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SELF-INJECTIVE QUOTIENT RINGS AND INJECTIVE QUOTIENT MODULES*

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1. Introduction

Until further notice, we assume that R is a ring (with unity) and S is a multiplicatively closed set of regular elements of R such that R satisfies the multiplicity condition with respect to S (for every a, s in R, s in S , there exist a_1, s_1 in R, s_1 in S such that $a s_1 = s a_1$). Let Q denote the (Asano's) quotient ring R_S of R . If 1 denotes the identity of $R, 1'$ the identity of Q , then $1.1' = 1.s.s^{-1} = (1.s) s^{-1} = s.s^{-1} = 1'$. Also $1.1' = 1$, because $1'$ is the identity of a bigger ring Q . So that the identities of the two rings coincide.

Let M be a (unital) right R -module. M is said to be S -free if $m s = 0, m \in M, s \in S$ implies $m = 0$. M is said to be S -divisible iff $M s = M \forall s \in S$. If M is both S -free and S -divisible R -module, then the module composition $M \times R \rightarrow M$ can be extended to $M \times Q \rightarrow M$ in one and only one way such that M becomes a Q -module, by defining $m(a.s^{-1}) = m'$ where m' is such that $m'.s = m.a$ (note that m' exists because of S -divisibility and is unique because of S -freeness). This composition is well defined, because if $a.s^{-1} = a_1.s_1^{-1}$, suppose $m'.s = m.a, m''.s_1 = m.a_1$. Now there exist $s_2, s_3, s_2 \in S$ such that $s.s_2 = s_1.s_3$. Then $a s_2 = a_1.s_3, m.a.s_2 = m.a_1.s_3, m'.s.s_2 = m''.s_1.s_3$ implies $m' = m''$. It is not very difficult to check that M is Q module with this composition.

However if M is S -free R -module, then there exists a Q -module M' such that $M_R \subset M'_R$ and $M' = MQ = \{m.s^{-1}; m \in M, s \in S\}$. This module M' is unique upto isomorphism over M . There are various construction for this module M' available in the literature.

Asano's construction. In $M \times Q$ define $(m, q) \sim (m', q')$ if there exists $s \in S$ such that $q.s \in R, q'.s \in R$ and $m(q.s) = m'(q'.s)$. The relation \sim can be verified to be an equivalence relation. In M' = the set of equivalence classes of $M \times Q$ define '+' and ' \cdot ' as follows: $\overline{(m, q)} + \overline{(m', q')} = \overline{(m(q.s) + m'(q'.s), s^{-1})}$ where $s \in S$ is such that $q.s \in R$ and $q'.s \in R$. $\overline{(m, q)} \cdot \overline{(m', q')} = \overline{(m, q q')}$. It can be verified that these compositions are well defined and M' is Q module. The mapping

* Part of the author's Doctoral Thesis.

$\sigma: m \rightarrow \overline{(m, 1)}$ is a R -isomorphism of M_R into M'_R so that identifying σM with M , we find that

$$\overline{(m, q)} = \overline{(m, a s^{-1})} = \overline{(m, a)} s^{-1} = \overline{(ma, 1)} s^{-1} = (m, a) s^{-1},$$

so that

$$M' = MQ = \{m s^{-1}: m \in M, s \in S\}.$$

Another construction of M' . Let $M' = M \otimes_R Q$. M' is a Q -module. The mapping $\sigma: m \rightarrow m \otimes 1$ embeds M_R into M'_R and

$$\sum_{i=1}^n (m_i \otimes q_i) = \sum_{i=1}^n (m_i \otimes a_i s_i^{-1}) = \sum_{i=1}^n (m_i \otimes a_i s_i^{-1}) s s^{-1},$$

where s is such that $a_i s_i^{-1} s \in R$ $i=1, 2, \dots, n$, so that

$$\begin{aligned} \sum_{i=1}^n (m_i \otimes q_i) &= \left(\sum_{i=1}^n m_i (a_i s_i^{-1} s) \right) \otimes s^{-1} = (m \otimes 1) s^{-1} \\ &= (\sigma m) s^{-1}, \quad \text{where } m = \sum_{i=1}^n m_i (a_i s_i^{-1} s). \end{aligned}$$

Note that if M is S -free R module, then $MQ = M$ iff M is S -divisible.

The starting point of this paper is the following result: If M is a Q -module, the M_Q is injective iff M_R is injective. This result is used to prove that if M is a Q module then injective dimension of $M_Q =$ injective dimension of M_R . The following corollary follows:

The right Global dim. of $Q \leq$ right Global dim. of R .

If M is any Q -module, then the injective hulls $E(M_Q)$ and $E(M_R)$ are seen to coincide. The following result is also proved: Every S -free S -divisible module over R is injective if and only if Q is semi-simple Artinian ring.

Similar results about quasi-injective modules have also been proved.

If M be an S -free R module, then the Q module M' (mentioned above) when regarded as a right R module is an essential extension of M . Necessary and sufficient condition on M such that M' becomes the injective hull of M are obtained. This result is applied to characterize rings whose classical quotient rings are quasi-Frobenius rings.

These results have also above been applied to obtain necessary conditions and sufficient conditions on a ring under which the Utumi's ring of quotients of the ring is an Asano's quotient ring. Necessary and sufficient conditions, when $R_v^\Delta = 0$, follow.

Hereditary orders in semi-simple Artinian rings have been characterized. These rings are found to 'resemble' Dedekind domains, which are precisely hereditary orders in commutative fields. A principal right ideal semi-prime ring is found to be a hereditary ring.

It is proved that if R is a semi-prime Goldie ring, and Q its semi-simple

Artinian classical right quotient ring, then Q_R is never projective except when $Q=R$.

Returning back to arbitrary R , with an Asano's quotient ring $Q=R_S$, if M be a projective R module, then trivially M is S -free. It is proved that $M' (=MQ)$ is a projective Q -module. This result is used to rederive our earlier result: right Global dim. $Q \leq \text{right Global dim. } R$.

Finally, necessary and sufficient conditions on R such that $Q(=R_S)$ becomes a hereditary ring are obtained. An immediate corollary thereof being: if R is right hereditary, then Q is also right hereditary.

