normal.*

Proof. Let $g = x_1^2 u' - x_2^2 v'$. Since $\pi(x_1^2 u') = \pi(x_2^2 v')$, we have $\sqrt{\pi(x_1^2 u')\pi(x_2^2 v')} = \pi(x_1^2 u')$; thus $\sqrt{\pi(u')\pi(v')} = \pi(x_1 u')/\pi(x_2)$. Hence, the monomial $\sqrt{\pi(u')\pi(v')}$ belongs to the quotient field of $K[\mathcal{A}]$ and is integral over $K[\mathcal{A}]$. Suppose that there exists a monomial w such that $\pi(w) = \pi(x_1 u')/\pi(x_2)$. It then follows that the binomial $g' = x_1 u' - x_2 w$ belongs to $I_{\mathcal{A}}$. Since the degree of g' is less than that of g, we have $g' = 0$. Hence, x_2 must divide u', which is impossible since g is irreducible. Thus, $K[\mathcal{A}]$ cannot be normal as required. □

Lemma 3.2. *If g is a circuit of $I_{\mathcal{A}}$, then there exists a combinatorial pure subring $K[\mathcal{B}]$ of $K[\Lambda(\mathcal{A})]$ with $I_{\mathcal{B}} = (\bar{g})$.*

Proof. Let $\{x_1, \ldots, x_m\}$ denote the support of g and $K[\mathcal{A}'] = K[f_1, \ldots, f_m]$. The defining ideal of $K[\mathcal{A}']$ is $I_{\mathcal{A}'} = I_{\mathcal{A}} \cap K[x_1, \ldots, x_m]$. First, we show that $I_{\mathcal{A}'} = (g)$. Let $g = x_1^p u - v$ with $x_1 \notin \mathrm{supp}(u)$. Let $h = x_1^q u' - v' \in I_{\mathcal{A}'}$ with $x_1 \notin \mathrm{supp}(u')$ be an irreducible binomial. Since both binomials $(x_1^p u)^q - v^q$ and $(x_1^q u')^p - v'^p$ belong to $I_{\mathcal{A}'}$, the binomial $(x_1^p u)^q v'^p - (x_1^q u')^p v^q$ belongs to $I_{\mathcal{A}'}$. Hence $u^q v'^p - u'^p v^q \in I_{\mathcal{A}'}$.

Since g is a circuit and since $x_1 \notin \mathrm{supp}(u^q v'^p - u'^p v^q)$, it follows that $u^q v'^p = u'^p v^q$. Since $\mathrm{supp}(u) \cap \mathrm{supp}(v) = \emptyset$ and $\mathrm{supp}(u') \cap \mathrm{supp}(v') = \emptyset$, we have $u^q = u'^p$ and $v^q = v'^p$. Hence, if $p \neq q$, say $p < q$, then we can find an integer $k > 1$ such that $h = x_1^q u' - v' = (x_1^{q'} u_1')^k - v_1'^k$. This is impossible since both binomials g and h are irreducible. Hence $p = q$ and we have $g = h$. Thus $I_{\mathcal{A}'} = (g)$ as desired.

Since $I_{\mathcal{A}'} = (g)$, the Graver basis of $I_{\mathcal{A}'}$ is equal to $\{g\}$. Now, let $K[\mathcal{B}]$ denote the subring $K[f_1 z_1, \ldots, f_m z_m, z_1, \ldots, z_m]$ of $K[\Lambda(\mathcal{A})]$. Then, $K[\mathcal{B}]$ is, in fact, a combinatorial pure subring of $K[\Lambda(\mathcal{A})]$. Since $K[\mathcal{B}]$ is the Lawrence lifting of $K[\mathcal{A}']$ and since the Graver basis of $I_{\mathcal{A}'}$ is $\{g\}$, the Graver basis of $I_{\mathcal{B}}$ is $\{\bar{g}\}$. Thus, in particular, $I_{\mathcal{B}} = (\bar{g})$ as desired. □

We say that a homogeneous semigroup ring $K[\mathcal{A}]$ is *unimodular* if all initial ideals of $I_{\mathcal{A}}$ are squarefree. It follows from [13, Remark 8.10] that $K[\mathcal{A}]$ is unimodular if and only if all triangulations of the configuration associated with $\mathcal{A}$ are unimodular. In addition, $K[\mathcal{A}]$ is unimodular if and only if all lexicographic initial ideals of $K[\mathcal{A}]$ are squarefree.

Even though the following criterion for $K[\mathcal{A}]$ to be unimodular must be well known, we will write its proof for the sake of completeness. A binomial $g = u - v$ is called *squarefree* if both the monomials u and v are squarefree.