2. Modules over Asano's quotient ring

2.1. Theorem. *Let R be a ring and S be a multiplicatively closed set of regular elements of R such that R satisfies the multiplicity condition with respect to S . Let Q denote the Asano's quotient ring R_S of R with respect to S . Let M be a module over Q . Then M is clearly a module over R , M is an injective Q -module if and only if M is an injective R -module.*

Proof. Let M be an injective Q module. Let $f: I_R \rightarrow M_R$, I a right ideal of R . We know that $IQ = \{a s^{-1} : a \in I, s \in S\}$ Define $f': IQ \rightarrow M$ as follows:

$$f'(a s^{-1}) = f(a) s^{-1}$$

f' is well defined for if

$a s^{-1} = a_1 s_1^{-1}$, $a, a_1 \in I$, $s, s_1 \in S$, then there exist $s_2 \in S$ such that (see Asano [1]) $s^{-1} s_2 \in R$ and $s_1^{-1} s_2 \in R$. Clearly $a(s^{-1} s_2) = a_1(s_1^{-1} s_2)$, therefore $f(a)(s^{-1} s_2) = f(a_1)(s_1^{-1} s_2)$.

Post multiplying by s_2^{-1} we get $f(a) s^{-1} = f(a_1) s_1^{-1}$.

We check that f' is a Q -homomorphism of IQ into M .

$$f'(a s^{-1} + a' s'^{-1}) = f'((a s^{-1} + a' s'^{-1}) s_1 s_1^{-1})$$

where s_1 is an element of S such that $s^{-1} s_1, s'^{-1} s_1 \in R$

$$\begin{aligned} &= (f(a s^{-1} s_1 + a' s'^{-1} s_1)) s_1^{-1} \\ &= (f(a) s^{-1} s_1 + f(a') s'^{-1} s_1) s_1^{-1} \\ &= f(a) s^{-1} + f(a') s'^{-1} \\ &= f'(a s^{-1}) + f'(a' s'^{-1}) \end{aligned}$$

$$\begin{aligned} f'(a s^{-1} \cdot r s_1^{-1}) &= f'(a r_1 s_2^{-1} s_1^{-1}) \quad \text{where } s^{-1} r = r_1 s_2^{-1} \\ &= f(a r_1) (s_1 s_2)^{-1} \\ &= f(a) r_1 s_2^{-1} s_1^{-1} \\ &= f(a) s^{-1} r s_1^{-1} \\ &= f'(a s^{-1}) \cdot r s_1^{-1} \end{aligned}$$

M being an injective Q -module there exists $m \in M$ such that

$$f'(x) = mx \quad \forall x \in IQ.$$

But $f'(x) = f(x) \quad \forall x \in I$. Therefore $f(x) = mx \quad \forall x \in I$.

Conversely suppose M_R is injective. Let $f: I \rightarrow M$, be Q -homomorphism, where I is a right ideal of Q . Then let $J = I \cap R$. J is a right ideal of R and $JQ = I$. Let f' denote the restriction of f to J . Then f' is clearly a R -homomorphism of J into M . As M is an injective R -module, there exists $m \in M$ such that $f'(x) = mx \quad \forall x \in J$. Any element of I is of the form xs^{-1} , $x \in J$, $s \in S$.

$$f(xs^{-1}) = f'(x)s^{-1} = m(xs^{-1})$$

Hence M is an injective Q -module.

2.2. Theorem. *Let M be a module over Q , then M_Q is quasi-injective iff M_R is quasi-injective.*

Proof. Assume that M is a quasi-injective module over Q . Let $f: N \rightarrow M$ be a R -homomorphism where N is a R -submodule of M . We know that $NQ = \{ns^{-1} : n \in N, s \in S\}$

Define a mapping $f': NQ \rightarrow M$

$$f'(ns^{-1}) = f(n)s^{-1}.$$

It can be checked that f' is well defined and f' is a Q -homomorphism of NQ into M . Also f' coincides with f on N . As M is Q -quasi-injective, therefore there exists an extension f'' of f' such that $f'' \in \text{Hom}_Q(M, M)$. Clearly $f'' \in \text{Hom}_R(M, M)$ and f'' is an extension of f .

Conversely suppose that M_R is quasi-injective. Let $f: N \rightarrow M$ be a Q -homomorphism, where N is Q -submodule of M . N is also R -submodule of M . There exists $g \in \text{Hom}_R(M, M)$ such that g coincides with f on N . We prove that g is in fact a Q -homomorphism.

$$g(mrs^{-1})s = g(mrs^{-1}s) = g(mr) = g(m)r.$$

Therefore

$$g(mrs^{-1}) = g(m)r s^{-1} \quad \forall r \in R, s \in S, m \in M.$$

2.3. Theorem. *Every S -divisible S -free module over R is injective iff $Q (=R_S)$ is a semi-simple Artinian ring.*

Proof. Let every S -free S -divisible module over R be injective. Then in order to prove that Q is semi-simple Artinian, we shall prove every module over Q is injective. Let M be any Q module, then M is S -free and S -divisible R -module ($ms=0$ implies $ms^{-1}=0$ $m=0$, $M s \supset (M s^{-1})s = M \supset M s$, $M = M s \quad \forall s$

$\in S$). Therefore M is an injective R -module. By theorem 2.1 M is an injective Q module. Hence Q is semi-simple Artinian, see Cartan-Eilenberg [3, page 11, Theorem 4.2].

Conversely let Q be a semi-simple Artinian ring, then if M be a S -free and S -divisible R -module, M can be regarded as a Q -module, the module composition $M \times Q \rightarrow M$ being such that it extends the original module composition $M \times R \rightarrow M$, in one and only one way (see introduction). Q being semi-simple Artinian, any module over Q is injective, therefore M is an injective Q -module. Consequently M is an injective R -module by theorem 2.1.

2.4. DEFINITION. Let M be a module over R . M is said to be a torsion free module if $mx=0, m \in M, x$ regular in R implies $m=0$. M is said to be a divisible module if $Mx=M \forall$ regular element x in M .

2.5. Corollary. (Levy, 1963) *If R be a ring having a classical right quotient ring Q , then every torsion free divisible R -module over R is injective if and only if Q is semi-simple Artinian.*

2.6. Theorem. *Let R be a ring and $Q(=R_S)$ be an Asano's quotient ring of R with respect to a set S of regular elements of R . Then every S -free S -divisible module over R is quasi-injective iff Q is semi-simple Artinian.*

Proof. In the proof we use the following result of Faith and Utumi: [6, page 169, Cor. 2.4]. A ring Q is semi-simple Artinian if and only if every module over Q is quasi-injective.

Assume that every S -free S -divisible R -module is quasi-injective. Let M be any module over Q . Then M is S -free, S -divisible module over R . Therefore M is a quasi-injective R -module. Therefore by theorem 2.2 M is a quasi-injective Q module. Hence Q is semi-simple Artinian.

Converse is proved in the previous theorem 2.3.

2.7. Corollary. *If R is a ring with a right classical quotient ring Q , then every torsion-free divisible R -module is quasi-injective if and only if Q is semi-simple Artinian.*

2.8. Theorem. *Let $Q(=R_S)$ be an Asano's quotient ring of R . Let M be a Q -module, then if E is the injective hull of M_Q then E_R = the injective hull of M_R .*

Proof. E_Q being an injective module, E_R is an injective module by 2.1. E_R is an essential extension of M_R , because if $0 \neq m \in E$, then $mQ \cap M \neq 0, 0 \neq mrs^{-1} \in M$ for some r in R, s in $S, 0 \neq m r \in M$. Therefore $mR \cap M \neq 0 \forall m \neq 0$ in E . Hence E_R is an injective hull of M_R .

2.9. Quasi-injective hull. The concept of quasi-injective hull of a module was introduced by Johnson and Wong [15] who proved that if M is any

module, then $M' = \Lambda M$, where $\Lambda = \text{Hom}_R(E, E)$, E being the injective hull of M is the unique minimal quasi-injective essential extension of M . Faith and Utumi [6] proved that M is in fact a unique minimal quasi-injective extension by observing that complement (closed) submodules of a quasi-injective module are quasi-injective, see Faith and Utumi [6, Corollary 2.2].

2.10. Theorem. *If $Q (=R_S)$ be an Asano's quotient ring of R and M be a Q -module, then quasi-injective hull of $M_Q =$ quasi-injective hull of M_R .*

Proof. Let E_Q be an injective hull of M_Q . Then E_R is an injective hull of M_R by 2.8. Now note that $\text{Hom}_Q(E, E) = \text{Hom}_R(E, E)$, because if $f \in \text{Hom}_R(E, E)$, then $f(mrs^{-1})s = f(mr) = f(m)r \forall r \in R, s \in S$, therefore $f(mrs^{-1}) = f(m)r s^{-1}$. Let $\Lambda = \text{Hom}_Q(E, E) = \text{Hom}_R(E, E)$. The quasi-injective hull of M_Q is $\Lambda M =$ the quasi-injective hull of M_R .

2.11. DEFINITION. Injective resolution of a module.
Injective dimension of a module.

Let R be an arbitrary ring. Let M_R be a module over R . An exact sequence

$$0 \rightarrow M \xrightarrow{\cdots} M_0 \xrightarrow{d_0} M_1 \xrightarrow{d_1} M_2 \xrightarrow{d_2} M_3 \rightarrow \cdots \rightarrow M_{n-1} \xrightarrow{d_{n-1}} M_n \rightarrow \cdots$$

where each M_i is injective is called an injective resolution of M . The least integer n such that kernel d_n is injective is called the injective dimension of M . If no such integer n exists, then the injective dimension of M is defined to be ∞ .

It has to be noticed that the injective dimension of a module is independent of the injective resolution.

2.12. Theorem. *Let $Q (=R_S)$ be an Asano's quotient ring of a ring R . If M be a module over Q , then injective dimension $(M_Q) =$ injective dimension (M_R) .*

Proof. Let

$$0 \rightarrow M \xrightarrow{\cdots} M_0 \xrightarrow{d_0} M_1 \xrightarrow{d_1} M_2 \rightarrow \cdots \rightarrow M_{n-1} \xrightarrow{d_{n-1}} M_n \rightarrow \cdots$$

be an injective resolution of the module M_Q . Then by 2.1 this is also an injective resolution of the module M_R . If $\ker d_n$ is never Q injective, then the $\ker d_n$ is never R -injective by 2.1, so that if $\dim M_Q = \infty$, then $\dim M_R = \infty$. If however $\dim M = n$, then $\ker d_n$ is Q injective and therefore it is R -injective. $\ker d_m, m < n$ cannot be R -injective, because if it is R -injective, then it will be Q injective by 2.1, which is a contradiction. Hence injective dimension $M_Q =$ injective dimension M_R .

2.13. DEFINITION. If R is any ring, then right Global dimension of $R =$ supremum of the injective dimension of all R modules.

2.14. Theorem. *If $Q (=R_S)$ be an Asano's quotient ring of R , then right Global dimension of $Q \leq$ right Global dimension of R .*

Proof. Immediate from 2.12.

3. Let R be a ring and $Q (=R_S)$ be the Asano's quotient ring of R with respect to a set S of regular elements of R . If M be an S -free module over R , then two known constructions for the 'quotient module' $M' (=MQ)$ have been outlined in the introduction. A still new construction for this module M , which seems to be more natural is given below. But before we give this construction we observe one lemma which in essence is due to Levy [18].

3.1. Lemma. *If S be a multiplicatively closed set of regular elements of a ring R , then R_S exists (i.e. R satisfies the multiplicity condition with respect to S) if and only if for every R -module M , $T(M) = \{m \in M, ms = 0 \text{ for some } s \in S\}$ is a submodule of M .*

Proof. This result can be proved exactly as in, Levy [18, theorem 1.4, page 134].

3.2. Lemma. *Every injective module is divisible.*

Proof. Let M be an injective R -module. Let a be a regular element of R . Let $m \in M$. The mapping $f: aR \rightarrow M$ such that $f(ar) = mr \forall r \in R$ can be realized by an element m' of M . $m'ar = mr \forall r \in R$. Hence $m'a = m$. Hence $Ma = M$.

We are now ready to give another construction of the quotient module MQ .