Proposition 3.3. *A homogeneous semigroup ring $K[\mathcal{A}]$ is unimodular if and only if every circuit of $I_{\mathcal{A}}$ is squarefree.*

Proof. First, suppose that every circuit of $I_{\mathcal{A}}$ is squarefree. Let a binomial $g = u - v$ belong to a reduced Gröbner basis of $I_{\mathcal{A}}$. Then, by virtue of [13, Lemma 4.10], we can find a circuit $g' = u' - v'$ of $I_{\mathcal{A}}$ with $\mathrm{supp}(u') \subset \mathrm{supp}(u)$ and $\mathrm{supp}(v') \subset \mathrm{supp}(v)$. Since g' is squarefree, it follows that u' divides u and v' divides v. Since g is primitive, we have $g = g'$. Hence, every reduced Gröbner basis consists of squarefree binomials.

Second, let $g = u - v$ be a circuit of $I_{\mathcal{A}}$ such that the monomial u is not squarefree. Since g belongs to the universal Gröbner basis of $I_{\mathcal{A}}$, it follows from [13, Corollary 7.9] that there exists a term order $<$ on $K[\mathbf{x}]$ with $in_<(g) = u$ such that g belongs to the reduced Gröbner basis of $I_{\mathcal{A}}$ with respect to $<$. Hence, the initial ideal $in_<(I_{\mathcal{A}})$ is not squarefree. □

We are now in the position to give a main result of this section.

Theorem 3.4. *Let $K[\mathcal{A}]$ be a homogeneous semigroup ring and $K[\Lambda(\mathcal{A})]$ its Lawrence lifting. Then, the following conditions are equivalent:*

(i) *$K[\mathcal{A}]$ is unimodular;*

(ii) *$K[\Lambda(\mathcal{A})]$ is unimodular;*

(iii) $K[\Lambda(\mathcal{A})]$ *is normal.*

Proof. First of all, (ii) $\Rightarrow$ (iii) is well known. Since the set of circuits of $I_{\Lambda(\mathcal{A})}$ coincides with the set of those binomials $\bar{g}$ such that g is a circuit of $I_{\mathcal{A}}$, we have (i) $\Leftrightarrow$ (ii) by Proposition 3.3.

In order to show that (iii) $\Rightarrow$ (i), suppose that $K[\mathcal{A}]$ is not unimodular. Then, by Proposition 3.3 again, we can find a circuit $g = u - v$ of $I_{\mathcal{A}}$ such that either u is not squarefree or v is not squarefree. Then, each of the monomials $u\bar{v}$ and $v\bar{u}$ of the circuit $\bar{g} = u\bar{v} - v\bar{u}$ of $I_{\Lambda(\mathcal{A})}$ is not squarefree. Now, Lemma 3.2 guarantees the existence of a combinatorial pure subring $K[\mathcal{B}]$ of $K[\Lambda(\mathcal{A})]$ with $I_{\mathcal{B}} = (\bar{g})$. Then, by Lemma 3.1, $K[\mathcal{B}]$ is not normal. Hence, $K[\Lambda(\mathcal{A})]$ is not normal by Proposition 1.2 as required. $\square$

REMARK 3.5. Let (P) be a ring-theoretical property which is inherited by (i) combinatorial pure subrings, (ii) localizations and (iii) rings R such that the Laurent polynomial ring $R[z_1, {z_1}^{-1}, \dots, z_m, {z_m}^{-1}]$ over R has the property (P). Then, if the Lawrence lifting $K[\Lambda(\mathcal{A})]$ of $K[\mathcal{A}]$ has the property (P), then, for any subset $\mathcal{B} \subset \mathcal{A}$, the ring $K[\mathcal{B}]$ has the property (P). In fact, $K[\Lambda(\mathcal{B})]$ has the property (P) since it is a combinatorial pure subring of $K[\Lambda(\mathcal{A})]$. Inverting all z_i occurring in $K[\Lambda(\mathcal{B})]$ we get $K[\mathcal{B}][z_1, {z_1}^{-1}, \dots, z_m, {z_m}^{-1}]$ which has the property (P). Hence, $K[\mathcal{B}]$ has the property (P), as desired.

We conclude this paper with some examples of homogeneous semigroup rings which are unimodular.

EXAMPLE 3.6. (a) Let $\mathcal{R}_K[L]$ denote the monomial ASL (algebra with straightening laws) associated with a finite distributive lattice L discussed in, e.g., [9]. Then, $\mathcal{R}_K[L]$ is unimodular if and only if L is planar. See also [1].

(b) Let $K[G]$ denote the homogeneous semigroup ring arising from a finite connected graph G studied in, e.g., [10] and [11]. Then, $K[G]$ is unimodular if and only if any two cycles of odd length of G possess a common vertex. In particular, $K[G]$ is unimodular if G is bipartite.

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