3.3. Let R be a ring and $Q (=R_S)$ be the Asano's right quotient ring of R with respect to S . Let M be a S -free right R module. Let E denote the injective hull of M_R . E is S -divisible by 3.2. E is S -free, because $T(E) = \{m \in E: ms = 0 \text{ for some } s \in S\}$ is a submodule of E by 3.1. But $T(E) \cap M = 0$ because M is S -free. Therefore $T(E) = 0$. Therefore E is an S -free S -divisible module, therefore E becomes a Q -module (see introduction) Let $M' = MQ \subset E$. Then M' is the required module over Q such that $M_R \subset M'_R$ and $M' = MQ$.

In view of this construction of MQ , a natural question arises when is $MQ = E(M)$? The following theorem answers this question.

3.4. Theorem. *Let R be a ring and $Q (=R_S)$ be an Asano's quotient ring of R . Let M be an S -free module over R . Then the 'quotient' module $M' (=MQ)$ is injective Q -module (or injective R -module see 2.1) iff every R -homomorphism $f: I \rightarrow M$, I a right ideal of R can be extended to a R -homomorphism $g: J \rightarrow M$, where J is a right ideal of R containing I and containing an element of S .*

Proof. Let M_R satisfy the condition. We prove that $M' = MQ$ is an

injective R -module. Let there be a R -homomorphism $f: I \rightarrow MQ$, where I is a right ideal of R .

Set $I' = \{x \in I: f(x) \in M\}$. I' is a right ideal of R , $I' \subset I$. We claim that for each $r \in I$, there exists $s \in S$ such that $rs \in I'$. Let $r \in I$. Then $f(r) \in MQ$. Therefore $f(r) = ms^{-1}$, where $m \in M$ and $s \in S$. Then $f(rs) = f(r)s = ms^{-1}s = m \in M$. $rs \in I'$. Let f' denote the restriction of f to I' . There exists by hypothesis $g: J \rightarrow M$, where J is a right ideal of R containing I and containing an element of S and $g(x) = f'(x) \forall x \in I'$. As J contains an element of S , $JQ = Q$. There exists an extension $g': JQ \rightarrow MQ$ of g defined as follows:

$$g'(js^{-1}) = g(j)s^{-1}.$$

The mapping g' is a Q -homomorphism. Let $g'(1) = m$. Let $r \in I$. There exists $s \in S$ such that $rs \in I'$.

$$f(r)s = f(rs) = f'(rs) = g(rs) = g'(rs) = g'(1).rs = (m.r).s$$

Therefore $f(r) = mr \forall r \in I$.

Conversely, suppose MQ be an injective R -module (injective Q -module). Let there be a R -homomorphism $f: I \rightarrow M$, where I is a right ideal of R . As MQ is R -injective and $M \subset MQ$, therefore there exists an element ms^{-1} of MQ such that

$$ms^{-1}x = f(x) \forall x \in I.$$

Let $J = sR + I$. Define $g: J \rightarrow M$ as follows:

$$g(sr + x) = ms^{-1}(sr + x) = mr + ms^{-1}x \forall r \in R, x \in I.$$

Surely g is a R -homomorphism of J into M . J contains I and an element s of S . Also $g(x) = ms^{-1}x = f(x) \forall x \in I$.

Self injective quotient rings

3.5. Corollary. *A quotient ring $Q (= R_S)$ of R is self injective (Q_R is injective) if and only if for every R -homomorphism $f: I \rightarrow R$, I a right ideal of R , there exists a R -homomorphism $g: J \rightarrow R$, where J is a right ideal of R containing I and an element of S and g is such that*

$$g(x) = f(x) \forall x \in I.$$

3.6. Corollary. *Let R be a ring having a classical right quotient ring Q . The ring Q is self injective (Q_R is injective or Q_R is the injective hull of R_R) iff every R -homomorphism*

$$f: I \rightarrow R,$$

where I is a right ideal of R can be extended to a R -homomorphism

$$g: J \rightarrow R$$

where J is a right ideal of R containing I and a regular element of R .

3.7. REMARK. Semi-prime right Goldie rings form a class of rings which possess self injective classical quotient rings. One is tempted to verify whether this condition is really satisfied by the class of rings.

If $f: I \rightarrow R$, then let J be a complement of I in R , so that $I \oplus J$ is an essential right ideal of R . $I \oplus J$ contains a regular element of R . Goldie [9, Theorem 3.9]. One can trivially extend f to $f': (I+J) \rightarrow R$ by defining $f(i+j)=f(i) \forall i \in I, j \in J$.

If Q is a classical quotient of a ring R , we say that R is an order in Q .

In view of the corollary 3.6 and the characterization of orders in perfect rings, orders in semi-primary rings, orders in Artinian rings and Noetherian orders in Artinian rings given in [10] and [14] we have the following four results:

Orders in self-injective perfect rings

3.8. Theorem. *A ring R is an order in a (left) perfect self injective ring iff*

- (i) $N(R)$, the upper nil radical of R is left T -nilpotent.
- (ii) $R/N(R)$ is a right Goldie ring.
- (iii) $a_\alpha R_\alpha$ is an essential right ideal of R_α , for every right regular element a_α in R_α and for every ordinal number α , where $R_\alpha=R/T_\alpha$, $a_\alpha=a+T_\alpha(R)$, $T_\alpha(R)$, an ideal of R defined as follows:

$$T_0(R) = 0, T_{\alpha+1}(R) = \{x: x \in N(R), xN(R) \subset T_\alpha(R)\},$$

for an ordinal number of the type $\alpha+1$.

$$T_\alpha(R) = \bigcup_{\beta < \alpha} T_\beta(R),$$

for a limit ordinal α .

- (iv) If $a+N(R)$ is regular in $R/N(R)$ then a is regular in R .
- (v) Every R -homomorphism $f: I \rightarrow R$, I a right ideal of R can be extended to a R -homomorphism $g: J \rightarrow R$, where J is a right ideal of R containing I and a regular element of R .

Orders in self-injective semi-primary rings

3.9. Theorem. *A ring R is an order in a self-injective semi-primary ring if and only if*

- (i) $N(R)$, the upper nil radical of R is nilpotent.
- (ii) $R/N(R)$ is a right Goldie ring.
- (iii) $a_i R_i$ is essential right ideal of R_i for every right regular element a_i in R_i and for every integer $i \geq 0$, where $R_i=R/T_i$, $a_i=a+T_i$, T_i being a two sided ideal of R defined as follows: $T_0=(0)$, $T_i=l(N(R)^i) \cap N(R)$ for $i \geq 1$,

- (iv) If $a+N(R)$ is regular in $R/N(R)$, then a is regular in R .
 (v) Every R -homomorphism $f: I \rightarrow R$, I a right ideal of R , can be extended to a R -homomorphism $g: J \rightarrow R$, where J is a right ideal of R containing I and a regular element of R .

DEFINITION. A ring R such that $I^r = I \forall$ right ideal I and $L^r = L \forall$ left ideal of L and R satisfies the d.c.c. on left and right ideals is called a quasi-Frobenius ring.

It is well known that a ring R is quasi-Frobenius iff R is Artinian and right self-injective see Faith [5], Eilenberg-Nakayama [4].

Orders in quasi-Frobenius rings

3.10. Theorem. A ring R is an order in a quasi-Frobenius ring iff

- (i) $N(R)$, the upper nil radical of R is nilpotent.
 (ii) $R/N(R)$ is a right Goldie ring.
 (iii) R/T_i has no infinite direct sum of right ideals for every $i \geq 0$, where T_i is defined as follows:

$$T_0 = (0), T_i = l(N(R)^i) \cap N(R) \quad i \geq 1.$$

- (iv) If $a+N(R)$ is regular in R , then a is regular in R .
 (v) Every R -homomorphism $f: I \rightarrow R$, I a right ideal of R , can be extended to $g: J \rightarrow R$, where J is a right ideal of R containing I and a regular element of R .

3.11. Theorem. A Noetherian ring R is an order in a quasi-Frobenius ring if and only if

- (i) If $a+N(R)$ is regular in $R/N(R)$, then a is regular in R .
 (ii) Every R -homomorphism $f: I \rightarrow R$, I a right ideal of R , can be extended to a R -homomorphism $g: J \rightarrow R$, where J is a right ideal of R containing I and a regular element of R .

(In connection with Theorem 3.10, the author wishes to point out that when this manuscript was ready, the author received an unpublished paper entitled 'Orders in quasi-Frobenius Rings' from Professor J.P. Jans wherein the following result is proved:

A ring R is an order in a quasi-Frobenius ring iff

- (i) R has no infinite direct sum of right ideals.
 (ii) $A_r(E(R), R)$ satisfies a.c.c. where $E(R)$ is the injective hull of R_R , $A_r(E(R), R) = \{S^\perp: S \subset E(R)\}$, $S^\perp = \{x \in R, Sx = 0\}$ for subsets S of $E(R)$.
 (iii) $T(E(R)/R) = E(R)/R$, where for any R -module M , $T(M) = \{m \in M: ma = 0 \text{ for some regular element } a \text{ in } R\}$.
 (iv) If M is a finitely generated (cyclic) R -module such that $T(M) = (0)$, then $E(M)$, the injective hull of $M \subset (E(R))^n$.

Semi-prime right Goldie rings belong to the class of rings R for which

$Q(R)$, the classical right quotient ring of R exists, $Q_R = E_R$, the injective hull of R_R and E_R is Σ -injective, see Faith [5, page 189, Corollary 3]. (An injective module M_R is said to be Σ -injective iff the direct sum of arbitrarily many copies of M_R is an injective R -module). The following theorem precisely determines this class of rings:

3.12. Theorem. *For a ring R , $Q(R)$ exists, $Q_R = E_R$, the injective hull of R_R , E_R is Σ -injective iff R is an order in a quasi-Frobenius ring.*

Proof. If R is an order in a quasi-Frobenius ring, then let $Q = Q(R)$, Q is quasi-Frobenius. Q_Q is injective. Therefore Q_R is injective by 2.1. $Q_R = E_R$, the injective hull of R_R . Now Q_Q is Σ -injective, because Q is quasi-Frobenius (Q is quasi-Frobenius iff every projective module over Q is an injective Q module, see Faith [5]). Therefore Q_R is Σ -injective by 2.1. Hence E_R is Σ -injective. Conversely let us assume $Q(R)$ exists, $Q_R = E_R$ and E_R is Σ -injective, then Q_R is injective and Q_R is Σ -injective. Then Q is self injective and Q_Q is Σ -injective by 2.1. Hence any free module over Q is injective. Therefore any projective Q module is Q injective. Hence Q is quasi-Frobenius.

3.13. Theorem. *If R has a classical right quotient ring Q , then Q is quasi-Frobenius iff every projective Q module is R -injective.*

Proof. Let Q be quasi-Frobenius. Let M be a projective module over Q , then by Faith [5], M is an injective Q module. Therefore by 2.1 M is an injective R module. Conversely suppose that any projective module over Q is an injective R module, then any projective module over Q is an injective Q module. Hence Q is quasi-Frobenius, Faith [5].

Faith and Walker [8] proved that a ring Q is quasi-Frobenius iff every injective module over Q is projective. From this and with the help of 2.1 we obtain the following theorem:

3.14. Theorem. *Let R be a ring having a classical right quotient ring Q . Then Q is quasi-Frobenius iff every torsion free injective R -module is a projective Q -module.*

Utumi's quotient ring and Johnson's quotient ring

3.15. Theorem. *Let R be a ring such that R has a multiplicatively closed set S of regular elements with respect to which R satisfies the multiplicity condition and every $f: I \rightarrow R$, I a right ideal of R , can be extended to $g: J \rightarrow R$, where J is a right ideal of R containing I and an element of S , then $Q (= R_S)$ is the Utumi's quotient ring of R .*

Proof. Lambek [16] proved that Q the Utumi's ring of quotients of R has the following characterization. $Q = \text{Bicommutant of } E_R$ where E_R is the injective

hull of R_R . In our case by 3.5 Q_Q is injective. Therefore by 2.1, Q_R is injective. Also $R_R \subset Q_R$. Therefore Q_R is the injective hull of R_R . Now $\text{Hom}_R(Q_R, Q_R) \cong Q$. Also $\text{Hom}_Q(Q, Q) \cong Q$. Therefore Q is the Utumi's quotient ring of R .

Conversely we have the following result:

3.16. Theorem. *If R is such that Q_R , where Q is the Utumi's quotient ring of R , is the injective hull of R_R (see, Lambek [17, page 95]) and Q is an Asano's quotient ring of R , then there exists a multiplicatively closed set S of regular elements of R such that R satisfies the multiplicity condition with respect to S and every R -homomorphism $f: I \rightarrow R$, I a right ideal of R , can be extended to $g: J \rightarrow R$, where J is a right ideal of R containing I and an element of S .*

Proof. Let $S = \{x \in R: x \text{ invertible in } Q\}$. Then $Q = R_S$ and the rest follows, because Q_R is injective by 3.5.

Combining the previous two theorems and noting that the Utumi's quotient ring of a ring R coincides with \hat{R} , the Johnson's maximal right quotient ring of R , if $R_r^\Delta = 0$, and that \hat{R}_R , where \hat{R} denotes the Johnson's maximal quotient ring of R , is the injective hull of R_R , we notice the following result:

3.17. Theorem. *If R is a ring with $R_r^\Delta = 0$, then \hat{R}_R , the Johnson's quotient ring of R is an Asano's quotient ring of R iff there exists a multiplicatively closed set S regular elements of R such that (i) R satisfies the multiplicity condition with respect to S and (ii) every R -homomorphism $f: I \rightarrow R$, I a right ideal of R , can be extended to a R -homomorphism $g: J \rightarrow R$, where J is a right ideal of R containing I and an element of S . In this case $\hat{R}_R = R_S$.*

3.18. Corollary. *If R is a ring such that $R_r^\Delta = 0$, and $Q(R)$ the classical right quotient ring of R exists, then $\hat{R} = Q(R)$ iff every $f: I \rightarrow R$, I a right ideal of R , can be extended to $g: J \rightarrow R$, where J contains I and a regular element of R .*

4. If R is a commutative integral domain and Q its field of quotients, then Q_R is projective implies $Q = R$, see Tsi-Che-Te [20, page 174]. The aim of the present section is to generalize this result from commutative integral domain to an arbitrary semi-prime right Goldie ring.

4.1. Theorem. *Let R be a semi-prime ring with the Goldie conditions on right ideals and let Q be its semi-simple Artinian classical right quotient ring. If Q_R is projective, then $R = Q$.*

Lemma. *Let R be a prime ring with right Goldie conditions and Q its classical right quotient ring. If there exists $0 \neq q$ in Q such that $q \cdot Q \subset R$, then $R = Q$.*

Proof. By a theorem of Faith and Utumi [7, Theorem 3, page 56], there exists a complete set $M = \{e_{ij}: i, j = 1, 2, \dots, n\}$ of matrix units in Q with the following property: If D is the centralizer of M in Q , then D is a division ring and

$$Q = \sum_{i,j=1}^n D e_{ij} \supset R \supset \sum_{i,j=1}^n F e_{ij}$$

where F is a right Ore-domain contained in $R \cap D$ and D is the right quotient field of F . (It is to be noted that F in general does not contain identity, see example given by Faith and Utumi [7, (C) page 60]) Let

$$q = \sum_{i,j=1}^n \alpha_{ij} e_{ij}, \alpha_{ij} \in D.$$

Suppose $\alpha_{kl} \neq 0$. Fix a non-zero element a of F . $a e_{rk} \in R$ for every $r=1, 2, \dots, n$. Also because $q Q \subset R$,

$$\left(\sum_{i,j=1}^n \alpha_{ij} e_{ij} \right) (a \cdot \alpha_{kl})^{-1} d e_{ls} \in R$$

for every $s=1, \dots, n$ and every $d \in D$. Therefore

$$a e_{rk} \left(\sum_{i,j=1}^n \alpha_{ij} e_{ij} \right) (a \alpha_{kl})^{-1} d e_{ls} \in R$$

for every $r, s=1, 2, \dots, n$ and every $d \in D$. But

$$a e_{rk} \left(\sum_{i,j=1}^n \alpha_{ij} e_{ij} \right) (a \alpha_{kl})^{-1} d e_{ls} = d e_{rs}.$$

Therefore $\sum_{i,j=1}^n D e_{rs} = Q \subset R$. Hence $Q = R$.

4.2. Lemma. *If Q be an Asano's right quotient ring of R and $f \in \text{Hom}_R(Q, R)$ then $f(q) = f(1) \cdot q$ for every $q \in Q$.*

Proof. Let $q = a b^{-1}$, a, b in R , b regular. Then

$$f(q)b = f(a b^{-1})b = f(a b^{-1}b) = f(a) = f(1)a.$$

Therefore $f(q) = f(1)a b^{-1} = f(1)q$.

Proof of the Theorem. A module M_R is projective iff there exist subsets $\{m_i\}_{i \in I}$ of M and $\{f_i\}_{i \in I}$ of $\text{Hom}_R(M, R)$, such that for each $m \in M$ $f_i(m) = 0$ for almost all i and $m = \sum_{i \in I} m_i f_i(m)$, see Bass [2, (4.8) page 477].

As Q_R is projective there exists subsets $\{b_i\}_{i \in I}$ of Q and $\{f_i\}_{i \in I}$ of $\text{Hom}_R(Q, R)$ with the above properties. Now

$$l = b_1 f_1(1) + b_2 f_2(1) + \dots + b_n f_n(1)$$

for finite subset $(1, 2, \dots, n)$ of I . By the above lemma 4.2 $f_k(1) Q \subset R$ for each $k=1, 2, \dots, n$. Let e_1, e_2, \dots, e_m the central idempotents in Q such that $e_1 Q, e_2 Q, \dots, e_m Q$ are the simple components of the semi-simple Artinian ring Q . Clearly for each $i=1, \dots, m$, there exists $k(i)$, $1 \leq k(i) \leq n$ such that $f(1)e_i \neq 0$. But

$$(f(1)e_i)e_i Q \subset e_i R \cap R.$$

But $e_i Q$ is the classical quotient ring of $e_i R$ and $e_i Q$ is a simple Artinian ring, therefore by the lemma $e_i Q = e_i R \subset R$. Therefore $e_i Q \cap R$ for every i . Therefore $Q = R$.

5. Hereditary orders in semi-simple Artinian rings

In this section right hereditary semi-prime right Goldie-rings have been characterized.

5.1. Theorem. *If R be a semi-prime right Goldie ring, and Q be its classical right quotient ring, then R is hereditary if and only if for every essential right ideal I of R , there exist b_1, b_2, \dots, b_n in I and $\alpha_1, \alpha_2, \dots, \alpha_n$ in Q such that $\sum_{i=1}^n b_i \alpha_i = 1$ and $\alpha_i I \subset R$ for every $i=1, 2, \dots, n$.*

Proof. Assume R is hereditary. Let I be an essential right ideal of R . I is a projective right R module. By the characterization of projective modules mentioned in the proof of Theorem 4.1, there exist subsets $\{b_j\}_{j \in J}$ of I and $\{f_j\}_{j \in J}$ of $\text{Hom}_R(I, R)$ such that for every $b \in I$, $f_j(b) = 0$ for almost all values of j and $b = \sum_{j \in J} b_j f_j(b)$. Now I being an essential right ideal of R , I contains a regular element, Goldie [9, Theorem 3.9]. Therefore $IQ = Q$. Each f_j has a unique extension f_j' in $\text{Hom}_Q(Q, Q)$. Let $f_j'(1) = \alpha_j$. Then $f_j'(q) = f_j'(1 \cdot q) = f_j'(1)q = \alpha_j(q)$ for every q in Q . Therefore $\alpha_j b = f_j(b) = f_j(b)$ for every $b \in I$. Consequently $\alpha_j I \subset R$. Now let a be a regular element of I .

$$a = \sum_{j=1}^n b_j f_j(a) = \left(\sum_{j=1}^n b_j \alpha_j \right) a$$

for some finite subset $(1, 2, \dots, n)$ of J . Therefore $1 = \sum_{i=1}^n b_i \alpha_i$. The sets (b_1, b_2, \dots, b_n) and $(\alpha_1, \alpha_2, \dots, \alpha_n)$ are the desired sets.

Conversely since every right ideal is a direct summand of an essential right ideal and a direct summand of a projective module is projective, it is sufficient to show that every essential right ideal is projective. Let I be an essential right ideal of R . There exist b_1, b_2, \dots, b_n in I and $\alpha_1, \alpha_2, \dots, \alpha_n$ in Q such that $\alpha_i I \subset R$ for every $i=1, 2, \dots, n$ and $\sum_{i=1}^n b_i \alpha_i = 1$. Define f_1, f_2, \dots, f_n such that $f_i(b) = \alpha_i b$, $\forall b \in I$, $f_i \in \text{Hom}_R(I, R)$. Also $b = \left(\sum_{i=1}^n b_i \alpha_i \right) b = \sum_{i=1}^n b_i (\alpha_i b) = \sum_{i=1}^n b_i f_i(b)$ $\forall b \in I$. Hence I is projective by the characterization of a projective module mentioned in the proof of Theorem 4.1. (The 'only if' part of this result is due to Levy [18]).

5.2. Corollary. (Levy, 1963) *A hereditary semiprime right Goldie ring is a right Noetherian ring.*

Proof. Since every right ideal is a direct summand of an essential right ideal, it is sufficient to show that every essential right ideal is finitely generated. Let I be an essential right ideal. There exist b_1, b_2, \dots, b_n in I and $\alpha_1, \alpha_2, \dots, \alpha_n$ in Q such that $\sum_{i=1}^n b_i \alpha_i = 1$ and $\alpha_i I \subset R$ $i=1, 2, \dots, n$. Therefore

$$I = \sum_{i=1}^n b_i (\alpha_i I) \subset \sum_{i=1}^n b_i R \subset I.$$

Hence $I = \sum_{i=1}^n b_i R$.

5.3. Corollary. *A commutative integral domain R is hereditary if and only if it is a Dedekind domain.*

Proof. It is easy to make the following observations (a) An ideal I of R is invertible in the semi-group of fractionary ideals if and only if there exist b_1, b_2, \dots, b_n in I and $\alpha_1, \alpha_2, \dots, \alpha_n$, in Q , the field of quotients of R such that $\sum_{i=1}^n b_i \alpha_i = 1$ and $\alpha_i I \subset R$ for every $i=1, 2, \dots, n$.

(b) Every ideal of R is an essential ideal. These two observations along with the theorem 6.1 proves the corollary.

5.4. Theorem. *A semi-prime principal right ideal ring is a hereditary ring.*

Proof. A semi-prime principal right ideal ring is clearly a semi-prime Goldie ring. By a remark made in the previous theorem 5.1 it is sufficient to prove that every essential right ideal of R is a projective R -module. Let I be an essential right ideal of R , then $I = aR$ for some a in R . The element a is regular, see Processi and Small [19, page 81]. Therefore $R_R \cong (aR)_R$ under the mapping $r \mapsto ar$. Hence $I (=aR)$ is a projective R -module.

6. We return to the study of modules over a ring R , which has an Asano's quotient ring $Q (=R_S)$. If M_R is a projective R -module, then $M' (=MQ)$ is proved to be a projective Q -module. This result is used to prove that if M is an S -free R -module then projective dimension $(MQ)_Q \leq$ projective dimension M_R .

6.1. Category of S -free modules. Let \mathfrak{F} denote the category of all S -free R -modules (with R -homomorphisms as the maps).

Let \mathfrak{D} denote the category of Q -modules (with Q -homomorphisms as maps).

If $M \in \mathfrak{F}$, then there exists M' (unique upto isomorphism over M) in \mathfrak{D} such that $M_R \subset M'_R$ and $M' = MQ$. If $M_1, M_2 \in \mathfrak{F}$ and $f: M_1 \rightarrow M_2$, then there exists unique Q -homomorphism $f': M_1 Q \rightarrow M_2 Q$, which extends f . This map f' is defined as follows:

$$f'(m_1 s^{-1}) = f(m_1) s^{-1} \quad \forall m_1 \in M_1, s \in S.$$

In fact the rule T :

$$\begin{aligned} T(M) &= MQ \\ T(f) &= f' \end{aligned}$$

is an additive covariant functor from the category \mathfrak{F} into the category \mathfrak{G} . T is seen below to be an exact functor:

Let

$$0 \rightarrow M_1 \xrightarrow{j} M \xrightarrow{\pi} M_2 \rightarrow 0$$

be an exact sequence in \mathfrak{F} . Then the sequence

$$0 \rightarrow M_1 Q \xrightarrow{j'} MQ \xrightarrow{\pi'} M_2 Q \rightarrow 0$$

is also an exact sequence.

Exactness at $M_1 Q$: $j'(m_1 s^{-1}) = 0$ implies $j(m_1) s^{-1} = 0$,
 $j(m_1) = 0$, $m_1 = 0$, $m_1 s^{-1} = 0$.

Exactness at MQ : $\pi' j'(m_1 s^{-1}) = \pi'(j'(m_1 s^{-1})) = \pi'(j(m_1) s^{-1})$
 $= (\pi j(m_1)) s^{-1} = 0$.

If $\pi'(m s^{-1}) = 0$, then $\pi(m) = 0$, $m = j(m_1)$ for some m_1 in M_1 .
 $j'(m_1 s^{-1}) = j(m_1) s^{-1} = m s^{-1}$.

Exactness at $M_2 Q$: If $m_2 s^{-1}$ be any element of $M_2 Q$, then there exists $m \in M$ such that $\pi(m) = m_2$, $\pi'(m s^{-1}) = m_2 s^{-1}$.

6.2. We observe in passing that a projective R -module is torsion free.

6.3. Theorem. *Let R be a ring and $Q (=R_S)$ be an Asano's right quotient ring of R . If M is a projective R -module, then MQ is a projective Q -module.*

Proof. Let

$$\begin{array}{ccccc} & & (MQ)_Q & & \\ & & \downarrow f & & \\ A_Q & \longrightarrow & B_Q & \longrightarrow & 0 \end{array}$$

π

be a diagram of Q -modules with exact row. Let $f_1: M \rightarrow B$ denote the restriction of f to M . f_1 is R -homomorphism of M into B . We therefore have the diagram

$$\begin{array}{ccccc} & & M_R & & \\ & & \downarrow f_1 & & \\ A_R & \longrightarrow & B_R & \longrightarrow & 0 \end{array}$$

π

of R -modules with exact row. Since M_R is projective there exists $g: M \rightarrow A$ such that $\pi g = f_1$. Let $g': MQ \rightarrow AQ (=A)$ denote the extension of $g: M \rightarrow A$. Then

$$\pi g'(m s^{-1}) = \pi(g(m) s^{-1}) = \pi g(m) s^{-1} = f_1(m) s^{-1} = f(m \cdot s^{-1}).$$

6.4. Corollary. *If M is a Q -module such that M_R is projective, then M_Q is also projective.*

6.5. DEFINITION. Projective dimension of a module
Global dimension of a ring

Let R be an arbitrary ring and M be a module over R . An exact sequence

$$\dots \rightarrow M_n \xrightarrow{d_n} M_{n-1} \rightarrow \dots \rightarrow M_2 \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \xrightarrow{\dots} M \rightarrow 0$$

where each M_i is projective is called a projective resolution of M . The smallest positive integer n such that kernel d_n is projective is called the projective dimension of the module M . If no such integer n exists, then the projective dimension of M is infinity. (It is known that the projective dimension of M is independent of the projective resolution). Right Global dimension of a ring R = supremum of the projective dimensions of all R -modules (see sec. 2).

It is well known that for any ring R , supremum of the injective dimensions of all R -modules = supremum of all the projective dimensions of all R -modules. Therefore the same term Right Global dimension is used for both the supremums.

6.6. Theorem. *Let R be a ring and $Q(=R_S)$ be an Asano's right quotient ring of R . If M is an S -free R -module, then projective dimension $(MQ)_Q \leq$ projective dimension M_R .*

Proof. Let

$$\dots \rightarrow M_n \xrightarrow{d_n} M_{n-1} \rightarrow \dots \rightarrow M_2 \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \xrightarrow{\dots} M \rightarrow 0 \quad (i)$$

be a projective resolution of M_R . If projective dimension $M_R = \infty$, we have nothing to prove. Let the projective dimension $M = n (< \infty)$. Then K_n , the kernel of d_n is projective R -module.

From the projective resolution (i) of M_R we get a projective resolution of $(MQ)_Q$

$$\dots \rightarrow M_n Q \xrightarrow{d'_n} M_{n-1} Q \rightarrow \dots \rightarrow M_1 Q \xrightarrow{d'_1} M_0 Q \xrightarrow{\dots} MQ \rightarrow 0$$

(We note that the sequence is exact because we have noticed that the functor T is an exact functor. Also each $M_i Q$ is a projective Q -module by theorem 6.3).

The kernel of $d'_n = (\text{kernel } d_n)Q$ which is a projective Q -module by theorem 6.3. Hence

projective dimension $(MQ)_Q \leq$ projective dimension M_R .

6.7. Corollary. *If M is a Q module, then projective dimension $M_Q \leq$ projective dimension M_R .*

6.8. Corollary. *Right Global dimension $Q \leq$ right Global dimension R .*

7. Hereditary quotient rings

Let R be a ring and S be a multiplicatively closed set of regular elements of R such that R_S exists. Denote R_S by Q . In this section we obtain necessary and sufficient conditions on R such that Q becomes a hereditary ring.

7.1. Theorem. *A quotient ring Q of R is hereditary if and only if for every exact sequence of R modules*

$$M \xrightarrow{\pi} N \longrightarrow 0$$

where M is S -free and injective and N is S -free, N is injective.

Proof. Let Q be hereditary. Let

$$M \xrightarrow{\pi} N \longrightarrow 0$$

be an exact sequence of R -modules, where M is S -free and injective and N is S -free. Since every injective module is divisible (see 3.2), M is S -divisible and therefore M is Q -module. Also because M is S -divisible, therefore N is also S -divisible, therefore N is also a Q -module. The map π is Q -homomorphism (see 6.1, $\pi' = \pi$ in this case). Now since M_R is injective, therefore M_Q is injective. As Q is hereditary N_Q is injective and therefore N_R is injective.

Conversely we prove that Q is hereditary. It is sufficient to show that for every exact sequence of Q -modules

$$M \xrightarrow{\pi} N \longrightarrow 0$$

where M is injective, N is injective. Now regarding the above sequence as a sequence of R -modules we note that M_R is S -free, M_R is injective and N is S -free. Therefore N_R is injective. Hence N_Q is injective.

7.2. Corollary. *If R is hereditary, then a quotient ring $Q (=R_S)$ of R is also hereditary.*

Proof. Since R is hereditary, for any exact sequence of R -modules

$$M \longrightarrow N \longrightarrow 0$$

where M is injective, N is injective. Therefore by the theorem 7.1 Q is hereditary.

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